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Area preserving monotone twist diffeomorphisms without non-Birkhoff periodic points

By

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Abstract

Let f be an area preserving monotone twist diffeomorphism on the annulus. In this paper, we prove the equivalence of the following three conditions: (i) the annulus is foliated by circles invariant under f. (ii) any periodic point of f is of Birkhoff type, and (iii) all iterations f^n are twist diffeomorphisms.

1. Introduction

Let $S^1 = \mathbf{R}/\mathbf{Z}$ be the circle and $A = S^1 \times [0,1]$ the closed annulus. Denote by S_a the level set $S^1 \times \{a\}$ for each $a \in [0,1]$. Let $\mathrm{Diff}^1_a(A)$ be the set of C^1 diffeomorphisms which preserve area, orientation and each component S_0 and S_1 of the boundary of A. For every point $z = (x,y)$ in a product space $X \times Y$, we write $[z]_1$ for the first coordinate x.

Definition 1.1. We call a diffeomorphism $f \in \mathrm{Diff}^1_a(A)$ *a monotone twist diffeomorphism* (or simply *a twist map*) if the inequality

$$\frac{\partial}{\partial y}[f(x,y)]_1 > 0$$

holds for any $(x,y) \in A$.

The above inequality is called *the twist condition.* Monotone twist diffeomorphisms are important objects in the theory of area preserving or symplectic mappings and there are many results on them. Basic results on monotone twist diffeomorphisms are summarized in Sections 9 and 13 of [4] for example.

We call a point z a periodic point if its orbit $\{g^n(z) \mid n \in \mathbf{Z}\}$ is finite. Denote by $\widetilde{A} = \mathbf{R}^1 \times [0,1]$ the universal cover of A, by $\pi : \widetilde{A} \to A$ the natural covering projection.

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One of the most important examples of twist maps is an integrable map. A twist diffeomorphism f is called *integrable* if there exists a C^1 function α from $[0,1]$ to S^1 such that $d\alpha/dy > 0$ and $f(x,y) = (x+\alpha(y), y)$ for all $(x,y) \in A$. The distinguished property of integrable twist maps is the existence of an invariant foliation $\{S_a\}_{a\in[0,1]}$. Perturbations of integrable twist maps have been studied by many researchers.

In this paper, we focus on the characterization of the integrability. First, it is easy to see that all iterations of an integrable twist map satisfy the twist condition. Hence, it is natural to ask whether the twist condition for all iterations characterizes integrable twist maps or not.

Second, the existence of invariant circles is related to the non-existence of so-called non-Birkhoff periodic points.

Definition 1.2. Let $f \in \mathrm{Diff}^1_a(A)$ be a monotone twist diffeomorphism and F its lift. We say an f-invariant compact subset Λ of A is *ordered* when $[F(z_1)]_1 \leq [F(z_2)]_1$ for any $z_1, z_2 \in \pi^{-1}(\Lambda)$ such that $[z_1]_1 < [z_2]_1$. A periodic point of f is called *of Birkhoff type* if its orbit is ordered.

Notice that whether an invariant set is ordered or not does not depend on the choice of a lift F. In particular, whether a periodic point is of Birkhoff or of non-Birkhoff does not depend on the choice of a lift F. In [2], Boyland and Hall have shown that an invariant circle with an irrational rotation number does not exist if and only if there exists a sequence of non-Birkhoff periodic points such that the rotation numbers of them converge to the irrational number.

It is easy to see that all periodic points are of Birkhoff type for every integrable twist map. Hence, it is natural to ask whether the condition that all periodic points are of Birkhoff type characterizes integrable twist maps or not.

The following main theorem asserts that the twist condition for all iterations or the non-existence of periodic points of non-Birkhoff type characterize the integrability of a twist map in a weak sense.

Theorem 1.1. *Let $f \in \mathrm{Diff}^1_a(A)$ be an area preserving monotone twist diffeomorphism. Then, the following three conditions are equivalent*:

(1) *there exists a homeomorphism h on A such that $f(h(S_a)) = h(S_a)$ for any $a \in [0,1]$,*

(2) *all periodic points of g are of Birkhoff type, and*

(3) *all iterations f^n of f satisfy the twist condition.*

One may ask whether the map is integrable if the above three conditions hold. Since the conjugation of an integrable map by an area preserving diffeomorphism does not preserve the foliation $\{S_a\}_{a\in[0,1]}$ in general, the above conditions does not imply that the map is integrable. Hence, a more suitable question is whether the above three conditions imply that the map is topologically conjugate to an integrable one or not. We do not know whether it is true or not so far. However, Mitsuhiro Shishikura has pointed out that the dynamics on invariant circles with a rational rotation number is the rigid rotation under the conditions in the main theorem. We discuss his observation in the appendix.

We remark that the proof of the main theorem fits the case that an area preserving monotone twist diffeomorphism on $S^1 \times \mathbf{R}$ preserves a band which has the form $\{(x,y) \in S^1 \times \mathbf{R} \mid x \in S^1, \gamma_1(x) \leq y \leq \gamma_2(x)\}$, where γ_1, γ_2 are continuous functions on S^1. Therefore, if there exist no periodic points of non-Birkhoff type then the band is foliated by invariant circles.

