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Renormalization and rigidity of polynomials of higher degree

By

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Abstract

Renormalization plays a very important role in the study of the dynamics of quadratic polynomials. We generalize renormalization to polynomials of any degree so that a lot of known properties still hold for the generalized renormalization. In particular, we generalize McMullen's result that any robust infinitely renormalizable quadratic polynomial carries no invariant line field.

1. Introduction

Renormalization is a phenomenon that a restriction of some iterate of a polynomial again behaves like a polynomial of lower degree. When a polynomial is renormalizable, the dynamics of the original polynomial comes down to the dynamics of the renormalization, which is in general simpler than that of the original polynomial.

However, when a polynomial is infinitely renormalizable, its renormalization is again infinitely renormalizable and it does not simplify the situation. McMullen considered some limit of infinite renormalizations and proved that robust infinitely renormalizable quadratic polynomial carries no invariant line field on its Julia set [Mc].

This result is a part of the following conjecture:

Conjecture 1.1 (No invariant line fields). *A rational map f carries no invariant line field on its Julia set except when f is double covered by an integral torus endomorphism.*

An *line field* on a set $E \subset \mathbb{C}$ is a Beltrami differential $\mu = \mu(z)d\overline{z}/dz$ supported on E with $|\mu| = 1$ on E. We say that f *carries an invariant line field on its Julia set* if there is a measurable Beltrami differential μ supported on a set of positive Lebesgue measure in the Julia set with $f^*\mu = \mu$.

When f is double covered by an integral torus endomorphism, the Julia set coincides with the whole sphere. Thus, in particular, this conjecture insists that no polynomial carries an invariant line field on its Julia set.

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It is known that Conjecture 1.1 implies the *density of hyperbolicity*, one of the central problems in complex dynamical systems (see [McSu] and [Mc]):

Conjecture 1.2 (Density of hyperbolicity). *In the space of all rational maps of degree d, the set of all hyperbolic maps forms an open and dense subset.*

The main theorem of this paper is the following:

Theorem 5.1 (Robust rigidity). *A robust infinitely renormalizable polynomial carries no invariant line field on its Julia set.*

McMullen has shown this theorem for quadratic polynomials. Our theorem is valid for polynomials of *any* degree. Furthermore, when the dynamics of a polynomial consists of essentially subhyperbolic dynamics and robust infinitely renormalizable ones, then this polynomial also carries no invariant line field on its Julia set (Corollary 5.2).

Let f be an infinitely renormalizable polynomial of degree $d > 1$ with connected filled Julia set. Roughly speaking, f is *robust* if infinitely many renormalizations of f have enough space around the small postcritical sets. (To define precisely, we must extract an infinitely renormalizable "part" of the polynomial, by using renormalization. See Section 4.3.)

If f is robust, the small postcritical sets of renormalizations of f shrink to a single point. So, there exists a Cantor set in the postcritical set of f which is forward invariant and measure zero. Furthermore, we properly rescale the renormalizations and obtain a proper map $g : U \to V$ as a limit.

However, if f carries an invariant line field on its Julia set, then g carries a *univalent* line field on V. Since the degree of g is greater than one, we can easily show g cannot carries any univalent invariant line field.

The paper follows almost the same pattern as McMullen's book which proves the robust rigidity in the quadratic case [Mc]. Some of the proofs in [Mc] can be applied quite similarly to our case, but by the existence of critical points outside the domain of renormalizations, a lot of statements and proofs are rather complicated.

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2. Preliminaries

2.1. Hyperbolic geometry

A Riemann surface is *hyperbolic* if its universal covering is isomorphic to the upper halfplane $\mathbb{H}$. The Schwarz-Pick lemma implies that any holomorphic map between hyperbolic Riemann surfaces does not increase the hyperbolic metric.

When a Riemann surface A satisfies $\pi_1(A) \cong \mathbb{Z}$, then A is isomorphic to either the cylinder $\mathbb{C}/\mathbb{Z}$, the punctured disk $\Delta^* = \{0 < |z| < 1\}$ or the standard annulus $A(R) = \{1 < |z| < R\}$ for some $R > 1$. In the last case, the *modulus* of A is defined by

$$\operatorname{mod}(A) = \frac{\log R}{2\pi}.$$

Otherwise, the modulus of A is defined to be infinity.

For a topological disk $V \subset \mathbb{C}$ and a subset $E \subset V$, let

$$\operatorname{mod}(E, V) = \sup \left\{ \operatorname{mod}(A) \;\middle|\; A \subset V \text{ is an annulus enclosing } E \right\}.$$

Theorem 2.1 (Collar theorem)**.** *Let γ be a simple closed geodesic on a hyperbolic surface. Then,*

- *The collar*

$$C(\gamma) = \left\{ x \;\middle|\; d(x, \gamma) < S(\ell(\gamma)) \right\}$$

is an embedded annulus, where

$$S(x) = \operatorname{arcsinh}\left(\frac{1}{\sinh(x/2)} \right),$$

d is the hyperbolic distance, and ℓ is the hyperbolic length.

- *If two simple closed geodesics γ_1 and γ_2 are disjoint, then $C(\gamma_1)$ and $C(\gamma_2)$ are disjoint.*
- *The modulus of collars $C(\gamma)$ tends from infinity to zero as the length of γ tends to zero to infinity.*

See, for example, [Bu, Chapter 4].

Lemma 2.2. *Let V be a hyperbolic surface which is topologically a disk and let $E \subset V$ be a subset with compact closure. Then*

$$\operatorname{diam}(E) \to 0 \Leftrightarrow \operatorname{mod}(E, V) \to \infty,$$
$$\operatorname{diam}(E) \to \infty \Leftrightarrow \operatorname{mod}(E, V) \to 0.$$

See [Mc, Theorem 2.4].

Lemma 2.3. *Let X be a hyperbolic surface. The logarithm of the injectivity radius is uniformly Lipschitz on X.*

See [Mc, Corollary 2.22].

Lemma 2.4. *Let X be a hyperbolic surface, and let x be a point on a loop $\delta \subset X$ which is homotopic to a geodesic γ. Then a lower bound on $\ell(\gamma)$ and an upper bound on $\ell(\delta)$ gives an upper bound on the distance from x to γ.*

See [Mc, Theorem 2.23].

Lemma 2.5. *Let X be a finitely connected hyperbolic surface with one cusp, whose remaining ends are cut off by geodesics $\gamma_1, \ldots, \gamma_n$, $n > 1$. Suppose the length of every γ_i is greater than $L > 0$. Then there are two distinct geodesics γ_j and γ_k such that the hyperbolic distance $d(\gamma_j, \gamma_k)$ is bounded by the constant which depends only on L.*

See [Mc, Theorem 2.24].

2.2. Dynamics of rational maps

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d > 1$. The *Fatou set* of f is the set of all $z \in \hat{\mathbb{C}}$ such that $\{f^n \mid n > 0\}$ is normal on some neighborhood of z. The *Julia set* $J(f)$ is the complement of the Fatou set. When f is a polynomial, the *filled Julia set* $K(f)$ is defined by:

$$K(f) = \left\{ z \in \mathbb{C} \,\middle|\, \{f^n(z)\}_{n>0} \text{ is bounded} \right\}.$$

Then the Julia set is equal to the boundary of $K(f)$.

Let $C(f)$ be the set of all critical points of f. The *postcritical set* $P(f)$ is the closure of the set of critical values of f^n, that is,

$$P(f) = \overline{\bigcup_{n>0} f^n(C(f))}.$$

Lemma 2.6. *For every point $x \in J(f)$ whose forward orbit does not intersect $P(f)$,*

$$\|(f^n)'(x)\| \to \infty$$

with respect to the hyperbolic metric on $\hat{\mathbb{C}} \setminus P(f)$.

See [Mc, Theorem 3.6].

Lemma 2.7. *Let f be a rational map satisfying $\#P(f) \geq 3$. Let γ be a path joining two points $x_1, x_2 \in \mathbb{C}$ such that $f(\gamma)$ does not intersect $P(f)$, and let ℓ be the parameterized length of $f(\gamma)$ in the hyperbolic metric on $\hat{\mathbb{C}} \setminus P(f)$. Then*:

$$\|f'(x_1)\|^{\alpha} \geq \|f'(x_2)\| \geq \|f'(x_1)\|^{1/\alpha},$$

where $\alpha = \exp(C_1 \ell)$ for a universal constant $C_1 > 0$; and

$$\frac{1}{C_2} \leq \frac{\|f'(x_1)\|}{\|f'(x_2)\|} \leq C_2,$$

where $C_2 > 0$ is a constant which depends only on ℓ and the injectivity radius of $\hat{\mathbb{C}} \setminus P(f)$ at $f(x_1)$.

See [Mc, Theorem 3.8].

Lemma 2.8. *Any rational map f of degree more than one satisfies one of the following*:

(1) *$J(f) = \hat{\mathbb{C}}$ and the action of f on $\hat{\mathbb{C}}$ is ergodic.*

(2) *the spherical distance $d(f^n(x), P(f)) \to 0$ for almost every x in $J(f)$ as $n \to \infty$.*

See [Mc, Theorem 3.9].

2.3. Polynomial-like maps

A *polynomial-like map* is a triple (f, U, V) such that $f : U \to V$ is a holomorphic proper map between disks in $\mathbb{C}$ and U is a relatively compact subset of V. The *filled Julia set* $K(f, U, V)$ is defined by

$$K(f, U, V) = \bigcap_{n=1}^{\infty} f^{-n}(V)$$

and the *Julia set* $J(f, U, V)$ is equal to the boundary of $K(f, U, V)$. We denote by $C(f, U, V)$ the set of all critical points and the *postcritical set* $P(f, U, V)$ is defined similarly as in the case of rational maps.

Every polynomial-like map (f, U, V) of degree d is hybrid equivalent to some polynomial g of degree d. That is, there is a quasiconformal conjugacy ϕ from f to g defined near their respective filled Julia sets and satisfies $\overline{\partial}\phi = 0$ on $K(f, U, V)$ (see [DH]).

For $m > 0$, let $\mathrm{Poly}_d(m)$ be the set of all polynomials of degree d and all polynomial-like maps of degree d with $\mathrm{mod}(U, V) > m$. We use the Carathéodory topology for the topology of $\mathrm{Poly}_d(m)$.

Lemma 2.9. $\mathrm{Poly}_d(m)$ *is compact up to affine conjugacy. Namely, any sequence* $(f_n, U_n, V_n) \in \mathrm{Poly}_d(m)$ *which is normalized so* $U_n \supset \{|z| < r\}$ *for some* $r > 0$ *and so the Euclidean diameter of* $K(f_n, U_n, V_n)$ *is equal to one, has a convergent subsequence.*

See [Mc, Theorem 5.8].

Moreover, if $(f, U, V) \in \mathrm{Poly}_d(m)$ has no attracting fixed point, then

$$\mathrm{diam}\, K(f, U, V) \le C\, \mathrm{diam}\, P(f, U, V)$$

for some C depending only on d and m in the Euclidean metric [Mc, Corollary 5.10].

The following two lemmas are used repeatedly in the next section.

Lemma 2.10. *For* $i = 1, 2$, *let* (f_i, U_i, V_i) *be polynomial-like maps of degree* d_i. *Assume* $f_1 = f_2 = f$ *on* $U = U_1 \cap U_2$. *Let* U' *be a component of* U *with* $U' \subset f(U') = V'$. *Then* $f : U' \to V'$ *is a polynomial-like map of degree* $d \le \min(d_1, d_2)$ *and*

$$K(f, U', V') = K(f_1, U_1, V_1) \cap K(f_2, U_2, V_2) \cap U'.$$

Moreover, if $d = d_i$, *then* $K(f, U', V') = K(f_i, U_i, V_i)$.

See [Mc, Theorem 5.11].

Lemma 2.11. *Let* f *be a polynomial with connected filled Julia set. For any polynomial-like restriction* (f^n, U, V) *of degree greater than one with connected filled Julia set* K_n,

(1) *the Julia set of* (f^n, U, V) *is contained in the Julia set of* f.
(2) *For any closed connected set* $L \subset K(f)$, $L \cap K_n$ *is also connected.*

See [Mc, Theorem 6.13].

We also need the following results:

Lemma 2.12. *Let* $f : U \to V$ *be a critically compact proper map of degree* $d > 1$. *There is a constant* M_d *(depending only on* d*) such that when* $\operatorname{mod}(P(f), V) > M_d$, *then one of the following holds*:

- f *has an attracting fixed point in* U.
- *there is a restriction* $f : U' \to V'$ *which is a polynomial-like map of degree* d *with connected Julia set.*

Here $P(f) \subset U' \subset U$, *and* U' *can be chosen so that*

$$\operatorname{mod}(U', V') > m_d(\operatorname{mod}(P(f), V)) > 0$$

where $m_d(x) \to \infty$ *as* $x \to \infty$.

See [Mc, Theorem 5.12].

Lemma 2.13. *Let* (U_n, u_n) *and* (V_n, v_n) *be sequences of pointed disks converging to* (U, u) *and* (V, v) *respectively. Assume neither* U *nor* V *are equal to* $\mathbb{C}$. *Let* $f_n : (U_n, u_n) \to (V_n, v_n)$ *be a sequence of proper maps of degree* d. *Then after passing to a subsequence,* f_n *converges to a proper map* $f : (U, u) \to (V, v)$ *of degree less than or equal to* d.

Furthermore, if there is a compact set $K \subset U$ *such that the critical points* $C(f_n) \subset K$ *for all* n *sufficiently large, then* f *has degree* d.

See [Mc, Theorem 5.6].

2.4. External rays

Let K be a connected compact subset of $\mathbb{C}$ which does not disconnect the plane. By the Riemann mapping theorem, there exists a unique conformal map

$$\phi : \left(\mathbb{C} \setminus \overline{\Delta}\right) \to (\mathbb{C} \setminus K)$$

with $\phi(z)/z \to \lambda > 0$ as $z \to \infty$. An *external ray* for K is a path

$$R_t = \left\{\phi(r \exp(2\pi i t)) \,\middle|\, 1 < r < \infty\right\}.$$

t is called the *angle* of the ray R_t.

We say R_t *lands at* some point $x \in K$ if $\lim_{r \searrow 1} \phi(r \exp(2\pi i t)) = x$. x is called a *landing point* and t is called an *external angle* or a *landing angle* for x.

It can be easily verified that landing points are dense in ∂K. Lindelöf's theorem implies that if some path γ in $\mathbb{C} \setminus K$ converges to $x \in \partial K$, then there exists an external ray landing at x from the same direction. Namely, if $\phi^{-1}(\gamma)$ converges to $\exp(2\pi i t)$, then R_t lands at x.

Let f be a monic polynomial of degree d. Then the conformal map

$$\phi : (\mathbb{C} \setminus \overline{\Delta}) \to (\mathbb{C} \setminus K(f))$$

normalized as above gives a conjugacy between z^d and f. Thus for any external ray R_t for $K(f)$, we have $f(R_t) = R_{dt}$. We say R_t is *periodic* if R_t is invariant by f^n for some $n > 0$. This implies that $d^n t \equiv t \bmod 1$. The angle of every periodic ray is rational and for every rational number t, the ray R_t is eventually periodic.

Theorem 2.14. *Every periodic ray land at a repelling or parabolic periodic point of f. Conversely, every repelling or parabolic periodic point x is a landing point of some ray. Moreover, every ray landing at x is periodic with the same period.*

See, for example, [Mi3, Section 18] or [St, Section 6.1].

For $k = 0, \ldots, d-2$, the ray $R_{k/(d-1)}$ is invariant by f, so it lands at some fixed point of f. Such fixed points are called β-fixed points and others are α-fixed points. Note that different invariant rays may land at the same fixed point. So the number of β-fixed points is at most $d-1$.

Since every polynomial-like map with connected filled Julia set is hybrid equivalent to some polynomial of the same degree, β-fixed points and α-fixed points of a polynomial-like map still make sense.

3. Renormalization

3.1. Definition of renormalization

Let f be a polynomial of degree d with connected Julia set. Fix a critical point $c_0 \in C(f)$.

Definition. f^n is called *renormalizable* about c_0 if there exist open disks $U, V \subset \mathbb{C}$ satisfying the following:

(1) c_0 lies in U.

(2) (f^n, U, V) is a polynomial-like map with connected filled Julia set.

(3) For each $c \in C(f)$, there is at most one i, $0 < i \leq n$, such that $c \in f^i(U)$.

(4) $n > 1$ or $U \not\supset C(f)$.

A *renormalization* is a polynomial-like restriction (f^n, U, V) as above. We call n the period of a renormalization (f^n, U, V).

First two conditions are the same as in the quadratic case. The others are needed to exclude trivial polynomial-like restrictions, that is, two times iteration of a renormalization and f itself. In the quadratic case, these two conditions are equivalent to the assumption that a renormalization is quadratic-like and $n > 1$, respectively.

Note that the degree of a renormalization of f is not greater than 2^d.

