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On the coefficient sheaf of equivariant elliptic cohomology for finite groups II

By

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1. Introduction

This is a continuation of the study [25] of the coefficient sheaf of Ginzburg, Kapranov and Vasserot's axiomatic equivariant elliptic cohomology for a finite group G. This cohomology theory, based on an elliptic curve E over a scheme S, is a cohomological functor with some natural properties from the homotopy category of finite G-CW-complexes to the category of coherent modules over the structure sheaf $\mathcal{O}_{M(E,G)}$ of the moduli scheme $M(E,G)$ (denoted by χ_G in [4]) of G-coverings of the elliptic curve E. (Here we restrict our attention to a finite group, not considering general compact Lie groups.)

In [25] we studied the case that S is a $\mathbf{Z}[1/(6|G|)]$-scheme, while this note deals with an opposite case; namely the case that G is a finite p-group P and that S is a $\mathbf{Z}/p^r$-scheme for a prime p greater than 3 and a positive integer r. More precisely, we consider the case that the elliptic curve E is the Weierstrass family $E_{\mathrm{univ}}\otimes\mathbf{Z}/p^r$ defined by the equation $y^2=4x^3-g_2x-g_3$ over the scheme $S=M(1)\otimes\mathbf{Z}/p^r=\mathrm{Spec}(\mathbf{Z}/p^r[g_2,g_3,\Delta^{-1}])$ ($\Delta=g_2^3-27g_3^2$). Then our main result constructs the moduli scheme $M(E_{\mathrm{univ}}\otimes\mathbf{Z}/p^r,P)$ as an affine scheme explicitly (Theorem 2.2) as in the case considered in [25], which provides a description of the group of the global sections of the coefficient sheaf of an equivariant elliptic cohomology based on $E_{\mathrm{univ}}\otimes\mathbf{Z}/p^r$ for a finite p-group P (Corollaries 2.3 and 2.4). A q-expansion of every (invariant) global section of the coefficient sheaf could be called a (generalized) mod p^r Thompson series. We hope that this paper together with [25] would shed some light on a (global) aspect of Hopkins-Kuhn-Ravenel character theory for elliptic cohomology from the view point of moduli of Galois coverings of elliptic curves.

In Section 2 we recall the definition of the coefficient sheaf of axiomatic equivariant elliptic cohomology and state our results. In Section 3 we study moduli problem of P-coverings of p-ordinary elliptic curves by similar method proving Theorem 2.3 in [25] and construct the moduli scheme $M((E_{\mathrm{univ}}\otimes\mathbf{Z}/p^r)^{\mathrm{ord}},P)$ as an affine scheme (Theorem 2.1). Here $(E_{\mathrm{univ}}\otimes\mathbf{Z}/p^r)^{\mathrm{ord}}=(E_{\mathrm{univ}}\otimes\mathbf{Z}/p^r)\times_{M(1)\otimes\mathbf{Z}/p^r}(M(1)\otimes\mathbf{Z}/p^r)^{\mathrm{ord}}$, $(M(1)\otimes\mathbf{Z}/p^r)^{\mathrm{ord}}=\mathrm{Spec}(\mathbf{Z}/p^r[g_2,$

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$g_3, \Delta^{-1}, E_{p-1}^{-1}]$) and E_{p-1} is the Eisenstein series. In Section 4 we prove Theorem 2.2 by using Theorem 2.1 with taking account of P-coverings of supersingular elliptic curves. In the Appendix we give a brief account of $\Gamma_1(n)^{\mathrm{arith}}$-modular forms over $\mathbf{Z}[1/6]$-algebras for the convenience of the reader.

2. Notation and statement of results

As in [25], for a finite group G, we denote by $\pi^1(X, G)$ the set of isomorphism classes of G-coverings of a locally noetherian scheme X. (As in [25] all schemes are assumed to be locally noetherian.)

