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Twining characters, Kostant's homology formula, and the Bernstein-Gelfand-Gelfand resolution

By

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Abstract

We give a new proof of the formulas for the twining character of the Verma module $M(\lambda)$ of symmetric highest weight λ and for the twining character of the irreducible highest weight module $L(\Lambda)$ of symmetric, dominant integral highest weight Λ over a symmetrizable generalized Kac-Moody algebra $\mathfrak{g}$, by using the Bernstein-Gelfand-Gelfand resolution of $L(\Lambda)$.

1. Introduction

In [FSS] and [FRS], they introduced a new type of character-like quantities, called twining characters, corresponding to a Dynkin diagram automorphism for certain highest weight modules over a symmetrizable (generalized) Kac-Moody algebra $\mathfrak{g}$. Moreover, they gave formulas (see Theorems 3.3 and 3.4) for the twining character of a Verma module $M(\lambda)$ of symmetric highest weight λ and for the twining character of an irreducible highest weight module $L(\Lambda)$ of symmetric, dominant integral highest weight Λ over $\mathfrak{g}$.

In the previous paper [N5], we obtained a formula of Kostant type for the twining characters of the Lie algebra homology modules $H_j(\mathfrak{n}_-, L(\Lambda))$, $j \geq 0$, of $\mathfrak{n}_-$ with coefficients in $L(\Lambda)$, where $\mathfrak{n}_-$ is the sum of all the negative root spaces of $\mathfrak{g}$, and then gave a new proof of the twining character formula for $L(\Lambda)$ as a corollary.

In this paper, we use an existence theorem in [N2] of a resolution of $L(\Lambda)$ of Bernstein-Gelfand-Gelfand type and an Euler-Poincaré principle to derive a formula expressing the twining character of $L(\Lambda)$ in terms of the twining characters of $M(\lambda)$'s. Then we immediately deduce the twining character formula for $L(\Lambda)$ and also that for $M(\lambda)$. Here we note that, unlike the case of an ordinary character, it is not at all easy to describe the twining character of the Verma module $M(\lambda)$ of symmetric highest weight λ. Thus our proof will cast

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new light on the connections among the twining character of $L(\Lambda)$, Kostant's homology formula, and the Bernstein-Gelfand-Gelfand resolution.

This paper is organized as follows. In Section 2 we recall the definition of a generalized Kac-Moody algebra and fix our notation. In Section 3, following [FSS] and [FRS], we review the definition of a twining character and the twining character formulas for $M(\lambda)$ and for $L(\Lambda)$. In Section 4 we recall briefly the twining character formula for $H_j(\mathfrak{n}_-, L(\Lambda))$, $j \geq 0$, which is the main result of [N5]. In Section 5 we give a (new) proof of the twining character formulas for $M(\lambda)$ and for $L(\Lambda)$, by using a resolution of $L(\Lambda)$ of Bernstein-Gelfand-Gelfand type.

2. Preliminaries and notation

2.1. Generalized Kac-Moody algebras.

Let $I = \{1, 2, \ldots, n\}$ be a finite index set, and let $A = (a_{ij})_{i,j\in I}$ be an $n \times n$ real matrix satisfying:

(C1) either $a_{ii} = 2$ or $a_{ii} \leq 0$ for all $i \in I$;
(C2) $a_{ij} \leq 0$ if $i \neq j \in I$, and $a_{ij} \in \mathbb{Z}$ for $j \neq i$ if $a_{ii} = 2$;
(C3) $a_{ij} = 0$ if and only if $a_{ji} = 0$ for $i, j \in I$.

Such a matrix $A = (a_{ij})_{i,j\in I}$ is called a GGCM. For a GGCM $A = (a_{ij})_{i,j\in I}$, there exists a triple $(\mathfrak{h}, \Pi = \{\alpha_i\}_{i\in I}, \Pi^\vee = \{h_i\}_{i\in I})$ satisfying:

(R1) $\mathfrak{h}$ is a finite-dimensional vector space over the complex numbers $\mathbb{C}$ such that $\dim_{\mathbb{C}} \mathfrak{h} = 2n - \operatorname{rank} A$;
(R2) $\Pi = \{\alpha_i\}_{i\in I}$ is a linearly independent subset of $\mathfrak{h}^* := \operatorname{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$, and $\Pi^\vee = \{h_i\}_{i\in I}$ is a linearly independent subset of $\mathfrak{h}$;
(R3) $\alpha_j(h_i) = a_{ij}$ for $i, j \in I$.

The generalized Kac-Moody algebra (GKM algebra) $\mathfrak{g} = \mathfrak{g}(A)$ associated to a GGCM $A = (a_{ij})_{i,j\in I}$ over $\mathbb{C}$ is the Lie algebra over $\mathbb{C}$ generated by the vector space $\mathfrak{h}$ above (called the Cartan subalgebra) and the elements e_i, f_i for $i \in I$ with the following defining relations:

(D1) $[h, h'] = 0$ for $h, h' \in \mathfrak{h}$;
(D2) $[h, e_i] = \alpha_i(h)e_i$, $[h, f_i] = -\alpha_i(h)f_i$ for $h \in \mathfrak{h}$ and $i \in I$;
(D3) $[e_i, f_j] = \delta_{ij}h_i$ for $i, j \in I$;
(D4) $(\operatorname{ad} e_i)^{1-a_{ij}}\, e_j = 0 = (\operatorname{ad} f_i)^{1-a_{ij}}\, f_j = 0$ if $a_{ii} = 2$ and $j \neq i$;
(D5) $[e_i, e_j] = 0 = [f_i, f_j]$ if $a_{ii}, a_{jj} \leq 0$ and $a_{ij} = 0 = a_{ji}$.

We have a root space decomposition of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$:

$$\mathfrak{g} = \left(\bigoplus_{\alpha\in\Delta_-} \mathfrak{g}_\alpha \right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha \right),$$

where $\Delta_+ \subset Q_+ := \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$ is the set of positive roots, $\Delta_- = -\Delta_+$ is the set of negative roots, and $\mathfrak{g}_\alpha$ is the root space of $\mathfrak{g}$ corresponding to a root

$\alpha \in \Delta = \Delta_- \sqcup \Delta_+$. We set

$$\mathfrak{n}_\pm := \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{g}_\alpha, \quad \mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}_+,$$

so that we have

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ = \mathfrak{n}_- \oplus \mathfrak{b}.$$

Note that $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i$, $\mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i$ for $i \in I$, so that $\Pi = \{\alpha_i\}_{i \in I} \subset \mathfrak{h}^*$ is the set of simple roots.

We set $I^{re} := \{i \in I \mid a_{ii} = 2\}$, $I^{im} := \{i \in I \mid a_{ii} \leq 0\}$, and call $\Pi^{re} := \{\alpha_i \in \Pi \mid i \in I^{re}\}$ the set of real simple roots, $\Pi^{im} := \{\alpha_i \in \Pi \mid i \in I^{im}\}$ the set of imaginary simple roots. For $i \in I^{re}$, let $r_i \in GL(\mathfrak{h}^*)$ be the simple reflection of $\mathfrak{h}^*$ given by:

$$r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i \quad \text{for} \quad \lambda \in \mathfrak{h}^*.$$

Then the Weyl group W of the GKM algebra $\mathfrak{g}$ is defined by

$$W := \langle r_i \mid i \in I^{re} \rangle \subset GL(\mathfrak{h}^*).$$

Note that W is a Coxeter group with the canonical generator system $\{r_i \mid i \in I^{re}\}$, whose length function is denoted by

$$\ell : W \to \mathbb{Z}.$$

We call $\Delta^{re} := W \cdot \Pi^{re}$ the set of real roots, and $\Delta^{im} := \Delta \setminus \Delta^{re}$ the set of imaginary roots. (Notice that $W \cdot \Pi^{im} \subset \Delta^{im}$.)

Throughout this paper, we assume that a GGCM $A = (a_{ij})_{i,j \in I}$ is symmetrizable, i.e., that there exist a diagonal matrix $D = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n)$ with $\varepsilon_i > 0$ for all $i \in I$ and a symmetric matrix $B = (b_{ij})_{i,j \in I}$ such that $A = DB$. Hence there exists a nondegenerate, symmetric, invariant bilinear form $(\cdot|\cdot)$ on $\mathfrak{g} = \mathfrak{g}(A)$. The restriction of this bilinear form $(\cdot|\cdot)$ to $\mathfrak{h}$ is again nondegenerate, so that it induces (through $\nu : \mathfrak{h} \to \mathfrak{h}^*$) a nondegenerate, symmetric, W-invariant bilinear form on $\mathfrak{h}^*$, which is also denoted by $(\cdot|\cdot)$.