We split the proof of the main theorem into three parts. In Section 2, we show that the existence of an invariant foliation implies the twist condition for all iterations. We devote Section 3 to show that the twist condition for all iterations implies the non-existence of non-Birkhoff periodic points. The proofs of them are elementary. Sections 4 and 5 are the main part of the proof of the theorem. In these sections, we show that the non-existence of non-Birkhoff periodic points implies the existence of an invariant foliation.

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2. Twist diffeomorphisms with a invariant foliation

In this section, we show that all iterations of a twist map also satisfy the twist condition if the map preserves a foliation.

For a function γ on a space X, we denote the graph of γ by $\Gamma(\gamma)$. Namely, let

$$\Gamma(\gamma) = \{(x, \gamma(x)) \mid x \in X\}.$$

Theorem 2.1 (Birkhoff theorem). *Let $f \in \mathrm{Diff}^1_a(A)$ be a monotone twist diffeomorphism and C an f-invariant continuously embedded circle which is homotopic to S_0. Then, there exists a continuous function γ on S^1 such that $C = \Gamma(\gamma)$. In particular C is an ordered set.*

Proposition 2.1 (Regularity Lemma). *Let $f \in \mathrm{Diff}^1_a(A)$ be a monotone twist diffeomorphism. Then, there exists a constant $K > 0$ such that*

$$|y_1 - y_2| \leq K|x_1 - x_2|$$

for all two points $(x_1, y_1), (x_2, y_2)$ in any f-invariant ordered set.

In particular, $[z_1]_1 \neq [z_2]_1$ for any two distinct points z_1, z_2 in an f-invariant ordered set.

We refer Section 13.2 of [4] for the proofs.

With the help of these results, we show the following.

Proposition 2.2. *Let $f \in \mathrm{Diff}^1_a(A)$ be a monotone twist diffeomorphism. Assume that there exists a homeomorphism h on A such that $f(h(S_a)) = h(S_a)$ for any $a \in [0,1]$. Then, all iterations f^n of f satisfy the twist condition.*

Proof. We identify the tangent space T_wA at $w \in A$ with the two dimensional Euclidean space $\{(u,v)_w \mid u, v \in \mathbf{R}\}$.

By Birkhoff theorem and Regularity Lemma, $h(S_a)$ is the graph of a Lipschitz function γ_a on S^1 for every $a \in A$. Define a function b on A by

$$b(x,y) = \limsup_{x' \to x+0} \frac{\gamma_a(x') - \gamma_a(x)}{x' - x}$$

if $(x,y) \in h(S_a)$. Since each γ_a is Lipschitz, b is a well-defined function on A.

For every $w \in A$, we define a half plane $H(w)$ and a subset $C(w)$ in T_wA by $H(w) = \{(u,v)_w \in T_wA \mid v > b(w)u\}$ and $C(w) = \{(u,v)_w \in H(w) \mid u > 0\}$. By the invariance of $\Gamma(\gamma_a) = h(S_a)$ and the twist condition, we have $Df(H(w)) = H(f(w))$ and $Df(0,1)_w \subset C(f(w))$. Hence, $Df(C(w))$ is a subset of $C(f(w))$ for all $w \in A$. In particular, $Df^n(0,1)_w \in C(f^n(w))$ for all $n \geq 1$. Therefore, we obtain that $(\partial/\partial y)[f^n(x,y)]_1 > 0$ for all $(x,y) \in A$ and $n \geq 1$. □

3. Twist maps of which all iterations are also twist maps

In this section, we show that all periodic points of a twist map are of Birkhoff type if all iterations of the map satisfy the twist condition.

Proposition 3.1. *Let $f \in \mathrm{Diff}^1_a(A)$ be a monotone twist diffeomorphism and N a positive integer. Assume that f^{N-1} and f^N satisfy the twist condition. Then, all periodic points of f of period N are of Birkhoff type.*

In particular, if all iterations f^n satisfy the twist condition, then all periodic points of f are of Birkhoff type.

Proof. The proof is by contradiction.

Assume that there exists a non-Birkhoff type periodic point w_0 with period N. Fix a lift F of f. Let Λ be the orbit of w_0. We choose an integer m so that $F^N(x_0, y_0) = (x_0 + m, y_0)$ for all $(x,y) \in \pi^{-1}(\Lambda)$. Since w_0 is of non-Birkhoff type, there exist two points $z_1, z_2 \in \pi^{-1}(\Lambda)$ such that $[z_1]_1 < [z_2]_1$ and $[F(z_1)]_1 \geq [F(z_2)]_1$.

Let $(x_i, y_i) = z_i$ and $(x_i^*, y_i^*) = F(z_i)$ for each $i = 1, 2$. By the twist condition, there exist an interval $[a_0, a_1]$ and a C^1 function l on $[a_0, a_1]$ such that $a_0 \leq x_2^* \leq a_1$ and $F(\{x_2\} \times [0,1]) = \Gamma(l)$. Notice that $F([x_2, \infty) \times [0,1]) = \{(x,y) \in \widetilde{A} \mid a_0 \leq x \leq a_1, y \leq l(x)\} \cup \{x > a_1\}$. Since $x_2^* = [F(z_2)]_1 \leq [F(z_1)]_1 = x_1^*$ and $F([x_2, \infty) \times [0,1])$ does not contain $(x_1^*, y_1^*) = F(x_1, y_1)$, we obtain that $a_0 \leq x_2^* \leq x_1^* \leq a_1$ and $l(x_1^*) < y_1^*$.