Let $E \to S$ be an elliptic curve over a locally noetherian scheme S equipped with a section $i : S \to E$ and ω_E denote the invertible sheaf $i^*\Omega^1_{E/S}$ on S. Let (Sch/S) and (Sets) denote the category of locally noetherian S-schemes and the one of sets respectively. Let

$$\pi^1_{E,G} : (\mathrm{Sch}/S) \longrightarrow (\mathrm{Sets})$$

be the functor defined by

$$\pi^1_{E,G}(T) = \pi^1(E_T, G) \ (\forall T \in (\mathrm{Sch}/S))$$

where $E_T = E \times_S T$. Let $M(E, G)$ denote the coarse moduli scheme in the sense of [25, Definition 2.1] (if it exists) and $p_{E,G} : M(E, G) \to S$ be the S-scheme structure on $M(E, G)$. Let $\omega_{E,G}$ denote the invertible sheaf $p^*_{E,G}\omega_E$ on $M(E, G)$. Then the coefficient sheaf of an equivariant elliptic cohomology $Ell^*_G(?)$, based on an elliptic curve E, on finite G-CW-complexes is defined by

$$Ell^k_G(pt) = \begin{cases} \omega_{E,G}^{\otimes - \frac{k}{2}} & (k \text{ even}), \\ 0 & (k \text{ odd}). \end{cases}$$

(As remarked in [25] we should have non-trivial $Ell^{\mathrm{odd}}_G(pt)$ in general if S is not a $\mathbf{Z}[1/|G|]$-scheme.)

For a fixed prime p (in this note, greater than 3), let $R^*_1(p^n) = R^*(\Gamma_1(p^n)^{\mathrm{arith}})$ be the graded ring of $\Gamma_1(p^n)^{\mathrm{arith}}$-modular forms ($\Gamma_{00}(p^n)^{\mathrm{arith}}$-modular forms in [10, Chapter II]) over $\mathbf{Z}[1/6]$ (see the Appendix). Particularly $R^*_1(1)$ is the graded ring of $\Gamma(1)$-modular forms over $\mathbf{Z}[1/6]$; $R^*_1(1) = R^*(1) = \mathbf{Z}[1/6][g_2, g_3, \Delta^{-1}]$ ($\Delta = g_2^3 - 27g_3^2$). Let $M_1(p^n) = M(\Gamma_1(p^n)^{\mathrm{arith}}) = \mathrm{Spec} R^*_1(p^n)$. For a fixed positive integer r, let $R^*_1(p^n)_r = R^*_1(p^n) \otimes \mathbf{Z}/p^r$ and $M_1(p^n)_r = M_1(p^n) \otimes \mathbf{Z}/p^r = \mathrm{Spec}(R^*_1(p^n)_r)$. Let $R^*_{\mathrm{ord},r} = \mathbf{Z}/p^r[g_2, g_3, \Delta^{-1}, E_{p-1}^{-1}]$, where E_{p-1} is the Eisenstein series, and $M_r^{\mathrm{ord}} = (M(1) \otimes \mathbf{Z}/p^r)^{\mathrm{ord}} = \mathrm{Spec} R^*_{\mathrm{ord},r}$. Let $E_{\mathrm{univ}} \otimes \mathbf{Z}/p^r$ and $E^{\mathrm{ord}}_{\mathrm{univ},r} = (E_{\mathrm{univ}} \otimes \mathbf{Z}/p^r)^{\mathrm{ord}}$ be the elliptic curves defined by the same Weierstrass equation $y^2 = 4x^3 - g_2x - g_3$ over $M(1)_r = M_1(1)_r$ and M_r^{ord} respectively.

Let $C_1(P)$ denote the quotient set of $\mathrm{Hom}(\mathbf{Z}/|P|, P)$ divided by the obvious conjugation action of P. Then we have an action of $(\mathbf{Z}/|P|)^\times$ on $C_1(P)$ by using a canonical action of $(\mathbf{Z}/|P|)^\times$ on $\mathbf{Z}/|P|$ given by

$$(\sigma, g) \mapsto g^\sigma \ (\forall \sigma \in (\mathbf{Z}/|P|)^\times, \forall g \in P).$$

Let $R^*(E^{\mathrm{ord}}_{\mathrm{univ},r}, P)$ denote the graded ring of all $(\mathbf{Z}/|P|)^\times$-equivariant maps $\mathrm{Map}_{(\mathbf{Z}/|P|)^\times}(C_1(P), R^*_1(|P|)_r)$ from $C_1(P)$ to $R^*_1(|P|)_r$ with obvious ring structure and grade. Here the action of $(\mathbf{Z}/|P|)^\times$ on $R^*_1(|P|)_r$ is a canonical one described in Section 3. Let $R^*(E_{\mathrm{univ}} \otimes \mathbf{Z}/p^r, P)$ be the graded subring of $R^*(E^{\mathrm{ord}}_{\mathrm{univ},r}, P)$ defined by $R^*(E_{\mathrm{univ}} \otimes \mathbf{Z}/p^r, P) = \{f \in R^*(E^{\mathrm{ord}}_{\mathrm{univ},r}, P) \mid f(e) \in R^*(1)\}$, where e denotes the conjugacy class of the trivial homomorphism.