2.2. Certain Lie algebra homology modules

For $\lambda \in \mathfrak{h}^*$, let

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}(\lambda)$$

be the Verma module of highest weight λ over $\mathfrak{g}$, where $U(\mathfrak{a})$ denotes the universal enveloping algebra of a Lie algebra $\mathfrak{a}$ and $\mathbb{C}(\lambda)$ is the one-dimensional (irreducible) $\mathfrak{h}$-module of weight λ on which $\mathfrak{n}_+$ acts trivially. We then define the $\mathfrak{g}$-module $L(\lambda)$ to be the unique irreducible quotient of $M(\lambda)$, that is,

$$L(\lambda) := M(\lambda)/J(\lambda),$$

where $J(\lambda)$ is the unique maximal proper submodule of $M(\lambda)$.

Let

$$P_+ := \{\Lambda \in \mathfrak{h}^* \mid \Lambda(h_i) \geq 0 \text{ for all } i \in I, \text{ and } \Lambda(h_i) \in \mathbb{Z} \text{ if } a_{ii} = 2\}$$

be the set of dominant integral weights. Now we recall the definition of the Lie algebra homology modules $H_j(\mathfrak{n}_-, L(\Lambda))$, $j \geq 0$, of $\mathfrak{n}_-$ with coefficients in $L(\Lambda)$ for $\Lambda \in P_+$. We denote by

$$\bigwedge^* \mathfrak{n}_- = \bigoplus_{j \geq 0} \bigwedge^j \mathfrak{n}_-$$

the exterior algebra of $\mathfrak{n}_-$, where $\bigwedge^j \mathfrak{n}_-$ is the homogeneous subspace of degree j. Notice that for each $j \geq 0$, the subspace $\bigwedge^j \mathfrak{n}_-$ is an $\mathfrak{h}$-module under the adjoint action since $[\mathfrak{h}, \mathfrak{n}_-] \subset \mathfrak{n}_-$. Let $\Lambda \in P_+$ and $j \in \mathbb{Z}_{\geq 0}$. We define the vector space $C_j(\mathfrak{n}_-, L(\Lambda))$ of j-chains by

$$C_j(\mathfrak{n}_-, L(\Lambda)) := \left(\bigwedge^j \mathfrak{n}_-\right) \otimes_{\mathbb{C}} L(\Lambda),$$

which is a tensor product of $\mathfrak{h}$-modules. Then the boundary operator d_j : $C_j(\mathfrak{n}_-, L(\Lambda)) \to C_{j-1}(\mathfrak{n}_-, L(\Lambda))$ is defined by

$$\begin{aligned} d_j(x_1 \wedge \cdots \wedge x_j \otimes v) &:= \sum_{i=1}^{j} (-1)^i (x_1 \wedge \cdots \wedge \check{x}_i \wedge \cdots \wedge x_j) \otimes x_i v \\ &+ \sum_{1 \leq r < t \leq j} (-1)^{r+t} ([x_r, x_t] \wedge x_1 \wedge \cdots \wedge \check{x}_r \wedge \cdots \wedge \check{x}_t \wedge \cdots \wedge x_j) \otimes v, \end{aligned}$$

where $x_1, \ldots, x_j \in \mathfrak{n}_-$, $v \in L(\Lambda)$, and the symbols $\check{x}_i$, $\check{x}_r$, $\check{x}_t$ indicate terms to be omitted. It is well-known that $\{C_j(\mathfrak{n}_-, L(\Lambda)), d_j\}_{j \geq 0}$ with $C_{-1}(\mathfrak{n}_-, L(\Lambda)) := \{0\}$ is a chain complex. The j-th homology of this chain complex is called the j-th Lie algebra homology of $\mathfrak{n}_-$ with coefficients in $L(\Lambda)$, denoted by $H_j(\mathfrak{n}_-, L(\Lambda))$. Note that for $j \geq 0$, the boundary operator $d_j : C_j(\mathfrak{n}_-, L(\Lambda)) \to C_{j-1}(\mathfrak{n}_-, L(\Lambda))$ commutes with the action of $\mathfrak{h}$, and hence $H_j(\mathfrak{n}_-, L(\Lambda))$ is an $\mathfrak{h}$-module in the usual way.

3. Twining character formula for $L(\Lambda)$

3.1. Twining characters.

We recall the definition of the twining character of a certain highest weight module, following [FRS] and [FSS] (see also [N4]).

Let $A = (a_{ij})_{i,j \in I}$ be a symmetrizable GGCM indexed by a finite set I. A bijection $\omega : I \to I$ such that

$$a_{\omega(i),\omega(j)} = a_{ij} \quad \text{for all} \quad i, j \in I$$

is called a (Dynkin) diagram automorphism, since such ω induces an automorphism of the Dynkin diagram of the GGCM $A = (a_{ij})_{i,j \in I}$ as a graph. Let N be the order of $\omega : I \to I$, and N_i the number of elements of the ω-orbit of $i \in I$ in I. We may (and will henceforth) assume that $\varepsilon_{\omega(i)} = \varepsilon_i$ for all $i \in I$ in the decomposition $A = DB$ with $D = \mathrm{diag}(\varepsilon_1, \dots, \varepsilon_n)$ (see [N4, Section 3.1]).

The diagram automorphism $\omega : I \to I$ can be extended (cf. [FSS, Section 3.2] and [K, Section 2.2]) to an automorphism of order N of the GKM algebra $\mathfrak{g} = \mathfrak{g}(A)$ associated to the GGCM $A = (a_{ij})_{i,j \in I}$ so that

$$\begin{cases} \omega(e_i) := e_{\omega(i)} & \text{for} \quad i \in I, \\ \omega(f_i) := f_{\omega(i)} & \text{for} \quad i \in I, \\ \omega(h_i) := h_{\omega(i)} & \text{for} \quad i \in I, \\ \omega(\mathfrak{h}) := \mathfrak{h}, & \\ (\omega(x)|\omega(y)) = (x|y) & \text{for} \quad x, y \in \mathfrak{g}. \end{cases}$$

Notice that this $\omega : \mathfrak{g} \to \mathfrak{g}$ extends to a unique algebra automorphism $\omega : U(\mathfrak{g}) \to U(\mathfrak{g})$ by

$$\omega(x_1 \cdots x_k) = \omega(x_1) \cdots \omega(x_k) \quad \text{for} \quad x_1, \dots, x_k \in \mathfrak{g}.$$

We call these two automorphisms ω also diagram automorphisms by abuse of notation.

The restriction of the diagram automorphism $\omega : \mathfrak{g} \to \mathfrak{g}$ to the Cartan subalgebra $\mathfrak{h}$ induces a dual map $\omega^* : \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$\omega^*(\lambda)(h) := \lambda(\omega(h)) \quad \text{for} \quad \lambda \in \mathfrak{h}^*, \ h \in \mathfrak{h}.$$

We set

$$(\mathfrak{h}^*)^0 := \{\lambda \in \mathfrak{h}^* \mid \omega^*(\lambda) = \lambda\},$$

and call an element of $(\mathfrak{h}^*)^0$ a symmetric weight. Note that we may (and will henceforth) take an element $\rho \in (\mathfrak{h}^*)^0$ (called a symmetric Weyl vector) such that

$$\rho(h_i) = (1/2) \cdot a_{ii} \quad \text{for all} \quad i \in I.$$

Let $\lambda \in (\mathfrak{h}^*)^0$ be a symmetric weight, and let $V(\lambda)$ be either the Verma module $M(\lambda)$ or the irreducible highest weight module $L(\lambda)$ of highest weight λ. Then there exists a unique linear automorphism $\tau_\omega : V(\lambda) \to V(\lambda)$ such that

$$\tau_\omega(xv) = \omega^{-1}(x)\tau_\omega(v) \quad \text{for} \quad x \in \mathfrak{g}, \ v \in V(\lambda),$$

and

$$\tau_\omega(v) = v \quad \text{for} \quad v \in V(\lambda)_\lambda,$$

where $V(\lambda)_\lambda$ is the (one-dimensional) highest weight space of $V(\lambda)$.

Remark 3.1. Because $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}(\lambda)$ by definition, we can take the linear automorphism $\omega^{-1} \otimes \mathrm{id} : U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}(\lambda) \to U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}(\lambda)$ for $\tau_\omega : M(\lambda) \to M(\lambda)$ above. Moreover, since this map $\omega^{-1} \otimes \mathrm{id} : M(\lambda) \to M(\lambda)$ stabilizes the unique maximal proper submodule $J(\lambda)$ of $M(\lambda)$, we can take for $\tau_\omega : L(\lambda) \to L(\lambda)$ above the linear map $M(\lambda)/J(\lambda) \to M(\lambda)/J(\lambda)$ induced from $\omega^{-1} \otimes \mathrm{id} : M(\lambda) \to M(\lambda)$.