Let b_0 be the number such that $F(x_2, b_0) = (x_1^*, l(x_1^*))$. Notice that $b_0 \in (y_2, 1)$ since $x_2^* \leq x_1^* < a_1$. By the twist condition for f^N, we have

$[F^N(x_2, b_0)]_1 \geq [F^N(x_2, y_2)]_1 = x_2 + m$. On the other hand, the twist condition for f^{N-1} and the inequality $l(x_1^*) < y_1^*$ imply that

$$\begin{aligned}[F^N(x_2, b_0)]_1 &= [F^{N-1}(x_1^*, l(x_1^*))]_1 \\ &< [F^{N-1}(x_1^*, y_1^*)]_1 = [F^N(x_1, y_1)]_1 = x_1 + m.\end{aligned}$$

Therefore, we obtain that $x_2 < x_1$. It contradicts that $x_1 = [z_1]_1 > [z_2]_1 = x_2$.

The latter part of the proposition is an immediate consequence of the former part. □

4. Invariant circles

In the rest of the paper, we prove that the annulus is foliated by invariant circles if all periodic points of a twist map are of Birkhoff type. In this section, we investigate invariant circles from the view point of rotation numbers.

We define a homeomorphism T on $\widetilde{A}$ by $T(x, y) = (x + 1, y)$. Remark that all lifts of a diffeomorphism on A commute with T.

Let $f \in \mathrm{Diff}^1_a(A)$ be a diffeomorphism which is not assumed to be a twist map. Fix a lift F of f. For every point $z \in \widetilde{A}$, we define the translation number $\tau(z, F)$ of z by

$$\tau(z, F) = \lim_{n\to\infty} \frac{1}{n}\{[F^n(z)]_1 - [z]_1\}$$

if the limit exists.

Let C be an f-invariant circle which is homotopic to S_0. By the theory of rotation numbers, the number $\tau(z, F)$ exists and does not depend on the choice of $z \in \pi^{-1}(C)$. We define *the rotation number* $\rho(C, F)$ with respect to a fixed lift F by $\rho(C, F) = \tau(z, F)$, where $z \in \pi^{-1}(C)$. It is easy to see that $\rho(C, T^q \circ F^p) = p \cdot \rho(C, F) + q$ for any integers p, q and that $\rho(C_1, F) = \rho(C_2, F)$ if two f-invariant circles C_1 and C_2 intersect with each other.

The following is an immediate corollary of Proposition 13.2.7 of [4] and Birkhoff theorem.

Proposition 4.1. *Let $f \in \mathrm{Diff}^1_a(A)$ be a monotone twist diffeomorphism. Then, the rotation number of invariant circles homotopic to S_0 is continuous with respect to Hausdorff metric.*

With the help of the area preserving condition, we can show that mutually disjoint invariant circles have different rotation numbers. We start with the following elementary lemma which does not require the area preserving condition.

Lemma 4.1. *Let $f \in \mathrm{Diff}^1_a(A)$ be a monotone twist diffeomorphism, and γ_1, γ_2 two continuous functions such that $\Gamma(\gamma_1)$ and $\Gamma(\gamma_2)$ are invariant under f. Fix a lift F of f.*

(1) *If $\gamma_1(x) \leq \gamma_2(x)$ for all $x \in S^1$, then $\rho(\Gamma(\gamma_1), F) \leq \rho(\Gamma(\gamma_2), F)$.*

(2) *If there exists a constant $\delta > 0$ such that $[F(z_1)]_1 - [z_1]_1 + \delta \leq [F(z_2)]_1 - [z_2]_1$ for all $z_1 \in \pi^{-1}(\Gamma(\gamma_1))$ and $z_2 \in \pi^{-1}(\Gamma(\gamma_2))$, then $\rho(\Gamma(\gamma_1), F) + \delta \leq \rho(\Gamma(\gamma_2), F)$.*

Proof. Assume that $\gamma_1(x) \leq \gamma_2(x)$ for any $x \in S^1$. Let $l_i = \gamma_i \circ \pi$ for each i.

First, we claim that $[F^n(x, l_1(x))]_1 \leq [F^n(x, l_2(x))]_1$ for any $x \in \mathbf{R}$ and $n \geq 0$. Once it is shown, it is easy to see that $\rho(\Gamma(\gamma_1)) \leq \rho(\Gamma(\gamma_2))$.

The proof of the claim is by induction. The case $n = 0$ is trivial. Assume that $[F^n(x, l_1(x))]_1 \leq [F^n(x, l_2(x))]_1$. Let $x_n = [F^n(x, l_1(x))]_1$. By the twist condition, $[F^{n+1}(x, l_1(x))]_1 = [F(x_n, l_1(x_n))]_1 \leq [F(x_n, l_2(x_n))]_1$. On the other hand, $[F(x_n, l_2(x_n))]_1 \leq [F^{n+1}(x, l_2(x))]_1$ since $x_n \leq [F^n(x, l_2(x))]_1$ and $\Gamma(l_2)$ is an ordered set. Therefore, we obtain that $[F^{n+1}(x, l_1(x))]_1 \leq [F^{n+1}(x, l_2(x))]_1$. It completes the proof of the claim.