With the above notation we have

Theorem 2.1.

$$M(E^{\mathrm{ord}}_{\mathrm{univ},r}, P) = \mathrm{Spec} R^*(E^{\mathrm{ord}}_{\mathrm{univ},r}, P).$$

Theorem 2.2.

$$M(E_{\mathrm{univ}} \otimes \mathbf{Z}/p^r, P) = \mathrm{Spec} R^*(E_{\mathrm{univ}} \otimes \mathbf{Z}/p^r, P).$$

(Note that a coarse moduli scheme does not commute with base change in general and hence that Theorem 2.1 is not an obvious consequence of Theorem 2.2.)

From these results we can describe the group of the global sections of coefficient sheaf of an equivariant elliptic cohomology based on $E^{\mathrm{ord}}_{\mathrm{univ},r}$ and $E_{\mathrm{univ}} \otimes \mathbf{Z}/p^r$ for a finite p-group P. But the arguments are completely analogous to those for $E_{\mathrm{univ}}[1/|G|]$ dealed with in [25] and so we will only give their statements. Let $(\mathbf{Ell}^{\mathrm{ord}}_r)^*_P(pt)$ and $(\mathbf{Ell} \otimes \mathbf{Z}/p^r)^*_P(pt)$ denote the group of all global sections of the coefficient sheaf of an equivariant elliptic cohomology based on $E^{\mathrm{ord}}_{\mathrm{univ},r}$ and $E_{\mathrm{univ}} \otimes \mathbf{Z}/p^r$ for a finite p-group P respectively. Then we have

Corollary 2.3. *For every integer k there are isomorphisms*

$$(\mathbf{Ell}^{\mathrm{ord}}_r)^{2k}_P(pt) \xrightarrow{\cong} R^*(E^{\mathrm{ord}}_{\mathrm{univ},r}, P)$$

and

$$(\mathbf{Ell} \otimes \mathbf{Z}/p^r)^{2k}_P(pt) \xrightarrow{\cong} R^*(E_{\mathrm{univ}} \otimes \mathbf{Z}/p^r, P).$$

The first isomorphism is $R^*(E^{\mathrm{ord}}_{\mathrm{univ},r}, P)$-linear and the second one is $R^*(E_{\mathrm{univ}} \otimes \mathbf{Z}/p^r, P)$-linear. Both isomorphisms are canonically determined by choosing a nowhere vanishing invariant differential on $E^{\mathrm{ord}}_{\mathrm{univ},r}$ and $E_{\mathrm{univ}} \otimes \mathbf{Z}/p^r$ respectively.

We can make $\omega_{E^{\mathrm{ord}}_{\mathrm{univ},r,P}}$ (resp. $\omega_{E_{\mathrm{univ}} \otimes \mathbf{Z}/p^r, P}$) into an $(R^0_{\mathrm{ord},r})^\times$ (resp. $R^0(1)^\times_r$)-equivariant invertible sheaf by choosing a nowhere vanishing invariant differential on $E^{\mathrm{ord}}_{\mathrm{univ},r}$ (resp. $E_{\mathrm{univ}} \otimes \mathbf{Z}/p^r$), say, $\omega_{\mathrm{univ}} = dx/y$. Then we have

Corollary 2.4. *The isomorphisms above, associated with ω_{univ}, induce isomorphisms*

$$(\mathbf{Ell}^{\mathrm{ord}}_r)^{2k}_P(pt)^{(R^0_{\mathrm{ord},r})^\times} \xrightarrow{\cong} R^{-k}(E^{\mathrm{ord}}_{\mathrm{univ},r}, P)$$

and

$$(\mathbf{Ell} \otimes \mathbf{Z}/p^r)^{2k}_P(pt)^{R^0(1)^\times_r} \xrightarrow{\cong} R^{-k}(E_{\mathrm{univ}} \otimes \mathbf{Z}/p^r, P)$$

respectively.

