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On the coefficient sheaf of equivariant elliptic cohomology for finite groups I

By

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1. Introduction

In [3] J. Devoto constructed equivariant elliptic cohomology for any finite group G of odd order from equivariant oriented cobordism whose coefficient ring is closely related to G-elliptic systems.

On the other hand V. Ginzburg, M. Kapranov and E. Vasserot, motivated by study of Yang-Baxter equation, proposed an axiomatic approach to equivariant elliptic cohomology on finite G-CW-complexes for any compact Lie group G in [4]. Their theory, based on an elliptic curve E over a scheme S, takes its values in the category of coherent modules over the structure sheaf $\mathcal{O}_{M(E,G)}$ of the moduli scheme $M(E,G)$ (denoted by χ_G in [4]) of certain semistable principal G_S-bundles over the elliptic curve E. Here $G_S = G_{\mathbf{Z}} \times_{\mathrm{Spec}\mathbf{Z}} S$ and $G_{\mathbf{Z}}$ is the $\mathbf{Z}$-group scheme associated with G. Some examples of such theories were provided by Grojnowski's construction of equivariant elliptic cohomology with complex coefficient for compact connected Lie groups (see the end of Section 1 of [4]) which were used, with suitable modification, by M. Ando [1] and I. Rosu [23].

The purpose of this note is to study relations between the coefficient sheaf of the above axiomatic equivariant elliptic cohomology for a finite group G and the Devoto's coefficient ring. In this note we only consider the case that the elliptic curve E is the Weierstrass family $E_{\mathrm{univ}}[1/|G|]$ defined by the equation $y^2 = 4x^3 - g_2x - g_3$ over the scheme $S = M(1)[1/|G|] = \mathrm{Spec}(\mathbf{Z}[1/(6|G|)][g_2, g_3, \Delta^{-1}])$ ($\Delta = g_2^3 - 27g_3^2$). (In the subsequent paper [28] we will study the case that G is a finite p-group and that S is a $\mathbf{Z}/p^r$-scheme for a prime p greater than 3 and a positive integer r.) In this case we can construct the moduli scheme $M(E_{\mathrm{univ}}[1/|G|], G)$ as an affine scheme explicitly in terms of (equivariant) modular forms (Theorem 2.3). This construction of the moduli scheme gives us our main result providing relations between global sections of the coefficient sheaf of axiomatic equivariant elliptic cohomology based on $E_{\mathrm{univ}}[1/|G|]$ for a finite group G and G-elliptic systems (Theorem 2.7). These results should be known to many people and in fact we can find a suggestion of such results in the Introduction of Devoto's paper [3]. But, as far as I know, there is no

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detailed account of it and, hopefully, this paper would give a complement to Section 3 of [3].

In Section 2 we recall some basic definitions related to equivariant elliptic cohomology and state our results. In Section 3 we study moduli problem of G-coverings of elliptic curves with naive level $|G|$ structure by using theory of etale fundamental groups and construct the moduli scheme $M(\tilde{E}_{\mathrm{univ}}, G)$. Here $\tilde{E}_{\mathrm{univ}} = E_{\mathrm{univ}}[1/|G|] \times_{M(1)[1/|G|]} M(|G|)$ and $M(|G|)$ is the prime ideal spectrum of the graded ring $R^*(\Gamma(|G|)^{\mathrm{naive}})$ of $\Gamma(|G|)^{\mathrm{naive}}$-modular forms over $\mathbf{Z}[1/(6|G|)]$. In Section 4 we construct the moduli scheme $M(E_{\mathrm{univ}}[1/|G|], G)$ as a quotient of $M(\tilde{E}_{\mathrm{univ}}, G)$ by a canonical action of $GL_2(\mathbf{Z}/|G|)$ obtained from a canonical action of $GL_2(\mathbf{Z}/|G|)$ on $R^*(\Gamma(|G|)^{\mathrm{naive}})$. We also describe the latter action of $GL_2(\mathbf{Z}/|G|)$ in terms of $\Gamma(|G|)^{\mathrm{arith}}$-modular forms over $\mathbf{Z}[1/(6|G|), e^{2\pi i/|G|}]$ and prove Theorem 2.7 by using a GAGA-type result. For the convenience of the reader we add two appendices. In Appendix A we give a brief account of $\Gamma(n)^{\mathrm{naive}}$ and $\Gamma(n)^{\mathrm{arith}}$-modular forms over $\mathbf{Z}[1/6]$-algebras, while in Appendix B we compute the etale fundamental group of an elliptic curve over an algebraically closed field.

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2. Basic definitions and statement of results

Let G be a finite group. (From now on we fix a finite group G.) A finite etale surjective morphism of locally noetherian schemes $Y \to X$ (In this note all schemes are assumed to be locally noetherian in order to quote several results from [17] and [19].) is said to be Galois with Galois group G (or a G-covering of X) if G acts on Y (say, on the right) as X-morphisms and the map

$$G_Y \longrightarrow Y \times_X Y \ ((y, g) \mapsto (y, yg) \text{ on points})$$

is an isomorphism. Here G_Y is the group scheme over Y given by $Y \times G = \coprod_{g \in G} Y_g$ ($Y_g = Y$ $(\forall g \in G)$) with obvious group scheme structure. We denote by $\pi^1(X, G)$ the set of isomorphism classes of G-coverings of X.

Let S be a locally noetherian scheme. An S-scheme E is said to be an elliptic curve over S if the morphism $E \to S$ is proper and smooth and its geometric fibers are all connected curves of genus one equipped with a section $i : S \to E$. (In this note a geometric point means a morphism from the prime ideal spectrum of an algebraically closed field.) Let ω_E denote the invertible sheaf $i^*\Omega^1_{E/S}$ on S. Let (Sch/S) and (Sets) denote the category of locally noetherian S-schemes and the one of sets respectively. Let

$$\pi^1_{E,G} : (\mathrm{Sch}/S) \longrightarrow (\mathrm{Sets})$$

be the functor defined by

$$\pi^1_{E,G}(T) = \pi^1(E_T, G) \ (\forall T \in (\mathrm{Sch}/S))$$

where $E_T = E \times_S T$.

Definition 2.1. A locally noetherian S-scheme M together with a natural transformation $\phi(?) : \pi^1_{E,G}(?) \to h_M(?) = (\mathrm{Sch}/S)(?, M)$ is said to be a *coarse moduli scheme* if

(1) For any locally noetherian S-scheme N together with a natural transformation $\psi : \pi^1_{E,G} \to h_N$ there is a unique S-morphism $\chi : M \to N$ such that $\psi = (\chi\circ)\phi$, where $\chi\circ$ denotes the natural transformation from h_M to h_N induced by χ in the obvious way.

(2) $\phi(\bar{s}) : \pi^1_{E,G}(\bar{s}) \to h_M(\bar{s})$ is bijective for every geometric point $\bar{s}$ of S.

(I could not find a reference for the existence of a coarse moduli scheme in the above sense for arbitrary elliptic curves and finite groups but we will construct it in our case explicitly in Section 4.) We should remark that a coarse moduli scheme M is unique up to canonical isomorphism, if it exists, by the first property. Let $M(E, G)$ denote the coarse moduli scheme in the above sense (if it exists) and $p_{E,G} : M(E, G) \to S$ be the S-scheme structure on $M(E, G)$. Let $\omega_{E,G}$ denote the invertible sheaf $p^*_{E,G}\omega_E$ on $M(E, G)$. Then the coefficient sheaf of an equivariant elliptic cohomology $Ell^*_G(?)$, based on an elliptic curve E, on finite G-CW-complexes is defined by

$$Ell^k_G(pt) = \begin{cases} \omega_{E,G}^{\otimes -\frac{k}{2}} & (k \text{ even}), \\ 0 & (k \text{ odd}). \end{cases}$$

Remark 2.2. A counter example to Hopkins-Kuhn-Ravenel conjecture given by I. Kriz [13] suggests that we should have non-trivial $Ell^{\mathrm{odd}}_G(pt)$ in general if S is not a $\mathbf{Z}[1/|G|]$-scheme.

Let $R^*(n) = R^*(\Gamma(n)^{\mathrm{naive}})$ be the graded ring of $\Gamma(n)^{\mathrm{naive}}$-modular forms over $\mathbf{Z}[1/6]$ [10, Chapter II] (see Appendix A) and $M(n) = \mathrm{Spec}R^*(n)$. In particular we have $R^*(1) = \mathbf{Z}[1/6][g_2, g_3, \Delta^{-1}]$ ($\Delta = g_2^3 - 27g_3^2$). Let E_{univ} be the elliptic curve over $M(1)$ defined by the Weierstrass equation $y^2 = 4x^3 - g_2x - g_3$ and $E_{\mathrm{univ}}[1/|G|] = E_{\mathrm{univ}} \times_{M(1)} M(1)[1/|G|]$, where $M(1)[1/|G|] = M(1) \times_{\mathrm{Spec}\mathbf{Z}[1/6]} \mathrm{Spec}\mathbf{Z}[1/(6|G|)]$. (We will often suppress $[1/|G|]$ from our notation.)

Let $C_2(G)$ denote the quotient set of $\widehat{C}_2(G) = \mathrm{Hom}(\mathbf{Z}/|G| \times \mathbf{Z}/|G|, G)$ divided by the obvious conjugation action of G. Then we have a left action of $GL_2(\mathbf{Z}/|G|)$ on $C_2(G)$ by using a canonical right action of $GL_2(\mathbf{Z}/|G|)$ on $\mathbf{Z}/|G| \times \mathbf{Z}/|G|$. (Here we regard every element of $\mathbf{Z}/|G| \times \mathbf{Z}/|G|$ as a row vector.) Let $R^*(E_{\mathrm{univ}}[1/|G|], G)$ denote the graded ring of all $GL_2(\mathbf{Z}/|G|)$-equivariant maps $\mathrm{Map}_{GL_2(\mathbf{Z}/|G|)}(C_2(G), R^*(|G|))$ from $C_2(G)$ to $R^*(|G|)$ with obvious ring structure and grade. Here the left action of $GL_2(\mathbf{Z}/|G|)$ on $R^*(|G|)$ is a canonical one described in Section 4.