Notation. For a renormalization $\rho = (f^n, U, V)$ and $i = 1, \ldots, n$ (or i may be regarded as an element of $\mathbb{Z}/n\mathbb{Z}$),

- Let $n(\rho) = n$, $U(\rho) = U$ and $V(\rho) = V$.
- The filled Julia set of ρ is denoted by $K(\rho)$, the Julia set by $J(\rho)$, and the postcritical set by $P(\rho)$.

• The *i-th small filled Julia set* is denoted by $K(\rho, i) = f^i(K(\rho))$ and the *i-th small Julia set* $J(\rho, i) = f^i(J(\rho))$.

• The *i-th small critical set* $C(\rho, i) = K(\rho, i) \cap C(f)$. Clearly, $C(\rho, i)$ may be empty for $0 < i < n$ (however, by definition, $C(\rho, n)$ is nonempty).

• $\mathcal{K}(\rho) = \bigcup_{i=1}^{n} K(\rho, i)$ is the union of the small filled Julia sets. Similarly, define $\mathcal{J}(\rho) = \bigcup_{i=1}^{n} J(\rho, i)$.

• $\mathcal{C}(\rho) = \bigcup_{i=1}^{n} C(\rho, i)$ is the set of critical points which appear in a renormalization ρ.

• $\mathcal{P}(\rho) = \overline{\bigcup_{k>0} f^k(\mathcal{C}(\rho))} \subset P(f) \cap \mathcal{K}(\rho)$.

• The *i-th small postcritical set* is denoted by

$$P(\rho, i) = K(\rho, i) \cap \mathcal{P}(\rho).$$

• Let $V(\rho, i) = f^i(U)$ and $U(\rho, i)$ be the component of $f^{i-n}(U)$ contained in $V(\rho, i)$. Then $(f^n, U(\rho, i), V(\rho, i))$ is a polynomial-like map of the same degree as (f^n, U, V). Moreover, it is also a renormalization of f if $C(\rho, i)$ is nonempty.

Let ρ and ρ' be renormalizations. Define an equivalence relation $\sim$ by

$$\rho \sim \rho' \Leftrightarrow n(\rho) = n(\rho') \quad \text{and} \quad K(\rho) = K(\rho').$$

This implies that the dynamics of ρ and ρ' are equal. Let

$$\mathcal{R}(f, c_0) = \{\text{renormalizations of some iterates of } f \text{ about } c_0\} / \sim .$$

and for a subset $C_R \subset C(f)$ containing c_0,

$$\mathcal{R}(f, c_0, C_R) = \left\{ \rho \in \mathcal{R}(f, c_0) \;\middle|\; \mathcal{C}(\rho) = C_R \right\}.$$

We confuse an element in $\mathcal{R}(f, c_0)$ with its representative and write like as $\rho = (f^n, U, V) \in \mathcal{R}(f, c_0)$.

The next proposition shows that this equivalence class is determined only by periods and small critical sets.

Proposition 3.1. *Let $\rho = (f^n, U^1, V^1)$ and $\rho' = (f^n, U^2, V^2)$ be renormalizations of the same period. When $C(\rho, i) = C(\rho', i)$ for any i $(1 \le i \le n)$, then their filled Julia sets are equal.*

Proof. By Lemma 2.11, $K = K(\rho) \cap K(\rho')$ is connected. Let U be the component of $U^1 \cap U^2$ containing K and let $V = f^n(U)$. Since V contains $f(K) = K$, V contains U. By Lemma 2.10, (f^n, U, V) is a polynomial-like map with filled Julia set K. By the assumption, the sets of critical points of these three maps are equal, so we have $K = K(\rho) = K(\rho')$. □

Proposition 3.2. *Let $\rho_n = (f^n, U_n, V_n)$ and $\rho_m = (f^m, U_m, V_m) \in \mathcal{R}(f, c_0)$. Then there exists $\rho_l = (f^l, U_l, V_l) \in \mathcal{R}(f, c_0)$ with filled Julia set $K(\rho_l) = K(\rho_n) \cap K(\rho_m)$, where l is the least common multiple of n and m.*

Remark 3.3. Even when two of the n, m and l is equal, the renormalizations ρ_n, ρ_m and ρ_l should be considered to be different.

Proof. By Lemma 2.11, $K = K(\rho_n) \cap K(\rho_m)$ is connected. Let

$$\tilde{U}_n = \left\{ z \in U_n \;\middle|\; f^{jn}(z) \in U_n \text{ for } j = 1, \dots, \frac{l}{n} - 1 \right\},$$
$$\tilde{U}_m = \left\{ z \in U_m \;\middle|\; f^{jm}(z) \in U_m \text{ for } j = 1, \dots, \frac{l}{m} - 1 \right\}.$$

Then $(f^l, \tilde{U}_n, V_n)$ and $(f^l, \tilde{U}_m, V_m)$ are both polynomial-like. Let U_l be the component of $\tilde{U}_n \cap \tilde{U}_m$ which contains K. Then by Lemma 2.10, $(f^l, U_l, f^l(U_l))$ is a polynomial-like map with filled Julia set K.

Now we should check the condition 3 of the definition of renormalization. Suppose $c \in C(\rho_l, i)$. Then $c \in C(\rho_l, j)$ is equivalent to $j \equiv i \pmod n$ and $j \equiv i \pmod m$, which means $j \equiv i \pmod l$. Therefore, $(f^l, U_l, f^l(U_l))$ is a renormalization with filled Julia set K. □

Definition. For $\rho \in \mathcal{R}(f, c_0)$, the *intersecting set* $I(\rho)$ is defined by

$$I(\rho) = K(\rho) \cap \left(\bigcup_{i=1}^{n(\rho)-1} K(\rho, i) \right).$$

When $n(\rho) = 1$, $I(\rho)$ is defined to be empty. We say ρ is *intersecting* if $I(\rho)$ is nonempty.

Although we now define intersecting "set", it consists of at most one point.

Proposition 3.4. *If $\rho = (f^n, U, V) \in \mathcal{R}(f, c_0)$ is intersecting, then $I(\rho)$ consists of only one point which is a repelling fixed point of f^n.*

Proof. Suppose $I(\rho)$ is nonempty. Then there exists some i $(1 \le i < n)$ such that $E = K(\rho) \cap K(\rho, i)$ is nonempty. E is connected by Lemma 2.11.

Let W be the component of $U \cap U(\rho, i)$ containing E. By Lemma 2.10, $f^n : W \to f^n(W)$ is polynomial-like. Furthermore, its degree is one (here we use the third condition in the definition of the renormalization). So by the Schwarz lemma, E consists of a single repelling fixed point x of f^n. Thus $I(\rho)$ consists of finite number of points.

Suppose that x_1, $x_2 \in I(\rho)$ and $x_1 \neq x_2$. For each $i = 1, \dots, n$ and $k = 1, 2$, $f^i(x_k)$ is contained in at least two $K(\rho, j)$'s. So the set

$$E = \{ f^i(x_k) \mid i = 1, \dots, n,\, k = 1, 2 \}$$

consists of at most n points.

Consider the graph which consists of vertices E and edges $e(i)$ joining $f^i(x_1)$ to $f^i(x_2)$ for $i = 1, \dots, n$. Since the number of edges is not less than that of vertices, the graph contains at least one cycle $\langle e(i_1), \ldots, e(i_k)\rangle$. Then $L = K(\rho, i_2) \cup \cdots \cup K(\rho, i_k)$ is connected. But $K(\rho, i_1) \cap L$ is disconnected (it consists of $f^{i_1}(x_1)$, $f^{i_1}(x_2)$ and finite number of points) and this contradicts Lemma 2.11. □

Although small Julia sets of a renormalization can meet at a repelling periodic point, the period of such point tends to infinity as the period of renormalization tends to infinity.

Theorem 3.5 (High periods). *For fixed $p > 0$, there are only finitely many $\rho \in \mathcal{R}(f, c_0)$ such that $K(\rho)$ contains a periodic point of period p.*

To prove Theorem 3.5, the next lemma is essential:

Lemma 3.6. *Let $\rho = (f^n, U, V) \in \mathcal{R}(f, c_0)$. Assume the filled Julia set $K(\rho)$ contains a fixed point x of f. Let q be the number of rays landing at x, p be the period of the rays landing at x and q_n be the number of components of $K(\rho) \setminus \{x\}$. Then $nq_n \leq q$ and $n|p$.*

Proof. First, note that $\mathcal{K}(\rho)\setminus\{x\}$ has exactly nq_n components. Thus nq_n external rays $R'_{t'_j}$ $(j = 1, \ldots, nq_n)$ for $\mathcal{K}(\rho)$ land at x.

Let $\psi : \mathbb{C} \to \mathbb{C}$ be the inverse of the unique solution of Schröder's equation

$$\psi(0) = x,$$
$$\psi(\lambda z) = f(\psi(z)),$$
$$\lambda = f'(x).$$

and let $\tilde{K} = \psi^{-1}(K(f))$ and $\tilde{\mathcal{K}_n} = \psi^{-1}(\mathcal{K}(\rho))$. By the forward invariance of $K(f)$ and $\mathcal{K}(\rho)$, we have

$$\lambda\tilde{K} = \tilde{K},$$
$$\lambda\tilde{\mathcal{K}}_n = \tilde{\mathcal{K}}_n.$$

Since q external rays for $K(f)$ land at x, $\mathbb{C} \setminus \tilde{K}$ has exactly q components, say $U_1, \ldots, U_q$. Each component U_i corresponds to the ray R_{t_i} landing at x (see [St, Section 6.1]).

For $j = 1, \ldots, nq_n$, we denote by γ the path component of $\psi^{-1}(R'_{t'_j})$ converging to 0. Let U'_j be the component of $\mathbb{C} \setminus \tilde{\mathcal{K}}_n$ containing γ. Each U_i lies in U'_j for some j. Indeed, since x is accessible from U_i, x is also accessible from the component U' of $\mathbb{C} \setminus \tilde{\mathcal{K}}_n$ which contains U_i. Therefore, U' must be one of U'_j's. Define a map $h : \{1, \ldots, q\} \to \{1, \ldots, nq_n\}$ by $U_i \subset U'_{h(i)}$.

Let $F : \{1, \ldots, q\} \to \{1, \ldots, q\}$ and $F' : \{1, \ldots, nq_n\} \to \{1, \ldots, nq_n\}$ be the map defined by

$$\lambda U_i = U_{F(i)},$$
$$\lambda U'_j = U'_{F'(j)}.$$

Note that since x is accessible from $\lambda U'_j$, this F' is well-defined. Since the map $z \mapsto \lambda z$ preserves the cyclic order of U_i's and U'_j's, F and F' are permutations and every element in $\{1, \dots, q\}$ has the same period p for F. Similarly, let p' be the period of $j \in \{1, \dots, nq_n\}$ for F'. Furthermore, by definition, $h \circ F = F' \circ h$, so $p'|p$.

We claim that h is surjective. Then we immediately obtain that $nq_n \leq q$. Suppose $U'_j \subset \tilde{K}$ for some $j \in \{1, \dots, nq_n\}$. Take sufficiently small $r > 0$ so that ψ is univalent on $B(r) = \{|z| < r\}$. Since accessible points from U'_j are dense in $\partial U'_j$, there exists an accessible point $y \in \partial U'_j \cap B(r)$. Then we can take a path $\gamma : [0,1] \to \overline{B(r) \cap U'_j}$ joining 0 and y such that $\gamma((0,1))$ lies in $B(r) \cap U'_j$. $\psi(\gamma)$ is a path in $K(f)$ joining x and $\psi(y)$. Since $\psi(y)$ lies in $K(\rho, i)$ for some i, this contradicts Lemma 2.11. Hence U'_j intersects $\mathbb{C} \setminus \tilde{K} = U_1 \cup \cdots \cup U_q$. Thus U'_j contains U_i for some i, so $h(i) = j$.

Now we will show $n|p$. Since $p'|p$, we need only show that $n|p'$. By the map $z \mapsto \lambda z$, every component E of $\tilde{\mathcal{K}}_n \setminus \{0\}$ has period p'. Since ψ is univalent on $B(r)$ and each component of $\mathcal{K}(\rho) \setminus \{x\}$ lies in a unique $K(\rho, i)$,

$$\psi(B(0, \lambda^{-p'} r) \cap E) \subset \psi(B(0, r) \cap E) \subset K(\rho, i)$$

for some i. Thus we have

$$\begin{aligned} f^{p'}(\psi(B(0, \lambda^{-p'} r) \cap E)) &= \psi(\lambda^{p'}(B(0, \lambda^{-p'} r) \cap E)) \\ &= \psi(B(0, r) \cap E) \subset K(\rho, i). \end{aligned}$$

Therefore, $(f^{p'}(K(\rho, i)) \cap K(\rho, i)) \setminus \{x\}$ is nonempty. By Proposition 3.4, we have $f^{p'}(K(\rho, i)) = K(\rho, i)$ and $n|p'$. □

Corollary 3.7. *Let q be the number of rays landing at some fixed point x. If $\rho \in \mathcal{R}(f, c_0)$ satisfies $n(\rho) > q$, then $K(\rho)$ does not contain x. In particular, if $n(\rho) > 1$, then $K(\rho)$ does not contain β-fixed point of f.*

Now we give the proof of Theorem 3.5.

Proof of Theorem 3.5. Since there are only finite number of critical points and periodic points of period p, it is enough to consider renormalizations about c_0 and one periodic point w of period p.

Suppose $w \in K(\rho_n)$ for some $\rho_n = (f^n, U_n, V_n) \in \mathcal{R}(f, c_0)$ with $n > p$ (if such a renormalization does not exist, then we are done). Since $K(\rho_n, p)$ contains $f^p(w) = w$, $I(\rho_n) = \{w\}$ and w is a repelling fixed point of f^n.

Let $\rho_m = (f^m, U_m, V_m) \in \mathcal{R}(f, c_0)$. Assume m is greater than n. Then by Proposition 3.2, there exists $\rho_l = (f^l, U_l, V_l)$ in $\mathcal{R}(f, c_0)$ such that l is the least common multiple of n and m and $K(\rho_l) = K(\rho_n) \cap K(\rho_m)$.

Let g be a polynomial hybrid equivalent to (f^n, U_n, V_n). The periodic point w of f corresponds to a repelling fixed point x of g. Let q be a number of rays landing at x. Then ρ_l corresponds to a renormalization of g of period l/n. So by Corollary 3.7, its filled Julia set does not contain x whenever $l/n > q$. Thus $K(\rho_m)$ does not contain w whenever $m > qn$. □

3.2. Simple renormalization

Since a repelling fixed point separates the filled Julia set into finite number of components, the number of components of $K(\rho) \setminus I(\rho)$ is finite. We classify renormalizations into two types; We say a renormalization is *simple* if $K(\rho) \setminus I(\rho)$ is connected, and *crossed* if it is disconnected. Let $\mathcal{SR}(f, c_0)$ be the set of all simple renormalizations in $\mathcal{R}(f, c_0)$. Similarly, $\mathcal{SR}(f, c_0, C_R)$ is the set of all $\rho \in \mathcal{SR}(f, c_0)$ with $\mathcal{C}(\rho) = C_R$.

In the next section, we will show any infinitely renormalizable polynomial has infinitely many simple renormalizations. So simple renormalizations play a very important role in the case of infinitely renormalizable polynomials.

However, there even exist finitely renormalizable polynomials which is not simply renormalizable. See [Mc, Section 7.4].

Proposition 3.8. *For two renormalizations $\rho_n = (f^n, U_n, V_n) \in \mathcal{R}(f, c_0)$ and $\rho_m = (f^m, U_m, V_m) \in \mathcal{SR}(f, c_0)$, either n divides m or m divides n.*

Proof. Let l be the greatest common divisor of n and m. Suppose the conclusion of the proposition is false. Then l is less than n and m.

Since $K(\rho_n)$ intersects $K(\rho_m)$ (both contain c_0), $f^i(K(\rho_n))$ intersects $f^i(K(\rho_m))$ for any $i > 0$. Thus $K(\rho_n, l)$ intersects $K(\rho_m, l)$, $K(\rho_n)$ intersects $K(\rho_m, l)$ and $K(\rho_n, l)$ intersects $K(\rho_m)$ (note that $f^{i+n}(K(\rho_n)) = f^i(K(\rho_n))$).

Therefore, $L = K(\rho_m) \cup K(\rho_n, l) \cup K(\rho_m, l)$ is connected. $K(\rho_n) \cap L$ is also connected by Lemma 2.11. Since L is closed and $K(\rho_n) \cap K(\rho_n, l)$ consists of at most one point, $K(\rho_n) \cap (K(\rho_m) \cup K(\rho_m, l))$ is also connected. Thus $K(\rho_n) \cap K(\rho_m) \cap K(\rho_m, l)$ is nonempty. By Proposition 3.4, we have $K(\rho_m) \cap K(\rho_m, l) = \{x\}$ and $x \in K(\rho_n)$.

