J. Math. Kyoto Univ. (JMKYAZ)
42-1 (2002), 41–55

Differential relations for modular forms of level five

By

Toshiyuki Mano*

Abstract

It is known that the ring of modular forms of rational number weights for $\Gamma(5)$ is isomorphic to a polynomial ring of two variables. In this paper, we describe differential relations between logarithmic derivatives of the generators of that ring.

1. Introduction

The origin of studies on differential equations for modular forms might go back to Jacobi ([7]). He deduced a differential equation satisfied by theta constants. Afterwards, Halphen [5] rewrote that equation in a form easier to deal with:

$$\begin{cases} \dfrac{d\omega_1}{d\tau} = \omega_1\omega_2 + \omega_1\omega_3 - \omega_2\omega_3, \\ \dfrac{d\omega_2}{d\tau} = \omega_1\omega_2 + \omega_2\omega_3 - \omega_1\omega_3, \\ \dfrac{d\omega_3}{d\tau} = \omega_1\omega_3 + \omega_2\omega_3 - \omega_1\omega_2. \end{cases} \tag{1}$$

This differential system has a special solution

$$\omega_1(\tau) = \frac{1}{2}\frac{d}{d\tau}\log\vartheta_2(0,\tau)^4, \quad \omega_2(\tau) = \frac{1}{2}\frac{d}{d\tau}\log\vartheta_3(0,\tau)^4,$$
$$\omega_3(\tau) = \frac{1}{2}\frac{d}{d\tau}\log\vartheta_4(0,\tau)^4.$$

Recall that $\vartheta_2^4, \vartheta_3^4, \vartheta_4^4$ are modular forms of level two and weight two. However, they are not algebraically independent. There is one linear relation between these three theta constants:

$$\vartheta_2^4 - \vartheta_3^4 + \vartheta_4^4 = 0.$$

Received March 20, 2000
Revised April 16, 2000

*The author was supported by JSPS Reseach Fellowship for Young Scientists.

Let $\mathrm{Mod}_{\Gamma(2)}^{\mathrm{even}} = \oplus_{k\in\mathbb{N}}\mathrm{Mod}_{\Gamma(2)}^{2k}$ be the ring of modular forms of even weights and level two. Then the following fact is known:

$$\mathrm{Mod}_{\Gamma(2)}^{\mathrm{even}} = \mathbb{C}[\vartheta_2^4, \vartheta_3^4, \vartheta_4^4]/(\vartheta_2^4 - \vartheta_3^4 + \vartheta_4^4).$$

Thus, $\mathrm{Mod}_{\Gamma(2)}^{\mathrm{even}}$ is isomorphic to a polynomial ring of two variables. Conversely, starting from the Halphen system (1), we can recover $\mathrm{Mod}_{\Gamma(2)}^{\mathrm{even}}$ by

$$\omega_1 - \omega_2 = \frac{\pi i}{2}\vartheta_4^4, \quad \omega_2 - \omega_3 = \frac{\pi i}{2}\vartheta_2^4, \quad \omega_3 - \omega_1 = -\frac{\pi i}{2}\vartheta_3^4.$$

Therefore we can deduce differential equations for any modular forms of level two from the Halphen system. In this way, the Halphen system as a differential ring characterizes modular forms of level two from analytic view point.

In this paper, we study the case of level five. Put

$$\alpha_1(\tau) = q^{1/20}\eta(\tau)^{-3/5}\vartheta_3\left(\frac{\tau+1}{2}, 5\tau\right) = q_0^{-3/5}\sum_{n\in\mathbb{Z}}(-1)^n q^{5n^2-n},$$
$$\alpha_2(\tau) = q^{9/20}\eta(\tau)^{-3/5}\vartheta_3\left(\frac{3\tau+1}{2}, 5\tau\right) = q_0^{-3/5}q^{2/5}\sum_{n\in\mathbb{Z}}(-1)^n q^{5n^2-3n},$$

where $q = e^{\pi i\tau}$ and $\eta(\tau) = q^{1/12}\prod_{n=1}^{\infty}(1-q^{2n})$, $q_0 = \prod_{n=1}^{\infty}(1-q^{2n})$. Recently, E. Bannai, M. Koike, A. Munemasa and J. Sekiguchi proved the following:

Theorem 1.1. *The above functions $\alpha_1(\tau)$ and $\alpha_2(\tau)$ are modular forms of weight $1/5$ for an appropriate automorphic factor for $\Gamma(5)$. Moreover, the ring of modular forms of level five is isomorphic to a polynomial ring of two variables generated by $\alpha_1(\tau)$ and $\alpha_2(\tau)$.*

This theorem may be considered as a refinement of F. Klein's works on modular functions. In fact, $\Lambda(\tau) = \alpha_2(\tau)/\alpha_1(\tau)$ is a *Hauptmodule* for $\Gamma(5)$ ([4]). In the present paper, we shall deduce the differential relations for logarithmic derivatives of these modular forms. Our main result is the following:

Theorem 1.2. *Put*

$$X_\infty = \frac{d}{d\tau}\log\alpha_1(\tau),$$
$$X_0 = \frac{d}{d\tau}\log\alpha_2(\tau),$$
$$X_{2k+1} = \frac{d}{d\tau}\log\left(\alpha_2(\tau) + \frac{1}{2}(1-\sqrt{5})\varepsilon^k\alpha_1(\tau)\right),$$
$$X_{2k+2} = \frac{d}{d\tau}\log\left(\alpha_2(\tau) + \frac{1}{2}(1+\sqrt{5})\varepsilon^k\alpha_1(\tau)\right),$$

where $k = 0,\dots,4$ and $\varepsilon = e^{2\pi i/5}$. Then $\{X_i\}_{i=\infty,0,\dots,10}$ satisfies the following *differential and algebraic relations*:

