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On the holomorphic rank-2 vector bundles with trivial discriminant over non-Kähler elliptic bundles

By

Marian Aprodu and Vasile Brînzănescu

1. Introduction

Let X be a smooth connected compact complex surface. A classical problem, in its simplified form, is to determine the pairs $(c_1, c_2) \in NS(X) \times \mathbf{Z}$, for which there exists a holomorphic rank-2 vector bundle E on X, having Chern classes $c_1(E) = c_1$ and $c_2(E) = c_2$.

For projective surfaces, the problem was solved by Schwarzenberger (cf. [13]): in this case, for any pair (c_1, c_2), there exists a smooth double covering $Y \xrightarrow{\varphi} X$, and a line bundle L over Y, such that $c_1(\varphi_* L) = c_1$ and $c_2(\varphi_* L) = c_2$.

In contrast to this situation, for non-projective surfaces, there is a natural *necessary* condition for the existence problem (cf. [2, Theorem 3.1], [6, Proposition 1.1], [8]):

$$\Delta(c_1, c_2) := \frac{1}{2}\left(c_2 - \frac{c_1^2}{4}\right) \geq 0.$$

In order to be able to find sufficient conditions for this existence problem, one tries to use the classical known methods for constructing vector bundles having given Chern classes c_1 and c_2. Serre's method, by far the most productive, brings out vector bundles whose Chern classes satisfy the inequality: $\Delta(c_1, c_2) \geq m(c_1)$ (cf. [2], see also [6], [11]), where

$$m(c_1) := -\frac{1}{2} \sup_{\mu \in NS(X)} \left(\frac{c_1}{2} - \mu\right)^2.$$

By using coverings of surfaces, we produced holomorphic vector bundles on primary Kodaira surfaces whose discriminants satisfy $0 \leq \Delta(c_1, c_2) < m(c_1)$, as direct images of line bundles (cf. [1], see also [15]). Unfortunately, this method is not very extensive and it is the aim of this Note to investigate its limits. We consider the case when X is a non-Kähler elliptic bundle over a curve B of genus at least 2. If $C \xrightarrow{\pi} B$ is a double covering, and $Y := C \times_B X \xrightarrow{\varphi} X$ denotes

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the canonically induced double covering, and one considers L a line bundle on Y, we compute the Chern classes of its direct image on X. We interpret the discriminant of the direct image as being the normalized relative degree of a morphism between the Prym variety of the covering, and the fiber of the elliptic fibration. The definition of the relative degree is introduced in the preliminary section. Consequently, we obtain necessary and sufficient conditions for a pair $(c_1, c_2) \in NS(X)$ with $\Delta(c_1, c_2) = 0$ be the Chern classes of a direct image of a line bundle over such an Y. We remark that there are concrete cases when this method is not capable to decide whether there exist or not holomorphic vector bundles with trivial discriminant. As Serre's method does not work either, in these cases we are not able to tell if there exists holomorphic rank-2 vector bundles or not.

2. Preliminaries

In this section we introduce the degree of a morphism between a polarised abelian surface and an elliptic curve, and give some properties of this notion. Although it seems to be a fairly natural definition, the lack of references concerning this notion leads us to give a more detailed account on the subject.

Throughout all this section, E is an elliptic curve.

Definition 2.1. Let (A, Θ) be an n-dimensional polarized abelian variety, and $\gamma : A \to E$ a morphism. We define the relative degree of γ with respect to Θ by:

$$\deg_\Theta(\gamma) := \begin{cases} \dfrac{1}{(n-1)!}(\{F_\gamma\}.\{\Theta\}^{n-1}) & \text{if} \quad \gamma \text{ is surjective,} \\ 0 & \text{if} \quad \gamma \equiv 0, \end{cases}$$

where, if γ is sujective, $\{F_\gamma\} \in H_{2n-2}(A, \mathbf{Z})$ denotes the class of a fiber of γ.

Remark 2.2. This terminology is justified by the fact that if (J_B, Θ_B) is the jacobian variety of a smooth curve B, and $\gamma : J_B \to E$ is induced by a curve morphism $B \xrightarrow{\pi} E$, then $\deg_{\Theta_B}(\gamma) = \deg(\pi)$.

Remark 2.3. If we change the polarization on A and set $\Theta' = N\Theta$, with $N > 0$ a positive integer, then $\deg_{\Theta'}(\gamma) = N^{n-1} \deg_\Theta(\gamma)$.

