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Some notes on Teichmüller shift mappings and the Teichmüller density

By

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Abstract

Kra [Kr] introduced a distance d_K on every hyperbolic Riemann surface R by means of Teichmüller shift mappings. Recently Gardiner and Lakic [GL2] defined a metric density λ, the Teichmüller density, on such a surface. The paper deals with some basic properties of the Teichmüller density λ and the distance d_K, giving some close relation between them. Particularly, it is shown that the distance function $d_K : R \times R \to \mathbb{R}$ is continuously differentiable off the diagonal, and the Teichmüller density λ is precisely the metric density of the infinitesimal form of the distance d_K and it is continuous on the whole surface R. Some related topics will also be discussed.

1. Introduction

When studying the self-maps of Riemann surfaces and the geometry of Teichmüller spaces, Kra [Kr] introduced a distance d_K on every hyperbolic Riemann surface R by means of Teichmüller shift mappings. It is known that the distance d_K has some close relation with the hyperbolic distance on R (see [EKK], [EL1], [EL2], [Ge], [Kr], [Kru3], [Liu], [Na1], [Re], [St2], [Te]). Recently, by using the infinitesimal Teichmüller norms of certain vector fields to Teichmüller spaces, Gardiner and Lakic [GL2] defined a metric density λ, the Teichmüller density, on such a surface R. They also use the Teichmüller density λ to study the hyperbolic density on R and characterize the uniform perfectness of closed sets in the complex plane $\overline{\mathbb{C}}$. In this note, we shall give some basic properties of the Teichmüller density λ and the distance d_K, giving some close relation between them. Particularly, we shall show that the Teichmüller density λ is precisely the metric density of the infinitesimal form of the distance d_K (see Theorems 1 and 2), that the distance function $d_K : R \times R \to \mathbb{R}$

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is continuously differentiable off the diagonal (see Theorem 6), and that the Teichmüller density λ is continuous on the whole surface R (see Theorem 4). We shall prove these results by studying the Teichmüller shift mappings and some holomorphic mappings from Riemann surfaces into Teichmüller spaces, and as by-products, we shall prove some results on the continuity of point shift differentials and extremal differentials (see Theorems 3 and 5).

2. Preliminaries

In this section, we shall give the precise definitions of Kra's distance d_K and the Teichmüller density λ. As already stated in Section 1, they are defined respectively by using the Teichmüller shift mappings and the infinitesimal Teichmüller norms of certain vector fields to Teichmüller spaces, and have close relation with the hyperbolic distance and the hyperbolic density, so we need to recall some basic definitions and fundamental results from hyperbolic geometry and quasiconformal Teichmüller theory.

We begin with the basic definitions and notations on hyperbolic density (and distance). Let $\Delta = \{z : |z| < 1\}$ denote the unit disk on the complex plane $\overline{\mathbb{C}}$. On Δ one can define the hyperbolic Poincaré metric $G_{H,\Delta} : T\Delta = \Delta \times \mathbb{C} \to \mathbb{R}$ by

$$G_{H,\Delta}(z, v) = \frac{|v|}{1 - |z|^2}, \tag{2.1}$$

with density $\rho_\Delta = \rho_\Delta(z)|dz|$ by

$$\rho_\Delta(z) = \frac{G_{H,\Delta}(z, v)}{|v|} = \frac{1}{1 - |z|^2}. \tag{2.2}$$

The Poincaré distance $d_{H,\Delta}(z_1, z_2)$ between two points z_1, z_2 induced by ρ_Δ is

$$d_{H,\Delta}(z_1, z_2) = \frac{1}{2} \log \frac{1 + \left|\dfrac{z_1 - z_2}{1 - \overline{z_1} z_2}\right|}{1 - \left|\dfrac{z_1 - z_2}{1 - \overline{z_1} z_2}\right|}. \tag{2.3}$$