$$
\left\{
\begin{aligned}
&\frac{dX_\infty}{d\tau} = \sum_{k=0}^{10} X_\infty X_k - X_\infty X_0 - X_1X_2 \\
&\quad -X_3X_4 - X_5X_6 - X_7X_8 - X_9X_{10}, \\
&\frac{dX_j}{d\tau} = X_\infty X_j + \sum_{k=0}^{10} X_jX_k - X_j^2 - X_\infty X_0 \\
&-X_1X_2 - X_3X_4 - X_5X_6 - X_7X_8 - X_9X_{10},
\end{aligned}
\right. \tag{2}
$$

where $j = 0, 1, \ldots, 10$ *and*

$$
\left\{
\begin{aligned}
&\frac{X_j - X_k}{a_j - a_k}\frac{X_l - X_n}{a_l - a_n} = \frac{X_j - X_n}{a_j - a_n}\frac{X_l - X_k}{a_l - a_k}, \\
&(X_j - X_\infty)\frac{X_l - X_k}{a_l - a_k} = \frac{X_j - X_k}{a_j - a_k}(X_l - X_\infty),
\end{aligned}
\right. \tag{3}
$$

where $j, k, l, n \in \{0, 1, \ldots, 10\}$ *are distinct to each other and* a_j*'s are certain constants defined later* (*see Section* 5).

Moreover we can prove that $X_\infty - X_0, X_\infty - X_1, \ldots, X_\infty - X_{10}$ are linearly independent modular forms of weight two for $\Gamma(5)$. However, from Theorem 1.1, the dimension of the linear space of these modular forms is eleven. Therefore the system of differential equations in Theorem 1.2 recovers the ring of modular forms of even weight for $\Gamma(5)$ in the same way as the case of level two.

The way to obtain Theorem 1.2 is similar to which Ohyama [9] adopted in the case level three. Roughly speaking, it consists of following two steps. The first step is to find the Picard-Fuchs equation for the elliptic modular surface of level five. Here it is important that we can descrive its periods explicitely in terms of modular forms. The second step is to apply Jacobi's method to our Picard-Fuchs equation, which is the idea in order to construct a nonlinear differential system from a second order linear differential equation ([10]). Harnad and McKay [6] is also an interesting application of this method.

2. Elliptic normal curves of fifth degree and elliptic modular surface of level five

In this section, we construct certain elliptic curves of fifth degree in a projective space by means of elliptic theta functions. These curves form a family of elliptic curves with varying the parameter contained in defining equations. It is known that this is an universal family with level five structure. What we need is the Picard-Fuchs equation for this family. Contents of this section follow from Bianchi [3] and Hulek [11].

For $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} | \Im\tau > 0\}$, let Γ_τ be the $\mathbb{Z}$-module of rank two generated by 1 and τ in $\mathbb{C}$ i.e. $\Gamma_\tau = \mathbb{Z}\cdot 1 + \mathbb{Z}\cdot\tau$. Then $C_\tau = \mathbb{C}/\Gamma_\tau$ is a complex torus of dimension 1. Now we recall definitions of elliptic theta functions:

$$\begin{aligned}
\vartheta_1(z,\tau) &= i\sum_{n\in\mathbb{Z}}(-1)^n v^{n-1/2}q^{(n-1/2)^2},\\
\vartheta_2(z,\tau) &= \sum_{n\in\mathbb{Z}} v^{n-1/2}q^{(n-1/2)^2},\\
\vartheta_3(z,\tau) &= \sum_{n\in\mathbb{Z}} v^{n}q^{n^2},\\
\vartheta_4(z,\tau) &= \sum_{n\in\mathbb{Z}}(-1)^n v^{n}q^{n^2},
\end{aligned}$$

where $q = e^{\pi i\tau}$ and $v = e^{2\pi iz}$. Put

$$x_j^{(\infty)}(z) = (-i)^j e^{2\pi i(2-j)z+\frac{j^2}{5}\pi i\tau}\prod_{k=0}^{4}\vartheta_1\left(z-\frac{j\tau+k}{5}\right),\quad j\in\mathbb{Z}. \tag{4}$$

Then we have $x_{j+5k}^{(\infty)}(z) = x_j^{(\infty)}(z)$ for any integer k. Moreover, by direct calculations, we have

$$x_j^{(\infty)}(z+1) = -x_j^{(\infty)}(z), \tag{5}$$

$$x_j^{(\infty)}(z+\tau) = -e^{-2\pi i(5z+\frac{\tau}{2})}x_j^{(\infty)}(z), \tag{6}$$

$$x_j^{(\infty)}(-z) = -e^{-8\pi iz}x_{-j}^{(\infty)}(z), \tag{7}$$

$$x_j^{(\infty)}\left(z-\frac{1}{5}\right) = -e^{\frac{2}{5}(j-2)\pi i}x_j^{(\infty)}(z), \tag{8}$$

$$x_j^{(\infty)}\left(z-\frac{\tau}{5}\right) = -e^{2\pi iz-\pi i\tau}x_{j+1}^{(\infty)}(z). \tag{9}$$

From (5) and (6), $x_j^{(\infty)}$'s are global sections of the same line bundle L_τ on C_τ. And we have $L_\tau \cong \mathcal{O}_{C_\tau}(5[0])$, where $[0]$ is the origin of C_τ since $\sum_{k=0}^{4}((j\tau+k)/5)\equiv 0 \pmod{\Gamma_\tau}$. After all $\{x_0^{(\infty)},\dots,x_4^{(\infty)}\}$ is a basis of $H^0(C_\tau, L_\tau)$. Therefore the holomorphic map $\rho^{(\infty)}$

$$\rho^{(\infty)}(z) = (x_0^{(\infty)}(z) : x_1^{(\infty)}(z) : x_2^{(\infty)}(z) : x_3^{(\infty)}(z) : x_4^{(\infty)}(z)) \tag{10}$$

