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On the global nilpotent cone of $\mathbf{P}^1$

By

Syu KATO*

1. Introduction

In Laumon [5], Laumon introduced the notion of a global nilpotent cone. It is a Lagrangian substack $Nilp$ of the cotangent stack of the moduli stack of vector bundles or more generally, principal G-bundles over a curve (See Faltings [3] or Ginzburg [4]). It is very important in the study of the Geometric Langlands Program, because it gives the characteristic variety of an automorphic sheaf. Actually, we can also realize it as the zero level set of the Hitchin Hamiltonians (See Donagi and Markman [2]). Therefore, it seems to be an interesting problem to describe the stratification induced by the global nilpotent cone. In this paper, we consider when a nontrivial nilpotent cotangent vector exists if the curve is $\mathbf{P}^1$ but add parabolic structure to enlarge the moduli space. We present a complete picture in rank two case in Corollary 3.7. We also present an analogous result for arbitrary rank case in Theorem 3.4. In particular, by Corollary 3.6, we obtain that the expected domain of an automorphic sheaf in $\mathbf{P}^1$ case should be contained in the product of flag varieties as expected (This is nontrivial since we consider it as a moduli stack, i.e. we can go to the boundary which has a different decomposition type). Moreover, a relation with the Bruhat stratification of flag varieties is obtained (Lemma 3.8).

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2. General facts and result

Let X be a smooth projective curve over an algebraically closed field k. We fix a finite subset S of closed points of X. Put $D := \sum_{x \in S}[x]$.

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Definition 2.1 (Seshadri [7]). A rank r bundle on X with (full) quasi-parabolic structure at S is a pair $(E, \{F_* E|_x\}_{x\in S})$ consisting of a rank r locally free sheaf E on X and a family of decreasing filtrations $\{F_* E|_x\}_{x\in S}$ of geometric fibers $E|_x$ at $x \in S$ such that

$$\begin{aligned} E|_x = F_{x,0}E|_x \supset F_{x,1}E|_x \supset \cdots \supset F_{x,r}E|_x = \{0\}, \\ \dim F_{x,i-1}E|_x \,/F_{x,i}E|_x = 1 \qquad \text{for all} \quad 1 \le i \le r. \end{aligned}$$

We may denote it simply by E_* and call it a (quasi) parabolic bundle. A rank r bundle on X with (full) parabolic structure is a quasi-parabolic bundle with extra parameters

$$0 \le \alpha_{E,x,0} < \alpha_{E,x,1} < \cdots < \alpha_{E,x,r-1} < 1$$

associated to $\{F_* E|_x\}_{x\in S}$.

Definition 2.2 (Yokogawa [9]). We define an extended filtration of the germ of a parabolic bundle E_* at $x \in S$ as follows.

$$F_{x,i}E_x = \begin{cases} \{\phi \in E_x \mid \phi|_x \in F_{x,i}E|_x\} & \text{for} \quad 1 \le i \le r, \\ F_{x,i+jr}E_x(j[x]) & \text{otherwise.} \end{cases}$$

Here we set $\alpha_{x,i+r} = \alpha_{x,i} + 1$.

Definition 2.3 (Seshadri [7] and Boden-Yokogawa [1]). A morphism of locally free sheaves with parabolic structures $(E, \{F_*\})$ and $(F, \{F_*\})$ at S is a morphism $f : E \to F$ of base sheaves satisfying the following condition for each $x \in S$.

$$\text{If } \alpha_{E,x,i} \ge \alpha_{F,x,j}, \text{ then we have } f(F_{x,i}E_x) \subset F_{x,j}F_x.$$

We may denote it by $f : E_* \to F_*$ and call it a parabolic morphism. We denote the set of parabolic morphisms by $\mathrm{Hom}_{par}(E_*, F_*)$. Similarly, a strong parabolic morphism $f : E_* \to F_*$ is a parabolic morphism such that

$$\text{If } \alpha_{E,x,i} = \alpha_{F,x,j}, \text{ then we have } f(F_{x,i}E_x) \subset F_{x,j+1}F_x \subset F_{x,j}F_x.$$

We denote the set of strong parabolic morphisms by $\mathrm{Hom}_{spar}(E_*, F_*)$.

Proposition 2.4 (Yokogawa [9], Faltings [3] Section V). *The dualspace of the set of infinitesimal deformations of a parabolic bundle E_* is the following.*

$$\mathrm{Hom}_{spar}(E_*, E_* \otimes \omega(D)).$$

Definition 2.5 (See Laumon [5] for usual bundle case). An element $\rho \in \mathrm{Hom}_{spar}(E_*, E_* \otimes \omega(D))$ is said to be nilpotent if and only if $\rho \otimes \mathrm{id} \circ \cdots \circ \rho : E \to E \otimes \omega(D)^{\otimes r}$ is zero. We call a nilpotent dual infinitesimal deformation by a nilpotent vector.