Since ρ_m is simple, x is a β-fixed point of ρ_m. By Proposition 3.2, there exists a renormalization $\rho_L = (f^L, U, V)$ with $K(\rho_L) = K(\rho_n) \cap K(\rho_m)$ for $L = nm/l$ (least common multiple of n and m). But L is greater than one and this contradicts Corollary 3.7. □

Proposition 3.9. *For two renormalizations $\rho_n = (f^n, U_n, V_n) \in \mathcal{R}(f, c_0)$ and $\rho_m = (f^m, U_m, V_m) \in \mathcal{SR}(f, c_0)$ with $n \geq m$ (then m divides n by Proposition 3.8) and $\mathcal{C}(\rho_n) \subset \mathcal{C}(\rho_m)$, $K(\rho_m)$ contains $K(\rho_n)$.*

Here, we should also consider ρ_n and ρ_m are different even if $n = m$ (but in this case, it turns out that these two renormalization are equal).

By Proposition 3.2, there exists a renormalization of period n whose filled Julia set is contained in $K(\rho_m)$. However this filled Julia set might be a proper subset of $K(\rho_n)$. This proposition asserts that $K(\rho_n)$ itself is contained in $K(\rho_m)$.

Proof. Assume $K(\rho_n)$ is not contained in $K(\rho_m)$. By Proposition 3.2, there exists a renormalization $\rho'_n = (f^n, U'_n, V'_n)$ with filled Julia set $K(\rho'_n) = K(\rho_m) \cap K(\rho_n)$, which is a proper subset of $K(\rho_n)$.

If $\mathcal{C}(\rho'_n) = \mathcal{C}(\rho_n)$, then $K(\rho'_n) = K(\rho_n)$ by Lemma 2.10 and this is a contradiction. Thus there exists a critical point $c_1 \in \mathcal{C}(\rho_n) \setminus \mathcal{C}(\rho'_n)$. Therefore, there exists i_0 with $1 \le i_0 < m$ such that $K(\rho_m, i) \cap K(\rho_n, j)$ is written as the filled Julia set of some polynomial-like map of degree greater than one whenever $i - j \equiv i_0 \bmod m$.

Let $k_1 > 0$ be the minimum number such that $\#(K(\rho_m) \cap K(\rho_n, k_1 i_0)) > 1$. Such k_1 exists and $k_1 i_0 \not\equiv 0 \pmod m$ because this condition is fulfilled when $k_1 = m - 1$.

Let

$$L = K(\rho_n) \cup \bigcup_{k=1}^{k_1} \left(K(\rho_m, ki_0) \cup K(\rho_n, ki_0)\right).$$

Then L is connected and

$$L \cap K(\rho_m) = \left(\bigcup_{k=0}^{k_1} (K(\rho_n, ki_0) \cap K(\rho_m))\right) \cup \left(\bigcup_{k=1}^{k_1} K(\rho_m, ki_0) \cap K(\rho_m)\right)$$

is also connected by Lemma 2.11. Then

$$(K(\rho_n) \cap K(\rho_m)) \cup (K(\rho_n, k_1 i_0) \cap K(\rho_m))$$

is connected because $K(\rho_n, ki_0) \cap K(\rho_m)$ consists of at most one point for $k = 1, \dots, k_1 - 1$ and so does $\bigcup_{k=1}^{k_1} K(\rho_m, ki_0) \cap K(\rho_m) \subset I(\rho_m)$.

Thus $K(\rho_n) \cap K(\rho_n, k_1 i_0) \cap K(\rho_m)$ is nonempty. Since it is contained in $I(\rho_n)$, so $I(\rho_n) = \{x\}$ where x is a repelling fixed point of f^n and

$$K(\rho_n) \cap K(\rho_n, k_1 i_0) \cap K(\rho_m) = \{x\}.$$

Quite similarly, there exists some $k_2 > 0$ such that

$$K(\rho_m) \cap K(\rho_m, -k_2 i_0) \cap K(\rho_n) = \{y\}$$

for some fixed point y of f^m. Since the period of y by f is strictly less than m, we have $y = x$. However, ρ_m is simple, so by Corollary 3.7, n must be equal to m.

Let ρ_n'' be a polynomial-like map of degree more than one such that

$$K(\rho_m) \cap K(\rho_n, k_1 i_0) = K(\rho_n'').$$

We will deduce the contradiction by using the fact that $K(\rho_n'), K(\rho_n'') \subset K(\rho_m)$ and that $K(\rho_n') \cap K(\rho_n'') = \{x\} = I(\rho_m)$. Let K be the component of $K(f) \setminus \{x\}$ containing the connected set $K(\rho_m) \setminus \{x\}$. For sufficiently small $r > 0$, there exists a component K' of $K \setminus B(x, r)$ which intersects $K(\rho_n')$ and $K(\rho_n'')$ where $B(x, r) = \{|z - x| < r\}$. (Otherwise, there exist two rays which land at x and which separate $K(\rho_n') \setminus \{x\}$ and $K(\rho_n'') \setminus \{x\}$. Note that f^m is conjugate to $z \mapsto (f^m)'(x) \cdot z$ near x.)

Let E be the union of components of $\overline{K' \setminus K(\rho_n')}$ which intersect $K(\rho_n'')$. Then $E \cap K(\rho_n')$ is nonempty. In fact, assume this set is empty. Then, since

$$F = \left(\overline{K' \setminus K(\rho_n'')} \setminus E\right) \cup (K(\rho_n') \cap K')$$

is closed and K' is equal to $E \sqcup F$, K' is disconnected and this is a contradiction.

We define $E' = E \cup K(\rho''_n)$. E' is connected and contained in $K(f)$. Moreover, $E' \cap K(\rho'_n) = (E \cap K(\rho'_n)) \cup \{x\}$ is disconnected and this contradicts Lemma 2.11. □

When we apply Proposition 3.9 to the case $m = n$ and $\mathcal{C}(\rho_m) = \mathcal{C}(\rho_n)$, then we conclude that a simple renormalization $\rho \in \mathcal{SR}(f, c_0)$ is completely determined by its period $n(\rho)$ and $\mathcal{C}(\rho)$. (cf. Proposition 3.1.)

For $c_0 \in C_R \subset C(f)$, define

$$\mathrm{sr}(f, C_R) = \left\{ n(\rho) \;\middle|\; \rho \in \mathcal{SR}(f, c_0, C_R) \right\}.$$

Clearly, it is independent of the choice of c_0. By the above two propositions, we have:

Theorem 3.10. *For $C_R \subset C(f)$, $\mathrm{sr}(f, C_R)$ is totally ordered with respect to division. Moreover, elements of $\mathcal{SR}(f, c_0, C_R)$ are uniquely determined by their period, and their filled Julia sets form a decreasing sequence.*

The next result will be used for constructing simple renormalizations.

Proposition 3.11. *Let $\rho_n = (f^n, U, V)$ a renormalization and let g be a polynomial hybrid equivalent to ρ_n. Suppose g^m is renormalizable. Then f has a renormalization of period $l = nm$ which is of the same type of the renormalization of g^m.*

Proof. Let $\psi : V \to \mathbb{C}$ be a hybrid conjugacy from ρ_n to g and let $\rho'_m = (g^m, U', V')$ be a renormalization of g^m. We may assume $V' \subset \psi(V)$. Then $\rho_l = (f^l, \psi^{-1}(U'), \psi^{-1}(V'))$ is a renormalization of f^l.

Now we will confirm ρ_l and ρ'_m are of the same type. First note that $K(\rho_l, i)$ lies in $K(\rho_n, j)$ if and only if $j \equiv i \bmod n$. Suppose $K(\rho_l) \cap K(\rho_l, i) = \{x\}$ for some $0 < i < l$. If $i \equiv 0 \bmod n$, then $K(\rho_l, i) \subset K(\rho_n)$. Since ψ is conformal map near $K(\rho_n)$, clearly ρ_l and ρ'_m are of the same type. Otherwise, $K(\rho_l, i)$ lies in $K(\rho_n, j)$ for some $0 < j < n$. Thus $I(\rho_n) = \{x\}$ and x is a fixed point of f^n. So $K(\rho_l) \cap K(\rho_l, n) = \{x\}$ and it is the previous case. □

3.3. Examples

Now we present an example of finitely renormalizable polynomials. Examples of infinitely renormalizable polynomials are given in Section 4.

Let

$$f(z) = z^3 - \frac{3}{4}z - \frac{\sqrt{7}}{4}i.$$

Then $C(f) = \{\pm 1/2\}$ and $\pm 1/2$ are both periodic of period 2. Let $W_\pm$ be the Fatou component which contains $\pm 1/2$, respectively. Each of them is a superattracting basin of period 2.

Every renormalization (f^n, U, V) must satisfy that U contains either W_- or W_+. So $n \leq 2$ and by symmetry, we will consider only the case $U \supset W_-$.

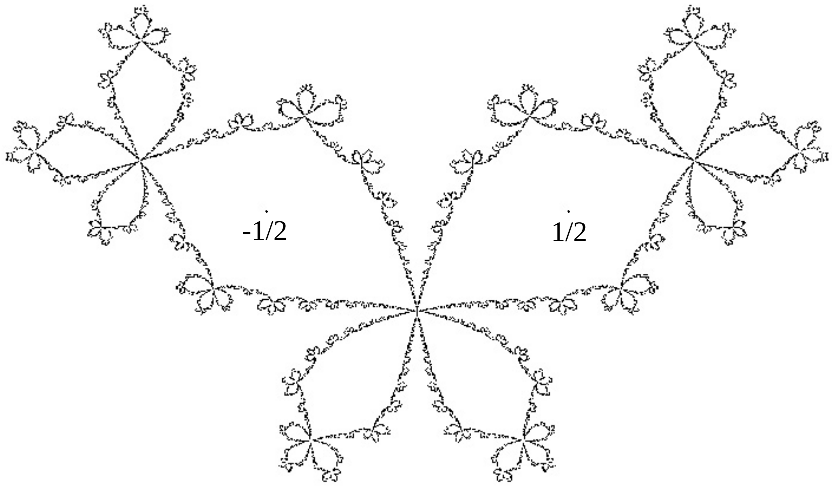

Figure 1. The Julia set of f

Case I. Let K be the connected component of the closure of $\bigcup_{n>0} f^{-n}(W_-)$ which contains W_- and let U_1 be a small neighborhood of K and U_1' the component of $f^{-1}(U_1)$ contained in U_1. Then $\rho_1 = (f, U_1', U_1)$ is a renormalization with filled Julia set $K(\rho_1) = K_1$ which is hybrid equivalent to $z \mapsto z^2 - 1$.

Case II. Let $U_{2,1}$ be a small neighborhood of W_- and $U_{2,1}'$ the component of $f^{-2}(U_{2,1})$ contained in $U_{2,1}$. Then $\rho_{2,1} = (f^2, U_{2,1}', U_{2,1})$ is a renormalization with filled Julia set $K(\rho_{2,1}) = \overline{W_-}$, which is hybrid equivalent to $z \mapsto z^2$.

Case III. Let $K_{2,2}$ be the connected component of

$$\overline{\bigcup_{n>0} f^{-2n}(W_- \cup W_+)}$$

which contains W_- and let $U_{2,2}$ be a small neighborhood of $K_{2,2}$ and $U_{2,2}'$ the component of $f^{-2}(U_{2,2})$ contained in $U_{2,2}$. Then $\rho_{2,2} = (f^2, U_{2,2}', U_{2,2})$ is a renormalization with filled Julia set $K_{2,2}$, which is hybrid equivalent to $z \mapsto -z^3 + (3/2)z$.

Case IV. Let $K_{2,3}$ be the connected component of

$$\overline{\bigcup_{n>0} f^{-2n}(W_- \cup f(W_+))}$$

which contains W_- and let $U_{2,3}$ be a small neighborhood of $K_{2,3}$ and $U'_{2,3}$ the component of $f^{-2}(U_{2,3})$ contained in $U_{2,3}$. Then $\rho_{2,3} = (f^2, U'_{2,3}, U_{2,3})$ is a renormalization with filled Julia set $K_{2,3}$, which is hybrid equivalent to $z \mapsto (z^2-1)^2 - 1 = z^4 - 2z^2$.

Similarly, we may consider

$$\overline{\bigcup_{n>0} f^{-2n}(W_- \cup f(W_-) \cup W_+)}$$

and construct a polynomial-like map (f^2, U, V) of degree 6. However, by definition, it is not a renormalization because $-1/2$ is contained in both U and $f(U)$.

Thus $\mathcal{R}(f, -1/2) = \{\rho_1, \rho_{2,1}, \rho_{2,2}, \rho_{2,3}\}$ and $\mathcal{SR}(f, -1/2) = \{\rho_1, \rho_{2,1}\}$.

4. Infinite renormalization

4.1. The Yoccoz puzzle

Let $O = \{x_1, \ldots, x_l\}$ be a repelling periodic orbit of f. Assume that each x_i does not lie in the forward orbit of any critical point and that two or more external rays land at x_1. Let $R_{t_1}, \ldots, R_{t_q}$ be the collection of all the external rays landing at one of the points in O. Fix $R > 0$ and let C_0 be the equipotential curve $C_0 = \{z \mid G(z) = R\}$ where G is Green's function for $K(f)$. C_0 is simple closed curve enclosing $K(f)$. Let

$$\Gamma_0 = \bigcup_{i=1}^{q} R_{t_i} \cup C_0 \cup O.$$

Then $\mathbb{C} \setminus \Gamma_0$ has a finite number of bounded components. Let $\Gamma_n = f^{-n}(\Gamma_0)$. The closure of a bounded component of $\mathbb{C} \setminus \Gamma_n$ is called a *puzzle piece* at depth n. Since $\bigcup R_{t_i}$ is invariant under f and $f^{-n-1}(C_0)$ lies inside of $f^{-n}(C_0)$, each piece at depth $n+1$ is contained in a unique piece at depth n and mapped onto some piece at depth n by f. Every point z in $K(f) \setminus \left(\bigcup_{k>0} f^{-k}(O)\right)$ is contained in a unique puzzle piece $P_n(z)$ at depth n. Clearly, $P_{n+1}(z) \subset P_n(z)$ and $P_n(f(z)) = f(P_{n+1}(z))$. The *tableau* of z is the two dimensional array $\{P_n(f^p(z))\}_{n,p\geq 0}$.

For $c_0 \in C(f)$, let p_1 be the minimum of such $p > 0$ that $P_n(f^p(c_0)) = P_n(c_0)$ for any n, if exists. We call p_1 the *period of the tableau* of c_0. And let p_2 be the minimum of such $p > 0$ that there exists a simple renormalization ρ about c_0 of period p satisfying $K(\rho) \setminus O$ is connected, if exists (note that if $p > l$, this condition is automatically fulfilled). We call p_2 the *renormalization period* of f about c_0. If such p does not exist, the corresponding value is defined to be ∞.

Lemma 4.1. *Let p_1 and p_2 as above. Then $p_1 = p_2$.*

The equality is understood that if p_1 or p_2 is finite, then the other is also finite and these values are equal.

Proof. First, we show f^{p_1} is simply renormalizable about c_0 assuming that p_1 is finite. Then we obtain $p_1 \geq p_2$.

There exists some $n > 0$ such that the pieces $P_n(c_0), \ldots, P_n(f^{p_1-1}(c_0))$ have disjoint interiors and if a critical point $c \in C(f)$ lies in $P_n(f^i(c_0))$ $(0 \leq i < p_1)$, then c lies in $P_{n+k}(f^i(c_0))$ for all $k > 0$. So, we have $f^k(c) \in P_n(f^{i+k}(c_0))$. By the assumption, $P_n(f^{p_1}(c_0)) = P_n(c_0)$. Thus $f^{p_1} : P_{n+p_1}(c_0) \to P_n(c_0)$ is a critically compact proper map (a proper map all of whose critical points do not escape).

Then by using the method of [Mi2, Lemma 2], we thicken these pieces and we obtain a polynomial-like map with connected Julia set. Furthermore, we thicken so slightly that any other critical points and critical values of f^k (for $1 \leq k < p_1$) do not lie in the thickened parts. So this polynomial-like map is indeed a renormalization. Since a pair of the small Julia sets can meet only at points in O and puzzle pieces are cut by the rays landing at points in O, it must be simple (see the proof of Corollary 3.7).

On the other hand, let $\rho = (f^{p_2}, U, V)$ be a simple renormalization about c_0 with $K(\rho) \setminus O$ connected. Then $K(\rho) \subset P_0(c_0)$. Since $K(\rho)$ is connected and forward invariant by f^{p_2}, $f^{-kp_2}(K(\rho))$ has a unique component containing $K(\rho)$ for any $k > 0$. But each component of $f^{-n}(K(\rho))$ is contained in a unique puzzle piece of depth n (because $f^{-n}(O)$ does not contain critical points). Therefore $K(\rho)$ is contained in one puzzle piece of depth kp_2, which must be $P_{kp_2}(c_0)$ because $c_0 \in K(\rho)$.