With the above notation we have

Theorem 2.3.

$$M\left(E_{\mathrm{univ}}\left[\tfrac{1}{|G|}\right], G\right) = \mathrm{Spec}R^*\left(E_{\mathrm{univ}}\left[\tfrac{1}{|G|}\right], G\right).$$

Now the invertible sheaf $\omega_{E_{\mathrm{univ}},G} = p^*_{E_{\mathrm{univ}},G} i^* \Omega^1_{E_{\mathrm{univ}}/M(1)}$ on $M(E_{\mathrm{univ}}[1/|G|], G)$ is trivial via an isomorphism obtained by choosing a nowhere vanishing invariant differential ω on $E_{\mathrm{univ}}[1/|G|]$, say $\omega_{\mathrm{univ}} = dx/y$. We denote the trivialization obtained by choosing a nowhere vanishing invariant differential ω by

$$\varphi_\omega : \omega_{E_{\mathrm{univ}},G} \xrightarrow{\cong} \mathcal{O}_{M(E_{\mathrm{univ}}[\frac{1}{|G|}],G)},$$

where $\mathcal{O}_{M(E_{\mathrm{univ}}[1/|G|],G)}$ is the structure sheaf of $M(E_{\mathrm{univ}}[1/|G|], G)$.

For every integer k let $\mathbf{Ell}[1/|G|]^{2k}_G(pt)$ denote the group of all global sections of the invertible sheaf $\omega^{\otimes -k}_{E_{\mathrm{univ}},G}$ on $M(E_{\mathrm{univ}}[1/|G|], G)$. Then we have

Corollary 2.4. *For every integer k there is an isomorphism of $R^*(E_{\mathrm{univ}}[1/|G|], G)$-modules*

$$\mathbf{Ell}\left[\tfrac{1}{|G|}\right]^{2k}_G (pt) \xrightarrow{\cong} R^*\left(E_{\mathrm{univ}}\left[\tfrac{1}{|G|}\right], G\right)$$

which is canonically determined by choosing a nowhere vanishing invariant differential on $E_{\mathrm{univ}}[1/|G|]$.

Next consider a canonical action of $R^0(1)^\times$ on $R^*(|G|)$ given by

$$(\lambda, f) \mapsto f(\tilde{E}_{\mathrm{univ}}, \lambda^{-1}\tilde{\omega}_{\mathrm{univ}}, \alpha_{\mathrm{univ}})\ (\forall \lambda \in R^0(1)^\times, \forall f \in R^*(|G|)).$$

Here $\tilde{E}_{\mathrm{univ}}$ is the elliptic curve $E_{\mathrm{univ}} \times_{M(1)} M(|G|)$ over $M(|G|)$, $\tilde{\omega}_{\mathrm{univ}}$ is the nowhere vanishing invariant differential on $\tilde{E}_{\mathrm{univ}}$ obtained from ω_{univ} and α_{univ} is a fixed naive level $|G|$ structure on $\tilde{E}_{\mathrm{univ}}$. (Note that $M(|G|)$ has a standard $M(1)$-scheme structure (see the begining of Section 4).) This action induces an action of $R^0(1)^\times$ on $R(E_{\mathrm{univ}}, G)$ in obvious way and hence we get a canonical action of $R^0(1)^\times$ on $M(E_{\mathrm{univ}}, G)$. We also have an action of $R^0(1)^\times$ on $\omega_{E_{\mathrm{univ}},G}$ given as follows. For any $\lambda \in R^0(1)^\times$ we have an isomorphism

$$\varphi_\lambda : \lambda^*\omega_{E_{\mathrm{univ}},G} \xrightarrow{\cong} \omega_{E_{\mathrm{univ}},G}$$

given by the composition

$$\lambda^*\omega_{E_{\mathrm{univ}},G} \xrightarrow{\varphi_{\omega_{\mathrm{univ}}}} \lambda^*\mathcal{O}_{M(E_{\mathrm{univ}},G)} \xrightarrow{\cong} \mathcal{O}_{M(E_{\mathrm{univ}},G)} \xrightarrow{\varphi^{-1}_{\lambda\omega_{\mathrm{univ}}}} \omega_{E_{\mathrm{univ}},G},$$

where the second isomorphism is the obvious one induced by $R^0(1)^\times$-action on $M(E_{\mathrm{univ}}, G)$. It is easy to see that these φ_λ make $\omega_{E_{\mathrm{univ}},G}$ an $R^0(1)^\times$-equivariant invertible sheaf. Let $\mathbf{Ell}[1/|G|]^{2k}_G(pt)^{R^0(1)^\times}$ denote the subgroup of $\mathbf{Ell}[1/|G|]^{2k}_G(pt)$ consisting of all invariant sections with respect to this action of $R^0(1)^\times$. Then we can easily prove that

Corollary 2.5. *The isomorphism in Corollary* 2.4 *obtained from $\varphi_{\omega_{\mathrm{univ}}}$ induces an isomorphism*

$$\mathbf{Ell}\left[\tfrac{1}{|G|}\right]^{2k}_G (pt)^{R^0(1)^\times} \xrightarrow{\cong} R^{-k}\left(E_{\mathrm{univ}}\left[\tfrac{1}{|G|}\right], G\right).$$

Next we restate the above result in analytic terms. Let $\mathcal{H} = \{\tau \in \mathbf{C} |\ \mathrm{Im}\tau > 0\}$. Then we have a standard left action of $SL_2(\mathbf{Z})$ on $\mathcal{H}$ given by

$$(A, \tau) \mapsto A\tau = \frac{a\tau + b}{c\tau + d} \ \left(\forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}), \forall \tau \in \mathcal{H}\right).$$

Let $F(\mathcal{H}, G)$ denote the $\mathbf{C}$-algebra of all $\mathbf{C}$-valued functions on $\widehat{C}_2(G) \times \mathcal{H}$. For every $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbf{Z})$ and $\tau \in \mathcal{H}$ let $j(A, \tau) = c\tau + d$. Then for every integer k we have a right action of $SL_2(\mathbf{Z}) \times G$ on $F(\mathcal{H}, G)$

$$F(\mathcal{H}, G) \times (SL_2(\mathbf{Z}) \times G) \longrightarrow F(\mathcal{H}, G) \ ((f, (A, g)) \mapsto f|[A, g]_k)$$

defined by

$$f|[A, g]_k(\theta, \tau) = j(A, \tau)^{-k} f(g(A\theta)g^{-1}, A\tau) \ (\forall \theta \in \widehat{C}_2(G), \forall \tau \in \mathcal{H}).$$

Here the left action of $SL_2(\mathbf{Z})$ on $\widehat{C}_2(G)$ is induced by the standard right action on $\mathbf{Z}/|G| \times \mathbf{Z}/|G|$.

Definition 2.6. Let $R^k_{\mathrm{an}}(E_{\mathrm{univ}}[1/|G|], G)$ denote the subgroup of $F(\mathcal{H}, G)$ consisting of all functions

$$f : \widehat{C}_2(G) \times \mathcal{H} \longrightarrow \mathbf{C}$$

satisfying:

(1) $f|[A, g]_k = f \ (\forall A \in SL_2(\mathbf{Z}), \forall g \in G)$.

(2) For every $\theta \in \widehat{C}_2(G)$ the function

$$f_\theta : \mathcal{H} \longrightarrow \mathbf{C} \ (\tau \mapsto f(\theta, \tau))$$

is holomorphic on $\mathcal{H}$ and meromorphic at $i\infty$ having Fourier expansion of the form $f_\theta(\tau) = \sum_{j \gg -\infty} f_{\theta,j} q^j$ with $q = e^{2\pi i \tau/|\theta(1,0)|}$.

(3) In the above expansion, $f_{\theta,j} \in \mathbf{Z}[1/(6|G|), e^{2\pi i/|\theta(1,0)|}] \ (\forall j)$ and $f_{\theta\sigma,j} = \sigma f_{\theta,j} \ (\forall \sigma \in (\mathbf{Z}/|G|)^\times, \forall j)$ where $(\mathbf{Z}/|G|)^\times$ $(= \mathrm{Gal}(\mathbf{Q}(e^{2\pi i/|G|})/\mathbf{Q}))$ acts on $\widehat{C}_2(G)$ via the multiplication on the first factor of $\mathbf{Z}/|G| \times \mathbf{Z}/|G|$ and on $\mathbf{Z}[1/(6|G|), e^{2\pi i/|\theta(1,0)|}]$ by the standrd Galois action.

(This definition should be compared with the definition of Devoto's coefficient ring [3, Definition 3.2].) These functions could be viewed as a kind of class functions in theory of G-elliptic systems (see [16] and [20]). They have certain integrality and Galois invariance properties (Property 3 in the above definition) which make them close to genuine G-elliptic systems to some extent (cf. [3, Remark 3.4]). Now our main result is

Theorem 2.7. *The isomorphism in Corollary* 2.5 *induces a canonical isomorphism*

$$\mathbf{Ell}\left[\tfrac{1}{|G|}\right]^{2k}_G (pt)^{R^0(1)^\times} \xrightarrow{\cong} R^{-k}_{\mathrm{an}}\left(E_{\mathrm{univ}}\left[\tfrac{1}{|G|}\right], G\right).$$

Remark 2.8. The above result might be viewed as an analogue for finite groups of Theorems 10.6 and 11.7 of Ando [1] in spite of lack of appropriate objects corresponding to positive energy representations of loop groups of compact connected Lie groups. It would be very important to find such objects for finite groups with full generality. (Ideally, such objects should recover positive energy representations of loop groups in some sense.)

