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On m-adic higher differentials and regularities of Noetherian complete local rings

By

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1. Introduction

In this paper, we try to construct a regularity criterion for a Noetherian complete local ring in terms of $\mathbf{m}$-adic higher differentials defined in [Y].

Let P be a ring, R a P-algebra and $\mathbf{m}$ an ideal of R. Then the $\mathbf{m}$-adic higher differential algebra $\widehat{D}_t(R/P, \mathbf{m})$ of R over P of length t is in a natural way a graded algebra, $\widehat{D}_t(R/P, \mathbf{m}) = \bigoplus_{n=0}^{\infty} \widehat{D}_t(R/P, \mathbf{m})_n$, and $\widehat{D}_t(R/P, \mathbf{m})_1$ coincides with the module $\widehat{D}_P(R)$ of $\mathbf{m}$-adic P-differentials in R, defined in [NS].

In [NS], Y. Nakai and S. Suzuki showed the following theorem.

Let $(R, \mathbf{m}, K)$ be a Noetherian complete local ring with $\mathrm{char}(K) = p > 0$, and let (P, pP, k) be a discrete valuation ring such that R dominates P. Then, under some assumptions of separability on the residue fields K and k, the following conditions are equivalent:

(1) R is a regular local ring and $p \notin \mathbf{m}^2$.

(2) $\widehat{D}_P(R)$ is a free R-module.

The main result of this paper is to prove that these conditions (1) and (2) are also equivalent with

(3) $\widehat{D}_t(R/P, \mathbf{m})$ is a polynomial ring over R for every $t\,(1 \leq t < \infty)$ (see Theorem 3.4).

In the equal characteristic case, we shall prove the following result in Theorem 3.1.

Let $(R, \mathbf{m}, K)$ be a Noetherian complete local ring containing a field k, K/k is separably generated, and $\mathrm{Tr.deg}(K/k)$ is finite. Then the following two conditions are equivalent:

(1) R is a regular local ring.

(2) $\widehat{D}_t(R/k, \mathbf{m})$ is a polynomial ring over R for every $t\,(1 \leq t < \infty)$.

Under the assumption that R is an essentially of finite type over k and K/k is separable (without the assumption that R is complete), U. Orbanz showed that these two conditions are equivalent [O, (4.2)].

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We remark that if one of the above two conditions is satisfied, then the module $\widehat{D}_k(R)$ is a finite free R-module. But the converse is not true in general (cf. [NS, Example, p. 473]).

2. Preliminaries

All rings in this paper are commutative rings with identity elements. A ring homomorphism will always mean a ring homomorphism which sends identity element to identity element. Let t be always a natural number.

Let P be a ring, R a P-algebra with a ring homomorphism $\rho : P \longrightarrow R$ and $\mathbf{m}$ an ideal of R.

(2.1) Let S be an R-algebra with a ring homomorphism $f : R \longrightarrow S$. For an integer $n > 0$, by a higher P-derivation of length t from R into S, we mean a sequence $\boldsymbol{D} = (D_0, D_1, \dots, D_t)$ of mappings $D_i : R \longrightarrow S$ such that

(1) $D_0 = f$,

(2) $D_i(a+b) = D_i(a) + D_i(b)$, $D_i(ab) = \sum_{j+k=i} D_j(a)D_k(b)$ for any $a, b \in R$ and $i \geq 0$,

(3) $D_i\rho = 0$ for every $i \geq 1$.

We denote the set of all higher P-derivations of length t from R into S by $HDer^t_P(R,S)$.

(2.2) Let A be an R-algebra and $\boldsymbol{d} = (d_0, d_1, \dots, d_t) \in HDer^t_P(R,A)$. Then A (together with $\boldsymbol{d}$) will called a higher differential algebra of R over P of length t, if the following conditions are satisfied:

(1) As an R-algebra, A is generated by the elements $\{d_i(a) \mid a \in R, 0 \leq i \leq t\}$.

(2) For any R-algebra V and for any $\boldsymbol{h} = (h_0, h_1, \dots, h_t) \in HDer^t_P(R,V)$, there exists a ring homomorphism $g : A \longrightarrow V$ such that $h_i = gd_i$ for every $i \geq 0$.

