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The Whitehead square of a lift of the Hopf map to a mod 2 Moore space

Dedicated to the memory of Professor Katsuo Kawakubo

By

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1. Introduction

We set $\mathbb{F} = \mathbb{R}$ (real), $\mathbb{C}$ (complex) or $\mathbb{H}$ (quaternion) with the usual norm and set $d = \dim_{\mathbb{R}} \mathbb{F}$. Let $\mathbb{F}P^n$ be the $n(\mathbb{F})$-dimensional $\mathbb{F}$-projective space. Let Q^n be the quaternionic quasi-projective space of dimension $4n-1$. Let $G_n(\mathbb{F})$ be the orthogonal group $O(n)$, the unitary group $U(n)$ or the symplectic group $Sp(n)$ respectively, according as $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. We denote by $\omega_n(\mathbb{F}) : S^{d(n+1)-2} \to G_n(\mathbb{F})$ the characteristic map for the standard sphere bundle over $G_{n+1}(\mathbb{F})/G_n(\mathbb{F}) = S^{d(n+1)-1}$. Let $c : Sp(n) \to U(2n)$ and $r : U(n) \to O(2n)$ be the canonical inclusions. We denote by $M^n = \Sigma^{n-2}\mathbb{R}P^2$ for $n \geq 2$ the Moore space of type $(\mathbb{Z}_2, n-1)$. Let $i_n : S^{n-1} \to M^n$ and $p_n : M^n \to S^n$ be the inclusion and collapsing maps respectively. Given an element $\alpha \in \pi_k(S^n)$, an element $\widehat{\alpha} \in \pi_k(M^n)$ is called a lift of α if $p_{n*}\widehat{\alpha} = \alpha$. Let $\iota_n \in \pi_n(S^n)$ be the identity map of S^n and let $\eta_n \in \pi_{n+1}(S^n)$ for $n \geq 2$ be the Hopf map. It is well-known that there exists a lift of η_3. We denote by $\widehat{\eta}_3 \in \pi_4(M^3)$ a lift of η_3 and $\widehat{\eta}_n = \Sigma^{n-3}\widehat{\eta}_3$ for $n \geq 3$.

The purpose of this note is to study the order of the Whitehead square of $\widehat{\eta}_n$. This paper is some kind of a byproduct of [9] but would be able to be read independently. Remark that in [9] the notation of the lift of η_n to M^n was the symbol $\tilde{\eta}_{n-1}$. In this paper we denote it by $\widehat{\eta}_n$.

It is proved in (i) of Theorem 4.1 of [9] that the Whitehead square of $\widehat{\eta}_n$ is of order 4 if n is odd. It is easy to see that $2[\widehat{\eta}_n, \widehat{\eta}_n] = 2\widehat{\eta}_n[\iota_{n+1}, \iota_{n+1}] = 0$ if n is even.

We summarize results about the order of $[\widehat{\eta}_n, \widehat{\eta}_n]$ as theorem, although our result is not complete.

Theorem 1.1. (1) $[\widehat{\eta}_{2n+1}, \widehat{\eta}_{2n+1}]$ *is of order* 4 *if* $n \geq 1$.
(2) $[\widehat{\eta}_{2n}, \widehat{\eta}_{2n}]$ *is non-trivial and of order* 2 *if* $n \neq 2^i - 1$.

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Remark 1.2. When $n = 2^i - 1$, the triviality of $[\widehat{\eta}_{2n}, \widehat{\eta}_{2n}]$ seems to be related to the stable order of Mahowald element [6] $\eta_j \in \pi_{2^j}(S^0)$.

Let $\nu_n \in \pi_{n+3}(S^n)$ for $n \geq 4$ be the Hopf map. Let $J : \pi_k(SO(n)) \to \pi_{k+n}(S^n)$ be the J-homomorphism. It is well-known [2], $\Sigma^3 J(rc\omega_n(\mathbb{H})) = [\iota_{4n+3}, \iota_{4n+3}]$. The proof of (2) in Theorem 1.1 is partially obtained by use of the following.

Theorem 1.3. $\eta_{4n+1} \circ \Sigma^2 J(rc\omega_n(\mathbb{H})) = [\iota_{4n+1}, \nu_{4n+1}]$.

2. Proof of Theorem 1.3

We need some preliminaries.