defines an embedding of C_τ into $\mathbb{P}^4(\mathbb{C})$. We shall determine the defining equations of $\rho^{(\infty)}(C_\tau)$ in $\mathbb{P}^4(\mathbb{C})$. By (7), (8) and (9), the actions of 5-torsion points on C_τ:

$$z\mapsto z-\frac{m+n\tau}{5}\qquad (m,n\in\mathbb{Z})$$

and the elliptic involution ι are extended to those on $\mathbb{P}^4(\mathbb{C})$ as follows:

$$\begin{aligned}
z\mapsto -z:&\ \ (X_0 : X_1 : X_2 : X_3 : X_4)\longmapsto (X_0 : X_4 : X_3 : X_2 : X_1),\\
z\mapsto z-\frac{1}{5}:&\ \ (X_0 : X_1 : X_2 : X_3 : X_4)\longmapsto (X_0 : \varepsilon X_1 : \varepsilon^2 X_2 : \varepsilon^3 X_3 : \varepsilon^4 X_4),\\
z\mapsto z-\frac{\tau}{5}:&\ \ (X_0 : X_1 : X_2 : X_3 : X_4)\longmapsto (X_1 : X_2 : X_3 : X_4 : X_0),
\end{aligned}$$

where $\varepsilon = e^{(2/5)\pi i}$. By these facts, we can show that the defining equations of $\rho^{(\infty)}(C_\tau)$ are given by

$$(11)\qquad \begin{cases} \varphi_0(X;a) = X_0^2 + aX_2X_3 - \dfrac{1}{a}X_1X_4 = 0, \\ \varphi_1(X;a) = X_1^2 + aX_3X_4 - \dfrac{1}{a}X_2X_0 = 0, \\ \varphi_2(X;a) = X_2^2 + aX_4X_0 - \dfrac{1}{a}X_3X_1 = 0, \\ \varphi_3(X;a) = X_3^2 + aX_0X_1 - \dfrac{1}{a}X_4X_2 = 0, \\ \varphi_4(X;a) = X_4^2 + aX_1X_2 - \dfrac{1}{a}X_0X_3 = 0, \end{cases}$$

where a is certain constant depending only on τ. Now let B_a be the variety in $\mathbb{P}^4(\mathbb{C})$ defined by (11). Then we have following propositions on B_a. For their proofs, see [1].

Proposition 2.1. *For each $a \in \mathbb{P}^1$, B_a is a curve in $\mathbb{P}^4$. If $a \in \mathbb{P}^1 \setminus \{0, \infty, -(1/2)(1 \pm \sqrt{5})\varepsilon^k\}$ where $\varepsilon = e^{2\pi i/5}$, $k = 0, \ldots, 4$, the curve B_a is a smooth elliptic curve. On the other hand, if $a \in \Lambda = \{0, \infty, -(1/2)(1 \pm \sqrt{5})\varepsilon^k\}$, B_a is a connected cycle of 5 lines which is denoted as type I_5 in Kodaira's notation ([8]). Above 12 points $\Lambda = \{0, \infty, -(1/2)(1 \pm \sqrt{5})\varepsilon^k\}$ can be identified with the 12 vertices of an icosahedron sitting inside $S^2 \cong \mathbb{P}^1$.*

Proposition 2.2. *For $a \neq a'$, two curves B_a and $B_{a'}$ intersect if and only if a and a' belong to opposite vertices of the icosahedron Λ. In this case, the two singular curves B_a and $B_{a'}$ intersect at vertices of cycles of 5 lines.*

We shall study B_a as a family of elliptic curves with parameter a.

Proposition 2.3. *Let us consider the union $S = \cup_{a \in \mathbb{P}^1} B_a$ of the family of elliptic curves. Then S is an irreducible surface in $\mathbb{P}^4$ and smooth outside 30 intersection points of the curves B_a and $B_{a'}$, $a, a' \in \Lambda$ in Proposition 2.2, and there, two smooth components of the surface S intersect transversely.*

Next, we recall the theory of elliptic modular surfaces. Let

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \middle| a \equiv d \equiv 1, b \equiv c \equiv 0 \pmod{n} \right\}$$

be the principal congruence subgroup of level n. When $n \geq 3$, we can construct an elliptic surface in the following manner. Let us define the semi-direct product $\Gamma(n) \ltimes \mathbb{Z}^2$ by

$$(\gamma, (m_1, m_2)) \cdot (\gamma', (m_1', m_2')) = (\gamma\gamma', (m_1, m_2)\gamma' + (m_1', m_2')).$$

It operates on $\mathbb{H} \times \mathbb{C}$ by

$$(\gamma, (m_1, m_2)) : (\tau, z) \mapsto (\gamma\tau, (z + m_1\tau + m_2)/(c\tau + d)),$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Take the quotient $S'(n) = \mathbb{H}\times\mathbb{C}/\Gamma(n)\ltimes\mathbb{Z}^2$. It has a canonical projection

$$S'(n) \to X'(n) = \mathbb{H}/\Gamma(n)$$

which induces an elliptic fibration. There are natural smooth compactifications $S(n)$ of $S'(n)$ and $X(n)$ of $X'(n)$ with holomorphic mapping $S(n) \to X(n)$ which is an extension of the mapping $S'(n) \to X'(n)$. The complex surface $S(n)$ with $S(n) \to X(n)$ is called the elliptic modular surface of level n. The surface $S(n)$ has singular fibers of type I_n over the points correspond to the cusps of $X(n)$, which is the points $X(n) \setminus X'(n)$. We are concerned with the case $n = 5$. The relation between the elliptic modular surface of level five and our family of elliptic curves is as follows:

Theorem 2.1. *The normalization $\tilde{S}$ of the surface S in Proposition* 2.3 *is isomorphic to the elliptic modular surface $S(5)$.*

Since the point $\rho^{(\infty)}(0) \in C_\tau$ (see (10)) satisfies the equalities (11), we obtain

$$a = a^{(\infty)}(\tau) = -\frac{x_1^{(\infty)}(0,\tau)}{x_2^{(\infty)}(0,\tau)}.$$

