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Estimates of invariant metrics on pseudoconvex domains with comparable Levi form

By

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Abstract

Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ and let $z_0 \in b\Omega$ be a point of finite type. We also assume that the Levi form of $b\Omega$ is comparable in a neighborhood of z_0. Then we get a quantity which bounds from above and below the Bergman metric, Caratheodory metric and Kobayashi metric in a small constant and large constant sense.

1. Introduction

The purpose of this paper is to estimate from above and below the values of the Bergman, Caratheodory and Kobayashi metrics for a vector X in a neighborhood of a boundary point z^0 of finite type with comparable Levi-form. In the rest of this paper, we let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth defining function r, i.e., $\Omega = \{z \in \mathbb{C}^n : r(z) < 0\}$, and let $\lambda_1(z), \dots, \lambda_{n-1}(z)$ be the eigenvalues of the Levi-form $\partial\overline{\partial}r$ of $b\Omega$ near a point $z^0 \in b\Omega$.

We say Ω has comparable Levi-form near z^0 if there are a constant $c > 0$ and a neighborhood U of z^0 such that

$$\lambda_k(z) \geq c \cdot \sum_{i=1}^{n-1} \lambda_i(z), \quad k = 1, 2, \dots, n-1, \quad z \in U. \tag{1.1}$$

For example, let $r(z) = 2\,\mathrm{Re}\, z_3 + (|z_1|^2 + |z_2|^2)^2$ be a defining function for a domain Ω in $\mathbb{C}^3$ near the origin. Then the Levi-form of $b\Omega$ satisfies (1.1) near the origin.

We first give the definition of each of the above metrics. Let X be a holomorphic tangent vector at a point z in Ω. Denote the set of holomorphic functions on Ω by $A(\Omega)$. Then the Bergman metric $B_\Omega(z; X)$, the Caratheodory

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metric $C_\Omega(z;X)$ and the Kobayashi metric $K_\Omega(z;X)$ are defined by

$$C_\Omega(z;X) = \sup\{|Xf(z)| : f \in A(\Omega), \quad \|f\|_{L^\infty(\Omega)} \le 1\},$$
$$K_\Omega(z;X) = \inf\left\{1/r : \exists\ f : D_r \subset \mathbb{C}^1 \to \mathbb{C}^n \text{ such that } f_*\left(\frac{\partial}{\partial t}|_0\right) = X\right\},$$
$$B_\Omega(z:X) = b_\Omega(z;X)/(K_\Omega(z,\overline{z}))^{1/2},$$

where D_r denotes the disc of radius r in $\mathbb{C}^1$, and

$$K_\Omega(z,\overline{z}) = \sup\{|f(z)|^2 : f \in A(\Omega), \quad \|f\|_{L^2(\Omega)} \le 1\},$$
$$b_\Omega(z;X) = \sup\{|Xf(z)| : f \in A(\Omega),\ f(z) = 0,\ \|f\|_{L^2(\Omega)} \le 1\}.$$

Let $z^0 \in b\Omega$ be a point of finite type m in the sense of D'Angelo [8]. Assuming that $|\partial r/\partial z_n(z)| \ge c_1 > 0$ in a neighborhood U of z_0, set

$$L_j = \frac{\partial}{\partial z_j} - \left(\frac{\partial r}{\partial z_n}\right)^{-1} \frac{\partial r}{\partial z_j}\frac{\partial}{\partial z_n}, \quad j = 1, 2, \dots, n-1, \text{ and}$$
$$L_n = \frac{\partial}{\partial z_n}.$$

Then $\{L_1, \dots, L_n\}$ form a basis of $\mathbb{C}T^{(1,0)}(U)$ provided U is sufficiently small. For any integer $j, k > 0$, set

$$\mathcal{L}_{j,k}\partial\overline{\partial}r(z) = \underbrace{L_1 \cdots L_1}_{(j-1) times} \underbrace{\overline{L}_1 \cdots \overline{L}_1}_{(k-1) times} \partial\overline{\partial}r(z)(L_1, \overline{L}_1)(z), \tag{1.2}$$

and define

$$C_l(z) = \max\{|\mathcal{L}_{j,k}\partial\overline{\partial}r(z)| : j + k = l\}. \tag{1.3}$$

Let $X = b_1L_1 + \cdots + b_nL_n := X' + b_nL_n$ be a holomorphic tangent vector at z and set

$$M(z;X) = |X'|\sum_{l=2}^{m} |C_l(z)|^{1/l}|r(z)|^{-1/l} + |b_n||r(z)|^{-1}, \tag{1.4}$$

where $|X'| = |b_1| + \cdots + |b_{n-1}|$. Then we can state our main result as follows

Theorem 1.1. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$. Let $z_0 \in b\Omega$ be a point of finite type m and assume that the Levi-form of $b\Omega$ is comparable in a neighborhood of z_0. Then there exist a neighborhood U about z_0 and positive constants c and C such that for all $X = b_1L_1 + \cdots + b_nL_n$ at $z \in U \cap \Omega$,*

$$cM(z;X) \le B_\Omega(z;X), \quad C_\Omega(z;X), \quad K_\Omega(z:X) \le CM(z;X). \tag{1.5}$$

Remark 1.2. Because $|C_m(z)| \geq c' > 0$ for all $z \in U \cap \Omega$, (1.5) says, in particular, that

$$B_\Omega(z;X), C_\Omega(z;X), K_\Omega(z;X) \gtrsim (|X'||r(z)|^{-1/m} + |b_n||r(z)|^{-1})$$

for a holomorphic vector field $X = b_1 L_1 + \cdots + b_n L_n$ at z.