Thus for any $n > 0$, choose k so that $kp_2 > n$. Then we have

$$K(\rho) \subset P_{kp_2}(c_0) \subset P_n(c_0).$$

Since $f^{p_2}(c_0) \in K(\rho)$, we have $P_n(f^{p_2}(c_0)) = P_n(c_0)$ and $p_1 \leq p_2$. □

4.2. Infinite simple renormalization

We now prove that an infinitely renormalizable polynomial has infinitely many simple renormalizations. For each $\rho \in \mathcal{R}(f, c_0, C_R)$, we construct a simple renormalization near the component of $\mathcal{K}(\rho)$ containing c_0 by using the Yoccoz puzzle. Then the period of the new simple renormalization is equal to the number of components of $\mathcal{K}(\rho)$, which tends to infinity.

First, we assume that every periodic point of f is repelling.

Theorem 4.2. *Let f be a polynomial. Assume every periodic point of f is repelling. If f is infinitely renormalizable, then f has infinitely many simple renormalizations.*

More precisely, if $\mathcal{R}(f, c_0, C_R)$ is infinite for some $C_R \subset C(f)$, then there exists some C'_R with $C_R \subset C'_R \subset C(f)$ such that $\mathcal{SR}(f, c_0, C'_R)$ is also infinite.

Proof. Suppose $\mathcal{R}(f, c_0, C_R)$ is infinite. For each $\rho = (f^n, U_n, V_n) \in \mathcal{R}(f, c_0, C_R)$, let κ be the number of components of $\mathcal{K}(\rho)$ and let μ be the *multiplicity* of the renormalization ρ, i.e. the number of the small Julia sets contained in each component of $\mathcal{K}(\rho)$. Then n is equal to $\kappa\mu$.

First, we claim κ tends to infinity as $n \to \infty$. Suppose $I(\rho)$ is nonempty (unless, we have $\kappa = n \to \infty$). By Proposition 3.4, it consists of one periodic point. κ is equal to its period and tends to infinity by Theorem 3.5.

Next we show that f^κ is simply renormalizable. Let $E_1, \ldots, E_\kappa$ be the components of $\mathcal{K}(\rho)$ indexed so that $f(E_j) = E_{j+1}$ and E_κ contains c_0. Take a repelling periodic orbit which does not disconnect any component of $\mathcal{K}(\rho)$ (we can choose an α-fixed point of f as such a periodic point. Indeed, since repelling periodic points have multiplicity one, there exist exactly $d = \deg f$ (repelling) fixed points and at most $(d-1)$ of them are β-fixed points).

We construct the Yoccoz puzzle by using this periodic orbit. Since $\mathcal{K}(\rho) = E_1 \cup \cdots \cup E_\kappa$ is forward invariant, each E_j is contained in a unique puzzle piece of each depth m. Therefore, the period of the tableau of c_0 is p, which divides κ. By Lemma 4.1, there exists a simple renormalization (f^p, U, V) with $K(f^p, U, V) \supset E_p, E_{2p}, \ldots, E_\kappa$.

Then repeating the above argument by replacing f with the polynomial hybrid equivalent to the renormalization (f^p, U, V), we eventually obtain a simple renormalization ρ' of f^κ (Note that even if $p = 1$, the degree of (f, U, V) is less than that of f, so we can eventually obtain a simple renormalization with $p > 1$). Since κ tends to infinity, we are done. Note that $\bigcup E_j \cap C(f) = C_R$, but, in general, $\mathcal{K}(\rho')$ is larger than $\bigcup E_j$ so $\mathcal{C}(\rho')$ may be strictly larger than C_R. □

Even when f has a non-repelling periodic point, we can deduce the same result from the preceding theorem.

Corollary 4.3. *An infinitely renormalizable polynomial has infinitely many simple renormalizations.*

Proof. Suppose $\#\mathcal{R}(f, c_0, C_R) = \infty$. Take $\rho \in \mathcal{R}(f, c_0, C_R)$ and let g be a polynomial hybrid equivalent to ρ.

By Proposition 3.2, g is also infinitely renormalizable. Furthermore, when g has infinitely many simple renormalizations, then so does f by Proposition 3.11. So, by the preceding theorem, we need only show that every periodic point of g is repelling.

Every critical point $c' \in C(g)$ corresponds to some critical point c of f^n which lies in $f^{-n+1}(C_R) \cap K(\rho)$. Since $\mathcal{R}(f, c_0, C_R)$ is infinite, the forward orbit of c cannot accumulate to non-repelling periodic points or the boundary of Siegel disks of f (if the period of a renormalization is greater than that of a bounded Fatou component Ω, then Ω cannot lies in the filled Julia set of this renormalization by Proposition 3.4). Thus the forward orbit of c' also cannot accumulate to non-repelling periodic points or the boundary of Siegel disks of g.

However, the closure of every Fatou component must intersect the postcritical set. Therefore, g does not have any non-repelling periodic point. □

In this case, we obtain that $\mathcal{SR}(f, c_0, \tilde{C}_R) = \infty$ for some $\tilde{C}_R \subset C_R$. Indeed, any critical point of g corresponds to some inverse image of a critical point of f in C_R. Thus the renormalization $\tilde{\rho}$ of f corresponding to some renormalization of g satisfies $\mathcal{C}(\tilde{\rho}) \subset C_R$. $\mathcal{C}(\tilde{\rho})$ may be strictly smaller than C_R because for $\rho' \in \mathcal{R}(f, c_0, C_R)$ with $n(\rho') > n(\rho)$, $K(\rho')$ is in general not contained in $K(\rho)$. In such a case, to obtain a renormalization of g from ρ', first we apply Proposition 3.2 and construct a renormalization ρ'' with $K(\rho'') = K(\rho) \cap K(\rho')$. then ρ'' corresponds to some renormalization of g with period $n(\rho'')/n(\rho)$. $\mathcal{C}(\rho'')$ is a proper subset of $\mathcal{C}(\rho')$ (if not so, it contradicts Proposition 3.1). So when we apply to ρ'' the above construction to obtain a simple renormalization $\hat{\rho}$ of f, then $\mathcal{C}(\hat{\rho})$ may be a proper subset of $C_R = \mathcal{C}(\rho)$ (in general, we only have $\mathcal{C}(\rho'') \subset \mathcal{C}(\hat{\rho}) \subset C_R$).

However, we can also see that $\#\,\mathrm{sr}(f, C'_R) = \infty$ for some $C'_R \supset C_R$. In the proof of Theorem 4.2, the assumption that every critical point of f is repelling is used only to find a periodic point where two or more rays land. But by the argument above, we can easily see that we can also find such a periodic point even when this assumption does not hold. Thus we can apply the proof of Theorem 4.2 to this case.

Remark 4.4. Do C'_R and $\tilde{C}_R$ coincide with C_R? If f satisfies the condition (4.1) below and is robust, C'_R is equal to C_R and $\tilde{C}_R$, see Corollary 4.14. However, in general, we do not have an answer to this question for now.

4.3. Robust infinite renormalization

Consider the following assumption for a polynomial f with connected Julia set and a subset $C_R \subset C(f)$ (note that this condition is trivially satisfied in the case of infinitely renormalizable quadratic polynomials):

$$\mathrm{sr}(f, C_R) \text{ is infinite and } f(C_R) = f(C(f)). \tag{4.1}$$

Remark 4.5. This assumption may seem rather strong and restrictive but it is not essential at all. That is to say, it corresponds to extracting an infinitely renormalizable "part" from an infinitely renormalizable polynomial. More precisely, assume that $\mathrm{sr}(f, C_R) = \infty$. For any $\rho = (f^n, U, V) \in \mathcal{SR}(f, c_0, C_R)$, let g be a polynomial hybrid equivalent to ρ. There exist some c'_0 and C'_R with $c'_0 \in C'_R \subset C(g)$ such that $\mathrm{sr}(g, C'_R)$ is infinite and each element ρ' of $\mathcal{SR}(g, c'_0, C'_R)$ corresponds to the renormalization in $\mathcal{SR}(f, c_0, C_R)$ with period $n(\rho') \cdot n$. Then we obtain $g(C'_R) = g(C(g))$ because the critical points of ρ is equal to $\bigcup_{i=0}^{n} f^{-i}(C(\rho, i)) \cap U$.

Let f be a polynomial satisfying (4.1). For each $n \in \mathrm{sr}(f, C_R)$, take a corresponding renormalization $\rho_n = (f^n, U_n, V_n) \in \mathcal{SR}(f, c_0, C_R)$. Then we have $\mathcal{P}(\rho_n) = P(f)$ for any $n \in \mathrm{sr}(f, C_R)$, and $\{K(\rho_n)\}_{n \in \mathrm{sr}(f, C_R)}$ forms a decreasing sequence by Theorem 3.10.

Proposition 4.6. *Let f, C_R and ρ_n as above. Then:*
(1) *All periodic points of f are repelling.*

(2) *The filled Julia set of f has no interior.*
(3) *There exist no periodic points in $\bigcap_{n\in\mathrm{sr}(f,C_R)}\mathcal{K}(\rho_n)$.*
(4) *there exist no periodic points in $P(f)$.*
(5) *For any $n \in \mathrm{sr}(f, C_R)$, $P(\rho_n, i)$ is disjoint from $K(\rho_n, j)$ if $i \neq j$.*

Proof. By the condition (4.1), all non-repelling periodic points must lie in $\mathcal{K}(\rho_n)$ for any $n \in \mathrm{sr}(f, c_0, C_R)$. However, then by Proposition 3.4, the period of a non-repelling periodic point is greater than n. So we have (1) and (2).

(3), (4) and (5) are easily derived from Theorem 3.5. Note that non-repelling periodic points are contained in $\mathcal{K}(\rho)$ and $P(f) = \mathcal{P}(\rho_n) \subset \mathcal{K}(\rho)$. □

For each $n \in \mathrm{sr}(f, C_R)$, let $\delta_n(i)$ be a simple closed curve which separates $K(\rho_n, i)$ from $P(f)\backslash P(\rho_n, i)$. Note that such curve exists by the fifth conclusion in the previous proposition. Since its homotopy class in $\mathbb{C} \setminus P(f)$ is uniquely determined, there exists a closed hyperbolic geodesic $\gamma_n(i)$ homotopic to $\delta_n(i)$ in $\mathbb{C} \setminus P(f)$. Let $\gamma_n = \gamma_n(n)$.

Proposition 4.7. *The geodesics $\gamma_n(i)$ ($n \in \mathrm{sr}(f, C_R)$, $1 \leq i \leq n$) are simple and mutually disjoint.*

Proof. A hyperbolic geodesic is simple if it is homotopic to some simple curve and two distinct hyperbolic geodesics are disjoint if they are homotopic to disjoint curves. Thus $\gamma_n(i)$ is simple because $\delta_n(i)$ is simple.

Take two geodesics $\gamma_a(i)$ and $\gamma_b(j)$. If $K(\rho_a, i)$ does not intersects $K(\rho_b, j)$, then we have nothing to prove. Suppose $K(\rho_a, i)$ intersects $K(\rho_b, j)$.

If $a = b$, then $K(\rho_a, i) \cap K(\rho_a, j)$ consists of a single repelling periodic point, which is not in $P(f)$. Since ρ_a is simple, this periodic point does not disconnect the filled Julia sets. Thus we can take representatives of the homotopy classes of $\gamma_a(i)$ and $\gamma_a(j)$ to be disjoint.

Otherwise, we may assume $a < b$. Then there exists some j' such that $K(\rho_a, j')$ contains $K(\rho_b, j)$. the case $j' \neq i$ is obviously obtained from the pervious case. So assume $K(\rho_b, j) \subset K(\rho_a, i)$. Then $P(\rho_b, j)$ is a proper subset of $P(\rho_a, i)$. Therefore, $\gamma_a(i) \neq \gamma_b(j)$ and we can take disjoint representatives, so they are disjoint. □

Proposition 4.8. *For $n \in \mathrm{sr}(f, C_R)$ and $1 \leq i \leq n$, $f^{-1}(\gamma_n(i))$ has a component α which is homotopic to $\gamma_n(i-1)$ on $\mathbb{C} \setminus P(f)$.*

Furthermore, $f^{-n}(\gamma_n)$ also has a component ζ which is homotopic to γ_n on $\mathbb{C} \setminus P(f)$.

Proof. $\delta_n(i)$ encloses $K(\rho_n, i)$ and is homotopic to $\gamma_n(i)$ on $\mathbb{C}\backslash P(f)$. Since f maps the connected set $K(\rho_n, i-1)$ to $K(\rho_n, i)$, there exists a component α' of $f^{-1}(\delta_n(i))$ which encloses $K(\rho_n, i-1)$. α' does not enclose other components of $f^{-1}(K(\rho_n, i-1))$ because $\delta_n(i)$ does not enclose any critical values of f which does not lie in $P(\rho_n, i)$. Thus, by definition, α' is homotopic to $\delta_n(i-1)$ on $\mathbb{C} \setminus P(f)$, which is homotopic to $\gamma_n(i-1)$.

Let $Q = f^{-1}(P(f))$. Then $f : \mathbb{C} \setminus Q \to \mathbb{C} \setminus P(f)$ is a covering map. Thus we can lift a homotopy between $\delta_n(i)$ and $\gamma_n(i)$ on $\mathbb{C} \setminus P(f)$ to a homotopy between α' and some component α of $f^{-1}(\gamma_n(i))$ on $\mathbb{C} \setminus Q$. Since $Q \supset P(f)$, α is homotopic to $\gamma_n(i-1)$.

Similar argument with f^n instead of f shows the existence of a component ζ of $f^{-n}(\gamma_n)$ homotopic to γ_n. □

Let $\ell(\cdot)$ denote the hyperbolic length in $\mathbb{C} \setminus P(f)$.

Lemma 4.9. *For $n \in \mathrm{sr}(f, C_R)$ and $1 \le i \le n$,*

$$\ell\left(\gamma_n(i)\right) \le d_n(i)\, \ell\left(\gamma_n(i+1)\right),$$

where $d_n(i)$ is the degree of the proper map $f : U(\rho_n, i) \to f(U(\rho_n, i))$ and $\gamma_n(n+1) = \gamma_n(1)$. In particular,

$$\frac{1}{2^d}\ell(\gamma_n) \le \ell\left(\gamma_n(i)\right) \le \ell(\gamma_n).$$

Proof. Let $Q = f^{-1}(P(f))$. The map $f : (\mathbb{C} \setminus Q) \to (\mathbb{C} \setminus P(f))$ is a covering map. So it is an isometry with respect to the respective hyperbolic metrics. Since the inclusion $(\mathbb{C} \setminus Q) \hookrightarrow (\mathbb{C} \setminus P(f))$ is a contraction,

$$\ell(\gamma_n(i)) \le \ell(\alpha) \le \ell_{\mathbb{C}\setminus Q}(\alpha) = d_n(i)\, \ell(\gamma_n(i+1)),$$

where α is the component of $f^{-1}(\gamma_n(i+1))$ homotopic to $\gamma_n(i)$ (see the preceding proposition) and $\ell_{\mathbb{C}\setminus Q}$ is the hyperbolic length on $\mathbb{C} \setminus Q$.

Since $\prod_{i=1}^{n} d_n(i)$ is uniformly bounded with respect to n (in fact, it is not greater than 2^d), the second conclusion holds. □

Definition. Let f be a polynomial of degree $d \ge 2$. We say f is *robust* if f satisfies the condition (4.1) for some $C_R \subset C(f)$ and

$$\liminf_{n \in \mathrm{sr}(f, C_R)} \ell(\gamma_n) < \infty.$$

Furthermore, we say a renormalization of any polynomial is *robust infinitely renormalizable* if it is hybrid equivalent to some robust infinitely renormalizable polynomial. See Remark 4.5.

Theorem 4.10. *Suppose f is robust. Then:*
(1) *the postcritical set $P(f)$ is a Cantor set of measure zero.*
(2) $\lim_{n\in\mathrm{sr}(f,C_R)} \left(\sup_{1\le i\le n} \operatorname{diam} P(\rho_n, i)\right) = 0.$
(3) *$f : P(f) \to P(f)$ is topologically conjugate to $\sigma : \Sigma \to \Sigma$, where*

$$\Sigma = \operatorname*{proj\,lim}_{n\in\mathrm{sr}(f,C_R)} \mathbb{Z}/n\mathbb{Z},$$

$$\sigma\left(\left(i_n\right)_{n\in\mathrm{sr}(f,C_R)}\right) = \left(i_n + 1\right)_{n\in\mathrm{sr}(f,C_R)}.$$

In particular, $f|_{P(f)}$ is a homeomorphism and the forward orbit of every point in $P(f)$ is dense in $P(f)$.

Proof. By the collar theorem, there exists the standard collar $A_n(i)$ about the geodesic $\gamma_n(i)$ on the hyperbolic surface $\mathbb{C} \setminus P(f)$. Note that the collar theorem asserts that these collars are mutually disjoint. Each annulus $A_n(i)$ separates $P(\rho_n, i)$ from the rest of the postcritical set.