Moreover we have the following explicit description of $a^{(\infty)}(\tau)$ from definitions of $x_1^{(\infty)}$ and $x_2^{(\infty)}$:

$$a^{(\infty)}(\tau) = q^{2/5}\frac{\vartheta_3\left(\frac{3\tau+1}{2}, 5\tau\right)}{\vartheta_3\left(\frac{\tau+1}{2}, 5\tau\right)} = q^{2/5}\frac{\sum_{n=-\infty}^{+\infty}(-1)^n q^{5n^2-n}}{\sum_{n=-\infty}^{+\infty}(-1)^n q^{5n^2-3n}}. \tag{12}$$

Thus, the function $a^{(\infty)}(\tau)$ is equal to $\Lambda(\tau)$, the *Hauptmodule* for $\Gamma(5)$ in Section 1. Values of $a^{(\infty)}$ at twelve representatives in $\Gamma(5)$-orbit of cusps are given by the following:

$$\begin{aligned}
a^{(\infty)}(\infty) &= 0, & a^{(\infty)}(0) &= \varepsilon + \varepsilon^4, \\
a^{(\infty)}(1) &= \varepsilon(\varepsilon + \varepsilon^4), & a^{(\infty)}(2) &= \varepsilon^2(\varepsilon + \varepsilon^4), \\
a^{(\infty)}(3) &= \varepsilon^3(\varepsilon + \varepsilon^4), & a^{(\infty)}(4) &= \varepsilon^4(\varepsilon + \varepsilon^4), \\
a^{(\infty)}(1/2) &= \varepsilon^3(\varepsilon^2 + \varepsilon^3), & a^{(\infty)}(-1/2) &= \varepsilon^2(\varepsilon^2 + \varepsilon^3), \\
a^{(\infty)}(3/2) &= \varepsilon^4(\varepsilon^2 + \varepsilon^3), & a^{(\infty)}(-3/2) &= \varepsilon(\varepsilon^2 + \varepsilon^3), \\
a^{(\infty)}(3/5) &= \infty, & a^{(\infty)}(5/3) &= \varepsilon^2 + \varepsilon^3.
\end{aligned}$$

Let us give a relation between $a^{(\infty)}(\tau)$ and the elliptic modular function $J(\tau)$ which gives an isomorphism from $\overline{\mathbb{H}/SL(2,\mathbb{Z})}$ to $\mathbb{P}^1(\mathbb{C})$. Put

$$H(u,v) = -(u^{20} + v^{20}) + 228(u^{15}v^5 - u^5v^{15}) - 494u^{10}v^{10}, \tag{13}$$

$$f(u,v) = uv(u^{10} + 11u^5v^5 - v^{10}), \tag{14}$$

and

$$\overline{H}(a^{(\infty)}(\tau)) = H(a^{(\infty)}(\tau),1), \quad \overline{f}(a^{(\infty)}(\tau)) = f(a^{(\infty)}(\tau),1).$$

Then we have

$$J(\tau) = \frac{\overline{H}(a^{(\infty)}(\tau))^3}{1728\overline{f}(a^{(\infty)}(\tau))^5}. \tag{15}$$

This is known as the icosahedral equation ([4]).

In the above construction, we may take other sections of the line bundle L_τ instead of $x_j^{(\infty)}$. Put

$$x_j^{(l)}(z) = (-\varepsilon^2)^j \prod_{k=0}^{4} \vartheta_1\left(z - \frac{k\tau + kl + j}{5}\right), \ j \in \mathbb{Z}, \ l = 0, \dots, 4. \tag{16}$$

These functions, for each l, give the same elliptic surface as (11). Then the parameters can be calculated in the similar way:

$$a^{(l)}(\tau) = \frac{\vartheta_1(\frac{1}{5}, \frac{\tau+l}{5})}{\vartheta_1(\frac{2}{5}, \frac{\tau+l}{5})}. \tag{17}$$

Moreover we have

$$a^{(l)}(\tau) = a^{(\infty)}\left(-\frac{1}{\tau+l}\right) = (\varepsilon^2+\varepsilon^3)\frac{a^{(\infty)}(\tau) - \varepsilon^{-l}(\varepsilon+\varepsilon^4)}{a^{(\infty)}(\tau) - \varepsilon^{-l}(\varepsilon^2+\varepsilon^3)}. \tag{18}$$

3. Periods of elliptic curves and Picard-Fuchs equations

The main purpose of this section is to find the Picard-Fuchs equation for the elliptic modular surface of level 5, $S(5) \to X(5)$ in the previous section. The 1-form

$$\kappa(a) = \frac{X_4 dX_0 - X_0 dX_4}{5a^3 X_1 X_3 - (2a^5+1)X_0 X_4} \tag{19}$$

is holomorphic on B_a and depends holomorphically on a for $a \notin \{0, \infty, -(1/2)(1 \pm \sqrt{5})\varepsilon^k\}$. We prepare several periods of elliptic curves given by

$$\kappa_1^{(l)}(a) = 5\int_{\gamma_1^{(l)}} \kappa(a), \quad \kappa_2^{(l)}(a) = 5\int_{\gamma_2^{(l)}} \kappa(a), \tag{20}$$

where $l = \infty, 0, \dots, 4$ and paths of integrals starts from the point $\rho^{(l)}(0)$ and $\gamma_1^{(l)}$ (resp. $\gamma_2^{(l)}$) ends at $\rho^{(l)}(1/5)$ (resp. $\rho^{(l)}(\tau/5)$) and satisfy $\kappa_2^{(l)}(a^{(l)}(\tau)) / \kappa_1^{(l)}(a^{(l)}(\tau)) = \tau$.