It is known that a higher differential algebra of R over P of length t exists and is uniquely determined up to isomorphism (cf. [KY] and [B]). We shall denote by $D_t(R/P)$ the higher differential algebra of R over P of length t, and $\boldsymbol{d}$ is called the associated derivation of $D_t(R/P)$.

For an integer $n \geq 0$, we denote by $D_t(R/P)_n$ the R-submodule of $D_t(R/P)$ generated by the elements

$$\{d_{n_1}(a_1) \cdots d_{n_s}(a_s) \mid a_i \in R, 0 \leq n_i \leq t, n_1 + \cdots + n_s = n \text{ for some } s \geq 1\}.$$

Then we have $D_t(R/P) = \displaystyle\bigoplus_{n=0}^{\infty} D_t(R/P)_n$ (cf. [KY]).

(2.3) Let $\mathbf{m}$ be an ideal of R. We say that R is an $\mathbf{m}$-adic ring or R has the $\mathbf{m}$-adic topology, if R has the topology with the fundamental system of neighborhoods of zero $\{\mathbf{m}^r \mid r = 1, 2, \cdots\}$. Let A be an R-algebra or an R-module. We say that A is an $\mathbf{m}$-adic R-algebra or an $\mathbf{m}$-adic R-module (or A

has the **m**-adic topology), if A has the topology with the fundamental system of neighborhoods of zero $\{\mathbf{m}^r A \mid r = 1, 2, \cdots\}$. The **m**-adic topology of A is Hausdorff (i.e. the set $\{0\}$ is closed) if and only if $\bigcap_{r=0}^{\infty} \mathbf{m}^r A = (0)$.

(2.4) An **m**-adic higher differential algebra of R over P of length t, denoted by $\widehat{D}_t(R/P, \mathbf{m})$, is defined as the R-algebra satisfying the following conditions (cf. [Y]):

(1) $\widehat{D}_t(R/P, \mathbf{m})$ is a Hausdorff **m**-adic R-algebra.

(2) There exists an element $\hat{\boldsymbol{d}} = (\hat{d}_0, \hat{d}_1, \ldots, \hat{d}_t) \in HDer_P^t(R, \widehat{D}_t(R/P, \mathbf{m}))$ ($\hat{\boldsymbol{d}}$ is called an associated derivation of $\widehat{D}_t(R/P, \mathbf{m})$ and $\hat{\boldsymbol{d}}$ is denoted by $\hat{\boldsymbol{d}}_{R/P}$ simply).

(3) $\widehat{D}_t(R/P, \mathbf{m})$ is generated over R by $\{\hat{d}_i(a) \mid a \in R, 0 \leq i \leq t\}$.

(4) For arbitrary Hausdorff **m**-adic R-algebra V and $(D_0, D_1, \ldots, D_t) \in HDer_P^t(R, V)$, then there exists a ring homomorphism $g : \widehat{D}_t(R/P, \mathbf{m}) \longrightarrow V$ such that $D_i = g\hat{d}_i$ for every $i \geq 0$.

It is known that an **m**-adic higher differential algebra of R/P of length t exists and is uniquely determined up to isomorphism and up to homeomorphism. Moreover $\widehat{D}_t(R/P, \mathbf{m})$ is given by $\widehat{D}_t(R/P, \mathbf{m}) = D_t(R/P)/\bigcap_{r=0}^{\infty} \mathbf{m}^r D_t(R/P)$ (cf. [Y]). If R is a field, then $\widehat{D}_t(R/P, (0)) = D_t(R/P)$.

(2.5)([Y]). Let $\hat{\boldsymbol{d}}_{R/P} = (\hat{d}_0, \hat{d}_1, \ldots, \hat{d}_t)$ be the associated derivation of $\widehat{D}_t(R/P, \mathbf{m})$. Then the following conditions hold:

(1) $\widehat{D}_t(R/P, \mathbf{m})$ is a graded R-algebra, $\widehat{D}_t(R/P, \mathbf{m}) = \bigoplus_{n=0}^{\infty} \widehat{D}_t(R/P, \mathbf{m})_n$, where $\widehat{D}_t(R/P, \mathbf{m})_n$ is the R-submodule of $\widehat{D}_t(R/P, \mathbf{m})$ generated by the homogeneous elements

$$\{\hat{d}_{n_1}(a_1) \cdots \hat{d}_{n_s}(a_s) \mid a_i \in R, 0 \leq n_i \leq t, n_1 + \cdots + n_s = n \text{ for some } s \geq 1\}.$$

(2) $\widehat{D}_t(R/P, \mathbf{m})_n = D_t(R/P)_n/\bigcap_{r=0}^{\infty} \mathbf{m}^r D_t(R/P)_n$.