3. Moduli of P-coverings of p-ordinary elliptic curves

In this section we will give a sketch proof of Theorem 2.1 without detailed arguments. It would be easy to complete all the arguments by consulting Sections 3 and 4 of [25]. Let $\tilde{E}^{\mathrm{ord}}_{\mathrm{univ},r} = E^{\mathrm{ord}}_{\mathrm{univ},r} \times_{M^{\mathrm{ord}}_r} M_1(|P|)_r$ and $R^*(\tilde{E}^{\mathrm{ord}}_{\mathrm{univ},r}, P) = \mathrm{Map}(C_1(P), R^*_1(|P|)_r)$, where the M^{ord}_r-scheme structure of $M_1(|P|)_r$ is a standard one (see below Proposition 3.2). (From now on we will often suppress subscript r from our notation.) Then we have

Theorem 3.1.

$$M(\tilde{E}^{\mathrm{ord}}_{\mathrm{univ},r}, P) = \mathrm{Spec} R^*(\tilde{E}^{\mathrm{ord}}_{\mathrm{univ},r}, P).$$

The proof of this result is completely parallel to that of Theorem 3.1 in [25]; Only changes needed are the following.

The $\mathbf{Z}/n \times \mathbf{Z}/n$-covering $E(\alpha)$ of an elliptic curve E with naive level n structure α played a fundamental role in [25]. Here we need to replace it by the $\mathbf{Z}/p^n$-covering $E(\iota)$ of an elliptic curve E with $\Gamma_1(p^n)^{\mathrm{arith}}$-structure ι over $\mathbf{Z}/p^r$-scheme constructed as follows.

Let E be an elliptic curve over a $\mathbf{Z}/p^r$-scheme T with $\Gamma_1(p^n)^{\mathrm{arith}}$-structure

$$\iota : \mu_{p^n} \hookrightarrow E[p^n].$$

Let $E' = E/\iota(\mu_{p^n})$ and

$$\pi : E \longrightarrow E'$$

be the projection. Dualizing, we get a homomorphism

$$\pi^t : E' \longrightarrow E$$

whose kernel is, by Cartier-Nishi duality (see [11, 2.8] and [6, Sections 2.6.3 and 2.6.4]), isomorphic to $(\mathbf{Z}/p^n)_T$ via an isomorphism

$$\iota' : (\mathbf{Z}/p^n)_T \xrightarrow{\cong} \mathrm{Ker}\pi^t$$

which is canonically determined by ι. It is easy to see that this

$$\pi^t : E' \longrightarrow E$$

is a $\mathbf{Z}/p^n$-covering of E with $\mathbf{Z}/p^n$-action on E' given by the composition

$$E' \times_T (\mathbf{Z}/p^n)_T \xrightarrow{1_{E'} \times \iota'} E' \times_T \mathrm{Ker}\pi^t \longrightarrow E',$$

where the second morphism is induced by the group scheme structure on E'. This is the desired $\mathbf{Z}/p^n$-covering $E(\iota)$. With this notation Proposition 3.7 in [25] is now replaced by

Proposition 3.2. *Let T be a connected $\mathbf{Z}/p^r$-scheme and E be an elliptic curve over T with $\Gamma_1(|P|)^{\mathrm{arith}}$-structure ι. Then there is a canonical map*

$$\theta(E,\iota) : \pi^1(E,P) \longrightarrow C_1(P)$$

such that:

(1) *It is bijective if $T = \mathrm{Spec}K$ with K algebraically closed field (of characteristic p).*

(2) *It is natural with respect to arbitrary base change $T' \to T$ with T' connected.*

The rest of the proof of Theorem 3.1 goes as that of Theorem 3.1 in [25] and is left to the reader.

Next to deduce Thereom 2.1 from 3.1 is again done by very similar argument of deducing Theorem 2.3 from Theorem 3.1 in [25] by replacing the $GL_2(\mathbf{Z}/n)$-covering

$$M(n) \longrightarrow M(1)$$

by the following $(\mathbf{Z}/p^n)^\times$-covering (and hence the relevant group $GL_2(\mathbf{Z}/n)$ by $(\mathbf{Z}/p^n)^\times$).