Proposition 3.1. *The period integrals $\kappa_1^{(\infty)}(a)$ and $\kappa_2^{(\infty)}(a)$ are linearly independent solutions of*

$$a(a^{10}+11a^5-1)\frac{d^2 f(a)}{da^2} + (11a^{10}+66a^5-1)\frac{df(a)}{da} + 25a^4(a^5+3)f(a) = 0. \tag{21}$$

Proof. Let us consider the auxiliary modular subgroup

$$\Gamma_1(5) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \middle| a \equiv d \equiv 1, c \equiv 0 \pmod 5 \right\}.$$

Then $\Gamma(5) \subset \Gamma_1(5)$, and we have canonical maps:

$$\mathbb{H} \to \mathbb{H}/\Gamma(5) \to \mathbb{H}/\Gamma_1(5).$$

Beukers [2] study the universal family of elliptic curves with 5-torsion points parameterized by $\mathbb{H}/\Gamma_1(5)$ and the Picard-Fuchs equation for it. When we identify $\overline{\mathbb{H}/\Gamma_1(5)}$ with $\mathbb{P}^1(\mathbb{C})$ by

$$t = t(\tau) = q^2 \frac{\vartheta_3\left(\frac{3\tau+1}{2}, 5\tau\right)^5}{\vartheta_3\left(\frac{\tau+1}{2}, 5\tau\right)^5}, \tag{22}$$

that family is presented by

$$y^2 = x^3 + \frac{1}{4}(1+6t+t^2)x^2 + \frac{1}{2}t(t+1)x + \frac{1}{4}t^2. \tag{23}$$

The Picard-Fuchs equation is given by

$$t(t^2+11t-1)\frac{d^2 f(t)}{dt^2} + (3t^2+22t-1)\frac{df(t)}{dt} + (t+3)f(t) = 0 \tag{24}$$

as $f(t) = \int(dx/y)$. If we chose independent solutions of (24) suitably, $f_1(t)$ and $f_2(t)$, then $\tau = f_2(t)/f_1(t)$ gives the inverse function of (22). We can complete the proof by comparing behaviors of (19) at each singular points with pull back of (24) by $t = a^5$. $\square$

Remark 3.1. Local exponents at each singular points of (24) are given in the following:

$$\begin{Bmatrix} 0, & \frac{1}{2}(-11 \pm 5\sqrt{5}), & \infty & \\ 0 & 0 & 1 & ;t \\ 0 & 0 & 1 & \end{Bmatrix}.$$

Moreover, let $\omega(\tau)$ be a modular form of weight one for $\Gamma_1(5)$ which vanishes at $\tau = 2/5$. Then $f_1(\tau) = \omega(\tau(t))$ and $f_2(t) = \tau\omega$ are fundamental system of solutions of (24) (see [2]). Along γ_0 which is a loop around $t = 0$, f_1 and f_2 are transformed as follows:

$$\begin{aligned} &f_1^{\gamma_0}(t) = f_1(t), \\ &f_2^{\gamma_0}(t) = f_2(t) + f_1(t) = (\tau+1)f_1(t), \end{aligned}$$

where $f_i^{\gamma_0}(t)$ denotes the analytic continuation of $f_i(t)$ along γ_0. Therefore $f_1(t)$ is expanded at $t = 0$ as

$$f_1(t) = c_0 + c_1 t + \cdots, \quad c_0 \neq 0.$$

Corollary 3.1. *Relations between $\kappa_1^{(l)}(\tau)$ defined in* (20) *are given as follows*

$$\kappa_1^{(l)}(a^{(l)}(\tau)) = c_l(a^{(\infty)}(\tau) - \varepsilon^{-l}(\varepsilon^2+\varepsilon^3))^5\kappa_1^{(\infty)}(a^{(\infty)}(\tau)), \tag{25}$$

where $l = 0, \ldots, 4$ and c_l is a nonzero constant.

Proof. By (20), $\kappa_1^{(l)}(a)$ is also a solution of the differential equation (21). We set $a = (\varepsilon^2+\varepsilon^3)(a^{(\infty)} - \varepsilon^{-l}(\varepsilon+\varepsilon^4))/(a^{(\infty)} - \varepsilon^{-l}(\varepsilon^2+\varepsilon^3))$. We change the independent variable a to $a^{(\infty)}$ in (21). It is evident that $\kappa_1^{(l)}((\varepsilon^2 + \varepsilon^3)(a^{(\infty)} - \varepsilon^{-l}(\varepsilon+\varepsilon^4))/(a^{(\infty)} - \varepsilon^{-l}(\varepsilon^2+\varepsilon^3)))$ is a solution of the new differential equation. Moreover it is checked easily that $(a^{(\infty)} - \varepsilon^{-l}(\varepsilon^2+\varepsilon^3))^5\kappa_1^{(\infty)}(a^{(\infty)})$ is also a solution of the new differential equation. Since both of these solutions are holomorphic at $a^{(\infty)} = 0$, they are different only constant multiplicity. Composing $a^{(\infty)} = a^{(\infty)}(\tau)$, we have $a(\tau) = a^{(l)}(\tau)$. Therefore we obtain

$$\kappa_1^{(l)}(a^{(l)}(\tau)) = c_l(a^{(\infty)}(\tau) - \varepsilon^{-l}(\varepsilon^2+\varepsilon^3))^5\kappa_1^{(\infty)}(a^{(\infty)}(\tau)), \tag{26}$$

for certain $c_l \neq 0$. □

Next, we give an expression of a composition $\kappa_1^{(l)}(a^{(l)}(\tau))$ as a function of τ.

Proposition 3.2. *We have the following expression of $\kappa_1^{(l)}(\tau) = \kappa_1^{(l)}(a^{(l)}(\tau))$ as a function of τ*:

$$\begin{aligned}\kappa_1^{(\infty)}(\tau) &= 2\pi q^4 \vartheta_1(2\tau, 5\tau)^5 \eta(\tau)^{-3} \\ &= 2\pi i q^{1/4}\vartheta_3\left(\frac{\tau+1}{2}, 5\tau\right)^5 \eta(\tau)^{-3}.\end{aligned} \tag{27}$$

$$\kappa_1^{(m)}(\tau) = 2\pi\frac{\sqrt{5}}{5^3}e^{-\frac{3}{4}\pi i}\vartheta_1\left(\frac{2}{5}, \frac{\tau+m}{5}\right)^5 \eta(\tau)^{-3} \quad (m = 0, \ldots, 4). \tag{28}$$

We need the following lemma obtained from the addition formulae of theta functions to prove Proposition 3.2.