Several authors found some results about these metrics for some pseudoconvex domains in $\mathbb{C}^n$, but in each case the lower bounds are different from the upper bounds [1], [5], [9], [10], [13], [14]. In [2], Catlin got a result similar to above theorem in $\mathbb{C}^2$, and Herbort [12] and the author [6] got the similar result independently for the domains of finite type with one degenerate eigenvalue.

To prove Theorem 1.1, we must get a complete geometric analysis of $b\Omega$ near z_0 as Catlin has employed in [2]. Then we construct a family of "maximal plurisubharmonic functions" which is a crucial ingredient to get a weighted estimates for $\overline{\partial}$ Neumann problem (Section 3).

2. Special coordinates and polydiscs

In this section we want to show that about each point z' in U, there is a special coordinates about z' and a polydisc of maximal size on which the function $r(z)$ changes by no more than some prescribed small number $\delta > 0$. We then construct a family of plurisubharmonic functions with maximal Hessian to push out the boundary of Ω.

Let α, β be multi-indices and let $\alpha' = (\alpha_1, \ldots, \alpha_{n-1}, 0)$, $\alpha'' = (0, \alpha_2, \ldots, \alpha_{n-1}, 0)$, etc. Also let ∂^α denote the holomorphic differential operator of order $|\alpha|$. We first construct special coordinates centered at $z' \in U$.

Proposition 2.1 ([7, Proposition 2.1]). *For each $z' \in U$ and positive integer m, there is a biholomorphism $\Phi_{z'} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $\Phi_{z'}^{-1}(z') = 0$, such that*

$$\begin{aligned}(2.1)\quad \rho(\zeta) := r(\Phi_{z'}(\zeta)) = r(z') + \operatorname{Re}\zeta_n + \sum_{\substack{j+k\leq m\\ j,k\geq 1}} a_{jk}(z')\zeta_1^j\overline{\zeta}_1^k \\ + \sum_{\substack{|\alpha'+\beta'|\leq m\\ |\alpha'|,|\beta'|\geq 1\\ 1\leq|\alpha''+\beta''|\leq m}} b_{\alpha'\beta'}(z')\zeta^{\alpha'}\overline{\zeta}^{\beta'} + \mathcal{O}(|\tilde{\zeta}|^{m+1} + |\zeta||\zeta_n|),\end{aligned}$$

where $\tilde{\zeta} = (\zeta_1, \ldots, \zeta_{n-1}, 0)$.

We now show how to define a polydisc around z' in ζ-coordinates. Set

$$\begin{aligned} A_l(z') &= \max\{|a_{jk}(z')| : j+k = l\}, \\ B_{l'}(z') &= \max\{|b_{\alpha'\beta'}(z')| : |\alpha'+\beta'| = l'\}, \qquad 2 \leq l, l' \leq m. \end{aligned}$$

For each $\delta > 0$, we define $\tau(z', \delta)$ by:

$$(2.2)\qquad \tau(z',\delta) = \min\{(\delta/A_l(z'))^{\frac{1}{l}},\ (\delta/B_{l'}(z'))^{\frac{1}{l'}} : 2 \leq l, l' \leq m\}.$$

Then the author showed that there are some constant $b_0 > 0$ and integers $j_0, k_0 \le m$ such that $|a_{j_0k_0}| \ge b_0 \cdot \delta \cdot \tau(z', \delta)^{-j_0-k_0}$ and hence

$$\tau(z', \delta)^{-1} \approx \sum_{l=2}^{m} (\delta / A_l(z'))^{-\frac{1}{l}}. \tag{2.3}$$

Define

$$R_\delta(z') = \{\zeta \in \mathbb{C}^n : |\zeta_k| \le \tau(z', \delta),\ 1 \le k \le n-1,\ |\zeta_n| \le \delta\}, \quad \text{and}$$
$$Q_\delta(z') = \{\Phi_{z'}(\zeta) : \zeta \in R_\delta(z')\}.$$

In order to study how $\tau(z', \delta)$ depends on $z \in Q_\delta(z')$, it is convenient to introduce an analogous quantity $\eta(z, \delta)$ that is defined more intrinsically. We take the frame $\{L_1, \dots, L_n\}$ defined on U, and let $\mathcal{L}_{j,k}\partial\overline{\partial}r(z)$ and $C_l(z)$ be defined as in (1.2) and (1.3) respectively. Define

$$\eta(z, \delta) = \min\{(\delta / C_l(z))^{\frac{1}{l}}; 2 \le l \le m\}. \tag{2.4}$$

Then we have the following important relations between $\eta(z, \delta)$ and $\tau(z, \delta)$ ([7, Section 2]).