Remark 3.2. Let V be an $\mathfrak{h}$-module admitting a weight space decomposition

$$V = \bigoplus_{\chi \in \mathfrak{h}^*} V_\chi$$

with finite-dimensional weight spaces V_χ, and let $f : V \to V$ be a linear map such that $f(hv) = \omega^{-1}(h) f(v)$ for $h \in \mathfrak{h}$, $v \in V$. Then it follows that

$$f(V_\chi) \subset V_{\omega^*(\chi)}$$

for all $\chi \in \mathfrak{h}^*$. Thus we define a formal sum:

$$\mathrm{Tr}_V\, f \exp := \sum_{\chi \in (\mathfrak{h}^*)^0} \mathrm{Tr}(f|_{V_\chi})\, e(\chi),$$

where V_χ is the χ-weight space of V for a symmetric weight $\chi \in (\mathfrak{h}^*)^0$.

Let $\lambda \in (\mathfrak{h}^*)^0$. The twining character $\mathrm{ch}^\omega(V(\lambda))$ of $V(\lambda)$ $(= M(\lambda), L(\lambda))$ is defined to be the formal sum

$$\mathrm{ch}^\omega(V(\lambda)) := \mathrm{Tr}_{V(\lambda)}\, \tau_\omega \exp = \sum_{\chi \in (\mathfrak{h}^*)^0} \mathrm{Tr}(\tau_\omega|_{V(\lambda)_\chi})\, e(\chi).$$

3.2. Twining character formulas for $M(\lambda)$ and for $L(\Lambda)$.

We review the twining character formulas for $M(\lambda)$ of symmetric highest weight λ and for $L(\Lambda)$ of symmetric, dominant integral highest weight Λ, which are the main results of [FSS] and [FRS].

We choose a set of representatives $\widehat{I}$ of the ω-orbits in I, and then introduce the following subset of $\widehat{I}$:

$$\breve{I} := \left\{ i \in \widehat{I} \;\middle|\; \sum_{k=0}^{N_i - 1} a_{i,\omega^k(i)} = 1,\ 2 \right\}.$$

We define the following subgroup of the Weyl group W:

$$\widetilde{W} := \{ w \in W \mid \omega^* w = w \omega^* \}.$$

We know from [FRS, Proposition 3.3] that the group $\widetilde{W}$ is a Coxeter group with the canonical generator system $\{w_i \mid i \in \check{I}\}$, where for $i \in \check{I}$,

$$w_i := \begin{cases} \prod_{k=0}^{N_i/2-1} \left(r_{\omega^k(i)}\, r_{\omega^{k+N_i/2}(i)}\, r_{\omega^k(i)}\right) & \text{if } \sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 1, \\ \prod_{k=0}^{N_i-1} r_{\omega^k(i)} & \text{if } \sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 2. \end{cases}$$

Here we note that if $\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 1$, then N_i is an even integer. We denote the length function of $\widetilde{W}$ by

$$\widehat{\ell} : \widetilde{W} \to \mathbb{Z}.$$

We also recall from [FRS, Equation (1) on p. 529] that for a symmetric weight $\lambda \in (\mathfrak{h}^*)^0$ and $i \in \check{I}$,

$$w_i(\lambda) = \lambda - \frac{2s_i(\lambda|\alpha_i)}{(\alpha_i|\alpha_i)} \sum_{k=0}^{N_i-1} \alpha_{\omega^k(i)},$$

where $s_i := 2/\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)}$.

Let $\Lambda \in P_+ \cap (\mathfrak{h}^*)^0$ be a symmetric, dominant integral weight. We denote by $\mathcal{S}(\Lambda)$ the set of sums of distinct, pairwise perpendicular, imaginary simple roots perpendicular to Λ. Then any element $\beta \in \mathcal{S}(\Lambda) \cap (\mathfrak{h}^*)^0$ can be written in the form $\beta = \sum_{i \in \widehat{I}} k_i \beta_i$, where $\beta_i := \sum_{k=0}^{N_i-1} \alpha_{\omega^k(i)} \in (\mathfrak{h}^*)^0$ and $k_i = 0, 1$ for $i \in \widehat{I}$. For such $\beta \in \mathcal{S}(\Lambda) \cap (\mathfrak{h}^*)^0$, we set

$$\widehat{\mathrm{ht}}(\beta) := \sum_{i \in \widehat{I}} k_i,$$

while we write $\mathrm{ht}(\alpha) := \sum_{i \in I} m_i$ for $\alpha = \sum_{i \in I} m_i \alpha_i \in Q_+$. Set for $(w, \beta) \in W \times \mathcal{S}(\Lambda)$,

$$(w, \beta) \circ \Lambda := w(\Lambda + \rho - \beta) - \rho,$$

where ρ is a (fixed) symmetric Weyl vector.

We have the following twining character formulas.

Theorem 3.3 ([FRS, Theorem 3.1]). *Let $\lambda \in (\mathfrak{h}^*)^0$ be a symmetric weight. Then*

$$\mathrm{ch}^\omega(M(\lambda)) = e(\lambda) \cdot \left(\sum_{\substack{w \in \widetilde{W} \\ \beta \in \mathcal{S}(0) \cap (\mathfrak{h}^*)^0}} (-1)^{\widehat{\ell}(w) + \widehat{\mathrm{ht}}(\beta)}\, e((w, \beta) \circ 0) \right)^{-1}.$$

Theorem 3.4 ([FRS, Theorem 3.1]). *Let $\Lambda \in P_+ \cap (\mathfrak{h}^*)^0$ be a symmetric, dominant integral weight. Then*

$$\mathrm{ch}^{\omega}(L(\Lambda)) = \frac{\displaystyle\sum_{\substack{w \in \widetilde{W} \\ \beta \in \mathcal{S}(\Lambda) \cap (\mathfrak{h}^*)^0}} (-1)^{\widehat{\ell}(w) + \widehat{\mathrm{ht}}(\beta)}\, e((w,\beta) \circ \Lambda)}{\displaystyle\sum_{\substack{w \in \widetilde{W} \\ \beta \in \mathcal{S}(0) \cap (\mathfrak{h}^*)^0}} (-1)^{\widehat{\ell}(w) + \widehat{\mathrm{ht}}(\beta)}\, e((w,\beta) \circ 0)}.$$

4. Twining character formula for $H_j(\mathfrak{n}_-, L(\Lambda))$

4.1. Setting.

Since the inverse $\omega^{-1} : \mathfrak{g} \to \mathfrak{g}$ of the diagram automorphism $\omega : \mathfrak{g} \to \mathfrak{g}$ stabilizes $\mathfrak{n}_-$, i.e., $\omega^{-1}(\mathfrak{n}_-) = \mathfrak{n}_-$, it induces an algebra automorphism

$$\bigwedge^* \omega^{-1} : \bigwedge^* \mathfrak{n}_- \to \bigwedge^* \mathfrak{n}_-$$

of the exterior algebra $\bigwedge^* \mathfrak{n}_-$ of $\mathfrak{n}_-$. The restriction of the $\bigwedge^* \omega^{-1} : \bigwedge^* \mathfrak{n}_- \to \bigwedge^* \mathfrak{n}_-$ to each homogeneous subspace $\bigwedge^j \mathfrak{n}_-$ for $j \geq 0$ is denoted by

$$\bigwedge^j \omega^{-1} : \bigwedge^j \mathfrak{n}_- \to \bigwedge^j \mathfrak{n}_-.$$

Let $\Lambda \in P_+ \cap (\mathfrak{h}^*)^0$ be a symmetric, dominant integral weight, and let $\tau_\omega : L(\Lambda) \to L(\Lambda)$ be the linear automorphism in Section 3.1. We define a linear automorphism

$$\Phi := \left(\bigwedge^* \omega^{-1}\right) \otimes \tau_\omega : \left(\bigwedge^* \mathfrak{n}_-\right) \otimes_{\mathbb{C}} L(\Lambda) \to \left(\bigwedge^* \mathfrak{n}_-\right) \otimes_{\mathbb{C}} L(\Lambda),$$

and for $j \geq 0$, we define a linear automorphism

$$\Phi_j := \left(\bigwedge^j \omega^{-1}\right) \otimes \tau_\omega : \left(\bigwedge^j \mathfrak{n}_-\right) \otimes_{\mathbb{C}} L(\Lambda) \to \left(\bigwedge^j \mathfrak{n}_-\right) \otimes_{\mathbb{C}} L(\Lambda).$$

Let $j \geq 0$. It is easily seen that

$$\Phi_j(hv) = \omega^{-1}(h)\Phi_j(v) \tag{4.1}$$

for $h \in \mathfrak{h}$ and $v \in (\bigwedge^j \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\Lambda)$. It also follows that for $h \in \mathfrak{h}$ and $v \in (\bigwedge^* \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\Lambda)$,

$$\Phi(hv) = \omega^{-1}(h)\Phi(v).$$

Moreover, we have the following commutative diagram for each $j \geq 0$:

$$\begin{array}{ccc} (\bigwedge^j \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\Lambda) & \xrightarrow{\Phi_j} & (\bigwedge^j \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\Lambda) \\ {\scriptstyle d_j}\downarrow & & \downarrow{\scriptstyle d_j} \\ (\bigwedge^{j-1} \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\Lambda) & \xrightarrow[\Phi_{j-1}]{} & (\bigwedge^{j-1} \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\Lambda), \end{array}$$

where $d_j : (\bigwedge^j \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\Lambda) \to (\bigwedge^{j-1} \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\Lambda)$ is the boundary operator in Section 2.2. Hence the linear automorphism $\Phi_j : (\bigwedge^j \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\Lambda) \to (\bigwedge^j \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\Lambda)$ induces in the usual way a linear automorphism

$$\overline{\Phi}_j : H_j(\mathfrak{n}_-, L(\Lambda)) \to H_j(\mathfrak{n}_-, L(\Lambda))$$

for $j \geq 0$. Notice that for $j \geq 0$ and $h \in \mathfrak{h}$, $v \in H_j(\mathfrak{n}_-, L(\Lambda))$,

$$\overline{\Phi}_j(hv) = \omega^{-1}(h)\overline{\Phi}_j(v)$$

by (4.1).