Lemma 3.1. *For the function $x_j(z) = x_j^{(l)}(z)$ for each $l = \infty, 0, \ldots, 4$ in* (4) *and* (16), *we obtain the following differential equations*:

$$-5a^2\frac{x_1}{x_0'}\frac{d}{dz}\left(\frac{x_0(z)}{x_4(z)}\right) = 5a^3\frac{x_1(z)}{x_4(z)}\frac{x_3(z)}{x_4(z)} - (2a^5+1)\frac{x_0(z)}{x_4(z)}, \tag{29}$$

$$-5a^2\frac{x_1}{x_0'}\frac{d}{dz}\left(\frac{x_1(z)}{x_4(z)}\right) = 5a^2\frac{x_2(z)}{x_4(z)}\frac{x_3(z)}{x_4(z)} - (2-a^5)\frac{x_1(z)}{x_4(z)}, \tag{30}$$

$$-5a^2\frac{x_1}{x_0'}\frac{d}{dz}\left(\frac{x_2(z)}{x_4(z)}\right) = -5a^2\frac{x_0(z)}{x_4(z)}\frac{x_1(z)}{x_4(z)} + (2-a^5)\frac{x_2(z)}{x_4(z)}, \tag{31}$$

$$-5a^2\frac{x_1}{x_0'}\frac{d}{dz}\left(\frac{x_3(z)}{x_4(z)}\right) = -5a^3\frac{x_0(z)}{x_4(z)}\frac{x_2(z)}{x_4(z)} + (2a^5+1)\frac{x_3(z)}{x_4(z)}, \tag{32}$$

where $x_1 = x_1(0,\tau)$, $x_0' = (dx_0/dz)(0,\tau)$.

Proof. We shall prove only the first equality. The addition formula for theta functions is given by

$$\begin{aligned} &x_0(z+w)x_4(z-w)x_1x_3 \\ &\quad = x_0(w)x_4(-w)x_1(z)x_3(z) - x_2(w)x_1(-w)x_0(z)x_4(z). \end{aligned} \tag{33}$$

Differentiating (33) by w and setting $w=0$, we have

$$\begin{aligned} &(x_0'(z)x_4(z) - x_0(z)x_4'(z))x_1x_3 \\ &\quad = x_0'x_4x_1(z)x_3(z) - x_2'x_1x_0(z)x_4(z) + x_1'x_2x_0(z)x_4(z). \end{aligned}$$

Thus we obtain

$$x_1x_3\frac{d}{dz}\left(\frac{x_0(z)}{x_4(z)}\right) = x_0'x_4\frac{x_1(z)}{x_3(z)}\frac{x_3(z)}{x_4(z)} + (x_1'x_2 - x_1x_2')\frac{x_0(z)}{x_4(z)}.$$

Moreover, differentiating (11) by z and setting $z=0$, we have

$$x_1'x_2 - x_1x_2' = -x_0'x_2\frac{2a^5+1}{5a^2}.$$

Therefore we obtain the equation (29). $\square$

Proof of Proposition 3.2. By comparing (29) with (19), the period of elliptic curves $\kappa_1(\tau)$ is given by

$$\kappa_1(\tau) = -\frac{1}{5a^2}\frac{x_0'}{x_1}. \tag{34}$$

We obtain equalities in Proposition 3.2 by direct calculations from (34) using the definition of $x_j^{(l)}$. $\square$

Corollary 3.2. *Functions $\kappa_1^{(\infty)}(\tau)$ and*

$${a^{(\infty)}}^5\kappa_1^{(\infty)}(\tau) = 2\pi i q^{9/4}\vartheta_3\left(\frac{3\tau+1}{2}, 5\tau\right)^5\eta(\tau)^{-3} \tag{35}$$

are modular forms of weight one for $\Gamma_1(5)$. Here a holomorphic function $f(\tau)$ on $\mathbb{H}$ is a modular form of weight one means that for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(5)$, f satisfies $f((a\tau+b)/(c\tau+d)) = (c\tau+d)f(\tau)$ and is holomorphic at each cusp. Moreover $\kappa_1^{(l)}(\tau)$ and

$${a^{(l)}}^5\kappa_1^{(l)}(\tau) = 2\pi\frac{\sqrt{5}}{5^3}e^{-\frac{l}{4}\pi i}\vartheta_1\left(\frac{1}{5}, \frac{\tau+l}{5}\right)^5\eta(\tau)^{-3} \quad (l=\infty, 0, \dots, 4) \tag{36}$$

are modular forms of weight one for $\Gamma(5)$.

Proof. For $\kappa_1^{(\infty)}(\tau)$, see Remark 3.1, or we can prove it directly by checking transformations of (27) by three generators of $\Gamma_1(5)$. For ${a^{(\infty)}}^5\kappa_1^{(\infty)}(\tau)$, since ${a^{(\infty)}}^5 = t$ is a modular function for $\Gamma_1(5)$ and $\kappa_1^{(\infty)}(a)$ has zero of fifth order at $a = \infty$, it is also a modular form for $\Gamma_1(5)$. For rests of functions, statements follow from that $\Gamma(5)$ is a normal subgroup of $SL(2, \mathbb{Z})$ and that $\kappa_1^{(l)}(\tau) = (1/(\tau + l))\kappa_1^{(\infty)}(-1/(\tau + l))$ and ${a^{(l)}}^5\kappa_1^{(l)}(\tau) = (1/(\tau + l))a^{(\infty)}(-1/(\tau + l))^5 \kappa_1^{(\infty)}(-1/(\tau + l))$. □

4. Jacobi's method

In this section, following Ohyama [10], we outline the method to obtain a differential system of Halphen type from a Fuchsian differential equation of second order.