3. Moduli of G-coverings of elliptic curves with naive level $|G|$ structure

The purpose of this section is to prove the following key result. Let $R^*(\tilde{E}_{\mathrm{univ}}, G) = \mathrm{Map}(C_2(G), R^*(|G|))$. Then we have

Theorem 3.1.

$$M(\tilde{E}_{\mathrm{univ}}, G) = \mathrm{Spec} R^*(\tilde{E}_{\mathrm{univ}}, G).$$

To prove the above theorem we first recall some basic facts about etale fundamental groups of schemes and their relations to Galois coverings. Let X be a connected scheme and $\bar{x} \to X$ be a geometric point. (Recall that all schemes are assumed to be locally noetherian in this note.) Let (FEt/X) denote the category of X-schemes finite etale over X. Let $(\pi_1(X, \bar{x})\text{-sets})$ denote the category of finite sets on which $\pi_1(X, \bar{x})$ acts continuously on the left. Here $\pi_1(X, \bar{x})$ is the etale fundamental group of X based on $\bar{x}$ which is a profinite group with usual inverse limit topology. Then we have

Theorem 3.2 ([19, 4.4.1] and [17, I 5.3])**.** (1) *There is an equivalence of categories*

$$F : (\mathrm{FEt}/X) \xrightarrow{\cong} (\pi_1(X, \bar{x})\text{-sets})$$

given by $F(Y) = (\mathrm{Sch}/X)(\bar{x}, Y)$ $(\forall Y \in (\mathrm{FEt}/X))$. *We call this F a fundamental functor on* (FEt/X) *based on* $\bar{x}$.

(2) *Let $\bar{x}' \to X$ be any other geometric point. Then there is a continuous isomorphism*

$$\pi_1(X, \bar{x}') \xrightarrow{\cong} \pi_1(X, \bar{x})$$

which is canonically determined up to an inner automorphism of $\pi_1(X, \bar{x})$.

Let $f : Y \to X$ be a morphism of connected schemes. Let $\bar{y} \to Y$ be a geometric point of Y and $f(\bar{y})$ denote the geometric point $\bar{y} \to Y \to X$ of X. Let F_X (resp. F_Y) be the fundamental functor on (FEt/X) (resp. (FEt/Y)) based on $f(\bar{y})$ (resp. $\bar{y}$). Then we have

Lemma 3.3 ([19, 5.1])**.** (1) *There is a unique continuous homomorphism*

$$f_* : \pi_1(Y, \bar{y}) \longrightarrow \pi_1(X, f(\bar{y}))$$

such that the diagram:

$$\begin{array}{ccc} (\mathrm{FEt}/X) & \xrightarrow{F_X} & (\pi_1(X, f(\bar{y}))\text{-sets}) \\ \downarrow & & \downarrow \\ (\mathrm{FEt}/Y) & \xrightarrow[F_Y]{} & (\pi_1(Y, \bar{y})\text{-sets}) \end{array}$$

is commutative where the left vertical arrow is given by $Z \mapsto Z \times_X Y$ ($\forall Z \in (\mathrm{FEt}/X)$) *and the right vertical arrow is induced by the above homomorphism in obvious way.*

(2) *Let* $\bar{y}' \to Y$ *be any other geometric point. Then the diagram*:

$$\begin{array}{ccc} \pi_1(Y, \bar{y}') & \xrightarrow{f_*} & \pi_1(X, f(\bar{y}')) \\ \cong\downarrow & & \downarrow\cong \\ \pi_1(Y, \bar{y}) & \xrightarrow[f_*]{} & \pi_1(X, f(\bar{y})) \end{array}$$

is commutative up to an inner automorphism of $\pi_1(X, f(\bar{y}))$.

We also have

Theorem 3.4 ([19, 6.3.2.1]). *Let* $f : Y \to X$ *be a proper separable morphism of connected schemes such that* $\mathcal{O}_X \xrightarrow{\cong} f_*\mathcal{O}_Y$. *Then there is an exact sequence of profinite groups*

$$\pi_1(Y_{f(\bar{y})}, \bar{y}) \xrightarrow{j_*} \pi_1(Y, \bar{y}) \xrightarrow{f_*} \pi_1(X, f(\bar{y})) \longrightarrow 1,$$

where $\bar{y} \to Y$ *is a geometric point and* $j : Y_{f(\bar{y})} = Y \times_X f(\bar{y}) \to Y$ *is the geometric fiber of* f *on* $f(\bar{y})$.

(In [19] this result is only stated in the case that $f(\bar{y}) = \mathrm{Spec}\overline{k(x)} \to X$ for some $x \in X$, but this restriction is unnecessary (see [19, 7.3.2]).)

Next we consider the relations between fundamental groups and Galois coverings. Let $Y \to X$ be a G-covering of connected X. Then we have an isomorphism

$$G_Y \xrightarrow{\cong} Y \times_X Y \ ((y, g) \mapsto (y, yg)).$$

Applying the fundamental functor F based on $\bar{x}$ we have an isomorphism in $(\pi_1(X, \bar{x})\text{-sets})$

$$F(Y) \times G \xrightarrow{\cong} F(Y) \times F(Y) \ ((y, g) \mapsto (y, yg)).$$

Thus by choosing a base point $y_0 \in F(Y)$ we have a continuous homomorphism

$$\rho(X, \bar{x}, Y) : \pi_1(X, \bar{x}) \longrightarrow G$$

defined by $\sigma y_0 = y_0[\rho(X,\bar{x},Y)(\sigma)]$ $(\forall\sigma \in \pi_1(X,\bar{x}))$.

Now replacing y_0 by any other element we get another continuous homomorphism which differs from the previous one only by an inner automorphism of G. Therefore the above $\rho(X,\bar{x},Y)$ gives a unique element $\bar{\rho}(X,\bar{x},Y)$ of $\mathrm{Rep}(\pi_1(X,\bar{x}),G)$ which is independent on the choice of a base point of $F(Y)$. Here $\mathrm{Rep}(\pi_1(X,\bar{x}),G)$ denotes the quotient set of $\mathrm{Hom}_{\mathrm{cont}}(\pi_1(X,\bar{x}),G)$ divided by the obvious conjugation action of G. Thus, as an immediate consequence of Theorem 3.2, we have

Lemma 3.5 (cf. [17, I 5.4]). (1) *With the above notation there is a canonical bijection*

$$\bar{\rho}(X,\bar{x}) : \pi^1(X,G) \xrightarrow{\cong} \mathrm{Rep}(\pi_1(X,\bar{x}),G)$$

given by $[Y] \mapsto \bar{\rho}(X,\bar{x},Y)$ $(\forall[Y] \in \pi^1(X,G))$.

(2) *Let* $\bar{x}' \to X$ *be any other geometric point. Then the diagram*:

$$\begin{array}{ccc} \pi^1(X,G) & \xrightarrow{\bar{\rho}(X,\bar{x})} & \mathrm{Rep}(\pi_1(X,\bar{x}),G) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \cong} \\ \pi^1(X,G) & \xrightarrow[\bar{\rho}(X,\bar{x}')]{} & \mathrm{Rep}(\pi_1(X,\bar{x}'),G) \end{array}$$

is commutative where the right vertical arrow is induced by any canonical isomorphism

$$\pi_1(X,\bar{x}') \xrightarrow{\cong} \pi_1(X,\bar{x}).$$

We also have the following result as a simple application of Lemma 3.3.

Lemma 3.6. *Let* $f : Y \to X$ *be a morphism of connected schemes. Then the diagram*:

$$\begin{array}{ccc} \pi^1(X,G) & \xrightarrow{\bar{\rho}(X,f(\bar{y}))} & \mathrm{Rep}(\pi_1(X,f(\bar{y})),G) \\ \downarrow & & \downarrow{\scriptstyle \circ f_*} \\ \pi^1(Y,G) & \xrightarrow[\bar{\rho}(Y,\bar{y})]{} & \mathrm{Rep}(\pi_1(Y,\bar{y}),G) \end{array}$$

is commutative where the left vertical arrow is given by $[Z] \mapsto [Z \times_X Y]$ $(\forall[Z] \in \pi^1(X,G))$ *and the right vertical arrow is induced by the canonical homomorphism*

$$f_* : \pi_1(Y,\bar{y}) \longrightarrow \pi_1(X,f(\bar{y})).$$

Now we will apply the above general theory to elliptic curves with naive level $|G|$ structure. Let n be the order of G and T be a connected $\mathbf{Z}[1/n]$-scheme. Let E be an elliptic curve over T with a naive level n structure

$$\alpha : (\mathbf{Z}/n \times \mathbf{Z}/n)_T \xrightarrow{\cong} E[n],$$

where $E[n]$ denotes the kernel of multiplication by n map $[n] : E \to E$. Then we have a standard $\mathbf{Z}/n \times \mathbf{Z}/n$-covering of E, which plays a fundametal role in the proof of Theorem 3.1, associated with α as follows. Let $E(n)$ denote the E-scheme $[n] : E \to E$ which is finite etale surjective morphism. There is a canonical action of $\mathbf{Z}/n \times \mathbf{Z}/n$ on $E(n)$ over E defined by the composition

$$E(n) \times_T (\mathbf{Z}/n \times \mathbf{Z}/n)_T \xrightarrow{1_{E(n)} \times \alpha} E(n) \times_T E[n] \longrightarrow E(n),$$

where the second morphism is induced by the group scheme structure on E. Then $E(n)$ with this $\mathbf{Z}/n \times \mathbf{Z}/n$-action is easily seen to be Galois with Galois group $\mathbf{Z}/n \times \mathbf{Z}/n$. We will denote the resulting $\mathbf{Z}/n \times \mathbf{Z}/n$-covering of E by $E(\alpha)$.