For $1 \leq m \leq n \leq \infty$, we set $\mathbb{F}P^n_m = \mathbb{F}P^n/\mathbb{F}P^{m-1}$ and $Q^\infty_m = Q^\infty/Q^{m-1}$. We set $G(\mathbb{F}) = G_\infty(\mathbb{F})$. Let $i_n(\mathbb{F}) : G_n(\mathbb{F}) \to G_{n+1}(\mathbb{F})$ be the inclusion and let $t : Q^n \to \Sigma\mathbb{C}P^{2n-1}$ be the natural map with a cofiber $\Sigma\mathbb{H}P^{n-1}$. For a based space X, ΩX stands for a loop space of X. Note that $\Omega(G(\mathbb{F})/G_n(\mathbb{F}))$ a homotopy fiber of the inclusion $i(\mathbb{F}) : G_n(\mathbb{F}) \to G(\mathbb{F})$. Then we show

Lemma 2.1. $c_*(\omega_n(\mathbb{H}))\eta_{4n+2} = n\omega_{2n}(\mathbb{C})\nu_{4n}$.

Proof. We denote by $i_Q : Q^\infty_{n+1} \to Sp/Sp(n)$ and $i_\mathbb{C} : \Sigma\mathbb{C}P^\infty_{2n} \to U/U(2n)$ the canonical inclusions respectively. We consider the following commutative diagram:

$$\begin{array}{ccccc}
\Omega Q^\infty_{n+1} & \xrightarrow{\Omega i_Q} & \Omega(Sp/Sp(n)) & \xrightarrow{\partial} & Sp(n) \\
\downarrow{\scriptstyle \Omega t} & & \downarrow{\scriptstyle \Omega c} & & \downarrow{\scriptstyle c} \\
\Omega\Sigma\mathbb{C}P^\infty_{2n} & \xrightarrow{\Omega i_\mathbb{C}} & \Omega(U/U(2n)) & \xrightarrow{\partial} & U(2n),
\end{array}$$

where the maps are canonical.

We know $\mathbb{H}P^{n+1}_n = S^{4n} \cup_{n\nu_{4n}} e^{4n+4}$. So, by using the cofiber sequence ([4]) $Q^\infty_{n+1} \to \Sigma\mathbb{C}P^\infty_{2n} \to \Sigma\mathbb{H}P^\infty_n$, we see the $(4n+5)$-skeleton of $\Sigma\mathbb{C}P^\infty_{2n}$ has the following cell structure:

$$(S^{4n+1} \vee S^{4n+3}) \cup_{n\nu_{4n+1} \vee \eta_{4n+3}} e^{4n+5}.$$

Since t restricted on S^{4n+3} is just the inclusion $S^{4n+3} \subset \Sigma CP^\infty_{2n}$, we have a relation

$$\Omega t_* j_* \eta_{4n+2} = nk_*\nu_{4n} \in \pi_{4n+3}(\Omega\Sigma\mathbb{C}P^\infty_{2n}), \tag{1}$$

where $j : S^{4n+2} \to \Omega Q^\infty_{n+1}$ and $k : S^{4n} \to \Omega\Sigma\mathbb{C}P^\infty_{2n}$ are the adjoint of the inclusions respectively. We note that $\omega_n(\mathbb{H}) = \partial_*\Omega i_{Q*}j$ and $\omega_{2n}(\mathbb{C}) = \partial_*\Omega i_{\mathbb{C}*}k$. Hence, by the above commutative diagram and (1), we have the assertion. This completes the proof. □

As is well known [4], for the projection $p : SO(4n) \to S^{4n-1}$, it holds that

$$p_*(r_* c_* \omega_n(\mathbb{H})) = (n+1)\nu_{4n-1}(n \geq 2).$$

Now consider the following commutative diagram up to sign:

$$\begin{array}{ccc} S^{4n-1} & \xrightarrow{E^{4n}} & \Omega^{4n} S^{8n-1} \\ \uparrow p & & \uparrow \Omega^{4n-1} H \\ SO(4n) & \xrightarrow{J} & \Omega^{4n} S^{4n} \\ \downarrow i & & \downarrow \Omega^{4n} E \\ SO(4n+1) & \xrightarrow{J} & \Omega^{4n+1} S^{4n+1}, \end{array} \tag{2}$$

where $E^k : S^m \to \Omega^k S^{m+k}$ is the k-fold suspension map, $H : \Omega S^m \to \Omega S^{2m-1}$ is the Hopf invariant and $J : SO(m) \to \Omega^m S^m$ is the J-map.

So we have

$$HJ(rc\omega_n(\mathbb{H})) = \pm(n+1)\nu_{8n-1}. \tag{3}$$

By Lemma 2.1, we have

$$(rc\omega_n(\mathbb{H}))\eta_{4n+2} = n(r\omega_{2n}(\mathbb{C}))\nu_{4n}. \tag{4}$$

We set $\alpha_n = J(rc\omega_n(\mathbb{H})) \in \pi_{8n+2}(S^{4n})$. Applying the composite $J \circ i_{4n}(R)$ to the above equation (4) and using the James-Whitehead theorem [2] which asserts $J\omega_n(\mathbb{R}) = [\iota_n, \iota_n]$, we obtain

Lemma 2.2. $(\Sigma\alpha_n)\eta_{8n+3} = n[\iota_{4n+1}, \nu_{4n+1}]$.