We begin with a differential equation

$$\frac{d^2y}{dz^2} + Q(z)y = 0 \tag{37}$$

with regular singular points at $z = a_0, a_1, \ldots, a_{m-1}, \infty$. Then we can write

$$Q(z) = \sum_{j=0}^{m-1} \frac{\alpha_j}{(z - a_j)^2} + \sum_{j=1}^{m-1} \frac{\beta_{j-1}}{(z - a_{j-1})(z - a_j)}. \tag{38}$$

Let u and v be two linearly independent solutions of (37). Wronskian of u and v is

$$W(u, v) = \begin{vmatrix} u & u_z \\ v & v_z \end{vmatrix} = uv_z - vu_z,$$

where $u_z = du/dz$ and so on. Since u and v satisfy (37), we have

$$\frac{dW}{dz} = uv_{zz} - vu_{zz} = 0.$$

Hence $W(u, v) = c$, where c is a non-zero constant. Let us consider the following multi-valued map defined by u and v:

$$\tau : \Delta = \mathbb{C} \setminus \{a_0, \ldots, a_{m-1}\} \to D \subset \mathbb{P}_1(\mathbb{C})$$
$$z \mapsto \tau(z) = \frac{v(z)}{u(z)}.$$

Then $(d\tau/dz) = W(u, v)/u^2 = c/u^2$. So we rewrite (37) by taking τ as a variable:

$$\frac{1}{u(\tau)}\frac{d^2u(\tau)}{d\tau^2} - 2\left(\frac{1}{u(\tau)}\frac{du(\tau)}{d\tau}\right)^2 + Q(z(\tau))\frac{u(\tau)^4}{c^2} = 0, \tag{39}$$

where $u(\tau) = u(z(\tau))$ and $z(\tau)$ is the inverse map of τ. Now we prepare Halphen variables for (37):

$$Y_\infty = \frac{d}{d\tau}\log u(\tau) = \frac{u_\tau}{u}, \tag{40}$$

$$Y_j = \frac{d}{d\tau}\log\left(\frac{u(\tau)}{z(\tau)-a_j}\right) = \frac{u_\tau}{u} - \frac{u^2}{c(z-a_j)}, \tag{41}$$

where $j = 0, 1, \dots, m-1$. Using (39), we can express $dY_k/d\tau$ in terms of polynomials of Y_k. More precisely we have

$$\frac{dY_k}{d\tau} = Y_k^2 - \sum_{j=0}^{m-1}\alpha_j(Y_j - Y_\infty)^2 - \sum_{j=1}^{m-1}\beta_{j-1}(Y_{j-1} - Y_\infty)(Y_j - Y_\infty), \tag{42}$$

where $k = \infty, 0, 1, \dots, m-1$. However, from definitions of Halphen variables, every four of them can not be algebraically independent. In fact, we have

$$(Y_j - Y_\infty)\frac{Y_l - Y_k}{a_l - a_k} = \frac{Y_j - Y_k}{a_j - a_k}(Y_l - Y_\infty), \tag{43}$$

$$\frac{Y_j - Y_k}{a_j - a_k}\frac{Y_l - Y_n}{a_l - a_n} = \frac{Y_j - Y_n}{a_j - a_n}\frac{Y_l - Y_k}{a_l - a_k}, \tag{44}$$

where $j, k, l, n = 0, 1, \dots, m-1$ and distinct to each other. We call the set of differential and algebraic equations defined by (42), (43) and (44) the generalized Halphen system associated to (37). We shall state important properties of this system. At first, for $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2, \mathbb{C})$ and holomorphic function $f(\tau)$ on some open set of $\mathbb{P}_1(\mathbb{C})$, put

$$f^A(\tau) = \frac{1}{(r\tau + s)^2} f\left(\frac{p\tau + q}{r\tau + s}\right) - \frac{r}{r\tau + s}. \tag{45}$$

Then we have the following proposition by direct calculations.

Proposition 4.1. *If a set of functions $\{Y_k(\tau)\}_{k=\infty,0,\dots,m-1}$ satisfies* (42), (43) *and* (44), *then for any $A \in SL(2, \mathbb{C})$, $\{Y_k^A(\tau)\}$ also satisfies* (42), (43) *and* (44).

Starting from the special solution defined by (40) and (41), we obtain general solutions by means of transformations in Proposition 4.1. More precisely, we have the following theorem.

Theorem 4.1. *Let $\{Y_k(\tau)\}$ be a set of solutions of the generalized Halphen system defined by* (40) *and* (41). *For any complex numbers y_k ($k = \infty, 0, \dots, m-1$) distinct to each other and satisfying* (43) *and* (44), *and for any $\tau_0 \in \mathbb{C}$, there exists $A \in SL(2, \mathbb{C})$ such that $Y_k^A(\tau_0) = y_k$.*

If $y_j = y_l$ for $j \neq l$, then the solutions Y_k's of the generalized Halphen system degenerate to rational functions of τ. In any case, we can completely solve the initial value problem for the generalized Halphen system.