Let R be a hyperbolic Riemann surface covered by the unit disk Δ. Then there is a Fuchsian group Γ such that $R = \Delta/\Gamma$. Let $\pi : \Delta \to R$ denote the canonical holomorphic projection. On R one can define the hyperbolic metric $G_{H,R} : TR \to \mathbb{R}$ by requiring that $\pi^* G_{H,R} = G_{H,\Delta}$, or precisely, for any $(z, v) \in T\Delta$, that

$$G_{H,R}(\pi(z), d_z\pi(v)) = G_{H,\Delta}(z, v), \tag{2.4}$$

with density $\rho_R = \rho_R(w)|dw|$ by

$$\rho_R(w(\pi(z)))|(w \circ \pi)'(z)| = \rho_\Delta(z) = \frac{1}{1 - |z|^2}, \tag{2.5}$$

where w is any local parameter on R. The hyperbolic distance $d_{H,R}(p_1, p_2)$ between two points p_1, p_2 induced by ρ_R is

$$d_{H,R}(p_1, p_2) = \inf\{d_{H,\Delta}(z_1, z_2) : \pi(z_1) = p_1, \pi(z_2) = p_2\}. \tag{2.6}$$

In the following, when there is no ambiguity, we set $d_{H,R} = d_H$, $G_{H,R} = G_H$, $\rho_R = \rho$.

Now we begin to define Kra's distance d_K by Teichmüller shift mappings. A (classical) Teichmüller shift mapping (see [Te]) is the uniquely extremal mapping T_δ which sends the zero point to $-\delta$ ($0 \le \delta < 1$) and fixes every boundary point of the unit disk Δ. It is a Teichmüller mapping with Beltrami coefficient μ_δ such that $\mu_0 = 0$, while for $\delta > 0$, $\mu_\delta = k_\delta |\phi_\delta|/\phi_\delta$, where $k_\delta > 0$, and ϕ_δ is a holomorphic function in $\Delta - \{0\}$, which has a first order pole at 0 and has unit L^1-norm.

In general, a Teichmüller shift mapping on a hyperbolic Riemann surface R is the uniquely extremal mapping T_{p_1,p_2} which sends p_1 to p_2 and is homotopic to the identity mapping modulo the ideal boundary ∂R. It is a Teichmüller mapping with Beltrami coefficient μ_{p_1,p_2} such that, for $p_2 = p_1$, $\mu_{p_1,p_2} = 0$, while for $p_2 \ne p_1$, $\mu_{p_1,p_2} = k_{p_1,p_2}|\phi_{p_1,p_2}|/\phi_{p_1,p_2}$, where $k_{p_1,p_2} > 0$, and ϕ_{p_1,p_2} is a holomorphic quadratic differential in $R - \{p_1\}$, which has a first order pole at p_1 and has unit L^1-norm. Following [St2], we call ϕ_{p_1,p_2} a point shift differential.

It is known that the Teichmüller shift mapping plays an important role in the theory of extremal quasiconformal mappings and Teichmüller spaces (see [EL2], [Ge], [Kr], [Re], [St2]) and in classical complex analysis (see [Kru3]) as well. As stated in the beginning, when studying the self-maps of Riemann surfaces and the geometry of Teichmüller spaces, Kra [Kr] introduced a distance d_K on every hyperbolic Riemann surface R by the Teichmüller shift mappings. The precise definition is as follows:

Definition 1. For any two points p_1 and p_2 in a surface S, Kra's distance d_K is defined as

$$d_K(p_1, p_2) = \frac{1}{2} \log \frac{1 + k_{p_1,p_2}}{1 - k_{p_1,p_2}}. \tag{2.7}$$

We have the following known results.

Theorem A. *The identity mapping id* : $(R, d_H) \to (R, d_K)$ *is not an isometry unless* $R = \overline{\mathbb{C}}_{pqr} = \overline{\mathbb{C}} - \{p, q, r\}$.

Remark 1. When R is of conformally finite type, Theorem A was proved independently by Kra [Kr] and Nag [Na1]. Later Liu [Liu] extended it to all hyperbolic Riemann surfaces of conformally infinite type with three exceptions: the cases when R is Δ, Δ with one puncture, or an annulus. Very recently, Earle and Lakic [EL2] gave an explicit and elementary proof of Theorem A in all cases.

Remark 2. It is a classical result of Teichmüller (see [Li]) that when R is the punctured sphere $\overline{\mathbb{C}}_{pqr}$, $d_K = d_H$. On the other hand, when R is simply connected, it is easy to see that d_K is uniquely determined by d_H. In fact, we have the following exact formula:

$$\log \frac{e^{d_K}+1}{e^{d_K}-1} = \mu\left(\frac{e^{2d_H}-1}{e^{2d_H}+1}\right),$$

where $\mu(r)$ is the conformal module of the Grötzsch ring domain whose boundary components are the unit circle and the line segment $\{x : 0 \le x \le r\}$. Noting that $\mu(r) = (1+o(1))\log(4/r)$ as $r \to 0$, we find that $d_K = d_H/2 + o(d_H)$ as $d_H \to 0$. For more details, see the papers [Re], [Te].