Next we will construct a canonical map

$$\theta(E, \alpha) : \pi^1(E, G) \longrightarrow C_2(G)$$

for the above E with α. Let $\bar{t} \to T$ be a geometric point. Then according to Appendix B we have an exact sequence of abelian groups

$$0 \longrightarrow C(n) \longrightarrow \pi_1(E_{\bar{t}}, i(\bar{t})) \xrightarrow{\rho(\alpha_{\bar{t}})} \mathbf{Z}/n \times \mathbf{Z}/n \longrightarrow 0$$

such that the image of $C(n)$ is contained in $n\pi_1(E_{\bar{t}}, i(\bar{t}))$. Here $j : E_{\bar{t}} \to E$ is the geometric fiber on $\bar{t}$ which is an elliptic curve over $\bar{t}$ with section i and naive level n structure $\alpha_{\bar{t}}$ induced by those of E and $\rho(\alpha_{\bar{t}}) = \rho(E_{\bar{t}}, i(\bar{t}), E_{\bar{t}}(\alpha_{\bar{t}}))$. Hence we have a bijection

$$\circ\rho(\alpha_{\bar{t}}) : C_2(G) = \mathrm{Rep}(\mathbf{Z}/n \times \mathbf{Z}/n, G) \xrightarrow{\cong} \mathrm{Rep}(\pi_1(E_{\bar{t}}, i(\bar{t})), G).$$

Now we define

$$\theta(E, \alpha) : \pi^1(E, G) \longrightarrow C_2(G)$$

by the composition

$$\pi^1(E, G) \xrightarrow{\bar{\rho}} \mathrm{Rep}(\pi_1(E, i(\bar{t})), G) \xrightarrow{\circ j_*} \mathrm{Rep}(\pi_1(E_{\bar{t}}, i(\bar{t})), G) \xrightarrow{\theta} C_2(G).$$

Here $\bar{\rho} = \bar{\rho}(E, i(\bar{t}))$ and $\theta = (\circ\rho(\alpha_{\bar{t}}))^{-1} = (\circ(\rho(E, i(\bar{t}), E(\alpha))j_*))^{-1}$. (In the defining equation of θ the second equality holds by Lemma 3.6.) We can show that this $\theta(E, \alpha)$ is independent on the choice of a geometric point $\bar{t}$ by routine diagram chasing with the aid of Lemmas 3.5 and 3.6 and Theorem 3.4. (We should remark that we can apply Theorem 3.4 to any elliptic curve over connected base (see [6, Corollary 1.9.12]).) By the above construction $\theta(E, \alpha)$ is bijective if $T = \mathrm{Spec}K$ with K algebraically closed field (of characteristic 0 or prime to n) and natural with respect to arbitrary base change $T' \to T$ with T' connected. Summarizing, we have shown that

Proposition 3.7. *Let T be a connected $\mathbf{Z}[1/|G|]$-scheme and E be an elliptic curve over T with naive level $|G|$ structure α. Then there is a canonical map*

$$\theta(E, \alpha) : \pi^1(E, G) \longrightarrow C_2(G)$$

such that:

(1) *It is bijective if* $T = \mathrm{Spec}K$ *with* K *algebraically closed field* (*of characteristic* 0 *or prime to* $|G|$).

(2) *It is natural with respect to arbitrary base change* $T' \to T$ *with* T' *connected.*

Now we will prove Theorem 3.1.

Proof of Theorem 3.1. Let $\tilde{M}_G = \mathrm{Spec}R^*(\tilde{E}_{\mathrm{univ}}, G)$ with $M(n)$-scheme structure obtained by regarding $R^*(n)$ as a subring of $R^*(\tilde{E}_{\mathrm{univ}}, G)$ consisting of all constant maps from $C_2(G)$ to $R^*(n)$. (Recall that n denotes the order of G.) Let $\tilde{\pi}^1$ (resp. $\tilde{h}$) denote the functor $\pi^1_{\tilde{E}_{\mathrm{univ}},G}$ (resp. $(\mathrm{Sch}/M(n))(?, \tilde{M}_G)$). Then it is clear that there is a unique natural transformation on $(\mathrm{Sch}/M(n))$

$$\tilde{\phi}(?) : \tilde{\pi}^1(?) \longrightarrow \tilde{h}(?)$$

such that, for any connected $M(n)$-scheme T, $\tilde{\phi}(T)$ coincides with

$$\tilde{\pi}^1(T) = \pi^1((\tilde{E}_{\mathrm{univ}})_T, G) \xrightarrow{\theta((\tilde{E}_{\mathrm{univ}})_T, (\alpha_{\mathrm{univ}})_T)} C_2(G) = \tilde{h}(T).$$

This natural transformation clearly satifies the second proprty in Definition 2.1. To prove that this $\tilde{\phi}$ satisfies the first property we first construct a right inverse of $\tilde{\phi}$. It is easy to see that there is a unique natural transformation

$$\tilde{\imath}(?) : \tilde{h}(?) \longrightarrow \tilde{\pi}^1(?)$$

such that $\tilde{\imath}(T)$ coincides, for any connected $M(n)$-scheme T with any geometric point $\bar{t} \to T$, with the composition

$$\tilde{h}(T) = C_2(G) = \mathrm{Rep}(\mathbf{Z}/n \times \mathbf{Z}/n, G) \xrightarrow{\circ\rho((\alpha_{\mathrm{univ}})_T)}$$

$$\mathrm{Rep}(\pi_1((\tilde{E}_{\mathrm{univ}})_T, i(\bar{t})), G) \xrightarrow{(\bar{\rho}_T)^{-1}} \pi^1((\tilde{E}_{\mathrm{univ}})_T, G) = \tilde{\pi}^1(T),$$

where $\rho((\alpha_{\mathrm{univ}})_T) = \rho((\tilde{E}_{\mathrm{univ}})_T, i(\bar{t}), (\tilde{E}_{\mathrm{univ}})_T((\alpha_{\mathrm{univ}})_T))$ and $\bar{\rho}_T = \bar{\rho}((\tilde{E}_{\mathrm{univ}})_T, i(\bar{t}))$. It is clear that this $\tilde{\imath}$ gives a right inverse of $\tilde{\phi}$. Now consider any $M(n)$-scheme N together with a natural transformation $\tilde{\psi} : \tilde{\pi}^1 \to \tilde{h}_N = (\mathrm{Sch}/M(n))(?, N)$. Let

$$\tilde{\chi} : \tilde{M}_G \longrightarrow N$$

be a unique $M(n)$-morphism such that

$$\tilde{\chi}\circ = \tilde{\psi}\tilde{\imath} : \tilde{h} \longrightarrow \tilde{h}_N.$$

For any connected $M(n)$-scheme T and any $[X] \in \tilde{\pi}^1(T)$ there is a finite etale surjective $M(n)$-morphism $f : T' \to T$ such that

$$\tilde{\pi}^1(f)([X]) = \tilde{\pi}^1(f)((\tilde{\imath}(T)\tilde{\phi}(T))([X]))$$

by the proposition below. Thus we have

$$\begin{aligned}(\tilde{\psi}(T)([X]))f &= (\tilde{\psi}(T')\tilde{\pi}^1(f))([X]) \\ &= (\tilde{\psi}(T')\tilde{\pi}^1(f)\tilde{\iota}(T)\tilde{\phi}(T))([X]) \\ &= ((\tilde{\psi}(T)\tilde{\iota}(T)\tilde{\phi}(T))([X]))f \\ &= (((\tilde{\chi}\circ)(T)\tilde{\phi}(T))([X]))f.\end{aligned}$$

Since

$$\circ f : (\mathrm{Sch}/M(n))(T, N) \longrightarrow (\mathrm{Sch}/M(n))(T', N)$$

is injective (see [19, 3.4.2.1] and [17, I 2.17]) we have

$$\tilde{\psi}(T)([X]) = ((\tilde{\chi}\circ)(T)\tilde{\phi}(T))([X]).$$

Therefore $\tilde{\psi}(T) = (\tilde{\chi}\circ)(T)\tilde{\phi}(T)$ for every connected $M(n)$-scheme T and hence $\tilde{\psi} = (\tilde{\chi}\circ)\tilde{\phi}$ on $(\mathrm{Sch}/M(n))$. The uniqueness of such a $\tilde{\chi}$ is trivial since we have a right inverse $\tilde{\iota}$ of $\tilde{\phi}$. This completes the proof of Theorem 3.1 assuming the following proposition. □

Proposition 3.8. *For any connected $M(n)$-scheme T and any $[X] \in \tilde{\pi}^1(T)$ there is a finite etale (necessarily) surjective $M(n)$-morphism $f : T' \to T$ with T' connected such that*

$$\tilde{\pi}^1(f)([X]) = \tilde{\pi}^1(f)((\tilde{\iota}(T)\tilde{\phi}(T))([X])).$$

Proof. By the argument preceding to Proposition 3.7 and the definition of $\tilde{\phi}$ we have

$$(\circ j_*)(\bar{\rho}_T([X])) = ((\circ j_*)(\circ\rho((\alpha_{\mathrm{univ}})_T)))(\tilde{\phi}(T)([X])),$$

where $j : (\tilde{E}_{\mathrm{univ}})_{\bar{t}} \to (\tilde{E}_{\mathrm{univ}})_T$ is the geometric fiber on a fixed geometric point $\bar{t}$ of T.