Next, we assume that there exists a positive constant $\delta > 0$ such that $[F(x_1, l_1(x_1))]_1 - x_1 + \delta \leq [F(x_2, l_2(x_2))]_1 - x_2$ for any $x_1, x_2 \in \mathbf{R}$. Then, for any $x \in \mathbf{R}$,

$$\begin{aligned}
[F^n(x, l_1(x))]_1 - x - n\delta &= \sum_{k=0}^{n-1} \big([F^{k+1}(x, l_1(x))]_1 - [F^k(x, l_1(x))]_1 + \delta\big) \\
&\leq \sum_{k=0}^{n-1} \big([F^{k+1}(x, l_2(x))]_1 - [F^k(x, l_2(x))]_1\big) \\
&= [F^n(x, l_2(x))]_1 - x.
\end{aligned}$$

Therefore, we obtain that $\rho(\Gamma(\gamma_1), F) \leq \rho(\Gamma(\gamma_2), F) - \delta$. □

Proposition 4.2. *Let $f \in \mathrm{Diff}^1_a(A)$ be a monotone twist diffeomorphism and γ_1, γ_2 are continuous functions on S^1 such that $\Gamma(\gamma_1)$ and $\Gamma(\gamma_2)$ are f-invariant. If $\gamma_1(x) < \gamma_2(x)$ for all $x \in S^1$, then $\rho(\Gamma(\gamma_1), F) < \rho(\Gamma(\gamma_2), F)$ for any lift F of f.*

Proof. Without loss of generality, we can choose a lift F of f so that $[F(z)]_1 > [z]_1$ for all $z \in \widetilde{A}$. Let $l_i = \gamma_i \circ \pi$ for each $i = 1, 2$ and $D_{z,z'} = \{(x, y) \mid [z]_1 \leq x \leq [z']_1, l_1(x) \leq y \leq l_2(x)\}$ for every $z, z' \in \widetilde{A}$. We denote the Lebesgue measure by Leb.

For every integer $n \geq 0$, we define a function α_n on $\widetilde{A}$ by $\alpha_n(z) = \mathrm{Leb}(D_{z,F^n(z)})$. It is easy to see that $\alpha_0(z) = 0$, $\alpha_n(T^m(z)) = \alpha_n(z)$, and $\alpha_n(z) = \sum_{k=0}^{n-1} \alpha_1(F^k(z))$ for any $m, n \geq 1$ and $z \in \widetilde{A}$.

We claim that there exists a constant $\delta > 0$ such that $\alpha_1(z_1) < \alpha_1(z_2)$ for any $z_1 \in \Gamma(l_1)$ and $z_2 \in \Gamma(l_2)$. Proof is by contradiction. Assume that there exist two points $z_1 \in \Gamma(l_1), z_2 \in \Gamma(l_2)$ such that $\alpha_1(z_1) \geq \alpha_1(z_2)$. Since $\alpha_1(T^m(z)) = \alpha_1(z)$ for any integer m and $z \in \widetilde{A}$, we can assume that $[z_1]_1 < 0 < [z_2]_1$ and $[F([z_1]_1, y)]_1 < 0 < [F([z_2]_1, y)]_1$ for any $y \in [0, 1]$ by replacing z_1 and z_2 by $T^{-m}(z_1)$ and $T^m(z_2)$ with a large integer m. Then, we have $\mathrm{Leb}(D_{F(z_1),F(z_2)}) = \mathrm{Leb}(D_{z_1,z_2}) + \alpha_1(z_2) - \alpha_1(z_1)$. By assumption, $\mathrm{Leb}(D_{F(z_1),F(z_2)}) \leq \mathrm{Leb}(D_{z_1,z_2})$. By the twist condition and the invariance of $\Gamma(l_1)$ and $\Gamma(l_2)$ under F, the set $F(D_{z_1,z_2})$ is a proper subset of $D_{F(z_1),F(z_2)}$. It contradicts that F is area preserving. The proof of the claim is completed.

Since $\alpha_1 \circ T = \alpha_1$ and $l_i(x+1) = l_i(x)$, a function α_1 has a minimum and a maximum on each $\Gamma(l_i)$. Hence, the above claim implies that there exists a constant $\delta > 0$ such that $\alpha_1(z_1) + \delta \leq \alpha_1(z_2)$ for any $z_1 \in \Gamma(l_1)$ and $z_2 \in \Gamma(l_2)$.

Let K be the Lebesgue measure of $\{(x,y) \in \widetilde{A} \mid 0 \leq x \leq 1, l_1(x) \leq y \leq l_2(x)\}$. Note that $m \leq [z']_1 - [z]_1 \leq m+1$ if $mK \leq \mathrm{Leb}(D_{z,z'}) \leq (m+1)K$ since $\mathrm{Leb}(D_{T^i(z),T^{i+1}(z)}) = K$ for every i. Choose a large integer N so that $N\delta \geq 2K$.

$$\alpha_N(z_2) - \alpha_N(z_1) = \sum_{k=0}^{N-1} \alpha_1(F^k(z_2)) - \alpha_1(F^k(z_1)) \geq N\delta \geq 2K$$

for every $z_1 \in \Gamma(l_1)$ and $z_2 \in \Gamma(l_2)$. It implies that $[F^N(z_1)]_1 - [z_1]_1 + 1 \leq [F^N(z_2)]_1 - [z_2]_1$. By Lemma 4.1, we obtain that $\rho(z_1, F^N) < \rho(z_2, F^N)$. Therefore, $\rho(\Gamma(\gamma_1), F) < \rho(\Gamma(\gamma_2), F)$ since $\rho(z_i, F^N) = N\rho(z_i, F^N)$ for each $i = 1, 2$. □

In the rest of this section, we show the uniqueness of the invariant circle with a given rotation number for twist maps without non-Birkhoff periodic points.