Remark 2.4. If we consider an isogeny of degree N, $A' \xrightarrow{\eta} A$, and set $\Theta' = \eta^*\Theta$, and $\gamma' = \gamma \circ \eta$, then $\deg_{\Theta'}(\gamma') = N \deg_\Theta(\gamma)$.

Lemma 2.5 (see also [1]). *Suppose Θ is of type $(d_1, \dots, d_n)$, and choose $\{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n\} \subset H_1(A, \mathbf{Z})$ a symplectic basis with respect to the polarization Θ. Then*

$$\deg_\Theta(\gamma) = \sum_{i=1}^{n} (d_1 \cdots d_{i-1} d_{i+1} \cdots d_n)(\gamma_*(\alpha_i).\gamma_*(\beta_i))$$

where $\gamma_* : H_1(A, \mathbf{Z}) \to H_1(E, \mathbf{Z})$ is the rational representation of γ.

Proof. Write $A = V/\Lambda$, with $\Lambda = H_1(A, \mathbf{Z})$, and consider the real coordinates $\{x_1, \dots, x_n, y_1, \dots, y_n\} \subset V$, which correspond to $\{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n\}$. In particular, $\{dx_1, \dots, dx_n, dy_1, \dots, dy_n\} \subset H^1_{DR}(A)$ is the dual basis of the chosen symplectic basis. Then, by means of [10] p. 107, the following identity holds in $H_2(A, \mathbf{Z})$:

$$\frac{1}{(n-1)!}\{\Theta\}^{n-1} = -\sum_{i=1}^{n}(d_1 \dots d_{i-1}d_{i+1} \dots d_n)(\alpha_i \star \beta_i),$$

where "$\star$" denotes the Pontrjagin product.

For any divisor D on A whose Chern class is represented, by means of Appell-Humbert theorem, by an alternating form $E_D : V \times V \to \mathbf{R}$, we have:

$$\frac{1}{(n-1)!}(\{D\}.\{\Theta\}^{n-1}) = -\sum_{i=1}^{n}(d_1 \dots d_{i-1}d_{i+1} \dots d_n)((\alpha_i \star \beta_i).\{D\})$$

and therefore (cf. [10, pp. 42–43, p. 104, p. 106]):

$$\frac{1}{(n-1)!}(\{D\}.\{\Theta\}^{n-1}) = \sum_{i=1}^{n}(d_1 \dots d_{i-1}d_{i+1} \dots d_n)E_D(\alpha_i, \beta_i). \tag{2.1}$$

If $\gamma \equiv 0$, the statement of the Lemma is trivial, so let us suppose γ is surjective. In this case, we consider $D = F_\gamma$ a fiber of the map $\gamma : A \to E$, then the associated alternating form is $E_D : V \times V \to \mathbf{R}$, defined by $E_D(u, v) = \gamma_*(u).\gamma_*(v)$ for all $u, v \in H_1(A, \mathbf{Z})$. Now apply the relation (2.1), and the proof of the Lemma is over.

Corollary 2.6. *Let (A_1, Θ_1), (A_2, Θ_2), be two polarized abelian varieties of dimensions n and m respectively, and $\gamma_i : A_i \to E$, $i = 1, 2$ be two morphisms. Set $A = A_1 \times A_2$ the product variety endowed with the naturally induced polarization $\Theta = \Theta_1 \times A_2 + A_1 \times \Theta_2$, and consider $\gamma = \gamma_1 \oplus \gamma_2$ the induced morphism. Then*

$$\deg_\Theta(\gamma) = \frac{(\Theta_2^m)}{m!}\deg_{\Theta_1}(\gamma_1) + \frac{(\Theta_2^n)}{n!}\deg_{\Theta_2}(\gamma_2).$$

3. The Chern classes of a direct image of a line bundle

Start with two curves B and C of genera g and $2g + r - 1$ respectively ($r \geq 0$), and $C \xrightarrow{\pi} B$ a double covering, ramified in $2r$ points. We shall use the following notation:

(J_B, Θ_B) the jacobian variety of B,
(J_C, Θ_C) the jacobian variety of C,
$i_B = \pi^* : J_B \longrightarrow J_C$ the pull-back morphism,
$\Theta'_B = i_B^*\Theta_C = 2\Theta_B$,