It is well known that the hyperbolic metric G_H is the infinitesimal form of the hyperbolic distance d_H, namely, for any $z \in \Delta$ and any number $v \in \mathbb{C}$, it holds that

$$G_H(\pi(z), d_z\pi(v)) = \lim_{t\to 0+} \frac{d_H(\pi(z), \pi(z+tv))}{t}. \tag{2.8}$$

Now we let G_K denote the infinitesimal form of the distance d_K, that is, for any $z \in \Delta$ and any number $v \in \mathbb{C}$, we have

$$G_K(\pi(z), d_z\pi(v)) = \lim_{t\to 0+} \frac{d_K(\pi(z), \pi(z+tv))}{t}. \tag{2.9}$$

Then, as stated in [EL2], a consequence of Theorems A and 5 of [EKL] is

Corollary B. *If R is not the punctured sphere $\overline{\mathbb{C}}_{pqr}$, then $d_K(p_1,p_2) < d_H(p_1,p_2)$ for any pair of distinct points p_1, p_2 in R. In addition, $G_K < G_H$ except at the points on the zero section.*

Remark 3. As remarked above, when R is the punctured sphere $\overline{\mathbb{C}}_{pqr}$, $G_K = G_H$, when R is simply connected, $G_K = G_H/2$. We shall show that $G_K > G_H/2$ whenever R is not simply connected, except at the points on the zero section.

Finally, we recall some basic definitions and some fundamental results from the theory of extremal quasiconformal mappings and Teichmüller spaces and define the Teichmüller density λ. For more details, see the papers [Ha], [Kru1], [RS2], [St1] and the books [Ga], [GL1].

Let $M(S)$ denote the unit ball of the space $Belt(S)$ of all essentially bounded Beltrami differentials $\mu = \mu(w)d\bar{w}/dw$ on a hyperbolic surface S, and $SA(S)$ the unit sphere of the space $A(S)$ of all holomorphic quadratic differentials $\phi = \phi(w)dw^2$ on S with finite L^1-norm

$$\|\phi\| = \iint_S |\phi| < +\infty.$$

For a given $\mu \in M(S)$, denote by f^μ the uniquely determined quasiconformal mapping on S with Beltrami coefficient μ and some normalized condition which can be specified from context. Two elements μ and ν in $M(S)$ are equivalent, and denoted by $\mu \sim \nu$, if f^μ is homotopic to f^ν (mod ∂S). Then $T(S) = M(S)/\sim$ is the Teichmüller space of S. Let Φ denote the canonical holomorphic projection $M(S) \to T(S)$, and $d_\mu\Phi$ denote the differential of Φ at the point $\Phi(\mu)$. We also denote by $M(\mu, S)$ the set of all elements $\nu \in M(S)$ equivalent to μ, and set

$$k(\mu) = \inf\{\|\nu\|_\infty : \nu \in M(\mu, S)\}. \tag{2.10}$$

We say that $\nu \in M(\mu, S)$ is extremal if $\|\nu\|_\infty = k(\mu)$. It is known that there always exists at least one extremal element in the class $M(\mu, S)$. A quasiconformal mapping f on S is said to be extremal if its Beltrami coefficient μ is extremal in its own class $M(\mu, S)$. The Teichmüller distance between two points $\Phi(\mu_1)$ and $\Phi(\mu_2)$ in $T(S)$ is defined as

$$d_T(\Phi(\mu_1), \Phi(\mu_2)) = \frac{1}{2}\log\frac{1 + k(\mu)}{1 - k(\mu)}, \tag{2.11}$$

where μ is the Beltrami coefficient of the mapping $f^{\mu_2} \circ (f^{\mu_1})^{-1}$.