Proposition 2.2. *Let $z \in Q_\delta(z')$. Then*

$$\begin{aligned} &\tau(z', \delta) \lesssim \eta(z, \delta) \lesssim \tau(z', \delta), \\ &\tau(z', \delta) \approx \tau(z, \delta). \end{aligned} \tag{2.5}$$

For $\epsilon > 0$, we let $\Omega_\epsilon = \{z; r(z) < \epsilon\}$ and set

$$S(\epsilon) = \{z : -\epsilon < r(z) < \epsilon\}.$$

The following theorem reflects the local geometry of the boundary of Ω near z^0, and shows the existence of one parameter family of plurisubharmonic functions with maximal Hessian.

Theorem 2.3 ([7, Theorem 3.2])**.** *For all small $\delta > 0$, there is a plurisubharmonic function $\lambda_\delta \in C^\infty(\Omega_\delta)$ with the following properties,*

(i) $|\lambda_\delta(z)| \le 1$, $z \in U \cap \Omega_\delta$,

(ii) *For all $L = \sum_{j=1}^n b_j L_j$ at $z \in U \cap S(\delta)$,*

$$\partial\overline{\partial}\lambda_\delta(z)(L, \overline{L}) \approx \tau(z, \delta)^{-2} \sum_{k=1}^{n-1} |b_k|^2 + \delta^{-2}|b_n|^2,$$

(iii) *If $\Phi_{z'}$ is the map associated with a given $z' \in U \cap S(\delta)$, then for all $\zeta \in R_\delta(z')$ with $|\rho(\zeta)| < \delta$,*

$$|\partial^\alpha \overline{\partial}^\beta (\lambda_\delta \circ \Phi_{z'})(\zeta)| \lesssim C_{\alpha,\beta} \delta^{-\alpha_n - \beta_n} \tau^{-|\alpha' + \beta'|}.$$

With this family of functions λ_δ, we shall construct for each $z' \in U \cap b\Omega$ and each small $\delta > 0$, a domain (locally defined in U) $\Omega_{z',\delta}$ which contains Ω such that the boundary of $\Omega_{z',\delta}$ is pushed out as far as possible, given the constraints that $d(z', b\Omega_{z',\delta}) < \delta$ and that $b\Omega_{z',\delta}$ is pseudoconvex. Since z' will be fixed, we will work in ζ-coordinates defined by $\Phi_{z'}(\zeta) = z$.

Let $A_l(z')$ be the quantities defined after Proposition 2.1. Set $\rho(\zeta) = r(\Phi_{z'}(\zeta))$ and set $U' = \{\zeta : \Phi_{z'}(\zeta) \in U\}$. For all small s and $\delta > 0$, define

$$J_\delta(z', \zeta) = \left[\delta^2 + |\zeta_n|^2 + \sum_{l=2}^{m} A_l(z')^2 |\tilde{\zeta}|^{2l}\right]^{1/2}, \tag{2.6}$$

where $\tilde{\zeta} = (\zeta_1, \dots, \zeta_{n-1}, 0)$, and set

$$W_{s,\delta}(z') = \{\zeta \in U' : |\rho(\zeta)| < sJ_\delta(\zeta)\}. \tag{2.7}$$

Set $J_\delta(z', \zeta) = J_\delta(\zeta)$ for a convenience. By adding up the weight functions in Theorem 2.3, we have the following theorem.

Proposition 2.4. *For each $z' \in U \cap b\Omega$ and each small $\delta > 0$, there exists a small real-valued function $H_{z',\delta}(\zeta)$ defined in $W_{s,\delta}(z')$ (where s is a small constant independent of z' and δ) such that*

(i) $-J_\delta(\zeta) \approx H_{z',\delta}(\zeta)$,

(ii) *for any $L = b_1 L'_1 + b_2 L'_2 + \cdots + b_n L'_n$ at ζ,*

$$\partial\overline{\partial} H_{z',\delta}(L, \overline{L})(\zeta) \approx J_\delta(\zeta)\left[\frac{|b_n|^2}{(J_\delta(\zeta))^2} + \frac{|b'|^2}{\tau(z', J_\delta(\zeta))^2}\right], \quad \textit{and}$$

$$|LH_{z',\delta}| \lesssim J_\delta(\zeta)\left(\frac{|b_n|}{J_\delta(\zeta)} + \frac{|b'|}{\tau(z', J_\delta(\zeta))}\right),$$

where $|b'| = |b_1| + \cdots + |b_{n-1}|$, and $L'_k = (\Phi_{z'}^{-1})L_k$, $k = 1, 2, \dots, n$.