(2.6)([Y]). Let $\widehat{D}_P(R)$ be the module of **m**-adic P-differentials in R, defined in [NS]. Then $\widehat{D}_t(R/P, \mathbf{m})_1$ coincides with $\widehat{D}_P(R)$ for every t.

(2.7) Let $\hat{\boldsymbol{d}}_{R/P} = (\hat{d}_0, \hat{d}_1, \ldots, \hat{d}_t)$ be the associated derivation of $\widehat{D}_t(R/P, \mathbf{m})$. For an ideal $\boldsymbol{a}$ of R, let $J_t(\boldsymbol{a})$ be the ideal of $\widehat{D}_t(R/P, \mathbf{m})$ generated by the elements $\{\hat{d}_i(a) \mid a \in \boldsymbol{a}, 0 \leq i \leq t\}$ and $\overline{J_t(\boldsymbol{a})}$ the closure of $J_t(\boldsymbol{a})$ in the **m**-adic R-algebra $\widehat{D}_t(R/P, \mathbf{m})$. Put $\overline{R} = R/\boldsymbol{a}$. Then we have

$$\widehat{D}_t(\overline{R}/P, \mathbf{m}\overline{R}) = \widehat{D}_t(R/P, \mathbf{m})/\overline{J_t(\boldsymbol{a})}.$$

Furthermore, we put $J_t(\boldsymbol{a})_n = \widehat{D}_t(R/P, \mathbf{m})_n \bigcap J_t(\boldsymbol{a})$. Then, for any integer $n \geq 0$, we have $\widehat{D}_t(\overline{R}/P, \mathbf{m}\overline{R})_n = \widehat{D}_t(R/P, \mathbf{m})_n/\overline{J_t(\boldsymbol{a})_n}$, where $\overline{J_t(\boldsymbol{a})_n}$ is the closure of $J_t(\boldsymbol{a})_n$ in the **m**-adic R-module $\widehat{D}_t(R/P, \mathbf{m})_n$.

(2.8) Suppose that R is a Noetherian local ring with the maximal ideal **m**. Assume that R is complete and there are finite elements $\{\bar{z}_1, \ldots, \bar{z}_s\}$ ($\bar{z}_i =$

$z_i + \mathbf{m}$) of $K := R/\mathbf{m}$ such that $D_t(K/P)$ is generated by the elements $\{D_i(\bar{z}_1), \dots, D_i(\bar{z}_s) \mid 1 \leq i \leq t\}$ as a K-algebra, where $(D_0, D_1, \dots, D_t)$ is the associated derivation of $D_t(K/P)$. Then $\widehat{D}_t(R/P, \mathbf{m})$ is the finitely generated R-algebra generated by the elements

$$\{\hat{d}_i(x_1), \dots, \hat{d}_i(x_n), \hat{d}_i(z_1), \dots, \hat{d}_i(z_s) \mid 1 \leq i \leq t\},$$

where $\mathbf{m} = (x_1, \dots, x_n)$ and $\boldsymbol{d}_{R/P} := (\hat{d}_0, \hat{d}_1, \dots, \hat{d}_t)$.

The proof is obtained from (2.7) and [M, Theorem 8.4] inductively.

(2.9) In (2.8), let us remove the condition that R is complete, instead, let us assume that $\widehat{D}_t(R/P, \mathbf{m})_i$ is a finitely generated R-module for every $i \geq 0$. In this case, we have the same conclusion as in (2.8).

The proof is obtained from (2.7) and Nakayama's lemma, inductively.