Lemma 4.2. *Let f be a homeomorphism on A, F a lift of f, and μ a Borel measure on A such that $\mu(U) > 0$ for all open subset U of A. Denote by B the subset $[0,\infty) \times [0,1]$ of $\widetilde{A}$. Suppose that f preserves μ, F has a fixed point, and $F(B)$ is a proper subset of B. Then, there exists an integer $p \geq 2$ and a point $z_0 \in \widetilde{A}$ such that $F^q(z_0) = T(z_0)$.*

Proof. Let $B_0 = [0,1) \times [0,1]$ and $D_n = F(B_0) \cap T^n(B_0)$ for every n. By assumption, $\mathrm{int}\, B_0 \setminus F(B_0) \neq \emptyset$. Let μ' be the pull back measure of μ on $\widetilde{A}$ by π. Since F preserves μ', we have $\mu'(F(B_0) \setminus B_0) > 0$. Hence, $F(B_0)$ is the disjoint union of D_n $(n \geq 0)$ and there exists an integer $n_0 \geq 1$ such that $\mu'(D_{n_0}) > 0$.

Since π maps both B_0 and $F(B_0)$ to A bijectively, the set B_0 is the disjoint union of $T^{-n}(D_n)$ $(n \geq 0)$. Since F and T preserve μ', we obtain that

$$\begin{aligned}
\int_{B_0} [F(z)]_1 d\mu' &= \int_{F(B_0)} [z]_1 d\mu' \\
&= \sum_{n \geq 0} \int_{T^{-n}(D_n)} ([z]_1 + n) d\mu' \\
&= \int_{B_0} [z]_1 d\mu' + \sum_{n \geq 1} n\mu'(D_n).
\end{aligned}$$

In particular, the integral of $[F(z)]_1 - [z]_1$ on B_0 is positive since $\mu'(D_{n_0}) > 0$.

Let $\varphi(w) = [F(z)]_1 - [z]_1$ for any $w = \pi(z) \in \widetilde{A}$. Note that the integral of φ on A is positive. By Birkhoff's ergodic theorem, there exist a point $w_1 \in A$ and a positive number $\delta > 0$ such that $\lim_{n\to\infty}(1/n)\sum_{k=0}^{n-1}\varphi(f^k(w_1)) = \delta$. It implies the existence of $z_1 \in \widetilde{A}$ such that

$$\lim_{n\to\infty}\frac{1}{n}\{[F^k(z_1)]_1 - [z_1]_1\} = \delta > 0.$$

Choose an integer $p \geq 2$ so that $0 < 1/p < \delta$ and $1/p \neq \rho(S_0, F), \rho(S_1, F)$. By assumption, F has a fixed point z_2. Let $G = T^{-1} \circ F^p$. Then, we have $\tau(z_1, G) = p\delta - 1 > 0 > -1 = \tau(z_2, G)$. By the fixed point theorem of Franks for annulus homeomorphisms in [3], there exists a fixed point z_0 of G. In particular, $F^p(z_0) = T(z_0)$. Since $1/p \neq \rho(S_0, F), \rho(S_1, F)$, we obtain that $\pi(z_0) \notin S_0 \cup S_1$. □

Lemma 4.3. *Let $f \in \mathrm{Diff}^1_a(A)$ be a twist diffeomorphism and F a lift of f. Assume that $F^{kp}(z_0) = T^{kq}(z_0)$ and $F^p(z_0) \neq T^q(z_0)$ for a point $z_0 \in \widetilde{A}$ and integers p, q and $k \geq 2$. Then, $\pi(z_0)$ is a non-Birkhoff periodic point of f.*

Proof. Let $G = T^{-q} \circ F^p$. Since $G^k(z_0) = z_0$ and $G(z_0) \neq z_0$, the orbit of z_0 for G is finite and contains at least two distinct points. Hence, there exists an integer n_0 such that $[G^{n_0+1}(z_0)]_1 = \max\{[G^n(z_0)]_1 \mid n \in \mathbf{Z}\}$. Notice that $G^{n_0+1}(z_0) \neq G^{n_0}(z_0)$, $[G^{n_0}(z_0)]_1 \leq [G^{n_0+1}(z_0)]_1$, and

$$[F^p(G^{n_0}(z_0))]_1 = [G^{n_0+1}(z_0)]_1 + q \geq [G^{n_0+2}(z_0)]_1 + q = [F^p(G^{n_0+1}(z_0))]_1.$$

Since $\pi \circ G^{n_0}(z_0)$ and $\pi \circ G^{n_0+1}(z_0)$ are contained in the orbit of a periodic point $\pi(z_0)$ of f, $\pi(z_0)$ is a non-Birkhoff periodic point of f. □

Proposition 4.3. *Let $f \in \mathrm{Diff}^1_a(A)$ be a monotone twist diffeomorphism and F its lift. Assume that C_1 and C_2 be f-invariant circles such that $\rho(C_1, F) = \rho(C_2, F)$. Then, either*

(1) *$C_1 = C_2$ or*

(2) *there exists a periodic point of non-Birkhoff type in the region bounded by C_1 and C_2.*

Proof. Assume that $C_1 \neq C_2$. Let γ_i be the function on S^1 satisfying that $\Gamma(\gamma_i) = C_i$ for each $i = 1, 2$. Without loss of generality, we could assume that $\gamma_1(x) \leq \gamma_2(x)$ for any $x \in S^1$.