Lemma 2.6. *Let $E_1 = (E, \{F_{x,*}\}_{x\in S})$ be a vector bundle with parabolic structure at S. Let $T \subset S$. Let $E_2 = (E, \{F_{x,*}\}_{x\in T})$ be a vector bundle with parabolic structure at T. If E_2 has a nontrivial nilpotent vector, then E_1 also has a nontrivial nilpotent vector.*

Proof. Let $\rho \in \mathrm{Hom}_{spar}(E_2, E_2 \otimes \omega(H))$ be a nilpotent vector of E_2. Here $H = \sum_{x\in T}[x]$. We have a natural injection $\mathrm{Hom}_{\mathcal{O}_{\mathbf{P}^1}}(E, E\otimes\omega(H)) \hookrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathbf{P}^1}}(E, E\otimes\omega(D))$. We denote the image of ρ by $\tilde{\rho}$. Put $R := S\backslash T$. We have

$$\tilde{\rho}|_{\mathbf{P}^1\backslash R} = \rho|_{\mathbf{P}^1\backslash R}\,.$$

Moreover, the above description says

$$\tilde{\rho}|_R = 0.$$

Therefore, we obtain the result. □

3. $\mathbf{P}^1$-specific description

In this section, we assume $X = \mathbf{P}^1$ and $\#S = N \geq 2$. We do not mention the parabolic parameters in this section, since any choice of the parameters gives the same result in our situation. Let $E_* = (E, \{F_{x,*}\})$ be a vector bundle with parabolic structure at S. We put $S = \{x_1, \ldots, x_N\}$.

Definition 3.1. Let $\vec{i} = (i_1, \ldots, i_N) \in \mathbb{Z}^N$. We define a vector bundle $F^{\vec{i}}E$ as follows.

$$F^{\vec{i}}E_x := \begin{cases} E_x & \text{if} \quad x \notin S, \\ F_{x_j, i_j}E_x & \text{if} \quad x = x_j \in S. \end{cases}$$

We denote $\mathbf{1} = (1, \ldots, 1)$ and $1_j = (0, \ldots, 0, \overbrace{1,}^{j} 0, \ldots, 0)$. Since we are dealing only with full (quasi-)parabolic structures, we have the following.

Corollary 3.2. *We have $F^{\vec{i}+r1_j}E([x_j]) \cong F^{\vec{i}}E$.*

Proposition 3.3. *For any $\vec{i} \in \mathbb{Z}^N$, we have natural embedding*

$$\mathrm{Hom}_{\mathcal{O}_{\mathbf{P}^1}}(F^{\vec{i}}E, F^{\vec{i}}E \otimes \omega) \subset \mathrm{Hom}_{spar}(E_*, E_* \otimes \omega(D))\,.$$

Moreover, the LHS consists of nilpotent vectors.

Proof. By the Grothendieck's theorem, we have a splitting of $F^{\vec{i}}E$. We have $\omega_{\mathbf{P}^1} \cong \mathcal{O}_{\mathbf{P}^1}(-2)$. Let $\rho \in \mathrm{Hom}_{\mathcal{O}_{\mathbf{P}^1}}(F^{\vec{i}}E, F^{\vec{i}}E\otimes\omega)$. Composing ρ sufficiently large number of times (say M), we have a morphism $\rho^{\otimes M} : F^{\vec{i}}E \to F^{\vec{i}}E \otimes \omega^{\otimes M}$. We can assume that any direct component of $F^{\vec{i}}E\otimes\omega^{\otimes M}$ has sufficiently small degree. Thus, ρ is nilpotent. We prove that ρ is strong parabolic. It suffices to prove

$$\rho(F^{\vec{i}-p\mathbf{1}}E) \subset F^{\vec{i}-(p-1)\mathbf{1}}E \otimes \omega(D)$$

for any $p \in \mathbb{Z}$. For $1 \leq p < r$, we have

$$F^{\vec{i}-p\mathbf{1}}E \subset F^{\vec{i}}E(D).$$

We have $\rho(F^{\vec{i}-p\mathbf{1}}E) \subset F^{\vec{i}}E \otimes \omega(D) \subset F^{\vec{i}-(p-1)\mathbf{1}}E \otimes \omega(D)$. Therefore, we obtain the result. □

Theorem 3.4. *For any dual infinitesimal symmetry ρ which satisfies* $\operatorname{rk}\operatorname{Im}\rho = 1$*, there exists $\vec{i} \in \mathbb{Z}^N$ such that*

$$\rho \in \operatorname{Hom}_{\mathcal{O}_{\mathbf{P}^1}}(F^{\vec{i}}E, F^{\vec{i}}E \otimes \omega).$$

Here $\operatorname{rk}E$ *is the rank of a vector bundle E.*

Proof. By assumption, $\rho(E)$ is a line bundle. Let $N \otimes \omega(D)$ be the maximal sub line bundle of $E \otimes \omega(D)$ such that $\rho(E) \subset N \otimes \omega(D)$. At each $x_p \in S$, we have an induced parabolic structure of N. Take $i_j \in [0, r-1]$ such that

$$\dim((F_{x_j,i_j}E \cap N + E(-D))/E(-D)) \neq \dim((F_{x_j,i_j+1}E \cap N + E(-D))/E(-D)).$$