(2.10) Let S be an R-algebra with a ring homomorphism $f : R \longrightarrow S$ and $\boldsymbol{n}$ an ideal of S such that $\mathbf{m}S \subset \boldsymbol{n}$. Then there exists the ring homomorphism $g_0 : \widehat{D}_t(R/P, \mathbf{m}) \longrightarrow \widehat{D}_t(S/P, \boldsymbol{n})$ such that $g_0 \hat{d}_i = \hat{h}_i f$, where $\hat{\boldsymbol{d}}_{R/P} := (\hat{d}_0, \hat{d}_1, \dots, \hat{d}_t)$ and $\hat{\boldsymbol{d}}_{S/P} := (\hat{h}_0, \hat{h}_1, \dots, \hat{h}_t)$. Thus there exists the ring homomorphism $g_1 : S \otimes_R \widehat{D}_t(R/P, \mathbf{m}) \longrightarrow \widehat{D}_t(S/P, \boldsymbol{n})$ such that $g_1(s \otimes x) = s \cdot g_0(x)$ for any $s \in S$ and $x \in \widehat{D}_t(R/P, \mathbf{m})$. Furthermore, g_1 induces the ring homomorphism

$$g : S \otimes_R \widehat{D}_t(R/P, \mathbf{m}) / \bigcap_{r=0}^{\infty} \boldsymbol{n}^r \big(S \otimes_R \widehat{D}_t(R/P, \mathbf{m})\big) \longrightarrow \widehat{D}_t(S/P, \boldsymbol{n}).$$

If S is 0-etale over R in the sense of [M] (i.e. S is both 0-smooth and 0-unramified over R), then g is an isomorphism.

Proof. Put $V = S \otimes_R \widehat{D}_t(R/P, \mathbf{m})$ and $\overline{V} = V / \bigcap_{r=0}^{\infty} \mathbf{m}^r V$. Let $f_1 : \widehat{D}_t(R/P, \mathbf{m}) \longrightarrow V$ be the ring homomorphism defined by $f_1(x) = 1 \otimes x$ ($x \in \widehat{D}_t(R/P, \mathbf{m})$) and $f_2 : V \longrightarrow \overline{V}$ the canonical mapping. Then we have $(f_2 f_1 \hat{d}_1, \dots, f_2 f_1 \hat{d}_t) \in HDer_P^t(R, \overline{V})$. Since S is 0-etale over R, there exists an element $(D_0, D_1, \dots, D_t) \in HDer_P^t(S, \overline{V})$ such that $f_2 f_1 \hat{d}_i = D_i f$ for every $i (0 \leq i \leq t)$. It follows that there is the ring homomorphism $f^* : \widehat{D}_t(S/P, \boldsymbol{n}) \longrightarrow \overline{V}$ such that $D_i = f^* \hat{d}_i$ for every i. Then it is easily verified that $f^* g = 1$ and $g f^* = 1$. Therefore g is an isomorphism. □

3. Regularity criteria

In this section, we shall show some regularity criteria of Noetherian complete local rings. In the proofs of our results we shall use similar techniques as in [NS] and [O] but with some modifications.

3.1. Equal characteristic case

Theorem 3.1. *Let $(R, \mathbf{m}, K)$ be a Noethrian complete local ring containing a field k. Assume that K is separably generated over k and* Tr.deg (K/k) *is finite. Then the following conditions are equivalent*:

(1) *R is a regular local ring.*

(2) *$\widehat{D}_t(R/k, \mathbf{m})$ is a polynomial ring over R for every $t (1 \le t < \infty)$.*

Under these conditions, let $\{x_1, \dots, x_n\}$ be a regular system of parameters of R and let $\{z_1 + \mathbf{m}, \dots, z_s + \mathbf{m}\}\,(z_i \in R)$ be a separating transcendence basis of K over k. Then $\widehat{D}_t(R/k, \mathbf{m})$ is the polynomial ring over R with variables $\{\hat{d}_i(x_1), \dots, \hat{d}_i(x_n), \hat{d}_i(z_1), \dots, \hat{d}_i(z_s) \mid 1 \le i \le t\}$, where $\hat{\boldsymbol{d}}_{R/k} := (\hat{d}_0, \hat{d}_1, \dots, \hat{d}_t)$.