Let $\mathcal{I}$ be the collection of the connected components of $\{x \in S^1 \mid \gamma_1(x) < y < \gamma_2(x)\}$. Since $C_1 \neq C_2$, the set $\mathcal{I}$ is not empty. By Proposition 4.2, $\Gamma(\gamma_1)$ and $\Gamma(\gamma_2)$ intersect with each other. Hence, each element of $\mathcal{I}$ is an open interval.

Fix an element $I \in \mathcal{I}$. Let $U = \{(x, y) \in A \mid x \in I, \gamma_1(x) < y < \gamma_2(x)\}$. Since $\Gamma(\gamma_1)$ and $\Gamma(\gamma_2)$ are invariant under f, there exists a sequence $\{I_n\}$ in $\mathcal{I}$ such that $f^n(U) = \{(x, y) \in A \mid x \in I_n, \gamma_1(x) < y < \gamma_2(x)\}$. It implies that either $f^n(U) = U$ or $f^n(U) \cap U = \emptyset$ for each n. Since f is area preserving and the area of U is positive, there exists an integer $p \geq 1$ such that $f^p(U) = U$.

Let $l_i = \gamma_i \circ \pi$ for each $i = 1, 2$. Fix a connected component of $\pi^{-1}(U)$ and let D be its closure. Then, there exists an interval $[a_0, b_0]$ such that $D = \{(x, y) \in \widetilde{A} \mid x \in [a_0, b_0], l_1(x) \leq y \leq l_2(x)\}$. Since $\pi(\operatorname{int} D) = U$ and $f^p(U) = U$, there exists an integer q such that $F^p(D) = T^q(D)$.

Let $c_0 = l_1(b_0) = l_2(b_0)$. By the invariance of $\Gamma(l_1)$ and $\Gamma(l_2)$, (b_0, c_0) is a fixed point of $T^{-q} \circ F^p$. By the fixed point theorem of Andrea in [1] and Poincaré's recurrence theorem, any orientation preserving homeomorphism on the plane which preserves a finite Borel measure has a fixed point. Since D is homeomorphic to the closed unit disk $\mathbf{D}^2 = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1\}$, there exists a fixed point $(x_0, y_0) \in \operatorname{int} D$ of $T^{-q} \circ F^p$.

Let $(x_n, y_n) = F^n(x_0, y_0)$ and $(b_n, c_n) = F^n(b_0, c_0)$ for each n. Since $F^n(D)$ is homeomorphic to $\mathbf{D}^2$, there exists a homeomorphism φ_n between $\mathbf{D}^2$ and $F^n(D)$ such that $\varphi(0, 0) = (x_n, y_n)$, $\varphi(\partial \mathbf{D}^2) = \partial D$, $\varphi(0, 1) = (b_n, c_n)$, and $\varphi([-1, 1] \times \{0\}) = \{x_n\} \times [l_1(x_n), l_2(x_n)]$. We can choose φ_p so that $\varphi_p = T^q \circ \varphi_0$.

Let $\pi'(\theta, r) = (r \cos(2\pi\theta), r \sin(2\pi\theta))$ for every $(\theta, r) \in A$. Define a homeomorphism g_n on A by $(\varphi_{n+1} \circ \pi') \circ g_n(\theta, r) = F \circ (\varphi_n \circ \pi'(\theta, r))$ for every $(\theta, r) \in A$. We remark that $g_n(1/4, 1) = (1/4, 1)$ since $F(b_n, c_n) = (b_{n+1}, c_{n+1})$.

Let $h_n(x, y) = \varphi_n \circ \pi' \circ \pi(x, y)$ for every $x, y \in \widetilde{A}$. Choose a lift G_n of g_n on $\widetilde{A}$ so that $G_n(1/4, 1) = (1/4, 1)$. Let $B = [0, \infty) \times [0, 1]$. We claim that $G_n(B)$ is a proper subset of B for all n. In fact, since F satisfies the twist condition, $F(\{x_n\} \times (y_n, l_2(x_n))) \subset F^{n+1}(D) \cap \{x > x_{n+1}\}$. It implies that $[G_n(0, y)]_1 \in (0, 1/2)$ since $G_n(1/4, 1) = (1/4, 1)$, $h_n(\{0\} \times [0, 1]) = \{x_n\} \times [y_n, l_2(x_n)]$, and $h_n(\{0\} \times [1/2, 1]) = \{x_n\} \times [l_1(x_n), y_n]$. Therefore, we obtain that $G_n(B)$ is a proper subset of B.