Proof. Set $N_1 = [\log_2(1/\delta)]$. Let $D_R = \{\zeta \in \mathbb{C}^n : |\zeta_i| < R, \quad i = 1, 2, \dots, n\}$, and let $\psi \in C_0^\infty(D_2 - D_{1/4})$ be a function that satisfies $\psi(\zeta) = 1$ for $\zeta \in D_1 - D_{1/2}$. For all k, $1 \le k$

$$\psi_k(\zeta) = \psi\left(\tau(z', 2^{-k})^{-1}\zeta_1, \dots, \tau(z', 2^{-k})^{-1}\zeta_{n-1}, 2^k\zeta_n\right),$$

and for $k = N_1$, set

$$\psi_{N_1}(\zeta) = \phi\left(\tau(z', 2^{-N_1})^{-1}\zeta_1, \dots, \tau(z', 2^{-N_1})^{-1}\zeta_{n-1}, 2^{N_1}\zeta_n\right),$$

where $\phi \in C_0^\infty(D_2)$ satisfies $\phi(\zeta) = 1$ for $\zeta \in D_1$. Combining (2.2) and (2.6), one obtains that

$$J_\delta(\zeta) \approx 2^{-k}, \quad \zeta \in \operatorname{supp} \psi_k. \tag{2.8}$$

For each $\delta > 0$, set $\lambda'_\delta = \lambda_\delta \circ \Phi_{z'}$, where λ_δ is the plurisubharmonic function as in Theorem 2.3. Choose N_0 so that $\lambda_{2^{-k}t}$ is well-defined for all $\zeta \in \operatorname{supp} \psi_k$ whenever $k \geq N_0$, and set

$$H_{z',\delta}(\zeta) = \sum_{k=N_0}^{N_1} 2^{-k}\psi_k(\zeta)\,(\lambda'_{2^{-k}t}(\zeta) - 2)\,.$$

Then $H_{z',\delta}$ is well-defined (fixed finite sum independent of z' and δ). From (2.7), (2.8) and from the fact that $H_{z',\delta}(\zeta) \approx -2^{-k}$ for $\zeta \in \operatorname{supp} \psi_k$, property (i) follows. Also the major part of the Hessian of $H_{z',\delta}$ will be $\partial\overline{\partial}\lambda_{2^{-k}t}(\zeta)$ and other error terms will be absorbed into $\partial\overline{\partial}\lambda_{2^{-k}t}(\zeta)$ for sufficiently small t. This fact together Theorem 2.3 prove property (ii). □

Set $\Omega_{z'} = \Phi_{z'}^{-1}(\Omega)$ and set $\Omega_{z',\epsilon} = \{\zeta \in \mathbb{C}^n : \rho(\zeta) < \epsilon\}$. Set $\lambda'_\delta = \lambda_\delta \circ \Phi_{z'}$. Then there is a fixed constant $b > 0$ (independent of z', δ) such that λ'_δ is defined and plurisubharmonic on $\Omega_{z',b\delta}$ and satisfies all the properties of Theorem 2.3 on $\Omega_{z'}$ instead of Ω.

Proposition 2.5. *For each sufficiently small $\delta > 0$, there is one parameter family of "maximal pushed-out" pseudoconvex domains $\{\Omega^\epsilon_{z',\delta}\}_{\epsilon>0}$ which contain $\Omega_{z'}$ near the origin.*

Proof. Let U'_1 be a small neighborhood of the origin with $U'_1 \subset\subset U' = \Phi_{z'}^{-1}(U)$. Then one has $|dH_{z',\delta}(\zeta)| \lesssim 1$ for $\zeta \in W_{s,\delta}(z')$ by the property (ii) of Proposition 2.4. Hence for all small $\epsilon > 0$, the function

$$\rho^\epsilon_{z',\delta}(\zeta) = \rho(\zeta) + \epsilon H_{z',\delta}(\zeta)$$

satisfies $\partial\rho^\epsilon_{z',\delta}/\partial\zeta_n \neq 0$ in U'_1 and therefore form a family of defining functions of hypersurfaces $\{\zeta : \rho^\epsilon_{z',\delta}(\zeta) = 0\}$ in $W_{s,\delta}(z')$. If we use the properties (i), (ii) of Proposition 2.4, it follows that the hypersurfaces defined by $\rho^\epsilon_{z',\delta}(\zeta)$ are pseudoconvex. □

Now we choose $\epsilon_0 > 0$ (independent of z' and δ) so that

$$\sup\{\rho(\zeta) : \zeta \in R_\delta(z') \;\text{ and }\; \rho^{\epsilon_0}_{z',\delta}(\zeta) \leq 0\} < b\delta,$$

where b is the small number before Proposition 2.5. Set $\rho_{z',\delta}(\zeta) = \rho^{\epsilon_0}_{z',\delta}(\zeta)$.