Proof. (1) ⇒ (2). Let $\{x_1, \dots, x_n\}$ be a regular system of parameters of R and $\{z_1 + \mathbf{m}, \dots, z_s + \mathbf{m}\}\,(z_i \in R)$ a separating transcendence basis of K over k. Then $\{z_1, \dots, z_s\}$ is algebraically independent over k, and $k[z_1, \dots, z_s] \cap \mathbf{m} = (0)$. Hence R contains the field $k(z_1, \dots, z_s) := k_0$. It follows that there exists a coefficient field L of R such that $R \supset L \supset k_0$, and hence we have that $R = L[[x_1, \dots, x_n]]$. Put $S = L[x_1, \dots, x_n]$ and $\boldsymbol{n} = (x_1, \dots, x_n)S$. Then R is the $\boldsymbol{n}$-adic completion of S and $\mathbf{m} = \boldsymbol{n}R$. In the similar way as the proof of (4.1) in [O], we have that $\widehat{D}_t(S/k, \boldsymbol{n})$ is the polynomial ring over S with variables $\{\hat{h}_i(x_1), \dots, \hat{h}_i(x_n), \hat{h}_i(z_1), \dots, \hat{h}_i(z_s) \mid 1 \le i \le t\}$, where $\hat{\boldsymbol{d}}_{S/k} := (\hat{h}_0, \hat{h}_1, \dots, \hat{h}_t)$. Therefore $R \otimes_S \widehat{D}_t(S/k, \boldsymbol{n})$ is a polynomial ring over R. Since it is Hausdorff with respect to the $\mathbf{m}$-adic topology, we have $\widehat{D}_t(R/k, \mathbf{m}) = R \otimes_S \widehat{D}_t(S/k, \boldsymbol{n})$ from (2.10). Thus $\widehat{D}_t(R/k, \mathbf{m})$ is the polynomial ring over R with variables $\{\hat{d}_i(x_1), \dots, \hat{d}_i(x_n), \hat{d}_i(z_1), \dots, \hat{d}_i(z_s) \mid 1 \le i \le t\}$, where $\hat{\boldsymbol{d}}_{R/k} := (\hat{d}_0, \hat{d}_1, \dots, \hat{d}_t)$.

(2) ⇒ (1). We can obtain the proof in almost the same way as the proof of [O, (4.2)]. Therefore we omit the proof. □

Corollary 3.2. *In Theorem* 3.1, *let us remove the condition that R is complete, instead, let us assume that $\widehat{D}_t(R/k, \mathbf{m})_i$ is a finitely generated R-module for every $i \ge 0$ and $t\,(1 \le t < \infty)$. In this case, we have the same conclusion as in Theorem* 3.1.

Proof. (1) ⇒ (2). Let $\{x_1, \dots, x_n\}$ be a regular system of parameters of R and $\{\bar{z}_1, \dots, \bar{z}_s\}$ $(\bar{z}_i = z_i + \mathbf{m})$ a separating transcendence basis of K over k. Then we have

$$\widehat{D}_t(R/k, \mathbf{m}) = R[\hat{d}_i(x_1), \dots, \hat{d}_i(x_n), \hat{d}_i(z_1), \dots, \hat{d}_i(z_s)],$$

by (2.9), where $\hat{\boldsymbol{d}}_{R/k} := (\hat{d}_0, \hat{d}_1, \dots, \hat{d}_t)$. Let R^* be the completion of R and we put $\mathbf{m}^* = \mathbf{m}R^*$. Then the assertion is obtained from the fact $\widehat{D}_t(R/k, \mathbf{m}) \subset \widehat{D}_t(R^*/k, \mathbf{m}^*)$.

(2) ⇒ (1). The proof follows from Theorem 3.1. □

3.2. Unequal characteristic case

In this paragraph, we shall treat exclusively the local rings of characteristic 0 with the residue field of prime characteristic p. For the sake of the proof of Theorem 3.4, we need the following lemma.

Lemma 3.3. *Let $(R, \mathbf{m}, K)$ be a complete local ring with $\mathrm{char}(K) = p > 0$, and let (B, pB, k) be a discrete valuation ring (DVR). Assume that $R \succ B$ (i.e. R dominates B), $p \notin \mathbf{m}^2$, and K is finite separably algebraic over k. Let $b \in K$ with $K = k(b)$. Then there are $a \in R$ and $f(X) \in B[X]$ which satisfy the following conditions (1) $a + \mathbf{m} = b$, (2)$B[a] := C$ is a DVR with the same prime element p as B, (3) $R \succ C \succ B$, (4) $K = C/pC$, (5) $f(X)$ is monic and irreducible, (6) $f(a) = 0$, and (7) $f'(a)$ is a unit in C.* □