Consider a sequence of nested annuli

$$\{A_n(i_n)\}_{n \in \mathrm{sr}(f, C_R)},$$

that is, for $m < n$, $A_n(i_n)$ lies in the bounded component of $\mathbb{C} \setminus A_m(i_m)$. By Lemma 4.9, $\ell(\gamma_n(i_n)) \le \ell(\gamma_n)$. Since $\mathrm{mod}(A_n(i_n))$ is a decreasing function of $\ell(\gamma_n)$ and $\liminf \ell(\gamma_n)$ is finite, the sum

$$\sum_{n \in \mathrm{sr}(f, C_R)} \mathrm{mod}\, A_n(i_n)$$

diverges to infinity. Thus the set $F = \bigcap F_n$ is totally disconnected and of measure zero, where F_n is the union of the bounded components of $\mathbb{C} \setminus (\bigcup_i A_n(i))$ (see [Mc, Theorem 2.16]). Clearly, F contains $P(f)$. Therefore, the postcritical set has measure zero. Furthermore, since each component of F_n intersects $P(f)$, we have $F = P(f)$. Each $P(\rho_n, i)$ lies in a single component of F_n. Since F is totally disconnected, the diameter of the largest component of F_n tends to zero as n tends to infinity, and so does $\sup_i \mathrm{diam}\, P(\rho_n, i)$.

For each $n \in \mathrm{sr}(f, C_R)$, let $\phi_n : P(f) \to \mathbb{Z}/n\mathbb{Z}$ be the map which sends $P(\rho_n, i)$ to $i \mod n$. These maps induce a continuous map $\phi : P(f) \to \Sigma$.

It is easy to confirm that ϕ is a conjugacy between $f|_{P(f)}$ and σ. Since Σ is a Cantor set, $P(f)$ is also a Cantor set. □

Corollary 4.11. *Suppose f is robust. If $n \in \mathrm{sr}(f, C_R)$ is sufficiently large, then $\#C(\rho_n, i)$ consists of at most one point for any i.*

Proof. Otherwise, we may assume $C(\rho_n) = \{c_0, \dots, c_r\}$, $r > 1$ for all sufficiently large $n \in \mathrm{sr}(f, C_R)$. Since $\mathrm{diam}(P(\rho_n, 1))$ tends to zero, all $f(c_j)$'s are equal.

Let m_j be the multiplicity of the critical point c_j. Then the degree of the proper map $f : U \to U(\rho_n, 1)$ is equal to $(\sum m_j) + 1$, but the cardinality of $f^{-1}(f(c_0))$ (counted with multiplicity) is not less than $\sum (m_j + 1)$, that is a contradiction. □

Corollary 4.12. *If f is robust, then $C_R \subset P(f)$ and every critical point in C_R is recurrent (its forward orbit accumulates to itself).*

Proof. Let $n \in \mathrm{sr}(f, C_R)$ be sufficiently large so that Corollary 4.11 holds. Then for $c \in C_R$, we have $C(\rho_n, i) = \{c\}$ for some i. Therefore, the inverse image of $f(c)$ by the proper map $f : U(\rho_n, i) \to f(U(\rho_n, i))$ consists only of c. Since $f(c) \in P(f)$ and $f : P(\rho_n, i) \to P(\rho_n, i+1)$ is a homeomorphism by Theorem 4.10, we have $c \in P(f)$.

Since the forward orbit of c is dense in $P(f)$, c is recurrent. □

Remark 4.13. We conjecture that when $\mathrm{sr}(f, C_R)$ is infinite (here we do not assume that f satisfies the condition (4.1)), then every critical point in C_R is recurrent. But up to now, we have proved it only in the robust infinitely renormalizable case.

The next corollary gives an answer of Remark 4.4 in the robust infinitely renormalizable case.

Corollary 4.14. *Suppose a renormalization* $\rho \in \mathcal{SR}(f, c_0, C_R)$ *is robust. Then every renormalization* ρ' *about* c_0 *satisfies* $\mathcal{C}(\rho') \supset C_R$.

In particular, if $\mathcal{R}(f, c_0, C'_R)$ *is infinite for some* $C'_R \subset C_R$*, then* $C'_R = C_R$.

Proof. By applying Theorem 4.10 to ρ, $\mathcal{P}(\rho)$ is a Cantor set and the forward orbit of c_0 is dense in $\mathcal{P}(\rho)$. Therefore, for any $\rho' \in \mathcal{R}(f, c_0)$, $\mathcal{K}(\rho')$ must contain $\mathcal{P}(\rho)$. By Corollary 4.12, $\mathcal{K}(\rho') \supset C_R$.

If $\mathcal{R}(f, c_0, C'_R)$ is infinite and $C'_R \subset C_R$, then for any $\rho' \in \mathcal{R}(f, c_0, C'_R)$, we have $C'_R = \mathcal{C}(\rho') \supset C_R$, so we have $C'_R = C_R$. □

4.4. Examples

Now we give some examples of infinitely renormalizable polynomials which have two or more critical points.

1. $f_1(z) = z^3 + 3c^2 z$ with $c = 0.907530\ldots$.

$g = -f_1$ have two renormalization of period 1. Both of them are hybrid equivalent to $h(z) = z^2 - 1.78644\ldots$, which is infinitely renormalizable with $\mathrm{sr}(h, \{0\}) = \{3^n\}$. (Indeed, g is regarded as a polynomial constructed by the intertwining surgery [EY] from two h's at their β-fixed points.)

f_1 has a renormalization of period 2 which is hybrid equivalent to h^2. Since the period of every renormalization of h is odd, f_1 has infinitely many renormalization of degree 4 (each renormalization of f_1 corresponds to some renormalization of h^2). The robustness of f_1 is easily verified because h is robust.

2. $f_2(z) = (z^2 + a)^2 + b$ with $a = -1.31434\ldots$ and $b = -0.459797\ldots$.

f_2 is infinitely renormalizable with $\mathrm{sr}(f, \{0, \sqrt{-a}\}) = \{2^n\}$ and the renormalization near 0 of period 2 is hybrid equivalent to $z \mapsto (z^2 + b)^2 + a$ and the renormalization near 0 of period 4 is hybrid equivalent to f itself.

This fact can be shown by using some results on dynamics of interval maps [MeSt]. When a polynomial is renormalizable in the real sense, then by using the method in the proof of Theorem 4.2, we can obtain a renormalization of the same period in the complex sense (in this case, we can always use an α-fixed point to construct the Yoccoz puzzle).

Let I be an interval. A map $f : I \to I$ is *k-unimodal* if it is of the form $g_k \circ \cdots \circ g_1$ where each $g_i : I \to I$ is unimodal and $g_i(\partial I) \subset \partial I$. We say a family of k-unimodal maps is *full* if for each k-unimodal map $g : I \to I$, there exists a map in this family which is essentially conjugate to g (conjugate when we "collapse" intervals with non-essential dynamics. For the precise definition, see [MeSt, Section II.4]).

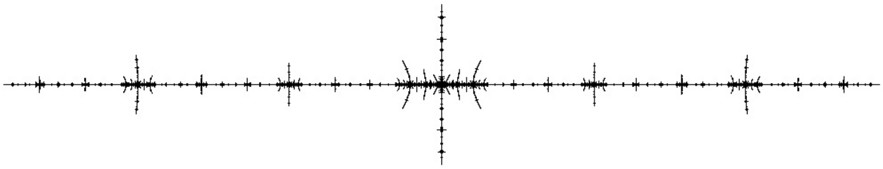

Figure 2. The Julia set of h

Figure 3. The Julia set of $f_1(z)$

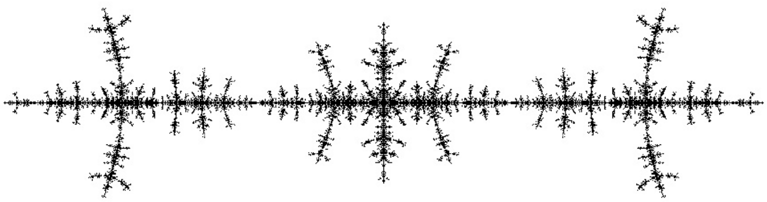

Figure 4. The Julia set of f_2

The fullness of family of biquadratic maps

$$\mathcal{BQ} = \{f : I \to I \mid f \text{ is polynomial of degree 4 and 2-unimodal}\}$$

can be derived easily from [MeSt, Theorem 5.1]. (Note that a k-unimodal map are regarded as a "reasonable" map in [MeSt, Section II.5].) Furthermore, the k-unimodal version of [MeSt, Theorem 5.2] is also valid (the proof is precisely the same). Here we restate this theorem only when a family is 2-unimodal and a renormalization contains two turning points.

Theorem 4.15 (cf. [MeSt, Theorem 5.2]). *Let $(f_\mu : I \to I)_{\mu \in \Delta}$ be a full family of 2-unimodal maps and let $\hat{f} : I \to I$ be a 2-unimodal map with three turning point $c_1 < c_2 < c_3$. Assume c_2 and c_i ($i = 1$ or 3) lies in the same periodic orbit of period p and $\hat{f}^{p_1}(c_2) = c_i$ $(0 < p_1 < p)$. We further assume $\hat{f}$ is not renormalizable with period less than p. Then there exists a connected subset $\Delta_0 \in \mathcal{BQ}$ such that for each $\mu \in \Delta_0$ we have the following:*

- *There exists a restrictive interval J_μ of period p which contains the second turning point of f_μ and $f^{p_1}(J_\mu)$ contains the i-th turning point of f_μ (and no other turning points of f_μ).*
- *The maps f_μ and $\hat{f}$ are $\approx$-combinatorially equivalent (the definition is given in* [MeSt, Section II.5.c]. *Roughly speaking, it specifies the order of the intervals $J_\mu, \dots, f_\mu^{p-1}(J_\mu)$).*

Furthermore, $(f_\mu^p : J_\mu \to J_\mu)_{\mu \in \Delta_0}$ is (when properly rescaled) again a full family of 2-unimodal maps.

By applying this theorem to the biquadratic family $\mathcal{BQ}$ repeatedly, we finally obtain the existence of an infinitely renormalizable biquadratic polynomial. Furthermore, f_2 is also robust. McMullen's argument for infinitely renormalizable quadratic polynomials [Mc, Chapter 11] is also valid for k-unimodal case.

5. Main results

In this section, we state our main results and the outline of the proof.

Theorem 5.1 (Robust rigidity). *A robust infinitely renormalizable polynomial carries no invariant line field on its Julia set.*

Since a hybrid equivalence preserves invariant line fields on the Julia set, we can easily apply the result to polynomials whose dynamics on its Julia set is essentially robust infinitely renormalizable. For example:

Corollary 5.2. *Let f be a polynomial of degree $d \geq 2$. Suppose every critical point $c \in C(f)$ satisfies one of the following:*

(1) *c is preperiodic.*

(2) *The forward orbit of c tends to an attracting cycle.*

(3) *There exists some simple renormalization ρ of f and $n > 0$ such that ρ is robust infinitely renormalizable and $f^n(c)$ lies in $J(\rho)$ and the forward orbit of c does not accumulate to $I(\rho)$.*

Then f carries no invariant line field on its Julia set.

Remark 5.3. If every critical point satisfies the assumption (1) or (2) of this corollary, then the polynomial is subhyperbolic. So this corollary means if the dynamics is locally subhyperbolic or eventually robust infinitely renormalizable (and other kind of dynamics does not happen), then there exists no invariant line field.

If a polynomial has only one critical point, it cannot have both subhyperbolic and robust infinitely renormalizable dynamics, so we do not need this corollary.

The corollary is an easy consequence of Theorem 5.1 and the following lemma:

Lemma 5.4. *Let f be a polynomial of degree $d \geq 2$. Suppose there exist simple renormalizations $\rho_1, \dots, \rho_l$ such that $\mathcal{P}(\rho_j)$'s are pairwise disjoint and every critical point $c \in C(f)$ satisfies one of the following:*

(1) *c is preperiodic.*

(2) *The forward orbit of c tends to an attracting cycle.*

(3) *There exist $n > 0$ and j such that $f^n(c)$ lies in $K(\rho_j)$ and the forward orbit of c does not accumulate to $I(\rho_j)$.*

Then almost every x in $J(f)$ eventually mapped onto $\bigcup \mathcal{J}(\rho_j)$ by f.

Proof. Let $P_J = J(f) \cap P(f)$. Let C_3 be the set of critical points which satisfy the condition (3) and let C_1 be the set of critical points which satisfy the condition (1) and which does not satisfy the condition (3). Then $P_J = P_1 \cup P_3$ where $P_i = \overline{\bigcup_{n>0} f^n(C_i)}$.

By Lemma 2.8, the Euclidean distance $d(f^n(x), P(f))$ tends to 0 for almost every $x \in J(f)$. Now we consider such $x \in J(f)$. Since there exist only countably many eventually periodic points, we may assume x is not eventually periodic.

Then x must be accumulate to P_3 because any point in P_1 is eventually mapped into a repelling periodic point of f. Thus for any $\epsilon > 0$, there exists $N > 0$, j and i such that $d(f^N(x), P_3 \cap K(\rho_j, i)) < \epsilon$ and $d(f^n(x), P_3) < \epsilon$ for any $n \geq N$. Because P_3 does not intersect $I(\rho_j)$, this implies that we have $d(f^n(x), P_3 \cap K(\rho_j, n - N + i)) < \epsilon$ for any $n > N$ when ϵ is sufficiently small. Therefore, the forward orbit of $f^{N+n-i}(x)$ by $f^{n(\rho_j)}$ does not escape from $U(\rho_j)$, so $f^{N+n-i}(x)$ must lie in $\mathcal{J}(\rho_j)$. ◻

Remark 5.5. If there exists no critical point satisfying the condition (3), then f is subhyperbolic and $J(f)$ is measure zero (since P_3 is empty, any point x in $J(f)$ cannot satisfy $d(f^n(x), P(f)) \to 0$).

Proof of Corollary 5.2. First we collect all renormalizations $\rho_1, \dots, \rho_l$ which appear in the assumption (3) of the corollary for some $c \in C(f)$. If $\mathcal{P}(\rho_i) \cap \mathcal{P}(\rho_j)$ is nonempty, then by applying Theorem 4.10 to ρ_i and ρ_j, we have $\mathcal{P}(\rho_i) = \mathcal{P}(\rho_j)$. This implies $\mathcal{C}(\rho_i) = \mathcal{C}(\rho_j)$ by Corollary 4.12. Then we can omit one of them by Proposition 3.9. Therefore, we assume that the postcritical sets are pairwise disjoint, so we can apply Lemma 5.4. Thus

$$E = \bigcup_{j=1}^{l} \bigcup_{k>0} f^{-k}(\mathcal{J}(\rho_j))$$

has full measure in $J(f)$. So we apply Theorem 5.1 to each ρ_j, f carries no invariant line field on E and so does on $J(f)$. □

In the rest of this section, we state the outline of the proof of Theorem 5.1. The proof is based on McMullen's proof in the quadratic case [Mc].

The proof is divided into two cases, whether $L = \liminf \ell(\gamma_n)$ is zero or positive. However, both proofs goes very similarly. We pass to a subsequence in $\mathrm{sr}(f, C_R)$ so that after rescaling properly, f^n converges to some proper map $g : U \to V$ of degree more than one near the small postcritical set $P(\rho_n)$. (We use some proper map $f^n : X_n \to Y_n$ constructed from $\rho_n \in \mathcal{SR}(f, c_0, C_R)$ to obtain good estimates. Only when the case L is sufficiently small, they are polynomial-like.)

Now suppose f carries an invariant line field μ on its Julia set. We will construct a g-invariant univalent line field ν on V. Then it is a contradiction because $U \cap V$ contains a critical point of g.

To construct ν, we will use the following two lemmas:

Lemma 5.6. *Suppose holomorphic maps $f_n : (U_n, u_n) \to (V_n, v_n)$ between pointed disks converge to some non-constant map $f : (U, u) \to (V, v)$ in the Carathéodory topology. If an f_n-invariant line field μ_n converges in measure to some line field μ, then μ is f-invariant.*

See [Mc, Theorem 5.14].

Lemma 5.7. *Suppose a measurable line field μ on $\mathbb{C}$ is almost continuous at a point x and $|\mu(x)| = 1$. Let $(V_n, v_n) \to (V, v)$ be a convergent sequence of pointed disks, and let $h_n : V_n \to \mathbb{C}$ be a sequence of univalent maps. Suppose $h_n'(v_n) \to 0$ and*

$$\sup \frac{|x - h_n(v_n)|}{|h_n'(v_n)|} < \infty.$$

Then there exists a subsequence such that $h_n^*(\mu)$ converges in measure to a univalent line field on V.

See [Mc, Theorem 5.16].

We take a point $x \in J(f)$ having some good properties. Then for infinitely many $n \in \mathrm{sr}(f, C_R)$, we take an inverse branch h_n of f^k for some k which sends some neighborhood of the small postcritical set univalently near x. Then $h_n^*(\mu) = \mu$ by f-invariance, so $h_n^*(\mu)$ is f^n-invariant line field on Y_n. We apply Lemma 5.7 and obtain a univalent line field μ on V. By Lemma 5.6, μ is g-invariant. However, since g has a critical point c in $U \cap V$, $\mu(c)$ must be equal to zero, and this is a contradiction.

The rest of this paper is devoted to prove Theorem 5.1. Many estimates in McMullen's proof can be applied similarly to our case. However, the main difficulty is to avoid critical points of f. For example, we need to construct a univalent map h_n by choosing an inverse branch of iterates of f, so we must check that the forward orbit x does not pass near critical points outside C_R.

