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On the general fiber of an algebraic reduction of a compact complex manifold of algebraic codimension two

By

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1. Introduction

Let Z be a compact complex manifold of dimension three and of algebraic dimension one. In 1969 S. Kawai [4] has shown that a (bimeromorphically) ruled surface of genus $g \geq 2$ never appears as a general fiber of an algebraic reduction of Z. Ueno subsequently conjectured that the result will still be true in the higher dimensional case where Z is of dimension n and of algebraic dimension $n-2$ for any $n \geq 3$ (cf. [5, Remark 12.5]). The proof of Kawai of the above result depends on his Proposition 2 in [4], which can be stated as follows. Let $f : Z \to Y$ be a fiber space of compact complex manifolds with $\dim Z = 3$ and $\dim Y = 1$. (Here by a *fiber space* we mean a surjective holomorphic map with connected fibers.) Suppose that a general fiber F has the Hodge numbers $h^{2,0}(F) = 0$ and $h^{1,0}(F) > 0$. Then there exist a fiber space $h : S \to Y$ of curves over Y and a meromorphic map $\beta : Z \to S$ such that $f = h\beta$ and that for a smooth fiber $Z_y, y \in Y$, of f, the induced map $\beta_y : Z_y \to S_y$ is identified with the Albanese map onto its image. However, there seem counterexamples to this proposition in the case where F is bimeromorphically a ruled surface of genus one (cf. Section 3) and indeed the proof of that proposition in [4] seems insufficient even in the general case. In the present note we shall remark that by a slight modification of Kawai's proof, at least the statement at the beginning concerning ruled surfaces of genus ≥ 2 can be shown to hold true, and in fact even in a generalized form conjectured by Ueno. Note that another consequence of [4, Proposition 2] was also used by another authors [1, (3.5)].

2. Theorem

The precise statement is as follows.

Theorem 2.1. *Let $f : Z \to Y$ be a fiber space of compact complex manifolds which gives an algebraic reduction of Y. Then the general fiber F of f is never bimeromorphically equivalent to a ruled surface of genus ≥ 2.*

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The result follows from Proposition 2.1 below as in [4]; in fact it is nothing but Kawai's Proposition 2 except for a restriction on the general fibers in question and for the fact that we also treat the higher dimensional case. We note that when Z is bimeromorphic to a Kähler manifold, the theorem together with the next proposition are known and easier to prove (cf. e.g., [2]).

Proposition 2.1. *Let $f : Z \to Y$ be a fiber space of compact complex manifolds. Suppose that the general fiber F of f is bimeromorphically a ruled surface of genus $g \geq 2$. Then there exist a flat fiber space $h : S \to Y$ of curves over Y and a meromorphic map β of Z onto S such that $f = h\beta$.*

Proof. Let V be a Zariski open subset of Y over which f is smooth. Then we can construct a smooth fiber space $AlbZ_V \to V$ of complex tori over V and a holomorphic map $\alpha_V : Z_V \to AlbZ_V$ over V such that on each fiber $Z_y, y \in V$, α gives the Albanese map of Z_y (cf. Kawai [4, proof of Proposition 2]). Let S_V be the image of α_V, which is a smooth fiber space of curves of genus g over V. Denote by $\beta_V : Z_V \to S_V$ the induced map which is a flat fiber space with general fiber a nonsingular rational curve. We have to show that the fiber space S_V can be compactified to a flat fiber space $S \to Y$ over the whole Y and that the morphism β_V extends to a meromorphic map β of Z onto S.

Let $D_{Z/Y}$ be the relative Douady space associated to f, which is a complex space over Y and whose points universally parametrize the compact subspaces of Z which are contained in some fiber of f. Since β_V is flat and surjective, S_V is considered as parametrizing (effectively) the subspaces of fibers of f over V, we may consider S_V as an irreducible component of $D_{Z/Y}|V$ and $\beta_V : Z_V \to S_V$ as the restriction of the universal family over $D_{Z/Y}$ to S_V. Then let S be the unique irreducible component of $D_{Z/Y}$ which contains S_V as a Zariski open subset with respect to the inclusion $D_{Z/Y}|V \subseteq D_{Z/Y}$. Then the restriction $Z^* \to S$ of the universal family to S gives a partial compactification of $\beta_V : Z_V \to S_V$ with respect to the natural inclusion $S_V \subseteq S$ and $Z_V \subseteq Z^*$. Thus if we can show that S is proper over Y, and that the inclusion $Z_V \to Z^*$ can be extened to a bimeromorphic map $Z \to Z^*$ we are done; the composition $Z \to Z^* \to S$ gives a desired meromorphic extension β of β_V.

The problem is then local with respect to Y. So, changing the notation we assume in what follows that Y is a polydisc with center the origin of $\boldsymbol{C}^d$, which we may eventually shrink. Take as in [4] a nonzero section w of the direct image sheaf $f_*\Omega^1_{Z/Y}$ identified with a relative holomorphic 1-form on Z. For any $y \in V$ the restriction w_y of w to the smooth fiber Z_y is the pull-back of a unique holomorphic 1-form $\bar{w}_y$ on S_y via the Albanese map β_y (restricted to its image) and for a general y, $\bar{w}_y$ vanishes at $2g-2(>0)$ points on S_y counted with mutliplicity, and hence, the zeroes of w_y on Z_y consists of a union of $2g-2$ fibers of the morphism $\beta_y : Z_y \to S_y$, which is a divisor D_y on Z_y with Iitaka dimension $\kappa(Z_y, D_y) = 1$. Thus the zero of w on Z contains a divisor D on Z which gives on Z_y the divisor D_y for general $y \in Y$.

