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Topological characterization of Bott map on BU

By

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1. Introduction

As in [1], if a Hopf space X, which is of finite type CW-complex and has the same cohomology ring as BU, is equipped with a map $\lambda : S^2 \wedge X \to X$ and λ satisfies some of the properties of Bott map $\beta : S^2 \wedge \mathrm{BU} \to \mathrm{BU}$, we see that X is an infinite loop space. Using this fact, they construct a homotopy equivalence $h : \mathrm{BU} \xrightarrow{\sim} X$. But they don't pursue h on the relation between λ, β and h. In this paper, we can see the perfect relation between λ, β and h with a help of [4]. Actually we construct a new Hopf equivalence $h' : \mathrm{BU} \xrightarrow{\sim} X$ which satisfies the following homotopy commutative diagram.

$$\begin{array}{ccc} S^2 \wedge \mathrm{BU} & \xrightarrow{h'|_{S^2} \wedge h'} & S^2 \wedge X \\ {\scriptstyle \beta}\big\downarrow & & \big\downarrow{\scriptstyle \lambda} \\ \mathrm{BU} & \xrightarrow[h']{} & X \end{array}$$

2. Characterization of BU

In this section, we recall the contents in [1] to prepare for the characterization of Bott map, and give some refinement.

Theorem 2.1. *Let $\mu : X \times X \to X$ be a Hopf space which is of finite type CW-complex and its cohomology be the following.*

$$H^*(X) = \mathbb{Z}[x_1, x_2, \ldots], \ |x_i| = 2i$$

There exist two maps with the following properties.

$$j : \mathbb{C}\mathrm{P}^\infty \to X, \qquad \lambda : S^2 \wedge X \to X$$

(1) $(\lambda \circ (1 \wedge j))^* : H^*(X) \to H^*(S^2 \wedge \mathbb{C}\mathrm{P}^\infty)$ *is epic.*
(2) $\mathrm{Ad}^2\, \lambda$ *is a Hopf map.*

Then we have the following homotopy equivalence.

$$\widetilde{\mathrm{Ad}}^2(\lambda) : X \xrightarrow{\simeq} \Omega^2 X\langle 2\rangle,$$

where $X\langle 2\rangle$ *is* 2*-connected fibre space of* X *and* $\widetilde{\mathrm{Ad}}^2(\lambda)$ *is a lift of* $\mathrm{Ad}^2\lambda$*.*

Proof. See [1]. □

Theorem 2.2. *Let* X *be the space as in Theorem* 2.1*. There exists a following Hopf equivalence.*

$$h : \mathrm{BU} \xrightarrow{\simeq} X$$

Proof. There exists a homotopy equivalence $h : \mathrm{BU} \xrightarrow{\simeq} X$ constructed in [1] in the following way.

Prepare the maps below (see [2]):

$$\begin{cases} \epsilon : \mathrm{BU} \to Q(\mathbb{C}\mathrm{P}^\infty) & \text{the Segal splitting,} \\ \xi_X : Q(X) \to X & \text{an infinite loop map.} \end{cases}$$

Then $h = \xi_X \circ Q(j) \circ \epsilon : \mathrm{BU} \xrightarrow{\simeq} X$ (see [1]). Since the Segal splitting ϵ is the loop map of the James-Miller splitting $\epsilon' : \mathrm{SU} \to Q(\Sigma\mathbb{C}\mathrm{P}^\infty)$ (see [3]), all of the maps above are loop maps and then h is a loop map. □

3. Characterization of Bott map

Let $S^2 \hookrightarrow \mathrm{BU}$ and $S^2 \hookrightarrow X$ be 2-skeleton of BU and X. Denote the universal bundle of $\mathrm{BU}(n)$ by ξ_n, the Hopf bundle on S^2 by η, of rank n trivial bundle by $\underline{n}$ and $\lim_n(\eta - \underline{1}) \hat{\otimes} (\xi_n - \underline{n}) \in \widetilde{\mathrm{K}}(S^2 \wedge \mathrm{BU})$ by ξ_∞.