$P = [\mathrm{Ker}(J_C \xrightarrow{\pi_*} J_B)]^0$ the Prym variety of the covering π,
$i_P : P \hookrightarrow J_C$ the natural inclusion map,
$\Theta_P = i_P^*\Theta_C$,
$\rho : J_B \times P \longrightarrow J_C$ the natural isogeny (cf. [12]),
$\Theta = \rho^*\Theta_C = \Theta'_B \times P + J_B \times \Theta_P$,
$N = \deg(\rho) = \begin{cases} 2^{2g} & \text{if } \pi \text{ is ramified,} \\ 2^{2g-1} & \text{if } \pi \text{ is unramified.} \end{cases}$

It is well-known (see, for example, [10, p. 368]) that Θ_P is of type $(1, \dots, 1, 2, \dots, 2)$; here the 1 appears $(r-1)$-times. In particular, we have

$$\frac{(\Theta_P^{g+r-1})}{(g+r-1)!} = \begin{cases} 2^g & \text{if} \quad \pi \text{ is ramified,} \\ 2^{g-1} & \text{if} \quad \pi \text{ is unramified.} \end{cases}$$

Consider next E be an elliptic curve, and $X \longrightarrow B$ a non-Kähler elliptic bundle over B with fiber E (see, for example, [9], [3], [5] for precise definitions). The fibered-product $Y = X \times_B C \to C$ is also a non-Kähler elliptic bundle over C with fiber E, and there is a naturally induced double covering $Y \xrightarrow{\varphi} X$.

If one considers now a line bundle L on Y, the push-forward sheaf $\varphi_* L$ is actually a rank-2 vector bundle on X. Denoting also by $\varphi_* : H^2(Y, \mathbf{Z}) \to H^2(X, \mathbf{Z})$ the map induced by the push-forward on homology and Poincaré dualities on Y and X, by applying the Riemann-Roch Theorem for φ one gets the following formulae for the Chern classes of the push forward:

$$\begin{aligned} c_1(\varphi_* L) &\equiv \varphi_* c_1(L) \,\mathrm{modulo}\, \mathrm{Tors}(NS(X)) \\ \Delta(\varphi_* L) &= \frac{1}{8}[(\varphi_* c_1(L))^2 - 2c_1^2(L)]. \end{aligned} \tag{3.1}$$

Besides, for any other line bundle L' on Y, we have the relation:

$$c_1(\varphi_*(L \otimes L')) = c_1(\varphi_* L) + \varphi_* c_1(L'). \tag{3.2}$$

We would like to give next algebraic interpretations to the first Chern class, and the discriminant of a push-forward bundle, similar to the ones of [1]. To do this, we use again the description given in [4] (see also [7]) for the Néron-Severi group modulo its torsion elements. More precisely, there is an isomorphism, $NS(X)/\mathrm{Tors}(NS(X)) \cong \mathrm{Hom}(J_B, E)$; similarly for Y. In the sequel, for an element $c_1 \in NS(X)$, we denote by $\widehat{c}_1$ its class modulo $\mathrm{Tors}(NS(X))$, which is an element in $\mathrm{Hom}(J_B, E)$; similarly for Y.

Lemma 3.1. *With the notation above, for any line bundle L on Y, $\mathcal{E} = \varphi_* L$ is a rank-2 vector bundle on X with $\widehat{c_1(\mathcal{E})} = \widehat{c_1(L)} \circ i_B$ and*

$$\Delta(\mathcal{E}) = \frac{2^g}{4N} \deg_{\Theta_P}(\widehat{c_1(L)} \circ i_P).$$

Proof. Let $\widehat{c} \in NS(Y)/\mathrm{Tors}\big(NS(Y)\big)$ be the class associated to $c_1(L) \in NS(Y)$ and $\widehat{c}_1 \in NS(X)/\mathrm{Tors}\big(NS(X)\big)$ the class associated to $c_1(\mathcal{E}) \in NS(X)$.

Then the formulae (3.1) read:

$$\widehat{c}_1 = \varphi_*(\widehat{c}) \quad \text{and} \quad \Delta(\mathcal{E}) = \frac{1}{8}(\widehat{c}_1^2 - 2\widehat{c}^2). \tag{3.3}$$

For the first assertion of the conclusion, we refer to [1]. The second assertion follows from Section 1, and the considerations above in the following simple way. Lemma 1.3 of [1] tells us that $\widehat{c}^2 = -2\deg_{\Theta_C}(\widehat{c})$, and $\widehat{c}_1^2 = -2\deg_{\Theta_B}(\widehat{c}_1)$, and thus

$$\Delta(\mathcal{E}) = \frac{1}{4}(2\deg_{\Theta_C}(\widehat{c}) - \deg_{\Theta_B}(\widehat{c}_1)).$$

By means of the Remark 2.4, $\deg_{\Theta_C}(\widehat{c}) = \deg_{\Theta}(\widehat{c}\circ\rho)/N$. Apply next the Corollary 2.6 to conclude.