4.2. Main result of [N5].

We define the twining character $\mathrm{ch}^{\omega}(H_j(\mathfrak{n}_-, L(\Lambda)))$ of the Lie algebra homology module $H_j(\mathfrak{n}_-, L(\Lambda))$ for each $j \geq 0$ by

$$\mathrm{ch}^{\omega}(H_j(\mathfrak{n}_-, L(\Lambda))) := \mathrm{Tr}_{H_j(\mathfrak{n}_-, L(\Lambda))} \overline{\Phi}_j \exp,$$

where $\overline{\Phi}_j : H_j(\mathfrak{n}_-, L(\Lambda)) \to H_j(\mathfrak{n}_-, L(\Lambda))$ is as in Section 4.1.

The following is a summary of the main result of [N5].

Theorem 4.1 (see [N5, Section 3.2]). *Let $\Lambda \in P_+ \cap (\mathfrak{h}^*)^0$ be a symmetric, dominant integral weight, and let $j \geq 0$. Then*

$$\mathrm{ch}^{\omega}(H_j(\mathfrak{n}_-, L(\Lambda))) = \sum_{\substack{w \in \widetilde{W} \\ \beta \in \mathcal{S}(\Lambda) \cap (\mathfrak{h}^*)^0 \\ \ell(w) + \mathrm{ht}(\beta) = j}} c_{(w,\beta)}\, e((w,\beta) \circ \Lambda),$$

where the scalar $c_{(w,\beta)} \in \mathbb{C}$ is defined by

$$c_{(w,\beta)} := \mathrm{Tr}(\overline{\Phi}_j|_{(H_j(\mathfrak{n}_-, L(\Lambda)))_{(w,\beta)\circ\Lambda}}).$$

Moreover, we have

$$\begin{aligned} c_{(w,\beta)} &= \mathrm{Tr}(\overline{\Phi}_j|_{(H_j(\mathfrak{n}_-, L(\Lambda)))_{(w,\beta)\circ\Lambda}}) \\ &= \mathrm{Tr}(\Phi_j|_{((\bigwedge^j \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\Lambda))_{(w,\beta)\circ\Lambda}}) \\ &= (-1)^{(\ell(w) + \mathrm{ht}(\beta)) - (\widehat{\ell}(w) + \widehat{\mathrm{ht}}(\beta))}. \end{aligned}$$

Remark 4.2. Here we recall from the proof of [N2, Proposition 3.3] the construction of a nonzero weight vector $v_{(w,\beta)} \in (\bigwedge^j \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\Lambda)$ of weight $\mu = (w,\beta) \circ \Lambda$. First we note that $w(\rho) - \rho = -\sum_{\alpha \in \Delta_w} \alpha$ and that the number of elements of the set Δ_w equals $\ell(w)$, where $\Delta_w := \{\alpha \in \Delta_+ \mid w^{-1}(\alpha) \in \Delta_-\}$. Second we write β in the form $\beta = \sum_{k=1}^{m} \alpha_{i_k}$, where $m = \mathrm{ht}(\beta)$, $\alpha_{i_k} \in \Pi^{im}$, and $i_r \neq i_t$ for $1 \leq r \neq t \leq m$. Now we take nonzero root vectors $F_k \in \mathfrak{g}_{-w(\alpha_{i_k})}$

for $1 \leq k \leq m$, $F_\alpha \in \mathfrak{g}_{-\alpha}$ for $\alpha \in \Delta_w$, and a nonzero weight vector $v_{w(\Lambda)} \in L(\Lambda)_{w(\Lambda)}$ of weight $w(\Lambda)$. Then we set

$$v_{(w,\beta)} := (F_1 \wedge \cdots \wedge F_m) \wedge \left(\bigwedge_{\alpha \in \Delta_w} F_\alpha \right) \otimes v_{w(\Lambda)} \in \left(\bigwedge\nolimits^j \mathfrak{n}_- \right) \otimes_{\mathbb{C}} L(\Lambda).$$

We know that the vector $v_{(w,\beta)} \in (\bigwedge^j \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\Lambda)$ is nonzero and of weight $\mu = (w, \beta) \circ \Lambda$. Moreover, we know that the image $\bar{v}_{(w,\beta)}$ of the vector $v_{(w,\beta)} \in (\bigwedge^j \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\Lambda)$ of weight μ by the natural quotient map $^- : (\bigwedge^j \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\Lambda) \to H_j(\mathfrak{n}_-, L(\Lambda))$ is nonzero, and hence that the μ-weight space $(H_j(\mathfrak{n}_-, L(\Lambda)))_\mu$ of $H_j(\mathfrak{n}_-, L(\Lambda))$ is spanned by the vector $\bar{v}_{(w,\beta)}$, i.e.,

$$(H_j(\mathfrak{n}_-, L(\Lambda)))_\mu = \mathbb{C}\, \bar{v}_{(w,\beta)}.$$

5. New proof of the twining character formulas

5.1. Construction of a resolution.

In order to give a new proof of the twining character formulas for $M(\lambda)$ and for $L(\Lambda)$, we recall from [N2] an existence theorem of a resolution of $L(\Lambda)$ of Bernstein-Gelfand-Gelfand type.

Theorem 5.1 ([N2, Theorem 3.4]). *Let $\Lambda \in P_+ \cap (\mathfrak{h}^*)^0$ be a symmetric, dominant integral weight. Then there exists an exact sequence of $\mathfrak{g}$-modules and $\mathfrak{g}$-module maps:*

$$0 \longleftarrow L(\Lambda) \xleftarrow{\partial_0} C_0(\Lambda) \xleftarrow{\partial_1} C_1(\Lambda) \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_p} C_p(\Lambda) \xleftarrow{\partial_{p+1}} \cdots,$$

where for each $p \geq 0$, the $\mathfrak{g}$-module $C_p(\Lambda)$ has an increasing $\mathfrak{g}$-module filtration of finite length

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{k_p} = C_p(\Lambda)$$

such that the quotient module V_i/V_{i-1} is isomorphic to a Verma module $M(\lambda_i)$ of highest weight λ_i for $1 \leq i \leq k_p$. Moreover, for each $p \geq 0$, the set of highest weights $\{\lambda_i \mid 1 \leq i \leq k_p\}$ is equal to the set

$$\{(w, \beta) \circ \Lambda \mid w \in W,\ \beta \in \mathcal{S}(\Lambda) \ \textit{ with } \ \ell(w) + \mathrm{ht}(\beta) = p\},$$

and $\lambda_i \neq \lambda_j$ if $1 \leq i \neq j \leq k_p$.