On the other hand there is a finite etale (necessarily) surjective $M(n)$-morphism $f : T' \to T$ with T' connected such that

$$(\circ i_* f_*)(\bar{\rho}_T([X])) = (\circ i_* f_*)((\circ\rho((\alpha_{\mathrm{univ}})_T))(\tilde{\phi}(T)([X]))) = \text{the trivial class}$$

since any two G-coverings of T can be simultaneously trivialized by some finite etale surjective base change.

Now applying Theorem 3.4 to $(\tilde{E}_{\mathrm{univ}})_T \to T$ we can easily prove that

$$(\circ\tilde{f}_*)(\bar{\rho}_T([X])) = (\circ\tilde{f}_*)((\circ\rho((\alpha_{\mathrm{univ}})_T))(\tilde{\phi}(T)([X])))$$

by the above two equalities, where $\tilde{f} : (\tilde{E}_{\mathrm{univ}})_{T'} \to (\tilde{E}_{\mathrm{univ}})_T$ is the canonical projection. Therefore we have

$$\tilde{\pi}^1(f)([X]) = \tilde{\pi}^1(f)((\tilde{\iota}(T)\tilde{\phi}(T))([X]))$$

as desired. □

4. $GL(2)$-action on $\Gamma(|G|)^{\text{naive}}$-modular forms and the proof of Theorems 2.3 and 2.7

In this section we will prove Theorems 2.3 and 2.7. First note that we have a canonical left $GL_2(\mathbf{Z}/n)$-action on $R^*(n)$ defined by

$$(A, f) \mapsto f(\tilde{E}_{\text{univ}}, \tilde{\omega}_{\text{univ}}, A^{-1}\alpha_{\text{univ}}) \ (\forall A \in GL_2(\mathbf{Z}/n), \forall f \in R^*(n)),$$

where $A^{-1}\alpha_{\text{univ}}$ is a naive level n structure on $\tilde{E}_{\text{univ}}$ given by the composition

$$(\mathbf{Z}/n \times \mathbf{Z}/n)_{M(n)} \xrightarrow{A^{-1}} (\mathbf{Z}/n \times \mathbf{Z}/n)_{M(n)} \xrightarrow{\alpha_{\text{univ}}} \tilde{E}_{\text{univ}}[n].$$

(Here n denotes the order of G as before.) This action induces a canonical right action on $M(n)$. We also have a canonical injection

$$R^*(1) \longrightarrow R^*(n) \ (f \mapsto f(\tilde{E}_{\text{univ}}, \tilde{\omega}_{\text{univ}})).$$

Hence we have a canonical morphism

$$M(n) \longrightarrow M(1)$$

and we can prove that this morphism is Galois with Galois group $GL_2(\mathbf{Z}/n)$ acting on $M(n)$ as defined above (cf. [12, 2.3.1 and 4.6] and [6, Section 2.6.2]). Now we also get an induced action of $GL_2(\mathbf{Z}/n)$ on $\tilde{M}_G = M(\tilde{E}_{\text{univ}}, G)$ given, on the coordinate ring, by

$$(A, f) \mapsto AfA^{-1} \ (\forall f \in R^*(\tilde{E}_{\text{univ}}, G)).$$

Let M_G be a quotient of $\tilde{M}_G$ by this action; explicitly given by $M_G = \mathrm{Spec}R^*(E_{\text{univ}}, G)$. Then M_G has a unique $M(1)$-scheme structure such that the diagram:

$$\begin{array}{ccc} \tilde{M}_G & \longrightarrow & M_G \\ \downarrow & & \downarrow \\ M(n) & \longrightarrow & M(1) \end{array}$$

is commutative.

Now we prove Theorem 2.3.

Proof of Theorem 2.3. Let π^1 denote the functor

$$\pi^1_{E_{\text{univ}}, G} : (\mathrm{Sch}/M(1)) \longrightarrow (\mathrm{Sets}).$$

For every $M(1)$-scheme T let $\tilde{T} = T \times_{M(1)} M(n)$ which is the $GL_2(\mathbf{Z}/n)$-covering of T obtained from $M(n) \to M(1)$ by the base change $T \to M(1)$.

(To distinguish $M_G \times_{M(1)} M(n)$ from $\tilde{M}_G = M(\tilde{E}_{\mathrm{univ}}, G)$ we denote the former by $(M_G)^\sim$.) Then we have a natural map

$$\times_{M(1)} M(n) : \pi^1(T) \longrightarrow \pi^1(\tilde{T}) = \tilde{\pi}^1(\tilde{T}) \ ([X] \mapsto [X \times_{M(1)} M(n)]),$$

where $\tilde{\pi}^1$ is as in Section 3. Let

$$\phi'(T) : \pi^1(T) \longrightarrow \tilde{h}(\tilde{T})$$

be the composition

$$\pi^1(T) \xrightarrow{\times_{M(1)} M(n)} \pi^1(\tilde{T}) = \tilde{\pi}^1(\tilde{T}) \xrightarrow{\tilde{\phi}(\tilde{T})} \tilde{h}(\tilde{T}),$$

where $\tilde{\phi}$ and $\tilde{h}$ are as in Section 3. To prove that every element of $\mathrm{Im}\phi'(T)$ is $GL_2(\mathbf{Z}/n)$-equivariant we need to describe the $GL_2(\mathbf{Z}/n)$-action on $\tilde{M}_G$ in terms of G-coverings. For every $M(n)$-scheme T and every $A \in GL_2(\mathbf{Z}/n)$ we denote by T^A the $M(n)$-scheme $T \longrightarrow M(n) \xrightarrow{A} M(n)$. For every $[X] \in \tilde{\pi}^1(T)$ let $[X]^A \in \tilde{\pi}^1(T^A)$ denote the isomorphism class of the same G-covering $X \to (\tilde{E}_{\mathrm{univ}})_T$ regarded as one of $(\tilde{E}_{\mathrm{univ}})_{T^A}$. (Note that $(\tilde{E}_{\mathrm{univ}})_{T^A}$ is the same as $(\tilde{E}_{\mathrm{univ}})_T$ when we disregard $M(n)$-scheme structure.) Then we have

Lemma 4.1. *For every $M(n)$-scheme T and $A \in GL_2(\mathbf{Z}/n)$ the diagram:*

$$\begin{array}{ccc}
\tilde{\pi}^1(T) & \xrightarrow{\tilde{\phi}(T)} & \tilde{h}(T) \\
{\scriptstyle [\ ]^A}\downarrow & & \downarrow{\scriptstyle A\circ} \\
\tilde{\pi}^1(T^A) & \xrightarrow{\tilde{\phi}(T^A)} & \tilde{h}(T^A)
\end{array}$$

is commutative where the right vertical arrow is induced by the action of A on $\tilde{M}_G$.

Proof. Easy by the construction of $\tilde{\phi}$, or $\theta(E, \alpha)$ in Proposition 3.7, and the fact that for every $M(n)$-scheme T and every $A \in GL_2(\mathbf{Z}/n)$ the elliptic curve $(\tilde{E}_{\mathrm{univ}})_{T^A}$ with naive level n structure $(\alpha_{\mathrm{univ}})_{T^A}$ is nothing but the elliptic curve $(\tilde{E}_{\mathrm{univ}})_T$ with naive level n structure $(A^{-1}\alpha_{\mathrm{univ}})_T$ when we disregard $M(n)$-scheme structure. □

Now it is easily seen that for any $M(1)$-scheme T every element of $\mathrm{Im}\phi'(T)$ is $GL_2(\mathbf{Z}/n)$-equivariant by using the above lemma and the fact that for every $[X] \in \pi^1(T)$ and $A \in GL_2(\mathbf{Z}/n)$, $[X \times_{M(1)} M(n)]^A \in \tilde{\pi}^1(\tilde{T}^A)$ is obtained from $[X \times_{M(1)} M(n)] \in \tilde{\pi}^1(\tilde{T})$ by the base change $1_T \times A : \tilde{T}^A \to \tilde{T}$. Hence we can consider $\phi'(T)$ as a map

$$\phi'(T) : \pi^1(T) \longrightarrow \tilde{h}(\tilde{T})^{GL_2(\mathbf{Z}/n)},$$

where $\tilde{h}(\tilde{T})^{GL_2(\mathbf{Z}/n)}$ denotes the subset of $\tilde{h}(\tilde{T})$ consisting of all $GL_2(\mathbf{Z}/n)$-equivariant morphisms. Therefore we have a natural transformation on $(\mathrm{Sch}/M(1))$

$$\phi(?) : \pi^1(?) \longrightarrow h(?) = (\mathrm{Sch}/M(1))(?, M_G)$$

defined by the composition

$$\pi^1(T) \xrightarrow{\phi'(T)} \tilde{h}(\tilde{T})^{GL_2(\mathbf{Z}/n)} \xrightarrow{\text{divide by } GL_2(\mathbf{Z}/n)} h(T),$$

where the second map sends every $\tilde{f} \in \tilde{h}(\tilde{T})^{GL_2(\mathbf{Z}/n)}$ to a unique element $f \in h(T)$ such that the diagram:

$$\begin{tikzcd} \tilde{T} \arrow[r, "\tilde{f}"] \arrow[d] & \tilde{M}_G \arrow[d] \\ T \arrow[r, "f"'] & M_G \end{tikzcd}$$

is commutative. It is easy to prove that this ϕ satisfies the second property in Definition 2.1 since we can easily show that every geometric point $\bar{x} \to M(1)$ can be factored into $\bar{x} \to M(n) \to M(1)$ and that the map

$$\tilde{h}(\bar{x}) \longrightarrow h(\bar{x})$$

induced by $\tilde{M}_G \to M_G$ is bijective. To prove that ϕ also satisfies the first property let N be any $M(1)$-scheme together with a natural transformation $\psi : \pi^1 \to h_N = (\mathrm{Sch}/M(1))(?, N)$. Then we have a natural transformation on $(\mathrm{Sch}/M(n))$

$$\tilde{\psi}(?) : \tilde{\pi}^1(?) \longrightarrow \tilde{h}_{\tilde{N}}(?) = (\mathrm{Sch}/M(n))(?, \tilde{N})$$

defined by the composition

$$\tilde{\psi}(T) : \tilde{\pi}^1(T) = \pi^1(T) \xrightarrow{\psi(T)} h_N(T) \xrightarrow{\times_{M(1)} M(n)} \tilde{h}_{\tilde{N}}(\tilde{T}) \xrightarrow{\circ e_T} \tilde{h}_{\tilde{N}}(T).$$