Two elements μ and ν in $Belt(S)$ are infinitesimally equivalent and denoted by $\mu \approx \nu$, if $\iint_S \mu\phi = \iint_S \nu\phi$ for all $\phi \in A(S)$. Then $\mu \approx \nu$ iff $d_0\Phi(\mu) = d_0\Phi(\nu)$. So $B(S) = Belt(S)/\approx$ is the tangent space of $T(S)$ at the base point $\Phi(0)$. We denote by $Belt(\mu, S)$ the set of all elements ν in $Belt(S)$ infinitesimally equivalent to μ and set

$$\|\mu\|_S = \inf\{\|\nu\|_\infty : \nu \in Belt(\mu, S)\}. \tag{2.12}$$

By the Hahn-Banach extension theorem and Riesz representative theorem from functional analysis theory, $\|\mu\|_S$ has another equivalent definition, namely,

$$\|\mu\|_S = \sup_{\phi \in SA(S)} \left| \iint_S \mu\phi \right|. \tag{2.13}$$

We say that $\nu \in Belt(\mu, S)$ is infinitesimally extremal if $\|\nu\|_\infty = \|\mu\|_S$. Again, there always exists at least one infinitesimally extremal element in the class $Belt(\mu, S)$.

Now we can define the Teichmüller density $\lambda = \lambda(w)|dw|$ introduced by Gardiner and Lakic [GL2] as follows:

Definition 2. For any point $p \in R$, choose some local parameter w on some neighborhood of p with $w(p) = w_0$. Choose some vector field $V(w)(\partial/\partial w)$ on R such that $V(w_0) = 1$, $V(\phi) =: \iint_R \overline{\partial} V\phi = 0$ for all $\phi \in A(R)$, that is, $\overline{\partial} V \in Belt(0, R)$. Then, with $R_p = R - \{p\}$,

$$\lambda(w_0) =: \|\overline{\partial} V\|_{R_p} = \sup_{\phi \in SA(R_p)} \left| \iint_R \overline{\partial} V\phi \right|. \tag{2.14}$$

We proceed to state some basic results on extremality and infinitesimal extremality of Beltrami differentials. Hamilton, Krushkal, Reich and Strebel showed that a Beltrami coefficient $\mu \in M(S)$ is extremal if and only if it is infinitesimally extremal, which can happen precisely when μ satisfies the Hamilton-Krushkal condition $\|\mu\|_\infty = \|\mu\|_S$. In this case, there exists a so-called Hamilton sequence (ϕ_n) in $SA(S)$ such that $\iint_S \mu\phi_n \to \|\mu\|_\infty$ as $n \to \infty$. Furthermore, either μ is Teichmüller, meaning as usual that $\mu = \|\mu\|_\infty |\phi|/\phi$ for some $\phi \in SA(S)$, or (ϕ_n) is degenerating in the sense that $\phi_n \to 0$ locally uniformly in S.

Now let $\mu \in M(S)$ and set

$$h(\mu) = \inf\{\|\nu|_{S-E}\|_\infty : \nu \in M(\mu, S), E \subset S \quad \text{compact}\}. \tag{2.15}$$

Clearly, $h(\mu) \le k(\mu) \le \|\mu\|_\infty$. If $h(\mu) < k(\mu)$, then, by the Frame Mapping Theorem, $M(\mu, S)$ contains a Teichmüller extremal $k(\mu)|\phi|/\phi$, and any Hamilton sequence for $k(\mu)|\phi|/\phi$ must converge in norm to ϕ. In this case, following [ELi], we call $\Phi(\mu)$ a Strebel point.

Let $\mu \in Belt(S)$. Following Earle-Gardiner [EG], let

$$\beta(\mu) = \sup \limsup_{n\to\infty} \left| \iint_S \mu\phi_n \right|, \tag{2.16}$$

where the supremum is taken over all degenerating sequences (ϕ_n) in $SA(S)$. Clearly, $\beta(\mu) \le \|\mu\|_S \le \|\mu\|_\infty$. If $\beta(\mu) < \|\mu\|_S$, then, by the Infinitesimal Frame Mapping Theorem, $Belt(\mu, S)$ contains a Teichmüller infinitesimal extremal $\|\mu\|_S|\phi|/\phi$ and any Hamilton sequence for $\|\mu\|_S|\phi|/\phi$ must also converge in norm to ϕ. In this case, we call $d_0(\Phi)(\mu)$ an infinitesimal Reich-Strebel point.