Bott map $\beta : S^2 \wedge \mathrm{BU} \to \mathrm{BU}$ is defined as a classifying map of ξ_∞.

Denote $c_1(\eta - \underline{1}) \in H^2(S^2)$ by α, a generator of $H^2(\mathbb{C}\mathrm{P}^\infty)$ by e, $c_n(\xi_\infty) \in H^{2n}(\mathrm{BU})$ by c_n and $s_n(c_1, c_2, \ldots, c_n) \in H^{2n}(\mathrm{BU})$ by s_n (the power sum symmetric polynomial). We know that s_n is a generator of $PH^{2n}(\mathrm{BU})$ and $\lambda^*(s_n) = n\alpha \otimes s_{n-1}$. (see [4])

Theorem 3.1. *Let* X *be the space as in Theorem* 2.1 *and* $h : \mathrm{BU} \xrightarrow{\simeq} X$ *be the Hopf equivalence in Theorem* 2.2*. Then we have a new Hopf equivalence* $h' : \mathrm{BU} \xrightarrow{\simeq} X$ *which satisfies the homotopy commutative diagram below.*

$$\begin{array}{ccc} S^2 \wedge \mathrm{BU} & \xrightarrow{h'|_{S^2} \wedge h'} & S^2 \wedge X \\ {\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle\lambda} \\ \mathrm{BU} & \xrightarrow[h']{} & X \end{array}$$

Proof. Let x_n and u_n be $h^*(u_n) = s_n$ and $h^*(x_i) = c_i$, and then we see $H^*(X) = \mathbb{Z}[x_1, x_2, \ldots]$.

Since $\mathrm{Ad}^2\lambda$ is a Hopf map, we have a homotopy commutative diagram below.

$$(1)\qquad \begin{array}{ccc} S^2\wedge(X\times X) & \xrightarrow{1\wedge\mu} & S^2\wedge X \\ \omega\downarrow & & \downarrow\lambda \\ X\times X & \xrightarrow[\mu]{} & X \end{array}$$

where $\omega=\mathrm{Ad}^{-2}(\mathrm{Ad}^2\lambda\times\mathrm{Ad}^2\lambda)$.

We see ω more clearly by the following factorization.

$$(2)\qquad \begin{array}{ccc} S^2\wedge(X\times X) & \xrightarrow{\omega} & X\times X \\ \triangle\wedge 1\downarrow & & \uparrow\lambda\times\lambda \\ (S^2\times S^2)\wedge(X\times X) & \xrightarrow[1\times T\times 1]{} & (S^2\wedge X)\times(S^2\wedge X) \end{array}$$

where $\triangle$ is a diagonal map and $T: S^2\times X\to X\times S^2$, $(s,x)\mapsto(x,s)$.

Denote $\lambda^*(u_n)$ by $\alpha\otimes v_{n-1}$ ($v_{n-1}\in H^{2n-2}(X)$), and we have the following from the diagram (1).

$$\begin{array}{ccc} \alpha\otimes\mu^*(v_{n-1}) & \longleftarrow & \alpha\otimes v_{n-1} \\ \omega^*\uparrow & & \uparrow \\ u_n\otimes 1+1\otimes u_n & \longleftarrow & u_n \end{array}$$

We also have the following from the diagram (2).

$$\begin{array}{ccc} \alpha\otimes(v_{n-1}\otimes 1+1\otimes v_{n-1}) & \xleftarrow{\omega^*} & u_n\otimes 1+1\otimes u_n \\ \uparrow & & \downarrow \\ \begin{array}{c}\alpha\otimes 1\otimes v_{n-1}\otimes 1\\ +\,1\otimes\alpha\otimes 1\otimes v_{n-1}\end{array} & \longleftarrow & \begin{array}{c}\alpha\otimes v_{n-1}\otimes 1\otimes 1\\ +\,1\otimes 1\otimes\alpha\otimes v_{n-1}\end{array} \end{array}$$

Then we see that v_{n-1} is primitive, because $H^*(S^2)$ and $H^*(X)$ are torsion free. Hence we have $v_{n-1}=\delta_n u_{n-1}$ for some $\delta_n\in\mathbb{Z}$.