6. Thin rigidity

In this section we will give the proof of Theorem 5.1 in the case $\liminf \ell(\gamma_n)$ is sufficiently small.

In this section, we always assume that a polynomial f satisfies the condition (4.1) and use the same notations as in Section 4.

We say a renormalization $\rho_n = (f^n, U_n, V_n)$ is *unbranched* if

$$V_n \cap P(f) = P(\rho_n).$$

Lemma 6.1. *There exists some $L > 0$ (depending only on d) satisfying the following*:

If $\ell(\gamma_n) < L$, then we can take an unbranched representative (f^n, U_n, V_n) of ρ_n with $\operatorname{mod}(U_n, V_n) > m(\ell(\gamma_n))$ where $m(\ell)$ is a positive function which tends to infinity as $\ell \to 0$.

Proof. Let A_n be the standard collar of γ_n in $\mathbb{C} \setminus P(f)$ and let B_n be the component of $f^{-n}(A_n)$ which has the same homotopy class in $\mathbb{C} \setminus P(f)$. Let D_n (resp. E_n) be the union of B_n (resp. A_n) and the bounded component of $\mathbb{C} \setminus B_n$ (resp. $\mathbb{C} \setminus A_n$). Then $f^n : D_n \to E_n$ is a critically compact proper map.

Then there exists some $M > 0$ such that if $\operatorname{mod}(P(\rho_n), E_n) > M$ then we can take $U'_n \subset D_n$ and $V'_n \subset E_n$ as follows: (f^n, U'_n, V'_n) is a renormalization and $\operatorname{mod}(U'_n, V'_n) > m(\operatorname{mod}(P(\rho_n), E_n))$ (see Lemma 2.12). Since $E_n \cap P(f) = P(\rho_n)$, (f^n, U'_n, V'_n) is unbranched.

Since $\operatorname{mod}(P(\rho_n), E_n) \geq \operatorname{mod} A_n$, there exists some $L > 0$ such that if $\ell(\gamma_n) < L$ then we can take an unbranched renormalization (f^n, U'_n, V'_n) with $\operatorname{mod}(U'_n, V'_n) > m(\ell(\gamma_n))$. □

Therefore, we will prove the following:

Theorem 6.2 (Polynomial-like rigidity). *Let f be a polynomial satisfying the condition* (4.1). *Suppose there exists some $m > 0$ such that (f^n, U_n, V_n) is unbranched with $\operatorname{mod}(U_n, V_n) > m$ for infinitely many $n \in \operatorname{sr}(f, C_R)$.*

Then f carries no invariant line field on its Julia set.

Corollary 6.3 (Thin rigidity). *There exists some $L > 0$ such that if a polynomial f satisfies the condition* (4.1) *and*

$$\liminf_{n \in \operatorname{sr}(f, C_R)} \ell(\gamma_n) < L,$$

then f carries no invariant line field on its Julia set.

Now we prepare some lemmas which will be used to prove Theorem 6.2.

Lemma 6.4. *Assume an unbranched renormalization* $\rho_n = (f^n, U_n, V_n)$ *satisfies that* $\mathrm{mod}(U_n, V_n) > m > 0$. *Let* E *be a component of* $f^{-1}(J(\rho_n, i))$ *which differs from* $J(\rho_n, i-1)$. *Then in the hyperbolic metric on* $\mathbb{C} \setminus P(f)$, *the diameter of* E *is bounded in terms of* m.

Note that E does not intersects $P(f)$ because $P(f) \subset \mathcal{J}(\rho_n)$.

Proof. Since $\mathrm{mod}(J(\rho_n), V_n) > \mathrm{mod}(U_n, V_n) > m$, $\mathrm{mod}(J(\rho_n, i), V_n(\rho_n, i))$ is greater than $m/2^d$.

Let W be the component of $f^{-1}(V(\rho_n, i))$ which contains E. Then $f : W \to V(\rho_n, i)$ is a branched covering of degree less than d. Note that all critical points of this map lie in E. Hence $\mathrm{mod}(E, W)$ is greater than $m/(2^d d)$. This implies that E is enclosed by an annulus whose modulus is bounded below, so the diameter of E with respect to the hyperbolic metric on W is bounded in terms of m by Lemma 2.2. By the Schwarz-Pick lemma, the hyperbolic diameter of E on $\mathbb{C} \setminus P(f)$ is also bounded. □

Lemma 6.5. *Suppose there exists some* $m > 0$ *such that* (f^n, U_n, V_n) *is unbranched with* $\mathrm{mod}(U_n, V_n) > m$ *for infinitely many* $n \in \mathrm{sr}(f, C_R)$.

Then f *is robust,* $P(f)$ *is measure zero and*

$$P(f) = \bigcap_{n \in \mathrm{sr}(f, C_R)} \mathcal{J}(\rho_n).$$

Proof. For $n \in \mathrm{sr}(f, C_R)$ with $\mathrm{mod}(U_n, V_n) > m$, let A_n be an annulus in $V_n \setminus U_n$ enclosing $K(\rho_n)$ with $\mathrm{mod}\, A_n > m$.

The length of the core curve (unique simple closed geodesic) of A_n is less than π/m in the hyperbolic metric on A_n. Since the core curve of A_n is homotopic to γ_n in $\mathbb{C} \setminus P(f)$, $\ell(\gamma_n)$ is also less than π/m by the Schwarz-Pick lemma. So, f is robust. By Theorem 4.10, the postcritical set is a Cantor set of measure zero and $\sup_i \mathrm{diam}\, P(\rho_n, i) \to 0$. Since $\mathrm{mod}(U_n, V_n) > m$, we have

$$\mathrm{diam}_E\, J(\rho_n, i) < C \cdot \mathrm{diam}_E\, P(\rho_n, i)$$

for some C which depends only on m and d where diam_E is the diameter with respect to the Euclidean metric. Thus $\sup \mathrm{diam}\, J(\rho_n, i) \to 0$ as well. Since $J(\rho_n, i)$ intersects $P(f)$, the theorem follows. □

Lemma 6.6. *Under the same assumption as Lemma* 6.5, *almost every* $x \in J(f)$ *satisfies the following*:

(1) *The forward orbit of* x *does not intersects* $P(f)$.

(2) $\|(f^n)'(x)\| \to \infty$ *with respect to the hyperbolic metric on* $\mathbb{C} \setminus P(f)$.

(3) *For each* $n \in \mathrm{sr}(f, C_R)$, *there exists some* $k > 0$ *such that* $f^k(x) \in \mathcal{J}(\rho_n)$.

(4) *For each* $k > 0$, *there exists some* $n \in \mathrm{sr}(f, C_R)$ *such that* $f^k(x) \notin \mathcal{J}(\rho_n)$.

Proof. Since $P(f)$ is measure zero, so is $\bigcup f^{-k}(P(f))$, which implies (1).

The second conclusion follows from Lemma 2.6.

By Lemma 2.8, $d(f^k(x), P(f)) \to 0$ for almost every $x \in J(f)$. When $f^k(x)$ is sufficiently close to $P(\rho_n, i)$, then $f^{k+1}(x)$ should also be close to $P(\rho_n, i+1)$, because $P(f) = \bigcup P(\rho_n, i)$ (here we use the fifth conclusion of Proposition 4.6). Therefore, $f^{k+nj-i}(x)$ lies in U_n for all $j > 0$ and this means that $f^{k+n-i}(x) \in J(\rho_n)$, so we proved (3).

By Lemma 6.5, $\mathrm{area}(\mathcal{J}(\rho_n))$ tends to zero. Therefore, $\bigcap_n f^{-k}(\mathcal{J}(\rho_n))$ is measure zero for any k and we have now proved (4). □

Proof of Theorem 6.2. Let

$$\mathrm{usr}(f, C_R, m) = \{n \in \mathrm{sr}(f, C_R) \mid \rho_n \text{ is unbranched and } \mathrm{mod}(U_n, V_n) > m\}.$$

Suppose $\#\,\mathrm{usr}(f, C_R, m) = \infty$ and there exists an f-invariant line field supported on $F \subset J(f)$ of positive Lebesgue measure.

Fix a point $x \in F$ which satisfies the conditions in Lemma 6.6 and where μ is almost continuous. For each $n \in \mathrm{usr}(f, C_R, m)$, let $k(n) \geq 0$ be the smallest number which satisfies $f^{k(n)+1}(x) \in \mathcal{J}(\rho_n)$ and assume $f^{k(n)+1}(x) \in J(\rho_n, i(n)+1)$. Note that $k(n)$ tends to infinity.

Let $n_0 = \min(\mathrm{usr}(f, C_R, m))$. Consider sufficiently large $n \in \mathrm{usr}(f, C_R, m)$ so that $k(n) > k(n_0)$. In particular, $k(n)$ is positive so $f^{k(n)}(x)$ does not lie in $\mathcal{J}(\rho_n)$.

Therefore, $f^{k(n)}(x)$ lies in a component E of $f^{-1}(J(\rho_n, i(n)+1))$, which is different from $J(\rho_n, i(n))$. Moreover, we have $f^{k(n)}(x) \in \mathcal{J}(\rho_{n_0})$ because $k(n) > k(n_0)$. Hence E lies in $\mathcal{J}(\rho_{n_0})$ and does not contain any critical points.

Since E is disjoint from the postcritical set, there exists a univalent branch $\tilde{h}_n$ of $f^{-k(n)-1}$ on $V(\rho_n, i(n)+1)$ which sends $f^{k(n)+1}(x)$ to x.

Let $j(n)$ be the smallest number which satisfies that $i(n) < j(n) \leq n$, that $C(\rho_n, j(n))$ is nonempty. Since $C(\rho_n, i)$ is empty for $i(n) < i < j(n)$, there exists a univalent inverse branch

$$V(\rho_n, j(n)) \xrightarrow{f^{-1}} V(\rho_n, j(n)-1) \xrightarrow{f^{-1}} \cdots \xrightarrow{f^{-1}} V(\rho_n, i(n)+1).$$

Let h_n be the composition of the map above and $\tilde{h}_n$. Namely, h_n is a univalent branch of $f^{-j(n)+i(n)-k(n)}$ on $V(\rho_n, j(n))$ which sends $f^{j(n)-i(n)+k(n)}(x)$ to x. Let $J_n^* = h_n(J(\rho_n, j(n)))$. Then $f^{k(n)}(J_n^*) = E$. By Lemma 6.4, the hyperbolic diameter of E in $\mathbb{C} \setminus P(f)$ is bounded in terms of m. Since $\|(f^k)'(x)\|$ tends to infinity,

$$\mathrm{diam}\, J_n^* \to 0$$

with respect to the hyperbolic metric on $\mathbb{C} \setminus P(f)$, by the Koebe distortion theorem.

There exists some $c \in C_R$, such that for infinitely many $n \in \mathrm{usr}(f, C_R, m)$, c lies in $J(\rho_n, j(n))$. Furthermore,

$$(f^n, U_n(j(n)), V_n(j(n)))$$

is also unbranched and satisfies

$$\operatorname{mod}\left(U_n\left(j(n)\right), V_n\left(j(n)\right)\right) > m/2^d.$$

Hence by replacing c_0, m, U_n and V_n with c, $m/2^d$, $U_n(j(n))$ and $V_n(j(n))$ respectively, we may suppose $j(n) = n$ for infinitely many $n \in \operatorname{usr}(f, C_R, m)$.

For such n, let

$$\begin{aligned} A_n(z) &= \frac{z - c_0}{\operatorname{diam}(J(\rho_n))}, \\ g_n &= A_n \circ f^n \circ A_n^{-1}, \\ y_n &= A_n(h_n^{-1}(x)). \end{aligned}$$

Then

$$(g_n, A_n(U_n), A_n(V_n)))$$

is a polynomial-like map with $\operatorname{diam}(J(g_n)) = 1$ and $\operatorname{mod}(A_n(U_n), A_n(V_n)) > m$. After passing to a subsequence, g_n converges to some polynomial-like map (or polynomial) (g, U, V) with $\operatorname{mod}(U, V) \geq m$ in the Carathéodory topology by Lemma 2.9.

Let $k_n = h_n \circ A_n^{-1}$ defined on $A_n(V_n)$. Then we have $k_n(y_n) = x$ and $\nu_n = k_n^*(\mu)$ is g_n-invariant line field on $A_n(V_n)$. Since $\operatorname{diam}(J(g_n)) = 1$, while $\operatorname{diam}(k_n(J(g_n))) = \operatorname{diam}(J_n^*) \to 0$, we have $k_n'(y_n) \to 0$ by the Koebe distortion theorem.

After passing to a further subsequence, y_n converges to some $y \in V$, because y_n lies in $J(g_n)$, which is surrounded by an annulus of definite modulus. By Lemma 5.7, there exists a further subsequence such that μ_n converges to a univalent line field μ on V. The critical point 0 lies in $J(g) \subset U \cap V$. However, this contradicts the fact that the univalent line field μ is g-invariant by Lemma 5.6.

Therefore, f carries no invariant line field on its Julia set. □

7. Thick rigidity

In this section, we will prove the other case of Theorem 5.1, which is the following:

Theorem 7.1 (Thick rigidity)**.** *Let f be robust. Suppose*

$$0 < \liminf_{n \in \operatorname{sr}(f, C_R)} \ell(\gamma_n) < \infty.$$

Then f carries no invariant line field on its Julia set.

In this section, we also assume a polynomial f satisfies the condition (4.1). Let

$$\operatorname{sr}(f, C_R, \lambda) = \left\{ n \in \operatorname{sr}(f, C_R) \;\middle|\; 1/\lambda < \ell(\gamma_n) < \lambda \right\}.$$

Assume $\# \operatorname{sr}(f, C_R, \lambda) = \infty$ for some $\lambda > 0$. For $n \in \operatorname{sr}(f, C_R, \lambda)$,

• Let ζ_n be the component of $f^{-n}(\gamma_n)$ homotopic to γ_n on $\mathbb{C} \setminus P(f)$ (the existence is guaranteed by Proposition 4.8).

• Let X_n and Y_n denote the disks enclosed by ζ_n and γ_n, respectively. Then $f^n : X_n \to Y_n$ is a proper map whose degree is equal to $\deg \rho_n$.

• Define $Y_n(i) = f^i(X_n)$ for $1 \le i \le n$. Then we have $Y_n(n) = Y_n$ and $Y_n(i) \cap P(f) = P(\rho_n, i)$.

• Let $\mathcal{Y}_n = \bigcup_{i=1}^n Y_n(i)$. Then $P(f)$ lies in $\mathcal{Y}_n$.

• Let B_n denote the largest Euclidean ball centered at c_0 which lies in $X_n \cap Y_n$. Note that by Corollary 4.12, $c_0 \in P(\rho_n) \subset X_n \cap Y_n$.

• Let $\zeta_n(i)$ be the component of $f^{-n}(\gamma_n(i))$ homotopic to $\gamma_n(i)$ on $\mathbb{C} \setminus P(f)$.

• Let $\tilde{Y}_n(i)$ be the disk enclosed by $\gamma_n(i)$ and let $\tilde{X}_n(i)$ be the disk enclosed by $\zeta_n(i)$.

In the rest of this paper, $A < C(\lambda)$ means that A is bounded above in terms of λ (and $d = \deg f$). All $C(\lambda)$'s are independent one another. And we denote by d_E and diam_E the Euclidean distance and diameter respectively.

Lemma 7.2. *For $n \in \operatorname{sr}(f, C_R, \lambda)$,*

$$\operatorname{diam}_E(X_n) \ge \operatorname{diam}_E(B_n) \ge C(\lambda) \operatorname{diam}_E(X_n),$$
$$\operatorname{diam}_E(Y_n) \ge \operatorname{diam}_E(B_n) \ge C(\lambda) \operatorname{diam}_E(Y_n).$$

Proof. The left inequalities are trivial, so we will prove the right inequalities.

By the collar theorem, there exists an annulus A in $\mathbb{C} \setminus P(f)$ whose core curve is γ_n and which satisfies $\operatorname{mod}(A) > m(\lambda) > 0$. Since c_0 lies in the bounded component of $\mathbb{C} \setminus A$,

$$r_1 = d_E(c_0, \gamma_n) \ge C_1 \operatorname{diam}_E(\gamma_n) = C_1 \operatorname{diam}_E(Y_n),$$

where $C_1 > 0$ depends only on λ.

Let $Q = f^{-n}(P(f))$. Then

$$f^n : (\mathbb{C} \setminus Q) \to (\mathbb{C} \setminus P(f))$$

is a covering map. Since the map f^n sends ζ_n to γ_n by degree $\le 2^d$, the hyperbolic length of ζ_n in $\mathbb{C} \setminus Q$ is not greater than $2^d \cdot \ell(\gamma_n)$, which is less than $2^d \cdot \lambda$. So by the same reason as in the case of γ_n,

$$r_2 = d_E(c_0, \zeta_n) \ge C_2 \operatorname{diam}_E(\zeta_n) = C_2 \operatorname{diam}_E(X_n),$$

where $C_2 > 0$ depends only on λ.