For ζ' near 0, define a polydisc $P_a(\zeta')$ by

$$\begin{aligned}(2.9)\quad P_a(\zeta') = \{\zeta \in \mathbb{C}^n : |\zeta_n - \zeta'_n| < aJ_\delta(\zeta'),\\ |\zeta_k - \zeta'_k| < \tau(z', aJ_\delta(\zeta')),\ 1 \leq k \leq n-1\}.\end{aligned}$$

Proposition 2.6. *There exist constants $a > 0$ and $d_1 > 0$ (independent of z', ζ' and δ) so that if $\zeta' \in \Omega_{z'}$ and $|\zeta'| < d_1$, then $\rho_{z',\delta}(\zeta) < 0$ for $\zeta \in P_a(\zeta')$.*

Proof. We may assume that $\zeta' \in b\Omega_{z'}$ (this will be the worst case). If a is sufficiently small (independent of z' and δ), then by virtue of (2.3)–(2.7), it follows that

$$J_\delta(\zeta) \approx J_\delta(\zeta'), \quad \zeta \in P_a(\zeta'). \tag{2.10}$$

By (2.10) and from the property (i) of Proposition 2.4, it follows that there exists a small constant $c > 0$, such that

$$H_{z',\delta}(\zeta) \leq -cJ_\delta(\zeta'), \quad \zeta \in P_a(\zeta'). \tag{2.11}$$

By a simple Taylor's theorem argument, we then obtain that

$$|\rho(\zeta)| \leq CaJ_\delta(\zeta'), \quad \zeta \in P_a(\zeta'). \tag{2.12}$$

Since $\rho_{z',\delta}(\zeta) = \rho(\zeta) + \epsilon_0 H_{z',\delta}(\zeta)$, it follows from (2.11) and (2.12) that, $\rho_{z',\delta}(\zeta) < 0$ provided a is chosen so that $a < c\epsilon_0/C$. This completes the proof. □

The existence of the following two-sided bumping family of pseudoconvex domains was shown by the author in [3], [4].

Theorem 2.7. *Let Ω be a smoothly bounded pseudoconvex domain and let $z_0 \in b\Omega$ be a point of finite type. Then there is a neighborhood V of z_0 and a family of smoothly bounded pseudoconvex domains $\{\Omega_t\}_{-1\leq t\leq 1}$ satisfying the following propoerties*;

(i) $\Omega_0 = \Omega$*,*

(ii) $\Omega_{t_1} \subset \Omega_{t_2}$ *if* $t_1 < t_2$*,*

(iii) $\{\partial\Omega_t\}_{-1\leq t\leq 1}$ *is a* C^∞*-family of real hypersurfaces in* $\mathbb{C}^n$ *and the points of* $\partial\Omega_t \cap V$ *are finite type,*

(iv) $D_t - D_{-t} \subset V$ *for all* t.

Remark 2.8. By virtue of the construction of $\Phi_{z'}$ and $\rho_{z',\delta}(\zeta)$, we can choose $d_1 > 0$ and a neighborhood $U \subset\subset V$ of z_0 (independent of z') so that $\rho_{z',\delta}$ is defined in $\{\zeta : |\zeta| < d_1\}$ and satisfies all the properties in this section for each $z' \in b\Omega \cap U$.

Set $\Omega_{t,z'} = \{\zeta \in \mathbb{C}^n : \Phi_{z'}(\zeta) \in \Omega_t\}$, where $\{\Omega_t\}$ is the family of domains as in Theorem 2.7. Set

$$\Omega_{z',\delta} = \{\zeta : |\zeta| < d_1 \text{ and } \rho_{z',\delta}(\zeta) < 0\}.$$

The construction of $\rho_{z',\delta}$ in this section shows that if $\zeta \in \overline{\Omega}_{z'}$ and if $d_1/2 < |\zeta| < d_1$, then $d(\zeta, b\Omega_{z',\delta}) \gtrsim J_\delta(\zeta, z')$. Since $A_m(z') + B_m(z') \gtrsim 1$ for all $z' \in U$, it follows from (2.3) and (2.4) that $J_\delta(\zeta, z') \gtrsim 1$, and hence there is a constant $c_1 > 0$ so that $d(\zeta, b\Omega_{z',\delta}) \geq c_1$, for $\zeta \in U \cap b\Omega$ and $d_1/2 < |\zeta| < d_1$. Choose $t = t_0$ sufficiently small so that

$$d(\zeta, b\Omega_{t_0,z'}) < c_1/2 \qquad \text{if } d_1/2 < |\zeta| < d_1.$$

Now define a domain $\tilde{\Omega}_{z',\delta}$ by

$$\tilde{\Omega}_{z',\delta} = \{\zeta \in \Omega_{t_0,z'} : |\zeta| \geq d_1\} \cup \{\Omega_{t_0,z'} \cap \Omega_{z',\delta}\}.$$

Since pseudoconvexity is a local condition, $\tilde{\Omega}_{z',\delta}$ is a pseudoconvex domain. By combining the properties of $\Omega_{z',\delta}$ and $\Omega_{t_0,z'}$, we obtain