5. Generalized Halphen system for modular forms of level five

We shall apply Jacobi's method to our Fuchsian differential equation (21). Regular singular points of (21) are $\{\infty, a_0 = 0, a_{2k+1} = -((1-\sqrt{5})/2)\varepsilon^k, a_{2k+2} = -((1+\sqrt{5})/2)\varepsilon^k\}_{k=0,\dots,4}$ and (21) can be rewritten to the form

$$\frac{d^2y}{da^2} + \left(\frac{1}{4}\sum_{j=0}^{10}\frac{1}{(a-a_j)^2} - \frac{1}{2}\sum_{j=0}^{4}\frac{1}{(a-a_{2j+1})(a-a_{2j+2})}\right) y = 0 \tag{46}$$

by the transformation of the unknown function

$$y = a^{1/2}(a^{10} + 11a^5 - 1)^{1/2} f. \tag{47}$$

However, we take Halphen variables different from ones in the previous section. We take as follows

$$X_\infty(\tau) = \frac{1}{5}\frac{d}{d\tau}\log \kappa_1^{(\infty)}(\tau), \tag{48}$$

$$X_0(\tau) = \frac{1}{5}\frac{d}{d\tau}\log a^{(\infty)}(\tau)^5 \kappa_1^{(\infty)}(\tau), \tag{49}$$

$$X_{2k+1}(\tau) = \frac{1}{5}\frac{d}{d\tau}\log \kappa_1^{(k)}(\tau), \tag{50}$$

$$X_{2k+2}(\tau) = \frac{1}{5}\frac{d}{d\tau}\log a^{(k)}(\tau)^5 \kappa_1^{(k)}(\tau), \tag{51}$$

for $k = 0, 1, \dots, 4$. Then X_k's are linear combinations of Y_k's, i.e.,

$$X_k = \frac{1}{2}\left(\sum_{j=\infty,0,\dots,10} Y_j - 10Y_k\right) \tag{52}$$

for $k = \infty, 0, \dots, 10$. Comparing (48–51) with (12), (25) and (27), we obtain Theorem 1.2. Moreover generic solutions of (2), (3) are given by a $SL(2,\mathbb{C})$-orbit of this special solution and other solutions are rational function of τ.

Finally, we study relations between the special solutions of our differential system and elements of the ring of modular forms.

Proposition 5.1. *For $X_j(\tau)$ defined in* (48), (49), (50) *and* (51)*, the differences between any two of them $X_j - X_k$ are modular forms of weight two for $\Gamma(5)$. More precisely, put*

$$\begin{aligned}\alpha^{(k)}(\tau) &= (a^{(\infty)}(\tau) - a_k)\alpha_1(\tau)\\ &= \alpha_2(\tau) - a_k\alpha_1(\tau) \quad (k = 0, 1, \dots, 10),\end{aligned}$$

where a_k was defined in Section 5. Then we have

$$X_j - X_\infty = -\frac{2\pi i}{5}\prod_{k\neq j}\alpha^{(k)}(\tau). \tag{53}$$

Proof. From (46), we can obtain

$$\frac{da^{(\infty)}(\tau)}{d\tau} = cy_1^2 = ca^{(\infty)}(a^{(\infty)10} + 11a^{(\infty)5} - 1)\kappa_1^{(\infty)2}, \tag{54}$$

where c is a nonzero constant. Therefore we have

$$X_j - X_\infty = \frac{1}{a^{(\infty)} - a_j}\frac{da}{d\tau} = c\prod_{k\neq j}(\alpha_2(\tau) - a_k\alpha_1(\tau)). \tag{55}$$

Now let us evaluate c. Put $u = q^{2/5} = e^{2\pi i\tau/5}$. Then $a^{(\infty)}(u)$ is a holomorphic function of u at $u = 0$. More precisely, from (12), we have

$$a^{(\infty)}(u) = u + a_2u^2 + \cdots,$$

and

$$\kappa_1^2 = 1 + k_2u + \cdots.$$

Therefore comparing coefficients of first order in (54), we have

$$-c = \frac{2\pi i}{5}. \qquad \square$$

Remark 5.1. Each modular form of $\alpha_1(\tau)$ and $\alpha^{(k)}(\tau)$ $(k = 0, 1, \ldots, 10)$ has zero at just one of twelve cusps of $\mathbb{H}/\Gamma(5)$. In fact, we can prove that $\alpha_1(\tau)^5\prod_k\alpha^{(k)}(\tau)^5$ is the cusp form of weight 12.

Moreover $dX_i/d\tau - X_i^2$ is a modular form of weight four for any i. So Theorem 1.2 gives relations between these modular forms and modular forms $X_j - X_k$'s.

Department of Mathematics
Graduate School of Science
Kyoto University
Kyoto 606-8502, Japan

References

[1] W. Barth, K. Hulek and R. Moore, Sioda's modular surface $S(5)$ and the Horrocks-Mumford bundle, Proceedings of the International Conference on Algebraic Vector Bundles over Algebraic Varieties, Bombay, 1984.

[2] F. Beukers, Irrationality of π^2, periods of an elliptic curve and $\Gamma_1(5)$, In: Approximations diophantiennes et nombres transcendants, Luminy, 1982, Prog. Math., **31** (1983), 47–66.

[3] L. Bianchi, Ueber die Normal formen dritter und fünfter stufe des elliptischen Integrales erster Gattung, Math. Ann., **17** (1880), 234–262.

[4] R. Fricke and F. Klein, Vorlesungen über die Theorie der automorphen Functionen I.

[5] G. Halphen, Sur une système d'équations différentielles, C. R. Acad. Sci., Paris, **92** (1881), 1101–1103.

[6] J. Harnad and J. McKay, Modular solutions to equations of generalized Halphen type, solv-int/9804006.

[7] C. G. J. Jacobi, Über die Differentialgleichung, welcher die Reihen $1 \pm 2q \pm 2q^4 \pm 2q^9 +$ etc., $2\sqrt[4]{q} + 2\sqrt[4]{q^9} + 2\sqrt[4]{q^{25}} +$ etc. Genüge Leisten, Crelles J, **36** (1848), 97–112.

[8] K. Kodaira, On compact analytic surfaces, II, Ann. of Math., **77** (1963), 563–626.

[9] Y. Ohyama, Differential equations for modular forms with level three, preprint.

[10] Y. Ohyama, Systems of nonlinear differential equations related to second order linear equations, Osaka J. Math., **33** (1996), 927–949.

[11] K. Hulek, Projective geometry of elliptic curves, Astérisque, **137** (1986).