Now we show that the ratio r_1/r_2 is bounded above and below. If we show this, then we are done because $\operatorname{diam}_E(B_n) = 2 \cdot \min(r_1, r_2)$.

Assume $r_1 \ge r_2/C_2 (\ge \operatorname{diam}_E(\zeta_n))$. Then $B = \left\{ z \;\middle|\; r_2/C_2 < |z - c_0| < r_1 \right\}$ is an annulus enclosing ζ_n and enclosed by γ_n. B is contained in $\mathbb{C} \setminus P(f)$ and the core curve of B is homotopic to γ_n on $\mathbb{C} \setminus P(f)$. By the Schwarz-Pick lemma, the hyperbolic length of the core curve of B on B is greater than $1/\lambda$. So

$$\lambda > \frac{\mathrm{mod}(B)}{\pi} = \frac{\log r_1 - \log r_2 + \log C_2}{2\pi^2},$$

which implies $r_1 < C_3 r_2$ for some $C_3 > 0$ which depends only on λ.

A similar argument shows that if $r_2 \geq r_1/C_1$, then $r_2 < C_4 r_1$ for some $C_4 > 0$. □

Remark 7.3. Quite similarly, we can also show that

$$\mathrm{diam}_E(Y_n(i)) \geq \mathrm{diam}_E(B_n(i)) \geq C(\lambda)\,\mathrm{diam}_E(Y_n(i)),$$
$$\mathrm{diam}_E(\tilde{Y}_n(i)) \geq \mathrm{diam}_E(B_n(i)) \geq C(\lambda)\,\mathrm{diam}_E(\tilde{Y}_n(i)),$$

where $B_n(i)$ is the largest Euclidean ball which lies in $Y_n(i) \cap \tilde{Y}_n(i)$ and centered at $f^i(c_0)$.

In particular, the Euclidean diameters of $Y_n(i)$ and $\tilde{Y}_n(i)$ are comparable.

Lemma 7.4.

$$P(f) = \bigcap_{n \in \mathrm{sr}(f, C_R)} \mathcal{Y}_n.$$

Proof. By Theorem 4.10, $\sup_i \mathrm{diam}(P(\rho_n, i))$ tends to zero as $n \to \infty$. So, because $Y_n(i) \supset P(\rho_n, i)$, it suffices to show that $\sup_i \mathrm{diam}(Y_n(i))$ also tends to zero. By Lemma 7.2 and Remark 7.3, it is equivalent to $\sup_i \mathrm{diam}(\tilde{Y}_n(i)) \to 0$.

But it is trivial since $\tilde{Y}_n(i)$ lies in some component of F_m for any $m < n$ in $\mathrm{sr}(f, C_R)$, where F_m is the same as in the proof of Theorem 4.10. □

Lemma 7.5. *Almost every $x \in J(f)$ has the following properties*:

(1) *The forward orbit of x does not intersect $P(f)$.*

(2) *$\|(f^n)'(x)\| \to \infty$ with respect to the hyperbolic metric on $\mathbb{C} \setminus P(f)$.*

(3) *For each $n \in \mathrm{sr}(f, C_R)$, there exists some $k > 0$ satisfying $f^k(x) \in \mathcal{Y}_n$.*

(4) *For each $k > 0$, there exists some $n \in \mathrm{sr}(f, C_R)$ satisfying $f^k(x) \notin \mathcal{Y}_n$.*

Proof. The properties (1) and (2) are the same as in Lemma 6.6. The fact that $\mathcal{Y}_n$ is a neighborhood of $P(f)$ implies the property (3). The property (4) follows from Lemma 7.4 (cf. Lemma 6.6). □

Lemma 7.6. *For $n \in \mathrm{sr}(f, C_R, \lambda)$, let*

$$A_n(z) = \frac{z - c_0}{\mathrm{diam}(B_n)}.$$

Then there exists a subsequence of $\mathrm{sr}(f, C_R, \lambda)$ such that

$$(A_n(X_n), 0) \to (X, 0),$$
$$(A_n(Y_n), 0) \to (Y, 0),$$
$$A_n^{-1} \circ f^n \circ A_n \to g,$$

in the Carathéodory topology where $g : (X, 0) \to (Y, g(0))$ is a proper map, $0 \in X \cap Y$, and $g'(0) = 0$.

Proof. Since $A_n(X_n)$ and $A_n(Y_n)$ contain the unit ball, there exists a subsequence such that

$$(A_n(X_n), 0) \to (X, 0),$$
$$(A_n(Y_n), 0) \to (Y, 0),$$

Since $\ell(\gamma_n) < \lambda$, the diameter of $P(\rho_n)$ in the hyperbolic metric on Y_n is bounded by the collar theorem. In particular, the distance between c_0 and $f^n(c_0)$ in the hyperbolic metric on Y_n is bounded above. Therefore, after passing to a further subsequence, $(A_n(Y_n), A_n(f^n(c_0))) \to (Y, y)$ for some $y \in Y$ (see [Mc, Theorem 5.2]).

By Lemma 7.2, X, $Y \neq \mathbb{C}$. So passing to a further subsequence, we may assume $A_n^{-1} \circ f^n \circ A_n \to g$ for some proper map $g : X \to Y$ by Lemma 2.13. Clearly, we have $(A_n^{-1} \circ f^n \circ A_n)'(0) \to g'(0) = 0$. □

Lemma 7.7. *For $n \in \mathrm{sr}(f, C_R)$,*

$$\ell(\gamma_n) \leq \ell(\zeta_n) \leq 2^d \ell(\gamma_n),$$
$$\ell(\gamma_n(i)) \leq \ell(\partial Y_n(i)) \leq 2^d \ell(\gamma_n).$$

Proof. The left inequalities follow from the fact that γ_n and $\gamma_n(i)$ are geodesics homotopic to ζ_n and $\partial Y_n(i)$, respectively.

The right inequalities follow from the fact that f expands the hyperbolic metric on $\mathbb{C} \setminus P(f)$. Note that f^n and f^{n-i} maps ζ_n and $\partial Y_n(i)$ to γ_n by degree $\leq 2^d$, respectively. □

Lemma 7.8. *For $n \in \mathrm{sr}(f, C_R, \lambda)$, ζ_n lies in a $C(\lambda)$-neighborhood of γ_n, and $\partial Y_n(i)$ lies in a $C(\lambda)$-neighborhood of $\gamma_n(i) = \partial \tilde{Y}_n(i)$ with respect to the hyperbolic metric on $\mathbb{C} \setminus P(f)$.*

Proof. This lemma is an easy consequence of Lemmas 4.9, 7.7 and 2.4. The condition $1/\lambda < \ell(\gamma_n) < \lambda$ gives lower bounds of γ_n and $\gamma_n(i)$, and upper bounds of ζ_n and $\partial Y_n(i)$. Thus these bounds give upper bounds of distances from every point on ζ_n and $\partial Y_n(i)$ to γ_n and $\gamma_n(i)$, respectively. □

Lemma 7.9. *For $n \in \mathrm{sr}(f, C_R, \lambda)$,*

$$\|(f^n)'(x)\| \leq C(\lambda) \qquad \textit{for any } x \in \zeta_n = \partial X_n,$$
$$\|(f^{n-i})'(x)\| \leq C(\lambda) \qquad \textit{for any } x \in \partial Y_n(i)$$

with respect to the hyperbolic metric on $\mathbb{C} \setminus P(f)$.

Proof. Since $f^n : \zeta_n \to \gamma_n$ is a proper map and its degree is at most 2^d,

$$2^d \ell(\gamma_n) \geq \int_{\zeta_n} \|(f^n)'(z)\| \, \rho(z)|dz|,$$

where $\rho(z)|dz|$ is the hyperbolic metric on $\mathbb{C} \setminus P(f)$. Because $\ell(\zeta_n) \geq \ell(\gamma_n)$, we have $\|(f^n)'(x_0)\| \leq 2^d$ for some $x_0 \in \zeta_n$.

For $x \in \zeta_n$, take a path η on ζ_n which joins x to x_0. Then

$$\ell(f^n(\eta)) \leq 2^d \ell(\gamma_n) \leq 2^d \lambda.$$

By Lemma 2.7, there exists some $\alpha > 0$, which depends only on λ, such that

$$\|(f^n)'(x)\| \leq \|(f^n)'(x_0)\|^\alpha \leq (2^d)^\alpha,$$

which implies the first inequality.

For $x \in \partial Y_n(i)$, take $x' \in \zeta_n$ with $f^i(x') = x$. Since f expands the hyperbolic metric on $\mathbb{C} \setminus P(f)$,

$$\|(f^{n-i})'(x)\| \leq \|(f^n)'(x'))\| \leq (2^d)^\alpha.$$

□

Lemma 7.10. *For any $n \in \mathrm{sr}(f, C_R, \lambda)$, there exist a disk Z_n and positive integers a_n and m_n with $Z_n \subset \mathbb{C} \setminus P(f)$ and $a_n, m_n \leq n$ such that*

(1) *$C(\rho_n, a_n)$ is nonempty,*

(2) *$f^{m_n} : Z_n \to \tilde{Y}_n(a_n)$ is univalent,*

(3) *$d(\partial \tilde{X}_n(a_n), \partial Z_n) < C(\lambda)$,*

(4) *$\ell(\partial Z_n) < \lambda$,*

(5) *$\mathrm{area}(Z_n) > 1/C(\lambda)$*

in the hyperbolic metric on $\mathbb{C} \setminus P(f)$.

Proof. By Lemma 2.5, there exists some i and j such that $d(\gamma_n(i), \gamma_n(j))$ is bounded in terms of λ independent of n. Therefore, $d(\partial Y_n(i), \partial Y_n(j))$ is also bounded above in terms of λ by Lemma 7.8. Let α be a geodesic of length bounded above in terms of λ joining $\partial Y_n(i)$ and $\partial Y_n(j)$ in $\mathbb{C} \setminus P(f)$.

Let $l > 0$ be the smallest number which satisfies that either $C(\rho_n, i - l)$ or $C(\rho_n, j - l)$ is nonempty. We may assume that $C(\rho_n, i - l)$ is nonempty (note that this assumption implies $i < j$ or $j \leq i - l$). Then the univalent inverse branch $f^{-l+1} : Y_n(i) \to Y_n(i - l + 1)$ extends to a univalent map on $Y_n(i) \cup \alpha \cup Y_n(j)$. Let η and W be the images of α and $Y_n(j)$ by this inverse branch, respectively (W may be equal to $Y_n(j - l + 1)$).

Let A be the component of $f^{-1}(Y_n(i - l + 1) \cup \eta \cup W)$ which contains $Y_n(i - l)$. Then

$$f : A \to Y_n(i - l + 1) \cup \eta \cup W$$

is a branched covering.

We claim there exists a component Z' of the interior of A which does not intersect $C(f) \cup P(f)$. Each component of the interior of A is mapped onto $Y_n(i-l+1)$ or W by f. These components are joined by the collection of paths $f^{-1}(\eta) \cap A$. Clearly, A cannot contain any essential loops. Since the degree of f is finite, there exist two or more "edges", that is, components of interior of A each of which touch only one component of $f^{-1}(\eta)$. They contains no critical point. Furthermore, at most one of them can intersect the postcritical set because $P(f) \cap A \subset P(\rho_n, i-l) \cup P(\rho_n, j-l)$ and $Y_n(i-l)$ cannot be an "edge". Let Z' be a component of interior of A which is an "edge" of A and which does not intersects $P(f)$.

Since the lengths of α, $\partial Y_n(i)$ and $\partial Y_n(j)$ are bounded in terms of λ and f is expanding with respect to the hyperbolic metric on $\mathbb{C} \setminus P(f)$, the hyperbolic distance between $\partial Z'$ and $\partial Y_n(i-l)$ on $\mathbb{C} \setminus P(f)$ is also bounded in terms of λ.

$f^l(Z')$ is equal to either $Y_n(i)$ or $Y_n(j)$. Let a_n and m' as follows: if $f^l(Z') = Y_n(i)$ (resp. $Y_n(j)$), then let a_n be the smallest number such that $i \le a_n \le n$ (resp. $j \le a_n \le n$) and that $C(\rho_n, a_n)$ is nonempty. Let $m' = a_n - i + l$ (resp. $a_n - j + l$). Then m' is the smallest integer satisfying that $f^{m'}(Z') \cap C(f)$ is nonempty. Therefore, $f^{-m'} : Y_n(a_n) \to Z'$ is univalent. This map can be extended univalently to $\tilde{Y}_n(a_n) \tilde{\to} \tilde{Z}$.

In the hyperbolic metric on $\mathbb{C} \setminus P(f)$, $\partial Y_n(a_n)$ is close to $\partial \tilde{Y}_n(a_n) = \gamma_n(a_n)$ by Lemma 7.8. So $\partial Z'$ is also close to $\partial \tilde{Z}$ because f is expanding. Hence the disk $\tilde{Z}$ is joined with $Y_n(i-l)$ by some path β' of length bounded above. Let

$$m'' = \begin{cases} (i-l) - a_n, & \text{if } i > a_n, \\ (i-l) - a_n + n, & \text{if } i \le a_n, \end{cases}$$

and let X be the component of $f^{-m''}(Y_n(i-l))$ which contains $P(\rho_n, a_n)$ (X coincide with $Y_n(a_n)$ if and only if $i > a_n$). Note that ∂X is close to $\gamma_n(a_n)$ in the hyperbolic metric on $\mathbb{C} \setminus P(f)$ (even if $i \le a_n$, the distance between ∂X and $\partial Y_n(a_n)$ is not greater than the distance between $\zeta_n = f^{n-a_n}(\partial X)$ and $\gamma_n = f^{n-a_n}(\partial Y_n(a_n))$, since f expands the hyperbolic metric on $\mathbb{C} \setminus P(f)$).

Let Z_n be a component of $f^{-m''}(\tilde{Z})$ which is joined with X by some path component β of $f^{-m''}(\beta')$. Then $\ell(\partial Z_n) \le \ell(Y_n(i)) < \lambda$. Since $\beta' \cup \tilde{Z}$ is disjoint from the postcritical set of f, $f^{m''}$ maps $\beta \cup Z_n$ univalently to $\beta' \cup \tilde{Z}$.

Let $m_n = m' + m''$. Note that since $i < j$ or $j \le i - l$, we have $0 < m_n \le n$. f^{m_n} sends Z_n univalently to $\tilde{Y}_n(a_n)$. Furthermore, $d(\partial \tilde{X}_n(a_n), Z_n)$ is bounded in terms of λ in the hyperbolic metric on $\mathbb{C} \setminus P(f)$, because $\partial \tilde{X}_n(a_n)$ and ∂X are both close to $\gamma_n(a_n)$, $\ell(\gamma_n(a_n)) < \lambda$ and $\ell(\beta) \le \ell(\beta')$ is bounded.

Finally, we will show that $\operatorname{area}(Z_n) > 1/C(\lambda)$. Let

$$E_1 = \{d(z, \gamma_n(a_n)) < 1\} \cap Y_n$$

in the hyperbolic metric on $\mathbb{C} \setminus P(f)$. The collar theorem guarantees the lower bound for the injectivity radius of any point on $\gamma_n(a_n)$, so it gives a lower bound for $\operatorname{area}(E_1)$. Since $f^{m_n} : Z_n \to \tilde{Y}_n(a_n)$ is univalent, there exists $E_2 \subset f(\tilde{Z})$ and $E_3 \subset Z_n$ such that $f^{m''+1}$ maps E_3 univalently to E_2 and $f^{m'-1}$ maps E_2 univalently to E_1.

We need only show that $\operatorname{area}(E_3)$ is bounded below. By Lemmas 7.9 and 2.7, $\|f^{m'-1}(z)\| < C(\lambda)$ on E_2 and $\|f^{m''+1}(z)\| < C(\lambda)$ on E_3. Moreover, for any $z \in E_3$, there exists a path η joining z to ∂X with the length $\ell(f^{m''+1}(\eta)) < C(\lambda)$. Indeed, the path consists of the geodesic joining z to ∂Z_n, the arc along $\partial \tilde{Z}$, and $f(\beta')$ satisfies this condition.