Proposition 2.9. *For all z' near z_0 and all δ, $0 < \delta < \delta_0$, the domain $\tilde{\Omega}_{z',\delta}$ has the following properties*;

(i) *$\tilde{\Omega}_{z',\delta}$ is a bounded pseudoconvex domain that contains $\Omega_{z'}$,*

(ii) *there is a constant $a > 0$ so that for all $\zeta' \in \Omega_{z'}$ with $|\zeta'| < d_1$, $P_a(\zeta') \subset \tilde{\Omega}_{z',\delta}$,*

(iii) *in the region $|\zeta| > d_1/2$, the boundaries $b\tilde{\Omega}_{z',\delta}$ are independent of δ and depend smoothly on z',*

(iv) *in the region $\{\zeta : d_1/2 < |\zeta| < d_1\}$, the boundaries $b\tilde{\Omega}_{z',\delta}$ are of finite type, uniformly in z', and δ.*

3. Metric estimates

In [11], K. T. Hahn got the following inequalities

$$C_\Omega(z;X) \leq B_\Omega(z;X), \quad K_\Omega(z;X).$$

Therefore the estimates for the lower bounds of $C_\Omega(z;X)$ will suffice for the lower bounds of $B_\Omega(z;X)$ and $K_\Omega(z;X)$.

Assume that $r(z) = -b\delta/2$ and let z' be the projection of z onto $b\Omega$, and $\Phi_{z'}$ be its associated map. Here $b > 0$ is the number before Proposition 2.5. Set $\zeta^\delta = (0, \ldots, 0, -b\delta/2) = (\zeta_1^\delta, \zeta_2^\delta, \ldots, \zeta_n^\delta)$. Then by virtue of (2.2), there is a small constant $c \leq b$ such that the polydisc

$$B_c = \{\zeta : |\zeta_n + b\delta/2| < c\delta, |\zeta_k| < c\tau(z',\delta), 1 \leq k \leq n-1\},$$

is contained in $\Omega_{z'}$. Let $Y = (\Phi_{z'}^{-1})_* X = b_1 L_1' + \cdots + b_n L_n'$ be a vector at ζ^δ, where $L_i' = (\Phi_{z'}^{-1})_* L_i$ for $i = 1, 2, \ldots, n$.

Set $\tau_n = \delta$ and $\tau_k = \tau(z',\delta)$, $1 \leq k \leq n-1$. Let k_0 be the minimum number such that

$$|b_{k_0}|\tau_{k_0}^{-1} = \max\{|b_k|\tau_k^{-1} : k = 1, 2, \ldots, n\}. \tag{3.1}$$

Set $v(\zeta) = \delta^{-1}(\zeta_n + b\delta/2)$ if $k_0 = n$, and $v(\zeta) = \tau(z',\delta)^{-1}\zeta_{k_0}$ otherwise. Since we may assume that $c \leq 1$, we have the inequality $\sup_B |v| \leq 1$. From the expansion in (2.1), one can see that $(\partial\rho/\partial\zeta_j)(\zeta^\delta) = 0$, $j = 2, \ldots, n$, and hence from (3.1), it follows that

$$|Yv(\zeta^\delta)| = \max\{|b_k|\tau_k^{-1} : k = 1, 2, \ldots, n\}, \tag{3.2}$$

provided that δ is sufficiently small.

Set $\phi(\zeta) = \lambda'_\delta(\zeta) + |\zeta|^2$ and set $\lambda(\zeta) = \chi(\phi(\zeta))$, where $\chi(t)$ is a smooth convex increasing function with $\chi'(t) \geq 1$. Using the standard $\overline{\partial}$-estimates on $\tilde{\Omega}_{z',\delta}$ with weight $e^{-\lambda(\zeta)}$, and from the estimate $\sum_{i,j=1}^n (\partial^2\phi/\partial\zeta_i\partial\overline{\zeta}_j)(\zeta)t_i\overline{t}_j \gtrsim \sum_{j=1}^n |t_i|^2\tau_i^{-2}$ for $\zeta \in B$, one has for any $g = \sum_{i=1}^n g_i d\overline{\zeta}_i$ satisfying $Sg = 0$, there is $u \in L^2_{0,0}(\tilde{\Omega}_{z',\delta}, \lambda)$ such that $\overline{\partial}u = g$ (in weak sense) and,

$$\|u\|^2_\lambda \lesssim \int_{\tilde{\Omega}_{z',\delta}-B} |g|^2 e^{-\lambda} + \int_B \sum_{i=1}^n \tau_i^2 |g_i|^2 e^{-\lambda} dV. \tag{3.3}$$