By investigating the construction of this resolution, following [N2] and [GL], we will give a new proof of Theorems 3.3 and 3.4. First we have the following exact sequence of $\mathfrak{g}$-modules and $\mathfrak{g}$-module maps:

$$0 \longleftarrow L(\Lambda) \xleftarrow{b_0} B_0(\Lambda) \xleftarrow{b_1} B_1(\Lambda) \xleftarrow{b_2} \cdots \xleftarrow{b_p} B_p(\Lambda) \xleftarrow{b_{p+1}} \cdots,$$

where for $p \geq 0$, the $\mathfrak{g}$-module $B_p(\Lambda)$ is defined by

$$B_p(\Lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \left(\left(\bigwedge^p (\mathfrak{g}/\mathfrak{b}) \right) \otimes_{\mathbb{C}} L(\Lambda) \right).$$

Furthermore, we have the following commutative diagram of $\mathfrak{g}$-modules and $\mathfrak{g}$-module maps for $p \geq 0$:

(5.1)

$$\begin{array}{ccc} (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \bigwedge^p (\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda) & \xrightarrow{\simeq} & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} ((\bigwedge^p (\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda)) \\ {\scriptstyle d_p \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle b_p} \\ (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \bigwedge^{p-1} (\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda) & \xrightarrow[\simeq]{} & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} ((\bigwedge^{p-1} (\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda)). \end{array}$$

Here the $\mathfrak{g}$-module map $d_p : U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\bigwedge^p (\mathfrak{g}/\mathfrak{b})) \to U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\bigwedge^{p-1} (\mathfrak{g}/\mathfrak{b}))$ is (well-) defined by

$$\begin{aligned} d_p(x \otimes \bar{y}_1 \wedge \cdots \wedge \bar{y}_p) &:= \sum_{i=1}^{p} (-1)^{i+1} (x y_i) \otimes \bar{y}_1 \wedge \cdots \wedge \check{\bar{y}}_i \wedge \cdots \wedge \bar{y}_p \\ &+ \sum_{1 \leq r < t \leq p} (-1)^{r+t} x \otimes \overline{[y_r, y_t]} \wedge \bar{y}_1 \wedge \cdots \wedge \check{\bar{y}}_r \wedge \cdots \wedge \check{\bar{y}}_t \wedge \cdots \wedge \bar{y}_p, \end{aligned}$$

where $x \in U(\mathfrak{g})$, $y_1, \ldots, y_p \in \mathfrak{g}$, and $^- : \mathfrak{g} \to \mathfrak{g}/\mathfrak{b}$ is the natural quotient map. Note that for $p = 0$, the map $d_0 : U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C} \to \mathbb{C}$ is defined by the condition that $d_0(x \otimes 1)$ is the constant term of $x \in U(\mathfrak{g})$.

Let $p \geq 0$. We define a linear automorphism Ψ_p of $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} ((\bigwedge^p (\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda))$ by

$$\Psi_p := \omega^{-1} \otimes \left(\left(\bigwedge^p \overline{\omega}^{-1} \right) \otimes \tau_\omega \right),$$

and a linear automorphism Ψ'_p of $(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\bigwedge^p (\mathfrak{g}/\mathfrak{b}))) \otimes_{\mathbb{C}} L(\Lambda)$ by

$$\Psi'_p := \left(\omega^{-1} \otimes \left(\bigwedge^p \overline{\omega}^{-1} \right) \right) \otimes \tau_\omega,$$

where $\tau_\omega : L(\Lambda) \to L(\Lambda)$ is the linear automorphism in Section 3.1, $\omega : U(\mathfrak{g}) \to U(\mathfrak{g})$ is the unique algebra automorphism of $U(\mathfrak{g})$ extending the diagram automorphism $\omega : \mathfrak{g} \to \mathfrak{g}$, and $\overline{\omega} : \mathfrak{g}/\mathfrak{b} \to \mathfrak{g}/\mathfrak{b}$ is the linear automorphism induced from $\omega : \mathfrak{g} \to \mathfrak{g}$.

Remark 5.2. Let $p \geq 0$. Then

$$\Psi_p(xv) = \omega^{-1}(x) \Psi_p(v) \quad \text{for } x \in \mathfrak{g},\ v \in U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \left(\left(\bigwedge^p (\mathfrak{g}/\mathfrak{b}) \right) \otimes_{\mathbb{C}} L(\Lambda) \right),$$

$$\Psi'_p(xv) = \omega^{-1}(x) \Psi'_p(v) \quad \text{for } x \in \mathfrak{g},\ v \in \left(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \left(\bigwedge^p (\mathfrak{g}/\mathfrak{b}) \right) \right) \otimes_{\mathbb{C}} L(\Lambda).$$

Lemma 5.3. *Let $p \geq 0$. Then the following diagram is commutative.*

$$\begin{array}{ccc} U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} ((\bigwedge^p(\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda)) & \xrightarrow{\Psi_p} & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} ((\bigwedge^p(\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda)) \\ {\scriptstyle b_p}\downarrow & & \downarrow{\scriptstyle b_p} \\ U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} ((\bigwedge^{p-1}(\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda)) & \xrightarrow[\Psi_{p-1}]{} & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} ((\bigwedge^{p-1}(\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda)). \end{array}$$

Proof. It immediately follows from the definitions of d_p and Ψ'_p for $p \geq 0$ that the following diagram commutes:

(5.2)

$$\begin{array}{ccc} (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \bigwedge^p(\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda) & \xrightarrow{\Psi'_p} & (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \bigwedge^p(\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda) \\ {\scriptstyle d_p \otimes \mathrm{id}}\downarrow & & \downarrow{\scriptstyle d_p \otimes \mathrm{id}} \\ (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \bigwedge^{p-1}(\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda) & \xrightarrow[\Psi'_{p-1}]{} & (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \bigwedge^{p-1}(\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda). \end{array}$$

In addition, by using the explicit form of the isomorphism

$$\left(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \bigwedge^p(\mathfrak{g}/\mathfrak{b})\right) \otimes_{\mathbb{C}} L(\Lambda) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \left(\left(\bigwedge^p(\mathfrak{g}/\mathfrak{b})\right) \otimes_{\mathbb{C}} L(\Lambda)\right)$$

described in the proof of [GL, Proposition 1.7], we can easily check that the following diagram is commutative:

(5.3)

$$\begin{array}{ccc} (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \bigwedge^p(\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda) & \xrightarrow{\simeq} & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} ((\bigwedge^p(\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda)) \\ {\scriptstyle \Psi'_p}\downarrow & & \downarrow{\scriptstyle \Psi_p} \\ (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \bigwedge^p(\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda) & \xrightarrow[\simeq]{} & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} ((\bigwedge^p(\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda)). \end{array}$$

The lemma now follows from the commutativity of these diagrams (5.2) and (5.3) together with the diagram (5.1). □

To explain the definition of $C_p(\Lambda)$ for $p \geq 0$, we need some more notation. The (generalized) Casimir operator Ω in [K, Chapter 2] is defined by

$$\Omega = 2\nu^{-1}(\rho) + \sum_{i=1}^{\dim_{\mathbb{C}} \mathfrak{h}} u^i u_i + 2 \sum_{\alpha \in \Delta_+} \sum_{i=1}^{\dim_{\mathbb{C}} \mathfrak{g}_\alpha} e_{-\alpha}^{(i)} e_{\alpha}^{(i)},$$

where $\{u^i\}_{i=1}^{\dim_{\mathbb{C}} \mathfrak{h}}$ and $\{u_i\}_{i=1}^{\dim_{\mathbb{C}} \mathfrak{h}}$ are dual bases of $\mathfrak{h}$ with respect to the bilinear form $(\cdot|\cdot)$, and for each $\alpha \in \Delta_+$, $\{e_{-\alpha}^{(i)}\}_{i=1}^{\dim_{\mathbb{C}} \mathfrak{g}_\alpha}$ and $\{e_{\alpha}^{(i)}\}_{i=1}^{\dim_{\mathbb{C}} \mathfrak{g}_\alpha}$ are bases of $\mathfrak{g}_{-\alpha}$

and $\mathfrak{g}_\alpha$ that are dual to each other with respect to $(\cdot|\cdot)$. Let V be a $\mathfrak{g}$-module admitting a weight space decomposition

$$V = \bigoplus_{\chi \in \mathfrak{h}^*} V_\chi$$

such that $\dim_{\mathbb{C}} V_\chi < \infty$ for all $\chi \in \mathfrak{h}^*$ and such that all weights of V lie in a set $\lambda - Q_+$ for some $\lambda \in \mathfrak{h}^*$. Then we know from [GL, Section 4] that the module V decomposes into a direct sum of $\mathfrak{g}$-modules

$$V = \bigoplus_{c \in \Theta(V)} V_{(c)},$$

where

$$\Theta(V) := \{c \in \mathbb{C} \mid \Omega(v) = cv \quad \text{for some } 0 \neq v \in V\}$$

and for $c \in \Theta(V)$,

$$V_{(c)} := \{v \in V \mid (\Omega - c)^n(v) = 0 \quad \text{for some } n \in \mathbb{Z}_{\geq 0}\}.$$

Lemma 5.4. *Let V be a $\mathfrak{g}$-module above. We further assume that there exists a linear automorphism $f : V \to V$ such that*

$$f(xv) = \omega^{-1}(x) f(v) \quad \textit{for } x \in \mathfrak{g},\ v \in V.$$