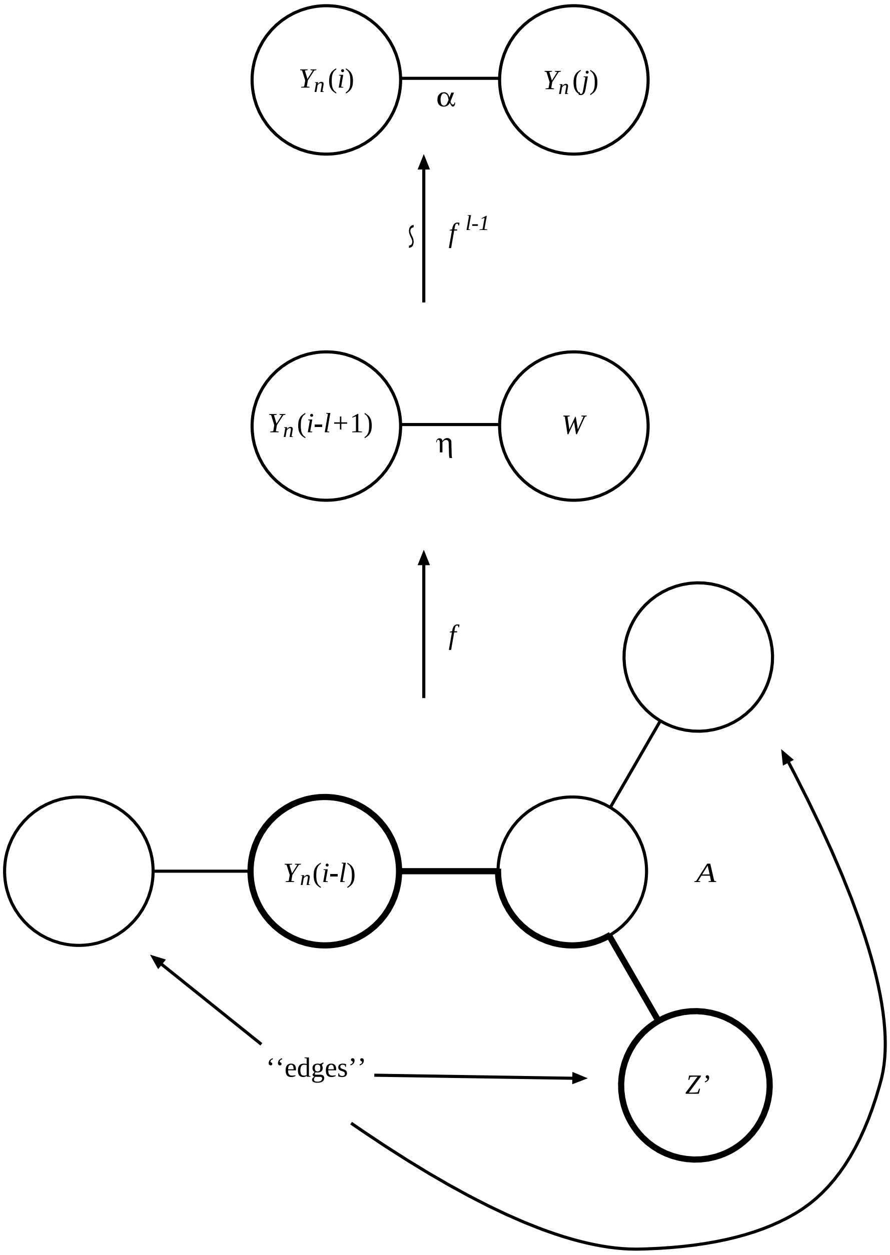

Figure 5. Construction of Z'

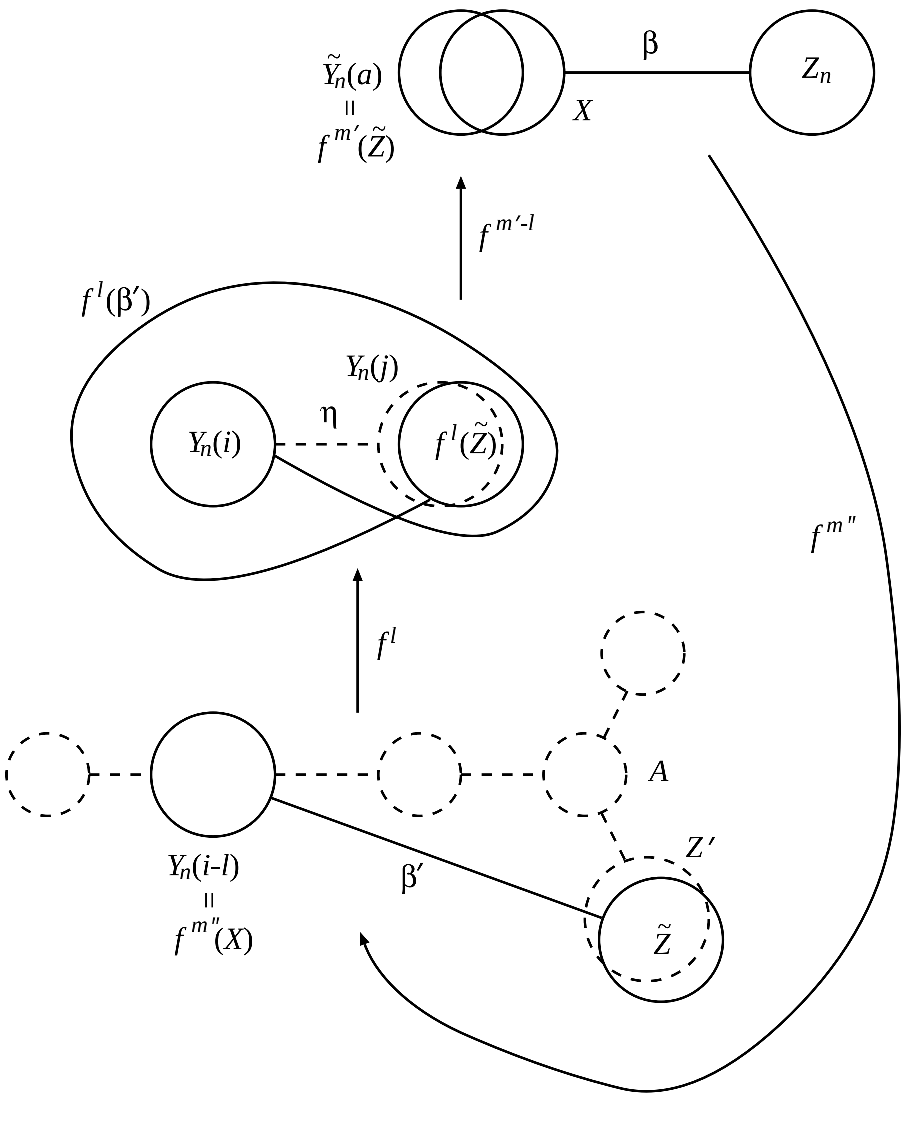

Figure 6. The case $f^l(Z') = Y_n(j)$

Therefore, f^{m_n} is not so expanding on E_3, and area(E_3) is bounded below in terms of λ. $\square$

Now we may assume a_n above is equal to n for infinitely many n. Indeed, there exists some $c \in C_R$ such that for infinitely many $n \in \mathrm{sr}(f, C_R, \lambda)$, $C(\rho_n, a_n) \ni c$. So we need only replace c_0, λ, X_n and Y_n with c, $2^d\lambda$, $\tilde{X}_n(a_n)$ and $\tilde{Y}_n(a_n)$ respectively and properly adjust the constants depending on λ (note that $\lambda/2^d < \ell(\gamma_n(i)) < \lambda$ by Lemma 4.9). Let

$$\mathrm{sr}_1(f, C_R, \lambda) = \left\{ n \in \mathrm{sr}(f, C_R, \lambda) \;\middle|\; a_n = n \right\}.$$

Then $\#\,\mathrm{sr}_1(f, C_R, \lambda) = \infty$ and for any $n \in \mathrm{sr}_1(f, C_R, \lambda)$,

(1) $f^{m_n} : Z_n \to Y_n$ is univalent,

(2) $d(\partial X_n, \partial Z_n) < C(\lambda)$,

(3) $\ell(\partial Z_n) < \lambda$,

(4) area$(Z_n) > 1/C(\lambda)$

in the hyperbolic metric on $\mathbb{C} \setminus P(f)$.

Proof of Theorem 7.1. Suppose f admits an invariant line field μ supported on a set of positive measure $E \subset J(f)$. Take a point $x \in E$ of almost continuity of μ which satisfies the properties in Lemma 7.5.

For each $n \in \mathrm{sr}_1(f, C_R, \lambda)$, let $k(n) \geq 0$ be the smallest number which satisfies that $f^{k(n)+1}(x) \in \mathcal{Y}_n$. Then $k(n)$ tends to infinity by the preceding lemma. Now let $n_1 = \min(\mathrm{sr}_1(f, C_R, \lambda))$ and we consider only sufficiently large n such that $k(n) > k(n_1)$. In particular, we have $k(n) > 0$ and $f^{k(n)}(x)$ does not lie in $\mathcal{Y}_n$.

First, we construct univalent maps $h_n : \tilde{Y}_n(j(n)) \xrightarrow{\sim} T_n \subset \mathbb{C}$. Let $i(n)$ be the number with $0 \leq i(n) < n$ and $f^{k(n)+1}(x) \in Y_n(i(n)+1)$.

Case I. $i(n) > 0$. Then $f^{k(n)}(x)$ lies in a component W_n of $f^{-1}(Y_n(i(n)+1))$. W_n lies in $\mathbb{C} \setminus P(f)$ because $f^{-1}(P(\rho_n, i(n)+1)) \cap P(f) = P(\rho_n, i(n))$ and $f^{k(n)}(x)$ does not lie in $Y_n(i(n))$, which contains $P(\rho_n, i(n))$. Moreover, W_n does not contain any critical point. Indeed, $C(f) \setminus C_R$ does not intersect $\mathcal{Y}_{n_1}$ and $f^{k(n)}(x)$ lies in $\mathcal{Y}_n$. Thus $\mathcal{Y}_n$ contains $W_n \cap f^{-1}(P(\rho_n, i(n)+1))$.

Let $j(n)$ be the smallest number which satisfies $i(n)+1 \leq j(n) \leq n$ and $C(\rho_n, j(n))$ is nonempty. Define h_n by choosing the univalent inverse branch

$$Y_n(j(n)) \xrightarrow{f^{i(n)-j(n)}} W_n \xrightarrow{f^{-k(n)}} \mathbb{C},$$

which sends $f^{k(n)+j(n)-i(n)}(x)$ to x. Since $\tilde{Y}_n(j(n)) \setminus Y_n(j(n))$ does not intersect the postcritical set of f, this map extends to a univalent map defined on $\tilde{Y}_n(j(n))$.

Case II. $i(n) = 0$ and $f^{k(n)}(x) \notin X_n \setminus Y_n$. Then $f^{k(n)}(x) \notin X_n$. Hence we can choose the univalent inverse branch h_n as in Case I.

Case III. $i(n) = 0$ and $f^{k(n)}(x)$ lies in $X_n \setminus Y_n$. Since $X_n \ni c_0$, we cannot take the inverse branch of $f : X_n \to Y_n(1)$. But by Lemma 7.10, there exists an disk close to ∂X_n, which maps univalently to Y_n by f^{m_n}. By Lemma 7.8, $f^{k(n)}(x)$ is close to ∂X_n in the hyperbolic metric on $\mathbb{C} \setminus P(f)$. Thus the distance from $f^{k(n)}(x)$ to ∂Z_n is also bounded in terms of λ. Take a path ζ_n from x to ∂Z_n with $\ell(\zeta_n) < C(\lambda)$. Define h_n by

$$Y_n(n) \xrightarrow{f^{-m_n}} Z_n \xrightarrow{f^{-k(n)}} \mathbb{C},$$

where the branch of $f^{-k(n)}$ is chosen so that its extension to $Z_n \cup \zeta_n$ sends $f^{k(n)}(x)$ to x. Let τ_n be the image of ζ_n by this branch of $f^{-k(n)}$. In this case, we set $j(n) = n$.

Let $T_n = h_n(\tilde{Y}_n(j(n)))$. There exists some $c \in C_R$ such that for infinitely many n, $C(\rho_n, j(n)) = \{c\}$. So just as the definition of $\mathrm{sr}_1(f, C_R, \lambda)$, by replacing c_0 by c and so on, and passing to a further subsequence, we may assume $j(n) = n$.

To complete the proof, we need the following lemma.

Lemma 7.11. *As $n \to \infty$,*

$$\operatorname{diam}(T_n) \to 0,$$
$$d(x, T_n) \leq C(\lambda) \operatorname{diam}(T_n).$$

in the hyperbolic metric on $\mathbb{C} \setminus P(f)$.

Proof. In Cases I and II, $f^{k(n)}$ maps T_n univalently to W_n and sends $x \in T_n$ into W_n. Moreover, since $\ell(\partial W_n) < \lambda$ and $W_n \subset \mathbb{C} \setminus P(f)$, $\operatorname{diam}(W_n)$ is bounded independent of n.

Therefore, the fact $\|(f^{k(n)})'(x)\| \to \infty$ and Lemma 2.7 implies

$$\min_{y \in \partial T_n} \|(f^{k(n)})'(y)\| \to \infty.$$

Hence $\ell(\partial T_n)$ tends to zero and so does $\operatorname{diam}(T_n)$. Since $x \in T_n$, the second inequality is trivial.

In Case III, $f^{k(n)}$ sends $T_n \cup \tau_n$ univalently to $Z_n \cup \zeta_n$. First, we claim that

$$\frac{1}{C(\lambda)} \leq \frac{\|(f^{k(n)})'(y)\|}{\|(f^{k(n)})'(x)\|} \leq C(\lambda)$$

for any y in $T_n \cup \tau_n$.

Let $r(y)$ be the injectivity radius of y in $\mathbb{C} \setminus P(f)$. If $z \in \gamma_n$, then $r(z)$ is bounded below in terms of λ by the collar theorem. Since $d(f^{k(n)}(x), \gamma_n)$ is bounded above and the logarithm of the injectivity radius is Lipschitz by Lemma 2.3 $r(f^{k(n)}(x))$ is bounded below in terms of λ. Since there exists an arc η joining y to x such that

$$\ell(f^{k(n)}(\eta)) \leq \ell(\zeta_n) + \operatorname{diam}(Z_n) < C(\lambda),$$

the claim follows from Lemma 2.7.

Therefore, we have

$$d(x,y) \leq \frac{C(\lambda)}{\|(f^{k(n)})'(x)\|}$$

for any $y \in T_n$. Since $\|(f^{k(n)})'(x)\|$ tends to infinity, $d(x,y)$ tends to zero. Thus $\operatorname{diam}(T_n) \to 0$.

By Lemma 7.10, $\operatorname{area}(Z_n) > 1/C(\lambda)$. So

$$\begin{aligned}\frac{1}{C(\lambda)} < \operatorname{area}(Z_n) &= \int_{T_n} \|(f^{k(n)})'(y)\|^2 \rho(y)^2 |dy|^2 \\ &\leq C(\lambda) \operatorname{area}(T_n) \|(f^{k(n)})'(x)\|^2,\end{aligned}$$

where $\rho(y)|dy|$ is the hyperbolic metric on $\mathbb{C} \setminus P(f)$. Since $\operatorname{diam}(T_n)$ is bounded, $\operatorname{area}(T_n) \leq C(\lambda) \operatorname{diam}(T_n)^2$. Therefore,

$$\frac{1}{\|(f^{k(n)})'(x)\|} \leq C(\lambda) \operatorname{diam}(T_n),$$

and for $y \in T_n$,

$$\begin{aligned}d(x, T_n) \leq d(x,y) &\leq \frac{C(\lambda)}{\|(f^{k(n)})'(x)\|} \\ &\leq C(\lambda) \operatorname{diam}(T_n).\end{aligned}$$

□

Now we will complete the proof of Theorem 7.1.

By passing to a further subsequence, we may also assume that Lemma 7.6 holds. Namely, $A_n \circ f^n \circ A_n^{-1} : (A_n(X_n), 0) \to (A_n(Y_n), A_n(f^n(c_0)))$ converges to a proper map $g : (X, 0) \to (Y, g(0))$ in the Carathéodory topology, $0 \in X \cap Y$ and $g'(0) = 0$, where $A_n(z) = (z - c_0)/\operatorname{diam}(B_n)$.

Let $k_n = h_n \circ A_n^{-1} : A_n(Y_n) \to T_n$. Then

$$1 \leq \operatorname{diam}_E(A_n(Y_n)) \leq C(\lambda)$$

by Lemma 7.2. Since $\partial Y_n = \gamma_n$ is a simple closed geodesic of length less than λ in $\mathbb{C} \setminus P(f)$, k_n can be extended univalently to an annulus about $A_n(Y_n)$ of modulus bounded below in terms of λ. Therefore, by the Koebe distortion theorem,

$$\frac{1}{C(\lambda)} |k_n'(0)| \leq \operatorname{diam}_E(T_n) \leq C(\lambda) |k_n'(0)|.$$

Since the Euclidean and hyperbolic metrics are comparable near x and $\operatorname{diam}(T_n)$ tends to 0 in the hyperbolic metric, $|k_n'(0)|$ also tends to zero.

Similarly,

$$\frac{|x - k_n(0)|}{|k_n'(0)|} \le C(\lambda)\frac{d(x, T_n) + \mathrm{diam}(T_n)}{\mathrm{diam}(T_n)} \le C(\lambda)$$

by Lemma 7.11. Therefore, by Lemma 5.7, there exists a further subsequence such that μ_n converges to a univalent line field μ on Y. But $0 \in X \cap Y$ is a critical point of $g : (X, 0) \to (Y, g(0))$ and g cannot carry any univalent line fields. So this is a contradiction.

Thus f admits no invariant line field on its Julia set. □

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