First note that we have a canonical $(\mathbf{Z}/p^n)^\times$-action on $R_1^*(p^n)$ defined by

$$(\sigma, f) \mapsto f(\tilde{E}^{\mathrm{ord}}_{\mathrm{univ}}, \tilde{\omega}_{\mathrm{univ}}, \sigma^{-1}\iota_{\mathrm{univ}})\ (\forall\sigma \in (\mathbf{Z}/p^n)^\times, \forall f \in R_1^*(p^n)),$$

where ι_{univ} is a fixed $\Gamma_1(p^n)^{\mathrm{arith}}$-structure on $\tilde{E}^{\mathrm{ord}}_{\mathrm{univ}}$ and $\sigma^{-1}\iota_{\mathrm{univ}}$ is the one given by the composition

$$\mu_{p^n} \xrightarrow{\sigma^{-1}} \mu_{p^n} \xrightarrow{\iota_{\mathrm{univ}}} \tilde{E}^{\mathrm{ord}}_{\mathrm{univ}}[p^n].$$

This action induces a canonical action on $M_1(p^n)$. We also have a canonical injection

$$R^*_{\mathrm{ord}} \longrightarrow R_1^*(p^n)\ (f \mapsto f(\tilde{E}^{\mathrm{ord}}_{\mathrm{univ}}, \tilde{\omega}_{\mathrm{univ}})),$$

since the existence of a $\Gamma_1(p^n)^{\mathrm{arith}}$-structure on an elliptic curve over a $\mathbf{Z}/p^r$-scheme implies that the curve is p-ordinary (fiber-by-fiber ordinary) and the Eisenstein series E_{p-1} gives Hasse invariant after mod p reduction (see [6, Section 2.9.1] and [9, 2.0 and 2.1]). Hence we have a canonical morphism

$$M_1(p^n) \longrightarrow M^{\mathrm{ord}}$$

and we can prove that this morphism is Galois with Galois group $(\mathbf{Z}/p^n)^\times$ acting on $M_1(p^n)$ as defined above (cf. [11, 4.9] and [6, Section 2.9.2]). Now we also get an induced action of $(\mathbf{Z}/p^n)^\times$ on $\tilde{M}^{\mathrm{ord}}_P = M(\tilde{E}^{\mathrm{ord}}_{\mathrm{univ}}, P)$ given, on the coordinate ring, by

$$(\sigma, f) \mapsto \sigma f \sigma^{-1}\ (\forall f \in R^*(\tilde{E}^{\mathrm{ord}}_{\mathrm{univ}}, P)).$$

Let M_P^{ord} be a quotient of $\tilde{M}_P^{\mathrm{ord}}$ by this action; explicitly given by $M_P^{\mathrm{ord}} = \mathrm{Spec} R^*(E_{\mathrm{univ}}^{\mathrm{ord}}, P)$. Then M_P^{ord} has a unique M^{ord}-scheme structure such that the diagram:

$$\begin{array}{ccc} \tilde{M}_P^{\mathrm{ord}} & \longrightarrow & M_P^{\mathrm{ord}} \\ \downarrow & & \downarrow \\ M_1(p^n) & \longrightarrow & M^{\mathrm{ord}} \end{array}$$

is commutative.

The rest of the proof is again left to the reader.

4. P-coverings of non-p-ordinary elliptic curves and the proof of Theorem 2.2

The purpose of this section is to prove Theorem 2.2. We begin with the following result about P-coverings of non-p-ordinary elliptic curves.

Proposition 4.1. *Let E be an elliptic curve over a connected $\mathbf{Z}/p^r$-scheme T and $X \to E$ be a P-covering of E. If E has a supersingular fiber then there is a finite etale (necessarily) surjective $\mathbf{Z}/p^r$-morphism $T' \to T$ with T' connected such that the P-covering $X_{T'} = X \times_E E_{T'} \to E_{T'} = E \times_T T'$ is trivial. In particular if E has a supersingular fiber then the P-covering $X_{\bar{t}} \to E_{\bar{t}}$ is trivial for every geometric point $\bar{t} \to T$.*

The proof of this result is very similar to that of Proposition 3.8 in [25] by the fact that

$$\mathrm{Hom}_{\mathrm{cont}}(\pi_1(E), P) = *$$

for any supersingular elliptic curve E over an algebraically closed field of characteristic p (see Appendix B of [25]); so we will omit it.