Proof. There exists a monic, irreducible polynomial $f(X) \in B[X]$ such that the polynomial $\bar{f}(X)$ is the minimal polynomial of b, where $\bar{f}(X) \in k[X]$ is obtained from $f(X)$ by reducing the coefficients modulo pB. By Hensel's lemma, there are $X - a$ and $f_1(X)$ in $R[X]$ such that $f(X) = (X - a)f_1(X)$ and $a + \mathbf{m} = b$. We put $C = B[a]$, then (C, pC) is a DVR and $C/pC = K$. Since $\bar{f}(X)$ is separable, $\bar{f}'(b) \neq 0$. It follows that $f'(a) + \mathbf{m} = \bar{f}'(b) \neq 0$. Therefore we have $f'(a) \notin \mathbf{m}$, and hence $f'(a) \notin \mathbf{m} \cap C = pC$. Thus $f'(a)$ is a unit in C. □

We are now ready to prove the main result.

Theorem 3.4. *Let $(R, \mathbf{m}, K)$ be a Noethrian complete local ring with $\mathrm{char}(K) = p > 0$, and let (P, pP, k) be a discrete valuation ring. Assume that R dominates P, K is separably generated over k, and $\mathrm{Tr.deg}(K/k)$ is finite. Then the following conditions are equivalent*:

(1) *R is a regular local ring and $p \notin \mathbf{m}^2$.*

(2) *$\widehat{D}_t(R/P, \mathbf{m})$ is a polynomial ring over R for every $t\,(1 \leq t < \infty)$.*

(3) *$\widehat{D}_P(R)$ is a free R-module.*

Under these conditions, let $\{x_2, \dots, x_n\}$ be a subset of $\mathbf{m}$ such that $\{p, x_2, \dots, x_n\}$ is a regular system of parameters of R and let $\{z_1 + \mathbf{m}, \dots, z_s + \mathbf{m}\}$ $(z_i \in R)$ be a separating transcendence basis of K/k. Then $\widehat{D}_t(R/P, \mathbf{m})$ is the polynomial ring over R with variables $\{\hat{d}_i(x_2), \dots, \hat{d}_i(x_n), \hat{d}_i(z_1), \dots, \hat{d}_i(z_s) \mid 1 \leq i \leq t\}$, where $\hat{\boldsymbol{d}}_{R/P} := (\hat{d}_0, \hat{d}_1, \dots, \hat{d}_t)$ is the associated derivation of $\widehat{D}_t(R/P, \mathbf{m})$.

Proof. (1) ⇒ (2). Let $\{z_1 + \mathbf{m}, \dots, z_s + \mathbf{m}\}$ be a separating transcendence basis of K/k. Then $\{z_1, \dots, z_s\}$ is algebraically independent over P. Put $S = P[z_1, \dots, z_s]_{(p)}\,(\subset R)$, then we have $\mathbf{m} \cap S = pS$, (S, pS) is a DVR and $S/pS = k(\bar{z}_1, \dots, \bar{z}_s)\,(\bar{z}_i := z_i + \mathbf{m})$. Thus S is a quasi-coefficient ring of R. Furthermore we can see that $\widehat{D}_t(S/P, pS)$ is the polynomial ring over S with variables $\{\hat{h}_i(z_1), \dots, \hat{h}_i(z_s) \mid 1 \leq i \leq t\}$, where $\hat{\boldsymbol{d}}_{S/P} := (\hat{h}_0, \hat{h}_1, \dots, \hat{h}_t)$. We consider the set Ω of all DVR (B, pB) which satisfy $R \succ B \succ S$ and $\widehat{D}_t(B/P, pB)$ is the polynomial ring over B with variables $\{\hat{q}_i(z_1), \dots, \hat{q}_i(z_s) \mid$

$1 \leq i \leq t\}$, where $\hat{\boldsymbol{d}}_{B/P} := (\hat{q}_0, \hat{q}_1, \ldots, \hat{q}_t)$. It is easy to see that Ω is an inductive set with respect to set theoretic inclusion. By Zorn's lemma, there exists a maximal element B of Ω. We put $L = B/pB$.