Then, as operators on V,

$$f \circ \Omega = \Omega \circ f.$$

Proof. Let $v \in V$. Then we have

$$\begin{aligned} f(\Omega(v)) &= 2f(\nu^{-1}(\rho)v) + \sum_{i=1}^{\dim_{\mathbb{C}} \mathfrak{h}} f(u^i u_i v) + 2 \sum_{\alpha \in \Delta_+} \sum_{i=1}^{\dim_{\mathbb{C}} \mathfrak{g}_\alpha} f(e_{-\alpha}^{(i)} e_\alpha^{(i)} v) \\ &= 2\omega^{-1}(\nu^{-1}(\rho)) f(v) + \sum_{i=1}^{\dim_{\mathbb{C}} \mathfrak{h}} \omega^{-1}(u^i)\omega^{-1}(u_i) f(v) \\ &\quad + 2 \sum_{\alpha \in \Delta_+} \sum_{i=1}^{\dim_{\mathbb{C}} \mathfrak{g}_\alpha} \omega^{-1}(e_{-\alpha}^{(i)})\omega^{-1}(e_\alpha^{(i)}) f(v). \end{aligned}$$

Recall that $(\omega(x)|\omega(y)) = (x|y)$ for $x, y \in \mathfrak{g}$, and $\omega(\mathfrak{h}) = \mathfrak{h}$. So, $\{\omega^{-1}(u^i)\}_{i=1}^{\dim_{\mathbb{C}} \mathfrak{h}}$ and $\{\omega^{-1}(u_i)\}_{i=1}^{\dim_{\mathbb{C}} \mathfrak{h}}$ are dual bases of $\mathfrak{h}$ with respect to $(\cdot|\cdot)$, and for $\alpha \in \Delta_+$, $\{\omega^{-1}(e_{-\alpha}^{(i)})\}_{i=1}^{\dim_{\mathbb{C}} \mathfrak{g}_\alpha}$ and $\{\omega^{-1}(e_\alpha^{(i)})\}_{i=1}^{\dim_{\mathbb{C}} \mathfrak{g}_\alpha}$ are bases of $\mathfrak{g}_{-\omega^*(\alpha)}$ and $\mathfrak{g}_{\omega^*(\alpha)}$ that are dual to each other with respect to $(\cdot|\cdot)$ since $\omega^{-1}(\mathfrak{g}_\alpha) = \mathfrak{g}_{\omega^*(\alpha)}$. In addition, $\omega^{-1}(\nu^{-1}(\rho)) = \nu^{-1}(\omega^*(\rho)) = \nu^{-1}(\rho)$. Because the Casimir operator Ω is independent of the choice of dual bases, we conclude that

$$f(\Omega(v)) = \Omega(f(v)).$$

This proves the lemma. □

We set for $p \geq 0$,

$$C_p(\Lambda) := (B_p(\Lambda))_{c_0},$$

where $c_0 := (\Lambda + \rho|\Lambda + \rho) - (\rho|\rho)$. It follows from Lemma 5.4 that for $p \geq 0$,

$$\Psi_p \circ \Omega = \Omega \circ \Psi_p$$

as operators on $B_p(\Lambda)$. Hence the linear automorphism $\Psi_p : B_p(\Lambda) \to B_p(\Lambda)$ stabilizes the $\mathfrak{g}$-submodule $C_p(\Lambda)$ of $B_p(\Lambda)$ for $p \geq 0$, that is,

$$\Psi_p(C_p(\Lambda)) = C_p(\Lambda).$$

Thus we obtain the exact sequence of Theorem 5.1. Note that the map $\partial_p : C_p(\Lambda) \to C_{p-1}(\Lambda)$ is the restriction of the map $b_p : B_p(\Lambda) \to B_{p-1}(\Lambda)$ for $p \geq 0$. In particular, the following diagram commutes for $p \geq 0$:

$$\begin{array}{ccc} C_p(\Lambda) & \xrightarrow{\Psi_p} & C_p(\Lambda) \\ {\scriptstyle \partial_p}\downarrow & & \downarrow{\scriptstyle \partial_p} \\ C_{p-1}(\Lambda) & \xrightarrow[\Psi_{p-1}]{} & C_{p-1}(\Lambda). \end{array}$$

Therefore we can apply an Euler-Poincaré principle to the exact sequence of Theorem 5.1 to obtain that

$$\mathrm{ch}^\omega(L(\Lambda)) = \sum_{p \geq 0} (-1)^p \, \mathrm{ch}^\omega(C_p(\Lambda)),$$

where $\mathrm{ch}^\omega(C_p(\Lambda))$ for $p \geq 0$ is defined by

$$\mathrm{ch}^\omega(C_p(\Lambda)) := \mathrm{Tr}_{C_p(\Lambda)} \Psi_p \exp.$$

5.2. New proof.

Now we compute the twining characters $\mathrm{ch}^\omega(C_p(\Lambda))$, $p \geq 0$. For this purpose, we have to modify the original construction of the $\mathfrak{g}$-module filtration of $C_p(\Lambda)$ for $p \geq 0$. By carefully reading the proof of [GL, Propositions 5.5 and 6.4], we see that for each $p \geq 0$, there exists a $\mathfrak{b}$-module filtration of $(\bigwedge^p(\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda)$

$$0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset \left(\bigwedge^p(\mathfrak{g}/\mathfrak{b})\right) \otimes_{\mathbb{C}} L(\Lambda)$$

such that:

- $((\bigwedge^p \overline{\omega}^{-1}) \otimes \tau_\omega)(N_i) \subset N_i$ for $i \geq 0$;
- $\mathfrak{n}_+ \cdot N_i \subset N_{i-1}$ for $i \geq 1$;
- $\dim_{\mathbb{C}} (N_i/N_{i-1}) < \infty$ for $i \geq 1$;

- $(\bigwedge^p(\mathfrak{g}/\mathfrak{b}))\otimes_{\mathbb{C}} L(\Lambda)=\bigcup_{i\geq 0} N_i$;
- $\bigoplus_{i\geq 1}(N_i/N_{i-1})\cong(\bigwedge^p \mathfrak{n}_-)\otimes_{\mathbb{C}} L(\Lambda)$ as $\mathfrak{h}$-modules.

(Notice that the N_i's are defined as in the proof of [GL, Proposition 5.5].) We write for $i\geq 1$,

$$N_i/N_{i-1}=\bigoplus_{k=1}^{l_i}\mathbb{C}\,\bar{v}_k,$$

where $v_k\in N_i$ is a weight vector of $(\bigwedge^p(\mathfrak{g}/\mathfrak{b}))\otimes_{\mathbb{C}} L(\Lambda)$ of weight λ_k, and $\bar{v}_k$ is its image by the natural quotient map $^-: N_i\to N_i/N_{i-1}$. We set

$$L_i:=U(\mathfrak{g})\otimes_{U(\mathfrak{b})} N_i$$

for $i\geq 0$. Then, by [GL, Proposition 1.10], we obtain a $\mathfrak{g}$-module filtration of $B_p(\Lambda)=U(\mathfrak{g})\otimes_{U(\mathfrak{b})}((\bigwedge^p(\mathfrak{g}/\mathfrak{b}))\otimes_{\mathbb{C}} L(\Lambda))$

$$0=L_0\subset L_1\subset L_2\subset\cdots\subset B_p(\Lambda)$$

such that:

- $\Psi_p(L_i)\subset L_i$ for $i\geq 0$;
- $L_i/L_{i-1}\cong U(\mathfrak{g})\otimes_{U(\mathfrak{b})}(N_i/N_{i-1})$ as $\mathfrak{g}$-modules for $i\geq 1$;
- $B_p(\Lambda)=\bigcup_{i\geq 0} L_i$.

Here we note that because N_i/N_{i-1} is a trivial $\mathfrak{n}_+$-module, the quotient $\mathfrak{g}$-module L_i/L_{i-1} is isomorphic to a direct sum of finitely many Verma modules $M(\lambda_k)$, $1\leq k\leq l_i$.

We set for $p\geq 0$,

$$V_i':=(L_i)_{c_0},$$

where $c_0=(\Lambda+\rho|\Lambda+\rho)-(\rho|\rho)$. Then, in the same way as [GL, Proposition 4.7], we get a $\mathfrak{g}$-module filtration of $C_p(\Lambda)$

$$0=V_0'\subset V_1'\subset V_2'\subset\cdots\subset C_p(\Lambda) \tag{5.4}$$

such that:

- $C_p(\Lambda)=\bigcup_{i\geq 0} V_i'$;
- the quotient $\mathfrak{g}$-module V_i'/V_{i-1}' for $i\geq 1$ is isomorphic to the direct sum of Verma modules $M(\lambda_k)$ with $1\leq k\leq l_i$ for which $(\lambda_k+\rho|\lambda_k+\rho)=(\Lambda+\rho|\Lambda+\rho)$.