Here the second map sends every $f \in h_N(T)$ to $f \times 1_{M(n)} \in \tilde{h}_{\tilde{N}}(\tilde{T})$ and the third map is induced by a canonical section e_T of $\tilde{T} \to T$ obtained from the diagonal morphism $M(n) \to M(n) \times_{M(1)} M(n)$. Thus we have a unique $M(n)$-morphism $\tilde{\chi} : \tilde{M}_G \to \tilde{N}$ such that $\tilde{\psi}(T) = (\tilde{\chi}\circ)\tilde{\phi}(T)$ for every $M(n)$-scheme T. It is not difficult to prove that for every $A \in GL_2(\mathbf{Z}/n)$, $A\tilde{\chi}A^{-1}$ also satisfies the above equality by using Lemma 4.1 and the definition of $\tilde{\psi}$. Hence $\tilde{\chi}$ is $GL_2(\mathbf{Z}/n)$-equivariant by using the uniqueness of $\tilde{\chi}$ satisfying the above equality. Thus, passing to quotient, we get an $M(1)$-morphism $\chi : M_G \to N$ and the equality $\psi(T) = (\chi\circ)\phi(T)$ for every $M(1)$-scheme T obtained from $\tilde{\chi}$ and the equality

$$\tilde{\psi}(\tilde{T})(\times_{M(1)} M(n)) = (\tilde{\chi}\circ)\tilde{\phi}(\tilde{T})(\times_{M(1)} M(n)) = (\tilde{\chi}\circ)\phi'(T)$$

respectively. (Note that we have

$$\psi(\tilde{T})(\times_{M(1)} M(n)) = (p_N\circ)(\times_{M(1)} M(n))\psi(T),$$

where $p_N : \tilde{N} \to N$ is the canonical projection.)

Conversely from such χ we get $\tilde{\chi} : \tilde{M}_G \to \tilde{N}$ as the composition

$$\tilde{M}_G \longrightarrow (M_G)^{\sim} \xrightarrow{\chi \times 1_{M(n)}} \tilde{N}$$

which can easily be shown to satisfy $\tilde{\psi}(T) = (\tilde{\chi}\circ)\tilde{\phi}(T)$ for every $M(n)$-scheme T. Here the first morphism is the canonical one obtained from $\tilde{M}_G \to M_G$ and $\tilde{M}_G \to M(n)$ by the universality of pull-back. Therefore the uniqness of such a χ is guaranteed by the uniqness of $\tilde{\chi}$. This completes the proof of Theorem 2.3. □

Next we will prove Theorem 2.7.

Proof of Theorem 2.7. Let $R^*_{\mathrm{arith}}(n) = R^*(\Gamma(n)^{\mathrm{arith}})[1/n, \zeta]$ be the graded ring of $\Gamma(n)^{\mathrm{arith}}$-modular forms over $\mathbf{Z}[1/6n, \zeta]$ [10, Chapter II] (see Appendix A). Here ζ denotes a primitive n-th root of unity $e^{2\pi i/n}$. Let $\bar{E}_{\mathrm{univ}}$ be the elliptic curve $E_{\mathrm{univ}} \times_{M(1)} M(\Gamma(n)^{\mathrm{arith}})[1/n, \zeta]$ over $M(\Gamma(n)^{\mathrm{arith}})[1/n, \zeta] = \mathrm{Spec} R^*_{\mathrm{arith}}(n)$ with nowhere vanishing invariant differential $\bar{\omega}_{\mathrm{univ}}$ obtained from ω_{univ} and arithmetic level n structure β_{univ}. (Here the $M(1)$-scheme structure on $M(\Gamma(n)^{\mathrm{arith}})[1/n, \zeta]$ is a standard one (cf. the begining of this section).) Now for any elliptic curve E over any $\mathbf{Z}[1/6n, \zeta]$-algebra B we may identify arithmetic level n structures on E with naive level n structures on E of determinant $\zeta \cdot 1 \in \mu_n^{\mathrm{prim}}(B)$ [10, 2.0.8]. Hence we have a unique naive level n structure $\bar{\alpha}_{\mathrm{univ}}$ on $\bar{E}_{\mathrm{univ}}$ corresponding to β_{univ}. We also have a unique arithmetic level n structure $\tilde{\beta}_{\mathrm{univ}}$ on $\tilde{E}_{\mathrm{univ}}$ corresponding to α_{univ} since we can make $R^*(n)$ into a $\mathbf{Z}[1/6n, \zeta]$-algebra via the ring homomorphism

$$\mathbf{Z}\left[\frac{1}{6n}, \zeta\right] \longrightarrow R^*(n) \ (\zeta \mapsto \det \alpha_{\mathrm{univ}}).$$

Therefore we have an isomorphism of graded $\mathbf{Z}[1/6n, \zeta]$-algebras

$$R^*(n) \xrightarrow{\cong} R^*_{\mathrm{arith}}(n) \ (f \mapsto f^{\mathrm{arith}})$$

defined by

$$f^{\mathrm{arith}} = f(\bar{E}_{\mathrm{univ}}, \bar{\omega}_{\mathrm{univ}}, \bar{\alpha}_{\mathrm{univ}})$$

whose inverse is

$$R^*_{\mathrm{arith}}(n) \xrightarrow{\cong} R^*(n) \ (f \mapsto f^{\mathrm{naive}})$$

defined by

$$f^{\mathrm{naive}} = f(\tilde{E}_{\mathrm{univ}}, \tilde{\omega}_{\mathrm{univ}}, \tilde{\beta}_{\mathrm{univ}}).$$

Next we investigate the behavior of the $GL_2(\mathbf{Z}/n)$-action on $R^*(n)$ under the above isomorphism. For every $A \in SL_2(\mathbf{Z}/n)$ we denote by $A\beta_{\mathrm{univ}}$ the arithmetic level n structure on $\bar{E}_{\mathrm{univ}}$ corresponding to $A\bar{\alpha}_{\mathrm{univ}}$. Then we have a canonical $SL_2(\mathbf{Z}/n)$-action on $R^*_{\mathrm{arith}}(n)$ given by

$$(A, f) \mapsto f(\bar{E}_{\mathrm{univ}}, \bar{\omega}_{\mathrm{univ}}, A^{-1}\beta_{\mathrm{univ}}) \ (\forall A \in SL_2(\mathbf{Z}/n), \forall f \in R^*_{\mathrm{arith}}(n)).$$

For every $\mathbf{Z}[1/6n, \zeta]$-algebra B and $\sigma \in (\mathbf{Z}/n)^\times$ we denote by B^σ the $\mathbf{Z}[1/6n, \zeta]$-algebra whose underlying $\mathbf{Z}[1/6n]$-algebra is B and the action of $\mathbf{Z}[1/6n, \zeta]$ on it is given by

$$(\zeta, b) \mapsto \zeta^\sigma \cdot b \ (\forall b \in B^\sigma = B).$$

Then for any $\Gamma(n)^{\mathrm{arith}}$-test object (E, ω, β) over B we denote by $(E, \omega, \beta)^\sigma$ the same $\Gamma(n)^{\mathrm{arith}}$-test object regarded as one over B^σ. Then we have a canonical $(\mathbf{Z}/n)^\times$-action on $R^*_{\mathrm{arith}}(n)$ (acting as $\mathbf{Z}[1/6n]$-algebra automorphisms) given by

$$(\sigma, f) \mapsto f((\bar{E}_{\mathrm{univ}}, \bar{\omega}_{\mathrm{univ}}, \beta_{\mathrm{univ}})^\sigma) \ (\forall \sigma \in (\mathbf{Z}/n)^\times, \forall f \in R^*_{\mathrm{arith}}(n)).$$

Now we have

Lemma 4.2. *For every $A \in SL_2(\mathbf{Z}/n)$, $\sigma \in (\mathbf{Z}/n)^\times$ and $f \in R^*(n)$ we have*

$$(Af)^{\mathrm{arith}} = A(f^{\mathrm{arith}})$$

and

$$(A_\sigma f)^{\mathrm{arith}} = \sigma(f^{\mathrm{arith}}),$$

where $A_\sigma = \left(\begin{smallmatrix} \sigma & 0 \\ 0 & 1 \end{smallmatrix}\right) \in GL_2(\mathbf{Z}/n)$.