Let $M_P = \mathrm{Spec} R^*(E_{\mathrm{univ}} \otimes \mathbf{Z}/p^r, P)$. Then it is easy to see that M_P is obtained by glueing $M_P^{\mathrm{ord}} = M(E_{\mathrm{univ}}^{\mathrm{ord}}, P)$ and $M(1)$ together along the open subscheme M^{ord}, i.e., $M_P = M(E_{\mathrm{univ}}^{\mathrm{ord}}, P) \cup_{M^{\mathrm{ord}}} M(1)$. Here M^{ord} is regarded as an open subscheme of $M(E_{\mathrm{univ}}^{\mathrm{ord}}, P)$ via the ring homomorphism

$$R^*(E_{\mathrm{univ}}^{\mathrm{ord}}, P) \longrightarrow R_{\mathrm{ord}}^* = R_1^*(|P|)^{(\mathbf{Z}/|P|)^\times} \ (f \mapsto f(e)).$$

Now we define a natural transformation

$$\psi(?) : \pi^1(?) = \pi^1_{E_{\mathrm{univ}} \otimes \mathbf{Z}/p^r, P}(?) \longrightarrow h(?) = (\mathrm{Sch}/M(1))(?, M_P)$$

as follows. For an $M(1)$-scheme T let $T^{\mathrm{supsing}} = \{t \in T | (E_{\mathrm{univ}})_t$ is supersingular$\}$. Then for a connected $M(1)$-scheme T with $T^{\mathrm{supsing}} = \emptyset$, i.e., T is actually an M^{ord}-scheme, we define

$$\psi(T) : \pi^1(T) \longrightarrow h(T)$$

by the composition

$$\pi^1(T) = \pi^1_{\mathrm{ord}}(T) \xrightarrow{\psi^{\mathrm{ord}}(T)} h^{\mathrm{ord}}(T) \longrightarrow h(T),$$

where

$$\psi^{\mathrm{ord}}(?) : \pi^1_{\mathrm{ord}}(?) = \pi^1_{E^{\mathrm{ord}}_{\mathrm{univ}}, P}(?) \longrightarrow h^{\mathrm{ord}}(?) = (\mathrm{Sch}/M^{\mathrm{ord}})(?, M^{\mathrm{ord}}_P)$$

is a natural transformation which makes M^{ord}_P into a coarse moduli scheme for π^1_{ord} and the second map is an obvious inclusion. If $T^{\mathrm{supsing}} \neq \emptyset$ then we define

$$\psi(T) : \pi^1(T) \longrightarrow h(T)$$

by the composition

$$\pi^1(T) \longrightarrow (\mathrm{Sch}/M(1))(T, M(1)) \hookrightarrow h(T),$$

where the middle set consits of a single element. Then we can easily prove that the above ψ uniquely extends to a natural transformation on $(\mathrm{Sch}/M(1))$ by Proposition 4.1 and it is clear that the resulting ψ is bijective on every geometric point of $M(1)$ by Theorem 2.1 and Proposition 4.1. To see that this (M_P, ψ) is actually a coarse moduli scheme for π^1 let N be an $M(1)$-scheme together with a natural transformation

$$\psi'(?) : \pi^1(?) \longrightarrow h'(?) = (\mathrm{Sch}/M(1))(?, N).$$

Let $N^{\mathrm{ord}} = N \times_{M(1)} M^{\mathrm{ord}}$ then for any M^{ord}-scheme T the natural map

$$(h')^{\mathrm{ord}}(T) = (\mathrm{Sch}/M^{\mathrm{ord}})(T, N^{\mathrm{ord}}) \longrightarrow h'(T)$$

is bijective. Therefore there is a unique natural transformation on $(\mathrm{Sch}/M^{\mathrm{ord}})$

$$(\psi')^{\mathrm{ord}}(?) : \pi^1(?) \longrightarrow (h')^{\mathrm{ord}}(?)$$

such that for any M^{ord}-scheme T, $\psi'(T)$ is factored into

$$\psi'(T) : \pi^1(T) \xrightarrow{(\psi')^{\mathrm{ord}}(T)} (h')^{\mathrm{ord}}(T) \xrightarrow{\cong} h'(T).$$

Thus there is a unique M^{ord}-morphism

$$\chi^{\mathrm{ord}} : M^{\mathrm{ord}}_P \longrightarrow N^{\mathrm{ord}}$$

such that

$$((\chi^{\mathrm{ord}} \circ) \psi^{\mathrm{ord}})(T) = (\psi')^{\mathrm{ord}}(T)$$

for any M^{ord}-scheme T. Let

$$\chi^{\mathrm{supsing}} : M(1) \longrightarrow N$$

denote the $M(1)$-morphism $((\psi'(M(1)))(\text{trivial class})$. Then it is easy to see that χ^{ord} and χ^{supsing} coincide on the open subscheme M^{ord} and hence they define an $M(1)$-morphism

$$\chi : M_P = M_P^{\mathrm{ord}} \cup_{M^{\mathrm{ord}}} M(1) \longrightarrow N.$$

Now we can easily prove that this χ is a unique $M(1)$-morphism with the property that $(\chi\circ)\psi = \psi'$ on $(\mathrm{Sch}/M(1))$ by using Theorem 2.1 and Proposition 4.1.