Here, by Lemma 5.4,

$$\Psi_p(V_i')\subset V_i'$$

for $i\geq 0$. Moreover, we know from [N2, Section 3.2] that

$$\bigoplus_{i\geq 0}\bigoplus_{\substack{1\leq k\leq l_i\\(\lambda_k+\rho|\lambda_k+\rho)=(\Lambda+\rho|\Lambda+\rho)}}\mathbb{C}(\lambda_k)\cong\bigoplus_{\substack{(w,\beta)\in W\times\mathcal{S}(\Lambda)\\ \ell(w)+\mathrm{ht}(\beta)=p}}\mathbb{C}((w,\beta)\circ\Lambda) \tag{5.5}$$

since $(\bigwedge^p(\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda) \cong (\bigwedge^p \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\Lambda)$ as $\mathfrak{h}$-modules. Now a suitable refinement of the sequence of the V_i''s gives the filtration of $C_p(\Lambda)$ for $p \geq 0$ in Theorem 5.1. From the Ψ_p-stable filtration (5.4), we immediately get that for $p \geq 0$,

$$\mathrm{ch}^{\omega}(C_p(\Lambda)) = \sum_{i \geq 1} \mathrm{ch}^{\omega}(V_i'/V_{i-1}').$$

Furthermore, it follows from the exactness of the functor $V \mapsto V_{(c)}$ for all $c \in \mathbb{C}$ that for $i \geq 1$,

$$\begin{aligned} \mathrm{ch}^{\omega}(V_i'/V_{i-1}') &= \mathrm{ch}^{\omega}((L_i)_{c_0}/(L_{i-1})_{c_0}) \\ &= \mathrm{ch}^{\omega}((L_i/L_{i-1})_{c_0}), \end{aligned}$$

where $c_0 = (\Lambda + \rho|\Lambda + \rho) - (\rho|\rho)$. Notice that the following diagram is commutative for $i \geq 1$:

$$\begin{array}{ccc} L_i/L_{i-1} & \xrightarrow{\ \simeq\ } & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (N_i/N_{i-1}) \\ {\scriptstyle \overline{\Psi}_p}\big\downarrow & & \big\downarrow {\scriptstyle \omega^{-1} \otimes \overline{((\bigwedge^p \overline{\omega}^{-1}) \otimes \tau_\omega)}} \\ L_i/L_{i-1} & \xrightarrow[\ \simeq\ ]{} & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (N_i/N_{i-1}), \end{array} \tag{5.6}$$

where

$$\overline{\Psi}_p : L_i/L_{i-1} \to L_i/L_{i-1}$$

is induced from $\Psi_p : L_i \to L_i$, and

$$\overline{\left(\bigwedge^p \overline{\omega}^{-1}\right) \otimes \tau_\omega} : N_i/N_{i-1} \to N_i/N_{i-1}$$

is induced from $(\bigwedge^p \overline{\omega}^{-1}) \otimes \tau_\omega : N_i \to N_i$. For simplicity of notation, we set for $i \geq 1$,

$$\begin{aligned} X_i &:= U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (N_i/N_{i-1}), \\ \Xi_p &:= \omega^{-1} \otimes \left(\overline{\left(\bigwedge^p \overline{\omega}^{-1}\right) \otimes \tau_\omega} \right) : X_i \to X_i. \end{aligned}$$

Because the linear automorphism $\Xi_p : X_i \to X_i$ commutes with the action of the Casimir operator Ω by Lemma 5.4, we deduce from the commutative diagram (5.6) that for $i \geq 1$,

$$\mathrm{ch}^{\omega}((L_i/L_{i-1})_{c_0}) = \mathrm{ch}^{\omega}((X_i)_{c_0}),$$

where

$$\mathrm{ch}^{\omega}((X_i)_{c_0}) := \mathrm{Tr}_{(X_i)_{c_0}} \Xi_p \exp.$$

Proposition 5.5. *Let $i \geq 1$. Then*

$$\mathrm{ch}^{\omega}((X_i)_{c_0}) = \sum_{\substack{1 \leq k \leq l_i \\ (\lambda_k+\rho|\lambda_k+\rho)=(\Lambda+\rho|\Lambda+\rho) \\ \omega^*(\lambda_k)=\lambda_k}} c_k \, \mathrm{ch}^{\omega}(M(\lambda_k)),$$

where the scalar $c_k \in \mathbb{C}$ is determined by

$$c_k := \mathrm{Tr}\left(\left(\left(\bigwedge^p \overline{\omega}^{-1} \right) \otimes \tau_\omega \right) \Big|_{((\bigwedge^p(\mathfrak{g}/\mathfrak{b}))\otimes_{\mathbb{C}} L(\Lambda))_{\lambda_k}} \right).$$

Proof. Since $N_i/N_{i-1} = \bigoplus_{k=1}^{l_i} \mathbb{C}\,\bar{v}_k$ is a trivial $\mathfrak{n}_+$-module for $i \geq 1$, it can be shown by using the Poincaré-Birkhoff-Witt theorem that

$$X_i = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (N_i/N_{i-1}) = \bigoplus_{k=1}^{l_i} U(\mathfrak{g})(1 \otimes \bar{v}_k), \tag{5.7}$$

where the $\mathfrak{g}$-submodule $U(\mathfrak{g})(1 \otimes \bar{v}_k)$ is isomorphic to the Verma module $M(\lambda_k) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}(\lambda_k)$ of highest weight λ_k. Because the Casimir operator Ω acts on the Verma module $M(\lambda_k)$ as the scalar $(\lambda_k + \rho|\lambda_k + \rho) - (\rho|\rho)$, we deduce from (5.7) that for $i \geq 1$,

$$(X_i)_{c_0} = \bigoplus_{\substack{1 \leq k \leq l_i \\ (\lambda_k+\rho|\lambda_k+\rho)=(\Lambda+\rho|\Lambda+\rho)}} U(\mathfrak{g})(1 \otimes \bar{v}_k).$$

Let $1 \leq k \leq l_i$ be such that $(\lambda_k + \rho|\lambda_k + \rho) = (\Lambda + \rho|\Lambda + \rho)$. Then we have for $x \in U(\mathfrak{g})$,

$$\begin{aligned} \Xi_p(x(1 \otimes \bar{v}_k)) &= \omega^{-1}(x)\, \Xi_p(1 \otimes \bar{v}_k) \\ &= \omega^{-1}(x) \left(1 \otimes \left(\overline{\left(\bigwedge^p \overline{\omega}^{-1} \right) \otimes \tau_\omega} \right) (\bar{v}_k) \right) \\ &= \omega^{-1}(x) \left(1 \otimes \overline{\left(\left(\bigwedge^p \overline{\omega}^{-1} \right) \otimes \tau_\omega \right) (v_k)} \right), \end{aligned} \tag{5.8}$$

where $((\bigwedge^p \overline{\omega}^{-1}) \otimes \tau_\omega)(v_k) \in (N_i)_{\omega^*(\lambda_k)}$. Here we recall from (5.5) that the weight λ_k of $(\bigwedge^p(\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda)$ with $1 \leq k \leq l_i$ such that $(\lambda_k + \rho|\lambda_k + \rho) = (\Lambda + \rho|\Lambda + \rho)$ can be written in the form $\lambda_k = (w, \beta) \circ \Lambda$ for a unique $(w, \beta) \in W \times \mathcal{S}(\Lambda)$, and that the multiplicity of the weight $\lambda_k = (w, \beta) \circ \Lambda$ in $(\bigwedge^p(\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda)$ is equal to one. Hence we deduce that

$$\dim_{\mathbb{C}} (N_i/N_{i-1})_{\omega^*(\lambda_k)} = 1 = \dim_{\mathbb{C}} (N_i/N_{i-1})_{\lambda_k}$$

since $\omega^*(\lambda_k)$ is also a weight of $N_i \subset (\bigwedge^p(\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda)$ such that

$$\begin{aligned} (\omega^*(\lambda_k) + \rho|\omega^*(\lambda_k) + \rho) &= (\omega^*(\lambda_k + \rho)|\omega^*(\lambda_k + \rho)) \\ &= (\lambda_k + \rho|\lambda_k + \rho) \\ &= (\Lambda + \rho|\Lambda + \rho). \end{aligned}$$

So $\overline{((\bigwedge^p \overline{\varpi}^{-1}) \otimes \tau_\omega)}(\bar{v}_k) \in (N_i/N_{i-1})_{\omega^*(\lambda_k)}$ implies that $\overline{((\bigwedge^p \overline{\varpi}^{-1}) \otimes \tau_\omega)}(\bar{v}_k) \in \mathbb{C}\,\bar{v}_m$ for a unique m with $1 \le m \le l_i$ such that $\lambda_m = \omega^*(\lambda_k)$. Thus we conclude that $\Xi_p(U(\mathfrak{g})(1 \otimes \bar{v}_k)) = U(\mathfrak{g})(1 \otimes \bar{v}_m)$ for a unique m with $1 \le m \le l_i$. Therefore, for $i \ge 1$,

$$\operatorname{ch}^\omega((X_i)_{c_0}) = \sum_{\substack{1 \le k \le l_i \\ (\lambda_k+\rho|\lambda_k+\rho)=(\Lambda+\rho|\Lambda+\rho) \\ \omega^*(\lambda_k)=\lambda_k}} \operatorname{ch}^\omega(U(\mathfrak{g})(1 \otimes \bar{v}_k)), \tag{5.9}$$

where

$$\operatorname{ch}^\omega(U(\mathfrak{g})(1 \otimes \bar{v}_k)) := \operatorname{Tr}_{U(\mathfrak{g})(1 \otimes \bar{v}_k)} \Xi_p \exp.$$