Since $\dim N|_x = 1$, i_j is unique for each $j = 1, \ldots, N$. Set $\vec{i} = (i_1, \ldots, i_N)$. We have $\rho(F^{\vec{i}}E) \subset N \otimes \omega = F^{\vec{i}+\mathbf{1}}E \otimes \omega(D) \cap N \otimes \omega(D)$. We have $\rho(F^{\vec{i}}E) \subset N \otimes \omega \subset F^{\vec{i}}E \otimes \omega$. □

Definition 3.5. A vector bundle $E \cong \oplus\mathcal{O}_{\mathbf{P}^1}(a_i)$ on $\mathbf{P}^1$ is said to split abnormally if there exists i and j such that $\|a_i - a_j\| > 1$.

Remark 1. Note that the condition $\operatorname{Hom}_{\mathcal{O}_{\mathbf{P}^1}}(E, E \otimes \omega) \neq \{0\}$ means that E splits abnormally. Hence, the conditions of the theorem are equivalent to the following.

- $F^{\vec{i}}E$ splits abnormally.

Corollary 3.6 (of Proposition 3.3)**.** *If the base sheaf E splits abnormally, every parabolic bundle of type $(E, \{F_*\})$ has a nonzero nilpotent vector.*

Proof. Apply Proposition 3.3 to the case $\vec{i} = (0, \ldots, 0)$. □

For rank two case, set $E_T = F^{\vec{i}}E$ where $i_x = 1$ (if $x \in T$) or 0 (if $x \notin T$).

Corollary 3.7. *In rank two case, the following three conditions are equivalent.*

(1) *There exists a nonzero nilpotent vector for* E_*.
(2) $\mathrm{Hom}_{\mathcal{O}_{\mathbf{P}^1}}(E_T, E_T \otimes \omega) \neq \{0\}$ *for some* $T \subset S$.
(3) E_T *splits abnormally for some* $T \subset S$.

Proof. Since a nilpotent cotangent vector cannot have full rank, the rank of any nonzero nilpotent vector is one. Therefore, we can apply Theorem 3.4 and Proposition 3.3. □

Consider $N = 2$ case. Let Z be the product of two flag varieties $\mathcal{F}$ of GL_r, which is viewed as the framed moduli space of parabolic structures on $\mathcal{O}_X^r$. Here the identification is as follows. We have a natural isomorphism $H^0(\mathcal{O}_X^r) \cong \mathcal{O}_X^r|_{x_1} \cong \mathcal{O}_X^r|_{x_2}$. Thus, we can associate each subspace $F_{x,i}\mathcal{O}_X^r|_x$ an inclusion $\iota_{x,i} : H^0(F_{x,i}\mathcal{O}_X^r|_x \otimes \mathcal{O}_X) \hookrightarrow H^0(\mathcal{O}_X^r)$. This assignment gives a point of Z. We call a GL_r-orbit in Z an extended Schubert cell.

Lemma 3.8. *Extended Schubert cells are the irreducible components of the stratification determined by the dimension of the set of nilpotent vectors.*

Proof. Let $S \not\ni \infty \in \mathbf{P}^1$ be a closed point. Let u be a rational function on $\mathbf{P}^1$ which has a unique pole of order one at ∞. Let z_1, z_2 be the coordinates of puncture points with respect to u. Let $\mathfrak{b}_i$ be a point of $\mathcal{F}$ corresponding to the filtration at z_i. $\mathfrak{b}_i$ is a Borel subalgebra of $\mathfrak{gl}_r$ which preserves the filtration. The fiber of cotangent bundle at that point is isomorphic to the nilpotent subalgebra $\mathfrak{n}_i = [\mathfrak{b}_i, \mathfrak{b}_i]$. The dual of the infinitesimal symmetry of a parabolic bundle E_* is isomorphic to

$$\left\{(n_1, n_2) \in \mathfrak{n}_1 \oplus \mathfrak{n}_2 \mid \sum \frac{n_i}{u - z_i} du \text{ is regular at } \mathbf{P}^1 \backslash S\right\}.$$

Since the above forms are regular outside of S, they are regular at ∞. It is of type $n/(u-z_1)du - n/(u-z_1)du$ for some $n \in \mathfrak{n}_1 \cap \mathfrak{n}_2$. These 1-forms are always nilpotent. We have $\dim GL_r.\ (\mathfrak{b}_1, \mathfrak{b}_2) = 2\dim \mathcal{F} - \dim(\mathfrak{n}_1 \cap \mathfrak{n}_2)$. Therefore, the locus with fixed dimensional nilpotent vectors is a union of equidimensional extended Schubert cells. Thus, we obtain the result. □

Graduate School of Mathematical Sciences
University of Tokyo
3-8-4, Komaba, Meguro-Ku
Tokyo 153-8914, Japan
e-mail: syuchan@ms.u-tokyo.ac.jp

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