Appendix A. Review of $\Gamma_1(n)^{\mathrm{arith}}$-moduli problem over $\mathbf{Z}[1/6]$

In this appendix we will give a brief account of $\Gamma_1(n)^{\mathrm{arith}}$-modular forms in the sense of Katz [10, Chapter II]. Our main references are [10, Chapter II], [11] and [6, Chapter I–III]; particularly Hida's recent book [6] contains most necessary information about scheme theory. For simplicity we exclude characteristic 2 and 3 which does not matter in this note.

Let E be an elliptic curve over a (not necessarily locally noetherian) scheme S. For a positive integer n let $E[n]$ denote the kernel of multiplication by n map on E:

$$[n] : E \longrightarrow E.$$

Then a $\Gamma_1(n)^{\mathrm{arith}}$-structure on E is an inclusion of group schemes over S:

$$\iota : \mu_{n,S} \hookrightarrow E[n].$$

For arbitrary scheme S a $\Gamma_1(n)^{\mathrm{arith}}$-test object over S is a triple (E, ω, ι) consisting of an elliptic curve E over S, a nowhere vanishing invariant differential ω on E and a $\Gamma_1(n)^{\mathrm{arith}}$-structure ι on E; particularly a $\Gamma_1(1)^{\mathrm{arith}}$-test object is nothing but a $\Gamma(1)$-test object (E, ω). Let $\mathcal{M}(\Gamma_1(n)^{\mathrm{arith}})_S$ denote the functor from (Sch/S) to (Sets) defined by

$$\begin{aligned}\mathcal{M}(\Gamma_1(n)^{\mathrm{arith}})_S(T) = {} & \text{the set of isomorphism classes of} \\ & \Gamma_1(n)^{\mathrm{arith}}\text{-test objects over } T.\end{aligned}$$

Then we have

Theorem A.1 ([10, 2.5]). *The functor $\mathcal{M}(\Gamma_1(n)^{\mathrm{arith}})_{\mathbf{Z}[1/6]}$ is representable by an affine $\mathbf{Z}[1/6]$-scheme $M(\Gamma_1(n)^{\mathrm{arith}})$.*

It is clear that for any $\mathbf{Z}[1/6]$-scheme S, the scheme $M(\Gamma_1(n)^{\mathrm{arith}})_S = M(\Gamma_1(n)^{\mathrm{arith}}) \times_{\mathrm{Spec}\mathbf{Z}[1/6]} S$ represents the functor $\mathcal{M}(\Gamma_1(n)^{\mathrm{arith}})_S$.

Now in Appendix A of [25] we have already seen that the functor $\mathcal{M}(\Gamma(n)^{\mathrm{arith}})_{\mathbf{Z}[1/6]}$ from $(\mathrm{Sch}/\mathbf{Z}[1/6])$ to (Sets), defined by

$$\begin{aligned}\mathcal{M}(\Gamma(n)^{\mathrm{arith}})_{\mathbf{Z}[\frac{1}{6}]}(T) = {} & \text{the set of isomorphism classes of} \\ & \Gamma(n)^{\mathrm{arith}}\text{-test objects over } T,\end{aligned}$$

is representable by an affine scheme $M(\Gamma(n)^{\mathrm{arith}}) = \mathrm{Spec}R^*(\Gamma(n)^{\mathrm{arith}})$. Then we have a canonical action of $\mu_{n,\mathbf{Z}[1/6]} = \mathrm{Hom}_{\mathbf{Z}[1/6]\text{-grp}}(\mathbf{Z}/n, \mu_n)$ on $M(\Gamma(n)^{\mathrm{arith}})$ defined by

$$(\lambda, (E, \omega, \beta)) \mapsto (E, \omega, \lambda^{-1}\beta),$$

where the $\Gamma(n)^{\mathrm{arith}}$-structure on E: $\lambda^{-1}\beta$ is given by