Proof. Clear by definitions. □

We also have

Lemma 4.3. *For every $f \in R^*_{\mathrm{arith}}(n)$ let $f(q) = \sum_{j \gg -\infty} f_j q^j$ denote the q-expansion $f(\mathrm{Tate}(q^n), \omega_{\mathrm{can}}, \beta_{\mathrm{can}}) \in \mathbf{Z}[1/6n, \zeta]((q))$ of f. Then we have*

$$(\sigma f)(q) = \sum_{j \gg -\infty} (\sigma f_j) q^j,$$

where $\sigma \in (\mathbf{Z}/n)^\times$ $(= \mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q}))$ acts on $\mathbf{Z}[1/6n, \zeta]$ by the standard Galois action.

Proof. Clear by using the fact that the $\Gamma(n)^{\mathrm{arith}}$-test object $(\mathrm{Tate}(q^n), \omega_{\mathrm{can}}, \beta_{\mathrm{can}})^\sigma$ is obtained from $(\mathrm{Tate}(q^n), \omega_{\mathrm{can}}, \beta_{\mathrm{can}})$ by the extension of scalars

$$\mathbf{Z}\left[\frac{1}{6n},\zeta\right]((q)) \longrightarrow \left(\mathbf{Z}\left[\frac{1}{6n},\zeta\right]((q))\right)^{\sigma} \left(\sum_j g_j q^j \mapsto \sum_j (\sigma g_j) q^j\right).$$

□

Now by using a GAGA-type result and q-expansion principle [10, 2.4 and 2.2.8] (cf. [9, 1.6, A1.1 and A1.2] and [6, Sections 2.4 and 3.1.1]) with the aid of Lemmas 4.2 and 4.3, it is not difficult to prove that the map

$$R^*(E_{\mathrm{univ}}, G) \longrightarrow F(\mathcal{H}, G)\ (f \mapsto f^{\mathrm{an}})$$

defined by

$$f^{\mathrm{an}}(\theta,\tau) = f([\theta^{\mathrm{op}}])^{\mathrm{arith}}(\mathbf{C}/\mathbf{Z}+\mathbf{Z}\tau, dz, \beta_\tau)\ (\forall\theta \in \hat{C}_2(G), \forall\tau \in \mathcal{H})$$

induces an isomorphism of graded rings

$$R^*(E_{\mathrm{univ}}, G) \xrightarrow{\cong} R^*_{\mathrm{an}}(E_{\mathrm{univ}}, G),$$

where

$$\beta_\tau : \mu_n(\mathbf{C}) \times \mathbf{Z}/n \xrightarrow{\cong} \frac{1}{n}(\mathbf{Z}+\mathbf{Z}\tau)/\mathbf{Z}+\mathbf{Z}\tau \left((\zeta^l, m) \mapsto \frac{l+m\tau}{n}\right)$$

and $[\theta^{\mathrm{op}}] \in C_2(G)$ denotes the equivalence class of $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\theta \in \hat{C}_2(G)$. (Note that $GL_2(\mathbf{Z}/n)$ is generated by elements A_σ ($\sigma \in (\mathbf{Z}/n)^\times$) and elements in $SL_2(\mathbf{Z}/n)$.) This completes the proof of Theorem 2.7. □

Appendix A. Review of $\Gamma(n)^{\mathrm{naive}}$ and $\Gamma(n)^{\mathrm{arith}}$-moduli problems over $\mathbf{Z}[1/6]$

In this appendix we will give a brief account of $\Gamma(n)^{\mathrm{naive}}$ and $\Gamma(n)^{\mathrm{arith}}$-modular forms in the sense of Katz [10, Chapter II]. Our main references are [10, Chapter II], [12] and [6, Chapter I–III]; particularly Hida's recent book [6] contains most necessary information about scheme theory. For simplicity we exclude characteristics 2 and 3 which does not matter in this note.

Let E be an elliptic curve over a (not necessarily locally noetherian) scheme S. For a positive integer n let $E[n]$ denote the kernel of multiplication by n map on E:

$$[n] : E \longrightarrow E.$$

Then a naive level n structure (or $\Gamma(n)^{\mathrm{naive}}$-structure) on E is an isomorphism of group schemes over S:

$$\alpha : (\mathbf{Z}/n \times \mathbf{Z}/n)_S \xrightarrow{\cong} E[n].$$

The existence of such an α implies that S is a $\mathbf{Z}[1/n]$-scheme and conversely if S is a $\mathbf{Z}[1/n]$-scheme then such an α always exists after some finite etale surjective base change (see [12, 2.3] and [6, Section 2.6.1]). Similarly an arithmetic level n structure (or $\Gamma(n)^{\text{arith}}$-structure) on E is an isomorphism of group schemes over S:

$$\beta : (\mu_n \times \mathbf{Z}/n)_S \xrightarrow{\cong} E[n]$$

under which e_n-paring on $E[n]$ (see [12, 2.8] and [6, Sections 2.6.3 and 2.6.4]) becomes the standard paring $\langle\ ,\ \rangle_{\text{std}}$ on $(\mu_n \times \mathbf{Z}/n)_S$ defined by the formula

$$\langle(\zeta_1, m_1), (\zeta_2, m_2)\rangle_{\text{std}} = \zeta_1^{m_2}/\zeta_2^{m_1}.$$

These two level structures on elliptic curves are closely related when the base scheme is a $\mathbf{Z}[1/n]$-scheme [10, 2.0.8].

For arbitrary scheme S a $\Gamma(n)^{\text{naive}}$-test object over S is a triple (E, ω, α) consisting of an elliptic curve E over S, a nowhere vanishing invariant differential ω on E and a $\Gamma(n)^{\text{naive}}$-structure α on E; particularly a $\Gamma(1)^{\text{naive}}$-test object is simply a pair (E, ω) which we call a $\Gamma(1)$-test object. Let $\mathcal{M}(\Gamma(n)^{\text{naive}})_S$ denote the functor from (Sch/S) to (Sets) defined by

$$\mathcal{M}(\Gamma(n)^{\text{naive}})_S(T) = \text{the set of isomorphism classes of } \Gamma(n)^{\text{naive}}\text{-test objects over } T.$$

We will denote $\mathcal{M}(\Gamma(1)^{\text{naive}})_S$ simply by $\mathcal{M}(1)_S$. Similarly we define the functor $\mathcal{M}(\Gamma(n)^{\text{arith}})_S$ for $\Gamma(n)^{\text{arith}}$. Then we have

Theorem A.1 ([10, 2.5]). *The functors $\mathcal{M}(\Gamma(n)^{\text{naive}})_{\mathbf{Z}[1/6n]}$ and $\mathcal{M}(\Gamma(n)^{\text{arith}})_{\mathbf{Z}[1/6]}$ are both representable by an affine $\mathbf{Z}[1/6n]$-scheme $M(\Gamma(n)^{\text{naive}})$ and an affine $\mathbf{Z}[1/6]$-scheme $M(\Gamma(n)^{\text{arith}})$ respectively.*

It is clear that for any $\mathbf{Z}[1/6n]$-scheme S, the scheme $M(\Gamma(n)^{\text{naive}})_S = M(\Gamma(n)^{\text{naive}}) \times_{\text{Spec}\mathbf{Z}[1/6n]} S$ represents the functor $\mathcal{M}(\Gamma(n)^{\text{naive}})_S$. Similarly for $\Gamma(n)^{\text{arith}}$.

Now we will give a proof of this result assuming the following result of representabilty for $n = 1$:

Theorem A.2 ([12, 2.2.6], [6, Theorem 2.2.3]). *The functor $\mathcal{M}(1)_{\mathbf{Z}[1/6]}$ is representable by an affine scheme $M(1) = \text{Spec}\mathbf{Z}[1/6][g_2, g_3, \Delta^{-1}]$ ($\Delta = g_2^3 - 27g_3^2$), with universal $\Gamma(1)$-test object $(E_{\text{univ}}, \omega_{\text{univ}}) = (y^2 = 4x^3 - g_2x - g_3, dx/y)$.*

First we prove the following lemma on isomorphisms of locally free group schemes. Let G and G' be locally free group schemes of rank m over S and we define the functor

$$\text{Isom}_{S\text{-grp}}(G, G')(?) : (\text{Sch}/S) \longrightarrow (\text{Sets})$$

by

$$\text{Isom}_{S\text{-grp}}(G, G')(T) = \text{the set of isomorphisms of } T\text{-group schemes: } G_T \xrightarrow{\cong} G'_T.$$

Lemma A.3. *The functor* $\mathrm{Isom}_{S\text{-grp}}(G, G')$ *is representable by a (possibly empty) S-scheme affine over S.*

Proof. The functor $\mathrm{Isom}_{S\text{-grp}}(G, G')$ is clearly local, i.e., determined by local data in Zariski topology, and hence we may assume that $S = \mathrm{Spec}R$, $G = \mathrm{Spec}A$ and $G' = \mathrm{Spec}A'$ where A and A' are commutative R-Hopf algebras whose underlying R-modules are free of rank m. Then it is sufficient to prove that the functor

$$\mathrm{Isom}_{R\text{-Hopf}}(A', A)(?) : (R\text{-alg}) \longrightarrow (\mathrm{Sets}),$$

defined by

$$\mathrm{Isom}_{R\text{-Hopf}}(A', A)(B) = \text{the set of isomorphisms of } B\text{-Hopf algebras:}$$
$$A' \otimes_R B \xrightarrow{\cong} A \otimes_R B,$$

is (co)representable by some R-algebra. By choosing R-module basis of A and A' we can identify $\mathrm{Isom}_{R\text{-Hopf}}(A', A)$ with a subfunctor of $GL_{m,R}(?) = \mathrm{Hom}_{R\text{-alg}}(R[x_{i,j}, \det(x_{i,j})^{-1} | 1 \le i, j \le m], ?)$ defined by zeros of suitable polynomials of $x_{i,j}$'s over R. Therefore $\mathrm{Isom}_R(A', A)$ is (co)representable by the quotient of $R[x_{i,j}, \det(x_{i,j})^{-1} | 1 \le i, j \le m]$ by the ideal generated by those polynomials. $\square$