4. Direct images with trivial discriminant

The aim of this section is to investigate some of the cases when the double covering method can produce vector bundle with trivial discriminant. More precisely, we have the following

Theorem 4.1. *Let $X \to B$ be a non-Kähler elliptic bundle, and consider $(c_1, c_2) \in NS(X) \times \mathbf{Z}$ such that $\Delta(c_1, c_2) = 0$, and $c_1 \notin 2NS(X)$. Then there exists a double covering $C \to B$, and a line bundle L over $Y = X \times_B C \to C$ such that, denoting by $Y \xrightarrow{\varphi} X$ the induced double covering, the rank-2 vector bundle $\varphi_* L$ has Chern classes c_1 and c_2 if and only if the map $\widehat{c}_1[2] : J_B[2] \to E[2]$ induced between the 2-torsion elements is not surjective.*

Proof. Let us remark first that, since the fibers of the elliptic fibration are mapped to fibers, the relation (3.2) shows that it would suffice to prove the statement of the theorem modulo torsion elements in the Néron-Severi group of X (see also [1]). We denote, as usual, the genus of B by g.

Suppose first that $\widehat{c}_1[2] : J_B[2] \to E[2]$ is not surjective. Then its kernel, which we call H_1, is a subgroup of $J_B[2]$ with 2^{2g-1} elements. It is easy to see that there exists an element $u \in H_1$ which is orthogonal with respect to the Weil pairing on the whole H_1. In particular, as the element u is of order 2, it gives rise to an unramified covering $C \xrightarrow{\pi} B$. By means of the general theory developped in [12] (mainly Corollary 1, p. 332), if we denote by P the Prym variety of the covering f, then the jacobian variety J_C of C is isomorphic to a quotient of the product $J_B \times P$. Moreover, the map $\tilde{c} : J_B \times P \to E$ defined by $\tilde{c}(x, y) = \widehat{c}_1(x)$ descends to map $\widehat{c} : J_C \to E$. The calculations of the Section 3 tell us that the direct image of a line bundle on $Y = X \times_B C \to C$ whose first Chern class modulo torsion equals $\widehat{c}$ has the first Chern class modulo torsion exactly $\widehat{c}_1$, and trivial discriminant.

Conversely, by means of the Lemma 3.1, and [12] (Corollary 1, p. 332), the trivial discriminant can be obtained, unless $\widehat{c} \in 2NS(X)$, out of unramified coverings only. Now choose such an unramified covering $C \xrightarrow{\pi} B$, which is defined by an element $u \in J_B[2]$, and a Chern class modulo torsion on $Y = X \times_B C \to C$, $\widehat{c} : J_C \to E$ such that $\widehat{c}_1 = \widehat{c} \circ i_B$. Let us denote by H_1 the group of elements in $J_B[2]$ which are orthogonal (with respect to the Weil pairing) on u. Then H_1 has exactly 2^{2g-1} elements, and $\widehat{c}_1$ must vanish on H_1, which shows that $\widehat{c}_1[2]$ cannot be surjective.

Example 4.2. To see that the case when the map $\widehat{c}_1[2] : J_B[2] \to E[2]$ is surjective really occurs, we choose B to be a curve of genus two, which admits a morphism of degree 2 to E, and $\widehat{c}_1$ induced by this double covering. In this case, the method used above, cannot produce rank-2 vector bundles with trivial discriminant. Moreover, since $m(c_1) = 1/2$, Serre's method cannot either decide whether there exists or not rank-2 holomorphic vector bundles with trivial discriminant.

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Institute of Mathematics "Simion Stoilow"
Romanian Academy
P.O.Box 1-764, RO-70700 Bucharest, Romania
e-mail: Marian.Aprodu@imar.ro

Université de Grenoble 1
Laboratoire de Mathématiques
UMR 5582 du CNRS, Institut Fourier BP 74
F-38402 Saint Martin d'Heres Cedex, France
e-mail: aprodu@mozart.ujf-grenoble.fr

Institute of Mathematics "Simion Stoilow"
Romanian Academy
P.O.Box 1-764, RO-70700 Bucharest, Romania
e-mail: Vasile.Brinzanescu@imar.ro

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