Proof of Theorem 1.3. By using the Barratt-Toda formula ([10], [1]) and (3), we have

$$\begin{aligned} \eta_{4n+1} \circ \Sigma^2\alpha_n - \Sigma\alpha_n \circ \eta_{8n+3} &= [\iota_{4n+1}, \iota_{4n+1}] \circ \Sigma^2 H(\alpha_n) \\ &= [\iota_{4n+1}, \iota_{4n+1}] \circ (n+1)\nu_{8n+1} \\ &= (n+1)[\iota_{4n+1}, \nu_{4n+1}]. \end{aligned}$$

So, by Lemma 2.2, we have the assertion. This completes the proof of Theorem 1.3.

Example 2.3.
(i) $\eta_9\sigma_{10}\nu_{17} = \bar{\nu}_9\nu_{17} = [\iota_9, \nu_9]$.
(ii) $\eta_{17}(\nu_{18}^* + \xi_{18}) = \omega_{17}\nu_{33} = [\iota_{17}, \nu_{17}]$.
(iii) $\eta_{21}\sigma_{22}^* = [\iota_{21}, \nu_{21}]$.

The second equalities of (i) and (ii) are obtained by [10]. The relation of (iii) is obtained by [7] and [8].

3. Proof of Theorem 1.1

First we show a formula which represents a relation between an absolute Whitehead product and a relative one.

Let X be a connected space and let $\varphi \in \pi_{n-1}(X)$. Then there exists a canonical extension of φ, $\bar{\varphi} : D^n \to X \cup_\varphi e^n$. Let $\Omega(X \cup_\varphi e^n, X)$ be the homotopy fiber of the inclusion $X \to X \cup_\varphi e^n$ and we denote the fiber inclusion map by ∂. Consider the following commutative diagram:

$$\begin{array}{ccccc} \Omega(D^n, S^{n-1}) & \xrightarrow{\partial} & S^{n-1} & \longrightarrow & D^n \\ \downarrow{\scriptstyle \Omega(\bar{\varphi},\varphi)} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \bar{\varphi}} \\ \Omega(X \cup_\varphi e^n, X) & \xrightarrow{\partial} & X & \longrightarrow & X \cup_\varphi e^n. \end{array} \tag{5}$$

Note that $\partial : \Omega(D^n, S^{n-1}) \to S^{n-1}$ is a homotopy equivalence. We denote the homotopy inverse of ∂ by $s : S^{n-1} \to \Omega(D^n, S^{n-1})$. Let $\gamma_n \in \pi_n(X \cup_\varphi e^n, X) \overset{\text{adj}}{\cong} \pi_{n-1}(\Omega(X \cup_\varphi e^n, X))$ be the characteristic map of the n-cell e^n. Note that the adjoint of γ_n is represented by the composite:

$$adj(\gamma_n) \ : \ S^{n-1} \xrightarrow{\ s\ } \Omega(D^n, S^{n-1}) \xrightarrow{\Omega(\bar{\varphi},\varphi)} \Omega(X \cup_\varphi e^n, X).$$

By definition,

$$\Omega p \circ adj(\gamma_n) = E : S^{n-1} \to \Omega S^n,$$
$$\partial \circ adj(\gamma_n) = \varphi : S^{n-1} \to X,$$

where $p : \Omega(X \cup_\varphi e^n, X) \to \Omega S^n$ is the canonical projection.

For an element $\beta \in \pi_k(S^{n-1})$, we denote $\beta_\varphi \in \pi_{k+1}(X \cup_\varphi e^n, X)$, the adjoint of the composite map $adj(\gamma_n) \circ \beta = \Omega(\bar{\varphi}, \varphi) \circ s \circ \beta$.

Then we show

Lemma 3.1. *Let $\beta \in \pi_k(S^{n-1})$. Then in $\pi_{n+k-1}(X \cup_\varphi e^n, X)$, it holds that*

$$[\gamma_n, \varphi \circ \beta] = \pm[\iota_{n-1}, \beta]_\varphi,$$

where $[\gamma_n, \varphi \circ \beta]$ is the relative Whitehead product and $[\iota_{n-1}, \beta]$ is the absolute one.