Denote $j^*(u_n)$ by $\theta_n e^n$ ($\theta_n\in\mathbb{Z}$).

Now we know Newton's formula as $u_n=\sum_{i=1}^{n-1}(-1)^{i-1}x_i u_{n-i}+(-1)^{n-1}nx_n$, then we have $\lambda^*(u_n)=(-1)^{n-1}n\lambda^*(x_n)$ and $(\lambda\circ(1\wedge j))^*(x_n)=\pm\alpha\otimes e^{n-1}$ by the fact that $\lambda^*(\text{decomposables})=0$ and that $(\lambda\circ(1\wedge j))^*: H^*(X)\to H^*(S^2\wedge\mathbb{C}\mathrm{P}^\infty)$ is epic. Therefore we see $n\mid\delta_n$.

Now we have the following.

$$\begin{aligned}(\lambda\circ(1\wedge j))^*(u_n)&=\delta_n(1\wedge j)^*(\alpha\otimes u_{n-1})\\ &=\delta_n\theta_{n-1}\alpha\otimes e^{n-1}\\ &=\pm n(\lambda\circ(1\wedge j))^*(x_n)\\ &=\pm n\alpha\otimes e^{n-1}\end{aligned}$$

Then we can tell $\delta_n = \epsilon_n n$. $(\epsilon_n = \pm 1)$

In the same way with the proof of theorem in [4], we see $\epsilon_{2n} = \epsilon_2$ and $\epsilon_{2n+1} = \epsilon_3$ for any n.

Let h' be the following.

$$h' = \begin{cases} h & \epsilon_2 = +1, \epsilon_3 = +1, \\ h \circ C & \epsilon_2 = +1, \epsilon_3 = -1, \\ h \circ I \circ C & \epsilon_2 = -1, \epsilon_3 = +1, \\ h \circ I & \epsilon_2 = -1, \epsilon_3 = -1, \end{cases}$$

where $I, C : \mathrm{BU} \to \mathrm{BU}$ are the homotopy inverse map and the conjugation map.

Since both I and C are Hopf equivalences, h' is a Hopf equivalence in any cases.

Replace α, x_n and u_n with α', x'_n and u'_n which are $h'|_{S^2}{}^*(\alpha') = \alpha$, $h'^*(x'_n) = c_n$ and $h'^*(u'_n) = s_n$, we have the following relation between (α, u_n) and (α', u'_n).

$$(\alpha', u'_n) = \begin{cases} (\alpha, u_n) & \epsilon_2 = +1, \epsilon_3 = +1, \\ (-\alpha, (-1)^n u_n) & \epsilon_2 = +1, \epsilon_3 = -1, \\ (\alpha, (-1)^{n-1} u_n) & \epsilon_2 = -1, \epsilon_3 = +1, \\ (-\alpha, -u_n) & \epsilon_2 = -1, \epsilon_3 = -1. \end{cases}$$

It is easily verified that $\lambda^*(u'_n) = n\alpha' \otimes u'_{n-1}$ in any cases, then we have the following for any n.

$$(\lambda \circ (h'|_{S^2} \wedge h'))^*(u'_n) = (h' \circ \beta)^*(u'_n)$$

Now we have $\lambda^*(u'_n) = (-1)^{n-1} n \lambda^*(x'_n)$ and $\beta^*(s_n) = (-1)^{n-1} n \beta^*(c_n)$, we see the following for any n.

$$n(\lambda \circ (h'|_{S^2} \wedge h'))^*(x'_n) = n(h' \circ \beta)^*(x'_n)$$

Since $H^*(S^2 \wedge \mathrm{BU})$ is torsion free, we finally see the following for any n.

$$(\lambda \circ (h'|_{S^2} \wedge h'))^*(x'_n) = (h' \circ \beta)^*(x'_n)$$

In other words,

$$\lambda \circ (h'|_{S^2} \wedge h') \simeq h' \circ \beta.$$

□

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