Proof of Theorem A.1. Since $(\mathbf{Z}/n \times \mathbf{Z}/n)_{M(1)[1/n]}$ and $(E_{\mathrm{univ}}[1/n])[n]$ are both locally free of rank n^2 over $M(1)[1/n]$ the functor $\Gamma(n)^{\mathrm{naive}}\text{-}\mathrm{Str}_{E_{\mathrm{univ}}[1/n]}(?)$, defined by

$$\Gamma(n)^{\mathrm{naive}}\text{-}\mathrm{Str}_{E_{\mathrm{univ}}[\frac{1}{n}]}(T) = \text{the set of } \Gamma(n)^{\mathrm{naive}}\text{-structures on } \left(E_{\mathrm{univ}}\left[\frac{1}{n}\right]\right)_T$$

for any $M(1)[1/n]$-scheme T, is representable by an affine $M(1)[1/n]$-scheme by the lemma above. Then it is clear that $\mathcal{M}(\Gamma(n)^{\mathrm{naive}})_{\mathbf{Z}[1/6n]}$ is represented by the same scheme regarded as a $\mathbf{Z}[1/6n]$-scheme. Similarly for $\mathcal{M}(\Gamma(n)^{\mathrm{arith}})_{\mathbf{Z}[1/6]}$ because when we identify the functor $\Gamma(n)^{\mathrm{arith}}\text{-}\mathrm{Str}_{E_{\mathrm{univ}}}(?)$ locally with a subfunctor of $GL_{n^2, M(1)}$, the condition on the e_n-paring can be expressed by polynomial equations and thus this functor is representable by an affine $M(1)$-scheme. $\square$

Now we have a canonical action of multiplicative group $\mathbf{G}_{m,\mathbf{Z}[1/6n]}$ on $M(\Gamma(n)^{\mathrm{naive}})$ given by

$$(\lambda, (E, \omega, \alpha)) \mapsto (E, \lambda^{-1}\omega, \alpha).$$

This action yields a coaction

$$\psi : R(\Gamma(n)^{\mathrm{naive}}) \longrightarrow \mathbf{Z}\left[\frac{1}{6n}\right][t, t^{-1}] \otimes R(\Gamma(n)^{\mathrm{naive}})$$

of the Hopf algebra associated with the affine group scheme $\mathbf{G}_{m,\mathbf{Z}[1/6n]}$ on the coordinate ring $R(\Gamma(n)^{\mathrm{naive}})$ of $M(\Gamma(n)^{\mathrm{naive}})$ and hence we get a grade on $R(\Gamma(n)^{\mathrm{naive}})$ defined by

$$R^k(\Gamma(n)^{\mathrm{naive}}) = \{f \in R(\Gamma(n)^{\mathrm{naive}}) |\ \psi(f) = t^{-k} \otimes f\}.$$

For any $\mathbf{Z}[1/6n]$-algebra R, let $R^*(\Gamma(n)^{\mathrm{naive}})_R = R^*(\Gamma(n)^{\mathrm{naive}}) \otimes R$ which is, by definition, the graded ring of $\Gamma(n)^{\mathrm{naive}}$-modular forms over R. Then for any $\Gamma(n)^{\mathrm{naive}}$-test object (E, ω, α) over any R-algebra B we have a unique R-algebra homomorphism

$$R^*(\Gamma(n)^{\mathrm{naive}})_R \longrightarrow B$$

classifying (E, ω, α) and we denote the image of an element f of $R^*(\Gamma(n)^{\mathrm{naive}})_R$ under this homomorphism, the *value* of f on (E, ω, α), by $f(E, \omega, \alpha)$. An element $f \in R^*(\Gamma(n)^{\mathrm{naive}})_R$ has degree, or *weight*, k if and only if for any $\Gamma(n)^{\mathrm{naive}}$-test object (E, ω, α) over any R-algebra B we have

$$f(E, \lambda^{-1}\omega, \alpha) = \lambda^k f(E, \omega, \alpha)\ (\forall \lambda \in B^\times).$$

Similarly we define the graded ring $R^*(\Gamma(n)^{\mathrm{arith}})_R$ of $\Gamma(n)^{\mathrm{arith}}$-modular forms over any $\mathbf{Z}[1/6]$-algebra R. In particular the ring of $\Gamma(1)$-modular forms over $\mathbf{Z}[1/6]$ is given by $R^*(1) = \mathbf{Z}[1/6][g_2, g_3, \Delta^{-1}]$ $(\Delta = g_2^3 - 27g_3^2)$ with $\deg g_2 = 4$ and $\deg g_3 = 6$.

Remark A.4. Over any $\mathbf{Z}[1/n]$-scheme, $\Gamma(n)^{\mathrm{naive}}$-structure on an elliptic curve is the same as Drinfeld style $\Gamma(n)$-structure but over general base scheme $\Gamma(n)^{\mathrm{arith}}$-structure is slightly different. For example, over $\mathbf{F}_p$, all supersingular elliptic curves are automatically excluded in $\Gamma(p)^{\mathrm{arith}}$-moduli problem (cf. [6, Section 2.9]).

Appendix B. Etale fundamental groups of elliptic curves over algebraically closed fields

The purpose of this appendix is to make a well known computation of the etale fundamental group of an elliptic curve over an algebraically closed field (cf. [5, IV Exercise 4.8]). Let E be an elliptic curve over an algebraically closed field K. Let Π denote the profinite group $\lim_n E[n](K)$ where the inverse limit is taken over all positive integers with respect to the map

$$[m] : E[mn](K) \longrightarrow E[n](K).$$

Then we will show that this Π gives the etale fundamental group of E.

Let

$$f : E' \longrightarrow E$$

be a finite etale morphism of K-schemes with E' connected. Then E' is a smooth proper curve over K whose genus is 1 by Hurwitz's theorem [5, IV 2.4] and hence we may assume that f is a homomorphism of elliptic curves over K by choosing a suitable section $\mathrm{Spec}K \to E'$ of $E' \to \mathrm{Spec}K$. Now by the pull-back square:

$$\begin{array}{ccc} \mathrm{Ker}f & \longrightarrow & E' \\ \big\downarrow & & \big\downarrow f \\ \mathrm{Spec}K & \xrightarrow[\text{0-section}]{} & E, \end{array}$$

$\mathrm{Ker}f$ is finite etale over K and thus the constant group scheme $(\mathrm{Ker}f)(K)_K$ defined by the K-valued points. Let $f^t : E \to E'$ be the dual homomorphism of f. Then f^t induces a surjection of group schemes over K

$$f^t : E[\deg f] = \mathrm{Ker}(ff^t) \twoheadrightarrow \mathrm{Ker}f.$$

Therefore we have a continuous action of Π on $F(E') = (\mathrm{Sch}/E)(0_E, E') = (\mathrm{Ker}f)(K)$ via this surjection in obvious way, where 0_E denotes a geometric point of E given by the 0-section.

Conversely let $E[n] \twoheadrightarrow C$ be a surjection of group schemes over K with C constant for some positive integer n. Then by self-duality of $E[n]$ (see [12, 2.8] and [6, Sections 2.6.3 and 2.6.4]) we get an inclusion

$$C^* \hookrightarrow E[n]$$

where C^* denotes the Cartier dual of C. Let E' be the quotient of E by the finite subgroup scheme given by C^* via this inclusion and

$$f : E' \longrightarrow E$$

be the dual of the projection $E \to E'$. Then by Cartier-Nishi duality (see [loc. cit.]) $\mathrm{Ker}f$ is canonically isomorphic to $(C^*)^* = C$ and in particular etale over K. Therefore f is finite etale over K since $f : E' \to E$ is $\mathrm{Ker}f$-torsor. By these observations it is easy to see that the functor

$$F : (\mathrm{FEt}/E) \longrightarrow (\mathrm{Sets})\ (E' \mapsto (\mathrm{Sch}/E)(0_E, E'))$$

yields an equivalence of categories

$$F : (\mathrm{FEt}/E) \xrightarrow{\cong} (\Pi\text{-sets}).$$

Now it is well known that for any prime p and any positive integer $n = p^a q$ with $(p, q) = 1$,

$$E[n](K) = \begin{cases} \mathbf{Z}/n \times \mathbf{Z}/n & \text{if } \ \mathrm{char}K = 0, \\ \mathbf{Z}/p^a \times \mathbf{Z}/q \times \mathbf{Z}/q & \text{if } \ \mathrm{char}K = p \text{ and } E \text{ is ordinary}, \\ \mathbf{Z}/q \times \mathbf{Z}/q & \text{if } \ \mathrm{char}K = p \text{ and } E \text{ is supersingular}. \end{cases}$$

(See [25, III 6.4 and V 3.1].) Hence we have shown that

Theorem B.1.

$$\pi_1(E, 0_E) = \begin{cases} \hat{\mathbf{Z}} \times \hat{\mathbf{Z}} & \text{if } \operatorname{char} K = 0, \\ \mathbf{Z}_p \times \prod_{l \neq p} (\mathbf{Z}_l \times \mathbf{Z}_l) & \text{if } \operatorname{char} K = p \text{ and } E \text{ is ordinary}, \\ \prod_{l \neq p} (\mathbf{Z}_l \times \mathbf{Z}_l) & \text{if } \operatorname{char} K = p \text{ and } E \text{ is supersingular.} \end{cases}$$

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