Proof. By the naturality of relative Whitehead products [3], we have a commutative diagram

$$\begin{array}{ccc} \pi_k(S^{n-1}) & \xrightarrow{\varphi_*} & \pi_k(X) \\ \downarrow{\scriptstyle [\gamma'_n,\ ]} & & \downarrow{\scriptstyle [\gamma_n,\ ]} \\ \pi_{n+k-1}(D^n, S^{n-1}) & \xrightarrow{(\bar{\varphi},\varphi)_*} & \pi_{n+k-1}(X \cup_\varphi e^n, X), \end{array}$$

where $\gamma'_n \in \pi_n(D^n, S^{n-1})$ is the obvious map. Therefore we have $[\gamma_n, \varphi\circ\beta] = (\bar\varphi,\varphi)_*[\gamma'_n,\beta]$. Since $\partial[\gamma'_n,\beta] = \pm[\iota_{n-1},\beta]$, the element $(\bar\varphi,\varphi)_*[\gamma'_n,\beta]$ is the adjoint of $\Omega(\bar\varphi,\varphi)\circ s\circ[\iota_{n-1},\beta]$, that is, $[\iota_{n-1},\beta]_\varphi$. This completes the proof. □

From now on we prove Theorem 1.1.

We show that if $n \neq 2^i - 1$, then $[\widehat\eta_{2n},\widehat\eta_{2n}]$ is non-trivial.

For $n = 2m$, we have $p_{4m}[\widehat\eta_{4m},\widehat\eta_{4m}] = [\eta_{4m},\eta_{4m}] = [\iota_{4m},\eta^2_{4m}] \neq 0$ by [5].

Suppose that $n = 2m+1$. We denote by $j : (M^n, *) \to (M^n, S^{n-1})$ the inclusion map. By Theorem 1.3, $\widehat\eta_{4m+1}\Sigma^2\alpha_m \in \pi_{8m+4}(M^{4m+1})$ is a lift of $[\iota_{4m+1},\nu_{4m+1}]$, where $\alpha_m = J(rc\omega_m(\mathbb{H})) \in \pi_{8m+2}(S^{4m})$. Since $\Sigma^3\alpha_m = [\iota_{4m+3},\iota_{4m+3}]$, we have

$$\Sigma(\widehat\eta_{4m+1}\Sigma^2\alpha) = \widehat\eta_{4m+2}[\iota_{4m+3},\iota_{4m+3}] = [\widehat\eta_{4m+2},\widehat\eta_{4m+2}].$$

Since it is easy to see that the following diagram commutes:

$$\begin{array}{ccc} S^n & \xrightarrow{adj(\widehat\eta_n)} & \Omega M^n \\ \downarrow{\scriptstyle \eta_{n-1}} & & \downarrow{\scriptstyle \Omega j} \\ S^{n-1} & \xrightarrow{adj(\gamma_n)} & \Omega(M^n, S^{n-1}), \end{array}$$

we have

$$\begin{aligned} j_*[\widehat\eta_{4m+2},\widehat\eta_{4m+2}] &= j\circ\widehat\eta_{4m+2}[\iota_{4m+3},\iota_{4m+3}] \\ &= adj(\Omega j\circ adj(\widehat\eta_{4m+2})\circ\Sigma^{-1}[\iota_{4m+3},\iota_{4m+3}]) \\ &= adj(adj(\gamma_{4m+2})\circ\eta_{4m+1}\circ\Sigma^{-1}[\iota_{4m+3},\iota_{4m+3}]) \\ &= adj(adj(\gamma_{4m+2})\circ\eta_{4m+1}\circ\Sigma^2\alpha_m) \\ &= adj(adj(\gamma_{4m+2})\circ[\iota_{4m+1},\nu_{4m+1}]) \quad \text{by Theorem 1.3} \\ &= [\iota_{4m+1},\nu_{4m+1}]_\varphi, \end{aligned}$$

where $\varphi = 2\iota_{4m+1}$.

By Lemma 3.1 for $\varphi = 2\iota_{4m+1}$ and $\beta = \nu_{4m+1}$, we have

$$j_*[\widehat\eta_{4m+2},\widehat\eta_{4m+2}] = [\iota_{4m+1},\nu_{4m+1}]_\varphi = \pm 2[\gamma_{4m+2},\nu_{4m+1}].$$

By Lemma 2.3 (ii) of [9], the order of $[\gamma_{4m+2},\nu_{4m+1}]$ is 4 for $m \neq 2^{i-1}-1$. Therefore we have proved that $[\widehat\eta_{4m+2},\widehat\eta_{4m+2}] \neq 0$ for $m \neq 2^{i-1}-1$.

This completes the proof of (2) in Theorem 1.1.

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