For $c \geq d > 0$, set $B_d = \{\zeta : |\zeta_i - \zeta_i^\delta| < d\tau_i, i = 1, 2, \ldots, n\}$. Since $\sum_{i,j=1}^n (\partial^2\phi/\partial\zeta_i\partial\overline{\zeta}_j)t_i\overline{t}_j \gtrsim \sum_{i=1}^n \tau_i^{-2}|t_i|^2$ on B_c, there is a small constant $0 < d \leq c$ (independent of $\tau_1, \ldots, \tau_n$) so that

$$\phi(\zeta) \geq \operatorname{Re} h(\zeta) + d\sum_{i=1}^n \tau_i^{-2}|\zeta_i - \zeta_i^\delta|^2, \qquad \zeta \in B_d,$$

where

$$h(\zeta) = 2\sum_{i=1}^n \frac{\partial\phi}{\partial\zeta_i}(\zeta^\delta)(\zeta_i - \zeta_i^\delta) + \sum_{i,j=1}^n \frac{\partial^2\phi}{\partial\zeta_i\partial\zeta_j}(\zeta^\delta)(\zeta_i - \zeta_i^\delta)(\zeta_j - \zeta_j^\delta).$$

Let $\psi \in C_0^\infty(B)$, where B is the unit polydisc in $\mathbb{C}^n$ such that $\psi(\zeta) = 1$ if $|\zeta_i| \leq 1/2$, $i = 1, 2, \ldots, n$. From (3.3), we conclude that if

$$\psi_d(\zeta) = \psi\left(\frac{\zeta_1}{d\tau(z',\delta)}, \cdots, \frac{\zeta_{n-1}}{d\tau_{n-1}}, \frac{\zeta_n + b\delta/2}{d\delta}\right),$$

and if $a = nd^3/8$, then

$$\operatorname{Re} h(\zeta) \leq -a, \quad \text{for } \zeta \in \{\zeta : \phi(\zeta) \leq a\} \cap \operatorname{supp} \overline{\partial}\psi_d.$$

Let χ be a smooth convex increasing function that satisfies $\chi(t) = 0$ for $t \leq a/2$ and $\chi''(t) > 0$ for $t > a/2$. Now define

$$\lambda_s(\zeta) = \phi(\zeta) + s^2\chi(\phi(\zeta)).$$

Then λ_s does not depend on s in B_e, for some fixed $0 < e \leq d$. Set

$$\alpha_s = \overline{\partial}(\psi_d v e^{sh}) = v e^{sh}\overline{\partial}\psi_d = \sum_{i=1}^n \alpha_{s,i} d\overline{\zeta}_i.$$

Following the standard weighted L^2 estimates for $\overline{\partial}$ as in [2], [6], it follow that for any $\epsilon_0 > 0$, there exists independent $s_0 > 0$ and a function u_{s_0} so that $\overline{\partial}u_{s_0} = \alpha_{s_0}$ and

$$|u_{s_0}(\zeta^\delta)| \lesssim \epsilon_0, \quad \left|\frac{\partial u_{s_0}}{\partial\zeta_k}(\zeta^\delta)\right|^2 \lesssim \epsilon_0\tau_k^{-2}, \quad 1 \leq k \leq n. \tag{3.4}$$

Therefore it follows from (3.1) and (3.4) that

$$|Yu_{s_0}(\zeta^\delta)| \lesssim \sqrt{\epsilon_0}\sum_{k=1}^n |b_k|\tau_k^{-1} \le n\sqrt{\epsilon_0}\ \max\,\{|b_k|\tau_k^{-1}:\ k=1,2,\dots,n\}.$$

Set $f = v\psi_d e^{s_0 h} - u_{s_0}$. Then f is holomorphic and it follows from (3.2) that

$$|Yf(\zeta^\delta)| \gtrsim\ \max\,\{|b_k|\tau_k^{-1} : k=1,2,\dots,n\}, \tag{3.5}$$

provided that ϵ_0 is sufficiently small.

Let us assume, for a moment, that $\sup_{\overline{\Omega}_{z'}} |f| \le C$, where C is independent of z' and δ. Then (3.5) and the definition of Caratheodory metric shows that

$$C_{\Omega_{z'}}(Y;\zeta^\delta) \ge C_{\tilde{\Omega}_{z',\delta}}(Y;\zeta^\delta) \gtrsim \max\,\{|b_k|\tau_k^{-1} : k=1,2,\dots,n\}. \tag{3.6}$$

On the other hand, the polydisc B about ζ^δ lies in $\Omega_{z'}$. So one can easily obtain that

$$C_{\Omega_{z'}}(\zeta^\delta;Y) \le C_B(\zeta^\delta;Y) = \max\{|b_k|\tau_k^{-1} : k=1,2,\dots,n\}. \tag{3.7}$$

Set $\Phi_{z'}(\zeta^\delta) = z$. Then by (1.3), (1.4) and (2.3)–(2.5), it follows that

$$\tau(z,\delta)^{-1} \approx \eta(z,\delta)^{-1} \approx \sum_{l=2}^m |C_l(z)|^{\frac{1}{l}}\cdot|r(z)|^{-\frac{1}{l}},$$

and hence one obtains that

$$\max\{|b_k|\tau_k^{-1} : k=1,2,\dots,n\} \approx \sum_{k=1}^n |b_k|\tau_k^{-1} \approx M(z;X). \tag{3.8}$$