Let $1 \le k \le l_i$ be such that $(\lambda_k+\rho|\lambda_k+\rho) = (\Lambda+\rho|\Lambda+\rho)$ and $\omega^*(\lambda_k) = \lambda_k$. We set

$$c_k := \operatorname{Tr}\left(\left(\left(\bigwedge^p \overline{\varpi}^{-1} \right) \otimes \tau_\omega \right) \Big|_{((\bigwedge^p(\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda))_{\lambda_k}} \right).$$

Then we have the following commutative diagram from Equation (5.8):

$$\begin{array}{ccc} U(\mathfrak{g})(1 \otimes \bar{v}_k) & \xrightarrow{\simeq} & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}(\lambda_k) \\ {\scriptstyle \Xi_p}\downarrow & & \downarrow{\scriptstyle c_k(\omega^{-1} \otimes \mathrm{id})} \\ U(\mathfrak{g})(1 \otimes \bar{v}_k) & \xrightarrow[\simeq]{} & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}(\lambda_k) \end{array}$$

since $((\bigwedge^p(\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda))_{\lambda_k} = \mathbb{C}\,v_k$ implies

$$\left(\left(\bigwedge^p \overline{\varpi}^{-1} \right) \otimes \tau_\omega \right)(v_k) = c_k v_k.$$

Thus it follows from Remark 3.1 that

$$\operatorname{ch}^\omega(U(\mathfrak{g})(1 \otimes \bar{v}_k)) = c_k \operatorname{ch}^\omega(M(\lambda_k)).$$

This together with (5.9) proves the proposition. □

Let $1 \le k \le l_i$ be such that $(\lambda_k+\rho|\lambda_k+\rho) = (\Lambda+\rho|\Lambda+\rho)$ and $\omega^*(\lambda_k) = \lambda_k$, and then write it in the form

$$\lambda_k = (w, \beta) \circ \Lambda$$

for a unique $(w, \beta) \in W \times \mathcal{S}(\Lambda)$ such that $\ell(w) + \operatorname{ht}(\beta) = p$. Then, as in the proof of [N5, Proposition 3.2.1], $\omega^*(\lambda_k) = \lambda_k$ if and only if $w \in \widetilde{W}$ and $\beta \in \mathcal{S}(\Lambda) \cap (\mathfrak{h}^*)^0$. Therefore, from the obvious commuting diagram:

$$\begin{array}{ccc} (\bigwedge^p(\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda) & \xrightarrow{\simeq} & (\bigwedge^p \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\Lambda) \\ {\scriptstyle (\bigwedge^p \overline{\varpi}^{-1}) \otimes \tau_\omega}\downarrow & & \downarrow{\scriptstyle \Phi_p = (\bigwedge^p \omega^{-1}) \otimes \tau_\omega} \\ (\bigwedge^p(\mathfrak{g}/\mathfrak{b})) \otimes_{\mathbb{C}} L(\Lambda) & \xrightarrow[\simeq]{} & (\bigwedge^p \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\Lambda), \end{array}$$

we see that the scalar c_k is equal to the scalar $c_{(w,\beta)}$ in Theorem 4.1, which equals $(-1)^{(\ell(w)+\mathrm{ht}(\beta))-(\widehat{\ell}(w)+\widehat{\mathrm{ht}}(\beta))}$.

Summarizing all the arguments above, we see that

(5.10)

$$\begin{aligned}
\mathrm{ch}^{\omega}(L(\Lambda)) &= \sum_{p\geq 0}(-1)^p\,\mathrm{ch}^{\omega}(C_p(\Lambda)) \\
&= \sum_{p\geq 0}(-1)^p \sum_{i\geq 1}\mathrm{ch}^{\omega}(V'_i/V'_{i-1}) \\
&= \sum_{p\geq 0}(-1)^p \sum_{i\geq 1}\mathrm{ch}^{\omega}((U(\mathfrak{g})\otimes_{U(\mathfrak{b})}(N_i/N_{i-1}))_{c_0}) \\
&= \sum_{p\geq 0}(-1)^p \sum_{i\geq 1}\sum_{\substack{1\leq k\leq l_i \\ (\lambda_k+\rho|\lambda_k+\rho)=(\Lambda+\rho|\Lambda+\rho) \\ \omega^*(\lambda_k)=\lambda_k}}\mathrm{ch}^{\omega}(U(\mathfrak{g})(1\otimes\bar{v}_k)) \\
&= \sum_{p\geq 0}(-1)^p \sum_{\substack{w\in\widetilde{W} \\ \beta\in\mathcal{S}(\Lambda)\cap(\mathfrak{h}^*)^0 \\ \ell(w)+\mathrm{ht}(\beta)=p}}(-1)^{(\ell(w)+\mathrm{ht}(\beta))-(\widehat{\ell}(w)+\widehat{\mathrm{ht}}(\beta))}\,\mathrm{ch}^{\omega}(M((w,\beta)\circ\Lambda)) \\
&= \sum_{\substack{w\in\widetilde{W} \\ \beta\in\mathcal{S}(\Lambda)\cap(\mathfrak{h}^*)^0}}(-1)^{\widehat{\ell}(w)+\widehat{\mathrm{ht}}(\beta)}\,\mathrm{ch}^{\omega}(M((w,\beta)\circ\Lambda)).
\end{aligned}$$

Here we note that for a symmetric weight $\lambda\in(\mathfrak{h}^*)^0$, the Verma module $M(\lambda)=U(\mathfrak{g})\otimes_{U(\mathfrak{b})}\mathbb{C}(\lambda)$ is isomorphic to $U(\mathfrak{n}_-)\otimes_{\mathbb{C}}\mathbb{C}(\lambda)$ as an $\mathfrak{h}$-module. Moreover, by Remark 3.1, we can apply [N5, Lemma 3.1.3] to deduce that

$$\mathrm{ch}^{\omega}(M(\lambda)) = e(\lambda)\cdot\mathrm{ch}^{\omega}(U(\mathfrak{n}_-)),$$

where

$$\mathrm{ch}^{\omega}(U(\mathfrak{n}_-)) := \mathrm{Tr}_{U(\mathfrak{n}_-)}\,\omega^{-1}\exp.$$

Therefore, by putting $\Lambda=0$ in Equation (5.10), we get that

$$1 = e(0) = \mathrm{ch}^{\omega}(U(\mathfrak{n}_-))\cdot\left(\sum_{\substack{w\in\widetilde{W} \\ \beta\in\mathcal{S}(0)\cap(\mathfrak{h}^*)^0}}(-1)^{\widehat{\ell}(w)+\widehat{\mathrm{ht}}(\beta)}\,e((w,\beta)\circ 0)\right),$$

and hence that for $\lambda\in(\mathfrak{h}^*)^0$,

$$\begin{aligned}
\text{(5.11)}\qquad \mathrm{ch}^{\omega}(M(\lambda)) &= e(\lambda)\cdot\mathrm{ch}^{\omega}(U(\mathfrak{n}_-)) \\
&= e(\lambda)\cdot\left(\sum_{\substack{w\in\widetilde{W} \\ \beta\in\mathcal{S}(0)\cap(\mathfrak{h}^*)^0}}(-1)^{\widehat{\ell}(w)+\widehat{\mathrm{ht}}(\beta)}\,e((w,\beta)\circ 0)\right)^{-1}.
\end{aligned}$$

We finally obtain from Equations (5.10) and (5.11) that

$$\operatorname{ch}^{\omega}(L(\Lambda)) = \frac{\displaystyle\sum_{\substack{w \in \widetilde{W} \\ \beta \in \mathcal{S}(\Lambda) \cap (\mathfrak{h}^*)^0}} (-1)^{\widehat{\ell}(w) + \widehat{\mathrm{ht}}(\beta)}\, e((w, \beta) \circ \Lambda)}{\displaystyle\sum_{\substack{w \in \widetilde{W} \\ \beta \in \mathcal{S}(0) \cap (\mathfrak{h}^*)^0}} (-1)^{\widehat{\ell}(w) + \widehat{\mathrm{ht}}(\beta)}\, e((w, \beta) \circ 0)}.$$

Thus we have given a new proof of Theorems 3.3 and 3.4.

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