Let $g = g_{p-1} \circ g_{p-2} \circ \cdots \circ g_0$, $G = G_{p-1} \circ G_{p-2} \circ \cdots \circ G_0$, and μ' the pull back measure of μ by $\varphi_0 \circ \pi'$. Notice that G is a lift of g and $(1/4, 1)$ is a fixed point of G. By the above claim, $G(B)$ is a proper subset of B. Since $(\varphi_n \circ \pi') \circ g = F^p \circ (\varphi_0 \circ \pi')$ and $\varphi_n = T^q \circ \varphi_0$, the map g preserves μ'. Hence, we apply Lemma 4.2 and obtain an integer $k \geq 2$ and a point $w_0 \in \operatorname{int} \widetilde{A}$ satisfying that $G^k(w_0) = T(w_0)$. Notice that $g(\pi(w_0)) \neq \pi(w_0)$ since $\tau(G, w_0) = 1/k$. Since $\varphi_0 = T^{-q} \circ \varphi_p$ maps $\operatorname{int} A$ to $\operatorname{int} D \setminus \{(x_0, y_0)\}$ homeomorphically, we obtain that $F^p \circ h_0(w_0) \neq T^q \circ h_0(w_0)$. By Lemma 4.3, a periodic point $\pi \circ h_0(w_0) \in \pi(D)$ of f is of non-Birkhoff type. □

5. Twist diffeomorphisms without non-Birkhoff periodic points

In this section, we show the existence of an invariant foliation for twist maps without non-Birkhoff periodic points.

First, we recall the result of Boyland and Hall in [2] on the existence of invariant circles.

Theorem 5.1 ([2, Theorem C]). *Let $f \in \mathrm{Diff}^1_a(A)$ be a monotone twist diffeomorphism and F a lift of f. If f possesses no invariant circles of irrational rotation number $\omega \in [\rho(S_0, F), \rho(S_1, F)]$, then for any rational number q/p sufficiently close to ω, there exists a non-Birkhoff periodic point with rotation number q/p.*

We combine the above theorem with the results in the last section.

Proposition 5.1. *Let $f \in \mathrm{Diff}^1_a(A)$ be a monotone twist diffeomorphism. Assume that all periodic points of f are of Birkhoff type. Then, there exists a homeomorphism h on A such that $f(h(S_a)) = h(S_a)$ for any $a \in [0, 1]$.*

Proof. Fix a lift F of f. Let $\rho_0 = \rho(S_0, F)$ and $\rho_1 = \rho(S_1, F)$. Notice that $\rho_0 < \rho_1$ by Proposition 4.2.

By Theorem 5.1, for any irrational number $\omega \in [\rho_0, \rho_1]$, there exists an invariant circle C_ω such that $\rho(C_\omega, F) = \omega$. Birkhoff theorem implies that $C_\omega = \Gamma(\gamma_\omega)$ for some function γ_ω on S^1. By Proposition 4.3, $C_\omega = \Gamma(\gamma_\omega)$ is the unique invariant circle with rotation number ω.

For any number $\alpha \in [\rho_0, \rho_1]$, we define two functions γ^-_α and γ^+_α on S^1 by

$$\gamma^-_\alpha(x) = \sup(\{\gamma_\omega(x) | \omega \notin \mathbf{Q}, \omega \le \alpha\} \cup \{0\}),$$
$$\gamma^+_\alpha(x) = \inf(\{\gamma_\omega(x) | \omega \notin \mathbf{Q}, \omega \ge \alpha\} \cup \{1\}).$$

For any irrational numbers $\omega_1, \omega_2 \in [\rho_0, \rho_1]$ with $\omega_1 < \omega_2$, we obtain that $\gamma_{\omega_1}(x) < \gamma_{\omega_2}(x)$ for all $x \in S^1$ by Proposition 4.2. Hence, $\gamma_\omega = \gamma^-_\omega = \gamma^+_\omega$ for any irrational number ω.

By Regularity Lemma and Proposition 4.1, all γ^-_α and γ^+_α are uniformly Lipschitz and $\rho(\Gamma(\gamma^-_\alpha), F) = \rho(\Gamma(\gamma^+_\alpha)) = \alpha$. By Proposition 4.3, we obtain that $\gamma^-_\alpha = \gamma^+_\alpha$. Let $\gamma_\alpha = \gamma^-_\alpha = \gamma^+_\alpha$ for every number $\alpha \in [\rho_0, \rho_1]$. By its definition, $\gamma_\alpha(x)$ is continuous with respect to α. Since $\rho(\Gamma(\gamma_\alpha)) = \alpha$ for any α, $\gamma_{\alpha_1}(x) \ne \gamma_{\alpha_2}(x)$ if $\alpha_1 \ne \alpha_2$.

Let $a(t) = (1 - t)\rho_0 + t\rho_1$ for every $t \in [0, 1]$. We define a map h on A by

$$h(x, y) = (x, \gamma_{a(y)}(x)).$$

We claim that h is a homeomorphism on A. Once it is shown, then $f(h(S_y)) = f(\Gamma(\gamma_{a(y)})) = \Gamma(\gamma_{a(y)}) = h(S_y)$ for any $y \in [0, 1]$ since $\Gamma(\gamma_{a(y)})$ is f-invariant.

Let $h_0(x, y) = (x, \gamma_y(x))$ for every $(x, y) \in S^1 \times [\rho_0, \rho_1]$. It is sufficient to show that h_0 is a homeomorphism between $S^1 \times [\rho_0, \rho_1]$ and A.

First, we show that h_0 is bijective. Fix a point $(x_0, y_0) \in A$. Since $\gamma_{\rho_0}(x_0) = 0$, $\gamma_{\rho_1}(x_0) = 1$ and $\gamma_\alpha(x_0)$ is strictly increasing and continuous with respect to α, there exists the unique number $\alpha_0 \in [\rho_0, \rho_1]$ such that $\gamma_{\alpha_0}(x_0) = y_0$. Then, $h_0(x_0, \alpha_0) = (x_0, y_0)$. Therefore, h_0 is bijective.