Then for some sufficiently large integer m the natural meromorphic map $q : Z \to \boldsymbol{P}(f_*\mathcal{O}_Z(mD))$ over Y has an image B which is generically of dimension one over Y, where $\boldsymbol{P}(f_*\mathcal{O}_Z(mD))$ is the projective fiber space associated to the coherent direct image sheaf $f_*\mathcal{O}_Z(mD)$. Moreover, the map q is holomorphic when restricted to Z_y for general y, giving the morphism $\beta_y : Z_y \to S_y$ above. More precisely, suppose that the last fact is true for $y \in U$ for some Zariski open subset U of Y which is contained in V and might be strictly smaller than V. Take the normalized graph $\hat{Z}$ of the meromorphic map q, and then take the flattening $\tilde{q} : \tilde{Z} \to \tilde{B}$ of the natural projection $\hat{q} : \hat{Z} \to B$. Then we have the universal morphism $\tau : \tilde{B} \to D_{Z/Y}$ over Y which gives an isomorphism onto S_U over U by the property of q. Then τ must also factors through S_V over V, and the image is nothing but the unique irreducible component S containing S_V. Thus since $\tilde{B}$ is proper over Y, so is S. Moreover, by construction we have the meromorphic map $\beta : Z \to S$ over Y as the composition of natural meromorphic maps over Y; $Z \to \hat{Z} \to \tilde{Z} \to \tilde{B} \to S$. It remains to show that this meromorphic map β coincides over V with the original map β_V. In fact, by construction β clearly coincides with β_V over U. Then as a meromorphic map from Z_V to S_V they must also coincide over the whole V. This completes the proof of Proposition 2.1 and hence of Theorem 2.1. □

The above argument can be readily generalized as follows. Let $f : Z \to Y$ be a fiber space of compact complex manifolds as above. For a general fiber $F = Z_y$ of f let $\alpha : F \to AlbF$ be the Albanese map of F and $F \to \bar{F} \to AlbF$ the Stein factorization of α. The dimension d of $\bar{F}$ is independent of the general fiber and we can relativize the map $F \to \bar{F}$ to obtain fiber spaces $h_V : S_V \to V$ and $\beta_V : Z_V \to S_V$ over V with $f_V = h_V\beta_V$ as before.

Proposition 2.2. *Suppose that the general fiber F of f has the properties that* 1) *α is not surjective, and* 2) *$F \to \bar{F}$ is flat. Then there exist a flat fiber space $h : S \to Y$ of relative dimension d over Y and a meromorphic map β of Z onto S with $f = h\beta$ such that the general fiber of f has a positive Kodaira dimension.*

Corollary 2.1. *A manifold F satisfying the above two conditions never appears as a general fiber of an algebraic reduction of a compact complex manifold.*

We use in the proof a section w of $f_*\Omega^d_{Z/Y}$ instead of $f_*\Omega^1_{Z/Y}$ (cf. the proof of [5, Theorem 10.3]). Because of the lack of immediate applications we omit a detailed proof here. But this generalization would somewhat clarify the nature of Theorem 2.1.

3. Example

For the counterexample to Proposition 2 of [4] we first note the following: Let (M, g) be a compact connected self-dual manifold and Z the associated twistor space. Z admits a C^∞ fibration $t : Z \to M$ with fibers isomorphic to a complex projective line $\boldsymbol{P}^1$. The fibers of t are called *twistor lines.* Any twistor line has the normal bundle which is isomorphic to $O(1) \oplus O(1)$ and gives rise to a complex four-dimensional family of curves on Z whose general members are called *complex twistor lines*. From this description we easily conclude the following:

Lemma 3.1. *Suppose that there exists a surjective meromorphic map $f : Z \to \boldsymbol{P}^1$ with connected fibers. Suppose further that f is factored as $f = hu$, where $h : T \to \boldsymbol{P}^1$ is a fiber space of curves with T smooth and $u : Z \to T$ is a surjective meromorphic map. Then T is algebraic.*

Proof. From the description of the normal bundle of a twistor line we see easily that at any point z of Z and for any tangent direction at z which is sufficiently near to that of the unique twistor line passing through z there exists a complex twistor line passing through z and with the given tangent direction at z. It follows that a general complex twistor line is mapped surjectively onto $\boldsymbol{P}^1$ by f. So if there exists a factorization as in the lemma, the image of a general twistor line by u gives a multi-section to the fiber space h. This implies that T is algebraic. $\square$

Now we consider the twistor space of a Hopf surface as described in [3]. Let M be a primary Hopf surface of the form $M = (\boldsymbol{C}^2 - \{0\})/\langle g \rangle$, where g acts by

$$(z, w) \to (re^{2\pi i m\theta} z, re^{2\pi i n\theta} w), \quad (z, w) \in \boldsymbol{C}^2 - \{0\}$$

where $r > 1$ is a real number, m and n with $(m, n) \neq (1, 1)$ are positive coprime integers, and θ is some non-rational real number. Then the associated twistor space Z is of algebraic dimension one and its algebraic reduction is given by a meromorphic map $f : Z \to Y$ such that the general fiber is a ruled surface of genus one (in general non-normal) [3, Theorem 3]. On the other hand, if $\hat{f} : \hat{Z} \to Y$ is a holomorphic model of f, $\hat{f}$ does not admit any factorization as in the above lemma although a general fiber of $\hat{f}$ is isomorphic to a ruled surface of genus one, for which we have $h^{2,0} = 0$ and $h^{1,0} = 1$. Indeed, otherwise we would have $1 = a(Z) \geq a(T) = 2$, a contradiction. Thus this gives a couterexample mentioned in Section 1.

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