From the invariant property of Caratheodory metric, and from (3.6)–(3.8), one obtains that

$$C_\Omega(z;X) = C_{\Omega_{z'}}(\zeta^\delta;Y) \approx M(z;X). \tag{3.9}$$

To show that $\sup_{\overline{\Omega}_{z'}} |f| \le C$, we use the fact that f is holomorphic in a larger domain $\tilde{\Omega}_{z',\delta}$. Assuming $\zeta \in \overline{\Omega}_{z'}$ and $|\zeta| < d_1$, it follows from Proposition 2.6 that $P_a \subset \tilde{\Omega}_{z',\delta}$. Since $|v\psi_d e^{s_0 h}| \lesssim 1$, the estimate (3.3) shows that $\int_{P_a(\zeta)} |f|^2 dV \lesssim \prod_{j=1}^n \tau_j^2$. Hence it follows that

$$|f(\zeta)|^2 \lesssim (Vol(P_a(\zeta))^{-1}\int_{P_a(\zeta)} |f|^2 dV \lesssim 1,$$

because $Vol(P_a(\zeta)) \gtrsim \prod_{j=1}^n \tau_j^2$. When $|\zeta| \ge d_1$, we use the Kohn's global regularity theory and some cut-off functions as Catlin did in [2]. Therefore we obtain that $\sup_{\overline{\Omega}_{z'}} |f| \lesssim 1$ and hence (3.9) follows.

To obtain an upper bound for the Bergman metric, we note that $\Omega_{z'}$ contains the polydisc B_c about ζ^δ. Thus by elementary estimates one has, for any $f \in L^2(\Omega_{z'}) \cap A(\Omega_{z'})$, that

$$\left|\frac{\partial f}{\partial \zeta_k}(\zeta^\delta)\right| \lesssim \tau_k^{-1} \prod_{j=1}^{n} \tau_j^{-1} \|f\|_{L^2(\Omega_{z'})},$$

for $k = 1, 2, \dots, n$. From (2.1) and (2.2), it follows that the coefficient $b(\zeta)$ of $\partial/\partial\zeta_n$ in L'_1 satisfies $|b(\zeta^\delta)| \lesssim \delta$ and $|(\partial\rho/\partial\zeta_j)(\zeta^\delta)| \lesssim \tau(z', \delta)$, for $j = 1, \dots, n-1$. Therefore, if $Y = \sum_{k=1}^{n} b_k L'_k$ is a vector at ζ^δ, then

$$b_{\Omega_{z'}}(\zeta^\delta; Y) \lesssim \left(\sum_{k=1}^{n} |b_k| \tau_k^{-1}\right) \prod_{j=1}^{n} \tau_j^{-1}. \tag{3.10}$$

In [7], the author showed that

$$K_{\Omega_{z'}}(\zeta^\delta, \overline{\zeta}^\delta) \approx \prod_{j=1}^{n} \tau_j^{-2} \approx \sum_{l=2}^{m} C_l(z)^{\frac{2(n-1)}{l}} |r(z)|^{-2-\frac{2(n-1)}{l}}. \tag{3.11}$$

Combining (3.10), (3.11) and the definition of $B_\Omega(z; X)$, one obtains that

$$B_\Omega(z; X) = B_{\Omega_{z'}}(\zeta^\delta; Y) \lesssim \sum_{k=1}^{n} |b_k| \tau_k^{-1},$$

and hence it follows that

$$C_\Omega(z; X) \approx B_\Omega(z; Y) \approx M(z; X). \tag{3.12}$$

Set

$$R = \min\{d_2 c \tau_k |b_k|^{-1} : k = 1, 2, \dots, n\}.$$

Then

$$f(t) = \left(b_1 t, \dots, b_{n-1} t, -\frac{b\delta}{2} + b_n t\right)$$

defines a map $f : D_R \longrightarrow B$ with $f_*((\partial/\partial t|_0) = X$. Hence

$$\begin{aligned} K_{\Omega_{z'}}(\zeta^\delta; Y) &\le K_B(\zeta^\delta; Y) \le R^{-1} \le \max\{|b_k|(c\tau_k)^{-1} :\ 1 \le k \le n\} \\ &\lesssim \max\{|b_k| \tau_k^{-1} :\ k = 1, 2, \dots, n\} \\ &\lesssim \sum_{k=1}^{n} |b_k| \tau_k^{-1} \lesssim C_{\Omega_{z'}}(\zeta^\delta; Y) = C_\Omega(z; X). \end{aligned}$$

Again from the invariant property of $K_\Omega(z; X)$ and (1.4), it follows that

$$K_\Omega(z; X) = K_{\Omega_{z'}}(\zeta^\delta; Y) \approx C_\Omega(z; X). \tag{3.13}$$

If we combine (3.12) and (3.13), it follows that

$$C_\Omega(z; X) \approx B_\Omega(z; X) \approx K_\Omega(z; X) \approx M(z; X),$$

and this proves our main theorem.

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