$$(\lambda^{-1}\beta)(\zeta, m) = \beta(\zeta\lambda^{-m}, m).$$

This action yields a coaction

$$\psi : R^*(\Gamma(n)^{\mathrm{arith}}) \longrightarrow \mathbf{Z}[\tfrac{1}{6}][t]/(t^n - 1) \otimes R^*(\Gamma(n)^{\mathrm{arith}})$$

of the Hopf algebra associated with the affine group scheme $\mu_{n,\mathbf{Z}[1/6]}$ on the coordinate ring $R^*(\Gamma(n)^{\mathrm{arith}})$ of $M(\Gamma(n)^{\mathrm{arith}})$ and let $R^*(\Gamma_1(n)^{\mathrm{arith}})$ denote the graded subring of $R^*(\Gamma(n)^{\mathrm{arith}})$ consisting of all primitive elements with respect to this coaction. Let $M(\Gamma_1(n)^{\mathrm{arith}})$ be a quotient of $M(\Gamma(n)^{\mathrm{arith}})$ by the action of $\mu_{n,\mathbf{Z}[1/6]}$ above; explicitly given by $M(\Gamma_1(n)^{\mathrm{arith}}) = \mathrm{Spec}R^*(\Gamma_1(n)^{\mathrm{arith}})$. Then we can prove that this $M(\Gamma_1(n)^{\mathrm{arith}})$ actually represents the functor $\mathcal{M}(\Gamma_1(n)^{\mathrm{arith}})_{\mathbf{Z}[1/6]}$ by the facts that: (1) The restriction of a universal $\Gamma(n)^{\mathrm{arith}}$-structure β_{univ}:

$$\beta_{\mathrm{univ}}|\mu_{n,R^*(\Gamma(n)^{\mathrm{arith}})} : \mu_{n,R^*(\Gamma(n)^{\mathrm{arith}})} \hookrightarrow (\mu_n \times \mathbf{Z}/n)_{R^*(\Gamma(n)^{\mathrm{arith}})} \xrightarrow{\cong} E_{\mathrm{univ}}[n] \otimes R^*(\Gamma(n)^{\mathrm{arith}})$$

descends to a $\Gamma_1(n)^{\mathrm{arith}}$-structure on $E_{\mathrm{univ}} \otimes R^*(\Gamma_1(n)^{\mathrm{arith}})$:

$$\iota_{\mathrm{univ}} : \mu_{n,R^*(\Gamma_1(n)^{\mathrm{arith}})} \hookrightarrow E_{\mathrm{univ}}[n] \otimes R^*(\Gamma_1(n)^{\mathrm{arith}}).$$

(2) Every $\Gamma_1(n)^{\mathrm{arith}}$-structure on any elliptic curve E over a scheme S extends to a $\Gamma(n)^{\mathrm{arith}}$-structure after some finite etale surjective base change $S' \to S$ by self-duality of $E[n]$ (see [11, 2.8 and 8.10] and [6, Sections 2.6.3 and 2.6.4]). (3) The set of such $\Gamma(n)^{\mathrm{arith}}$-structures is principal homogeneous under the action of $\mu_n(S')$ above. We omit the details.

For any $\mathbf{Z}[1/6]$-algebra R, the graded ring $R^*(\Gamma_1(n)^{\mathrm{arith}})_R = R^*(\Gamma_1(n)^{\mathrm{arith}}) \otimes R$ is, by definition, the graded ring of $\Gamma_1(n)^{\mathrm{arith}}$-modular forms over R. For any $\Gamma_1(n)^{\mathrm{arith}}$-test object (E, ω, ι) over any R-algebra B we have a unique R-algebra homomorphism

$$R^*(\Gamma_1(n)^{\mathrm{arith}})_R \longrightarrow B$$

classifying (E, ω, ι) and we denote the image of an element f of $R^*(\Gamma_1(n)^{\mathrm{arith}})_R$ under this homomorphism, the *value* of f on (E, ω, ι), by $f(E, \omega, \iota)$.

Remark A.2. Over any $\mathbf{Z}[1/n]$-scheme, $\Gamma_1(n)^{\mathrm{arith}}$-structure on an elliptic curve is the same as Drinfeld style $\Gamma_1(n)$-structure but slightly different in general. For example, over $\mathbf{F}_p$, all supersingular elliptic curves are automatically excluded in $\Gamma_1(p)^{\mathrm{arith}}$-moduli problem (cf. [6, Section 2.9]).

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