Second, we show that h_0 is continuous. We can choose a constant $K > 0$ so that $|\gamma_\alpha(x_1) - \gamma_\alpha(x_2)| \le K|x_1 - x_2|$ for any $x_1, x_2 \in S^1$ and $\alpha \in [\rho_0, \rho_1]$ since γ_α are uniformly Lipschitz. Hence, for any points $(x, \alpha), (x', \alpha')$,

$$\begin{aligned}|\gamma_\alpha(x) - \gamma_{\alpha'}(x)| &\le |\gamma_\alpha(x) - \gamma_{\alpha'}(x)| + |\gamma_{\alpha'}(x) - \gamma_{\alpha'}(x')| \\ &\le |\gamma_\alpha(x) - \gamma_{\alpha'}(x)| + K|x - x'|\end{aligned}$$

Since $\gamma_\alpha(x)$ is continuous with respect to α, $\gamma_{\alpha'}(x')$ converges to $\gamma_\alpha(x)$ as (x', α') goes to (x, α). It implies the continuity of h_0.

At last, we show that h_0 is an open map. Fix $(x_0, \alpha_0) \in S^1 \times [\rho_0, \rho_1]$ and a neighborhood U_0 of (x_0, α_0). There exists $\epsilon > 0$ such that the set

$U = \{(x, \alpha) \mid |x - x_0| < \epsilon, |\alpha - \alpha_0| < \epsilon\}$ is contained in U_0. We show that $h_0(U)$ is a neighborhood of $h_0(x_0, \alpha_0)$. Once it is shown, the map h_0 is open map and the proof is completed.

Recall that there exists $K > 0$ such that $|\gamma_\alpha(x') - \gamma_\alpha(x)| < K|x' - x|$ for all $x, x' \in S^1$ and $\alpha \in [\rho_0, \rho_1]$. Take $\delta > 0$ so that $2K\delta < \min\{\gamma_{\alpha_0+\epsilon}(x_0) - \gamma_{\alpha_0}(x_0), \gamma_{\alpha_0}(x_0) - \gamma_{\alpha_0-\epsilon}(x_0)\}$. If $|x - x_0| < \delta$, then

$$\begin{aligned}\gamma_{\alpha_0+\epsilon}(x) - \gamma_{\alpha_0}(x_0) &\geq |\gamma_{\alpha_0+\epsilon}(x_0) - \gamma_{\alpha_0}(x_0)| - |\gamma_{\alpha_0+\epsilon}(x) - \gamma_{\alpha_0+\epsilon}(x_0)| \\ &> 2K\delta - K\delta = K\delta.\end{aligned}$$

By the same calculation, we also obtain that $\gamma_{\alpha_0-\epsilon}(x) < \gamma_{\alpha_0}(x_0) - K\delta$. Since $\gamma_\alpha(x)$ is increasing with respect to α for any x, we have $h_0(U) = \{(x, y) \mid |x - x_0| < \epsilon, \gamma_{\alpha_0-\epsilon}(x) < y < \gamma_{\alpha_0+\epsilon}(x)\}$. Hence, $h_0(U)$ contains a neighborhood $\{(x, y) \mid |x - x_0| < \delta, |y - \gamma_{\alpha_0}(x_0)| < K\delta\}$ of $(x_0, \gamma_{\alpha_0}(x_0))$. □

Main theorem follows immediately from Propositions 2.2, 3.1 and 5.1.

Appendix A. The rigid rotations on invariant circles with a rational rotation number

As mentioned in the introduction, we do not know whether the three conditions in the main theorem imply that the map is topologically conjugate to an integrable one or not. However, Mitsuhiro Shishikura has pointed out the following.

Lemma A.1. *Let f be an area preserving monotone twist diffeomorphism on A and h a homeomorphism on A such that $f \circ h(S_a) = h(S_a)$ for all $a \in [0, 1]$. Then, $f|_{h(S_{y_0})}$ is topologically conjugate to the rigid rotation if the rotation number on $h(S_{y_0})$ is rational.*

Proof. Assume that the rotation number of f on $h(S_{y_0})$ is a rational number p/q. Notice that the rotation number of f on $h(S_y)$ is strictly increasing with respect to y by Proposition 4.2.

We identify the map $h \circ f \circ h^{-1}|_{S_{y_0}}$ with a homeomorphism g on S^1. Then, there exists a lift G of g such that $G^q - p$ has a fixed point. If $G^q(x) - p > x$ for some $x \in \mathbf{R}$, then every small perturbation of $G^q - p$ either satisfies that $G^q(x) - p > x$ for all $x \in S^1$ or has a fixed point by the mean value theorem. Hence, the rotation number of $h^{-1} \circ f \circ h|_{S_y}$ is not less than p/q for all y close to y_0. It contradicts that the rotation number of $h^{-1} \circ f \circ h|_{S_y}$ is strictly increasing. Therefore, we obtain that $G^q(x) - p \leq x$ for all x. By the same argument, we can show that $G^q(x) - p \geq x$ for all x. Therefore, $G^q(x) = x + p$, and hence, $h^{-1} \circ f \circ h|_{S_{y_0}}$ is topologically conjugate to the rigid rotation. □

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