We shall show that $L = K$. If $L \neq K$, then there exists an element $b \in K - L$. Then the field $L(b)$ is separably algebraic over L. Under the same notations as in Lemma 3.3, there are $a \in R$ and $f(X) \in B[X]$ which satisfy conditions in Lemma 3.3. Then, since $f'(a)$ is a unit in $C = B[a]$, C is 0-etale over B. It follows from (2.10) that

$$C \otimes_B \widehat{D}_t(B/P, pB) / \bigcap_{r=0}^{\infty} (pC)^r \big(C \otimes_B \widehat{D}_t(B/P, pB)\big) = \widehat{D}_t(C/P, pC).$$

Since $\widehat{D}_t(B/P, pB)$ is a polynomial ring over B, we see that $C \in \Omega$ and $C \supsetneq B$. This is a contradiction. Therefore we have $L = K$. Let B^* be the pB-adic completion of B. Then we have that $B^* \in \Omega$, and thus B is complete. Therefore B is a coefficient ring of R.

Since $p \notin \mathbf{m}^2$, there exists a subset $\{x_2, \ldots, x_n\}$ of $\mathbf{m}$ such that $\mathbf{m} = (p, x_2, \ldots, x_n)$ $(n = \dim(R))$. Then we have $R = B[[x_2, \ldots, x_n]]$ $(\simeq B[[X_2, \ldots, X_n]])$. We put $A = B[x_2, \ldots, x_n]$, $\boldsymbol{a} = (p, x_2, \ldots, x_n)A, Q = P[x_2, \ldots, x_n]$ and $\boldsymbol{q} = (p, x_2, \ldots, x_n)Q$. Then we have $A = B \otimes_P Q$ and $\widehat{D}_t(Q/P, \boldsymbol{q})$ is the polynomial ring over Q with variables $\{\hat{r}_i(x_2), \ldots, \hat{r}_i(x_n) \mid 1 \leq i \leq t\}$, where $\hat{\boldsymbol{d}}_{Q/P} := (\hat{r}_0, \hat{r}_1, \ldots, \hat{r}_t)$. Since $\widehat{D}_t(B/P, pB) \otimes_P \widehat{D}_t(Q/P, \boldsymbol{q})$ is a polynomial ring over B, we see that $\widehat{D}_t(A/P, \boldsymbol{a}) = \widehat{D}_t(B/P, pB) \otimes_P \widehat{D}_t(Q/P, \boldsymbol{q})$ (cf. [KY, Proposition 5]) and thus $\widehat{D}_t(A/P, \boldsymbol{a})$ is a polynomial ring over A. Let A^* be the $\boldsymbol{a}$-adic completion of A. Then we see that $R = A^* \supset A$ and hence $\widehat{D}_t(R/P, \mathbf{m}) = A^* \otimes_A \widehat{D}_t(A/P, \boldsymbol{a})$. Cosequently, $\widehat{D}_t(R/P, \mathbf{m})$ is the polynomial ring over R with variables $\{\hat{d}_i(x_2), \ldots, \hat{d}_i(x_n), \hat{d}_i(z_1), \ldots, \hat{d}_i(z_s) \mid 1 \leq i \leq t\}$, where $\hat{\boldsymbol{d}}_{R/P} := (\hat{d}_0, \ldots, \hat{d}_t)$.

(2) $\Rightarrow$ (3). We consider the case of $t = 1$. By (2.8), $\widehat{D}_1(R/P, \mathbf{m})$ is a polynomial ring over R with finite variables. Therefore we see that $\widehat{D}_1(R/P, \mathbf{m})_1$ $(= \widehat{D}_P(R))$ is a finite free R-module.

(3) $\Rightarrow$ (1). Since R is complete, $\widehat{D}_P(R)$ is finitely generated. Thus the assertion follows from Theorem 9 in [NS]. □

Corollary 3.5. *In Theorem* 3.4*, let us remove the condition that R is complete, instead, let us assume that $\widehat{D}_t(R/P, \mathbf{m})_i$ is a finitely generated R-module for every $i \geq 0$ and t $(1 \leq t < \infty)$. In this case, we have the same conclusion as in Theorem* 3.4.

Proof is obtained in almost the same way as the proof of (3.4) but with some modifications.

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