We will make essential use of the following results:

Proposition C. *If $M(\mu, S)$ contains a Teichmüller extremal $k(\mu)|\phi|/\phi$, then*

$$\frac{1 + k(\mu)}{1 - k(\mu)} \le \iint_S |\phi| \frac{|1 + \mu\frac{\phi}{|\phi|}|^2}{1 - |\mu|^2}. \tag{2.17}$$

Proposition D. *For any Beltrami coefficient $\mu \in Belt(S)$, if $\mu_t = t\mu + o(t)$ uniformly in S, then*

$$k(\mu_t) = |t|\|\mu\|_S + o(t) = |t| \sup_{\phi\in SA(S)} \left| \iint_S \mu\phi \right| + o(t). \tag{2.18}$$

From (2.18) one can deduce that the infinitesimal form $G_T : TT(S) \to \mathbb{R}$ of the Teichmüller distance d_T is

$$\begin{aligned} G_T(\Phi(\mu), d_\mu\Phi(\nu)) &= \inf\left\{ \left\| \frac{\tilde\nu}{1-|\mu|^2} \right\|_\infty : d_\mu\Phi(\tilde\nu) = d_\mu\Phi(\nu) \right\} \\ &= \sup_{\phi\in SA(f^\mu(S))} \left| \iint_{f^\mu(S)} \left(\frac{\nu}{1-|\mu|^2} \frac{\partial f^\mu}{\overline{\partial f^\mu}} \right) \circ (f^\mu)^{-1} \phi \right|. \end{aligned} \tag{2.19}$$

3. The infinitesimal form G_K and the Teichmüller density λ

In this section, we shall show that the Teichmüller density λ is precisely the metric density of the infinitesimal form G_K. We first give some expressions of the infinitesimal form G_K and give a lower bound of G_K as a supplement to Corollary B.

Recall that $p = \pi(\zeta)$ is the universal covering from Δ onto R. In the following we shall use $\zeta = \pi^{-1}$ as a local parameter. Let $p \in R$ be given. Choose $z_0 \in \Delta$ with $\pi(z_0) = p$. Then there exists some $r > 0$ such that π is injective in $\Delta_r = \{\zeta : |\zeta - z_0| < r\}$. Set $D_r = \pi(\Delta_r)$. Then for any $q \in D_r$ there exists a unique $z \in \Delta_r$ with $\pi(z) = q$. Now we define

$$\tilde{f}(\zeta) = \zeta + \frac{z - z_0}{r}(r - |\zeta - z_0|), \qquad \zeta \in \Delta_r, \tag{3.1}$$

and set

$$f = \chi_{D_r}\pi \circ \tilde{f} \circ \pi^{-1} + \chi_{R-D_r} id, \tag{3.2}$$

where χ denotes the characteristic function of a set. Clearly, f is homotopic to the Teichmüller shift mapping $T_{p,q}$ modulo ∂R_p.

Now a direct computation will show that the Beltrami coefficient of $\tilde{f}$ is

$$\tilde{\mu}(\zeta) = \frac{(z - z_0)(\zeta - z_0)}{(z - z_0)\overline{(\zeta - z_0)} - 2r|\zeta - z_0|} = -\frac{z - z_0}{2r}\frac{\zeta - z_0}{|\zeta - z_0|} + o(|z - z_0|). \tag{3.3}$$

By definition, the Beltrami coefficient of f is $\mu = \chi_{\Delta_r}\tilde{\mu}$. So we get from (3.3) that

$$\mu(\zeta) = -\frac{z - z_0}{2r}\frac{\zeta - z_0}{|\zeta - z_0|}\chi_{\Delta_r} + o(|z - z_0|). \tag{3.4}$$

Now we can apply Proposition D to μ and obtain

$$k_{p,q} = \frac{|z - z_0|}{2r} \sup_{\phi \in SA(R_p)} \left| \iint_{\Delta_r} \frac{\zeta - z_0}{|\zeta - z_0|}\phi \right| + o(|z - z_0|). \tag{3.5}$$

Noting that $d_H(p, q) = d_H(z_0, z)$, we conclude that

$$|z - z_0| = (1 - |z_0|^2)d_H(p, q) + o(d_H(p, q)). \tag{3.6}$$

Consequently,

$$k_{p,q} = \frac{(1 - |z_0|^2)d_H(p, q)}{2r} \sup_{\phi \in SA(R_p)} \left| \iint_{\Delta_r} \frac{\zeta - z_0}{|\zeta - z_0|}\phi \right| + o(d_H(p, q)). \tag{3.7}$$

So

$$\begin{aligned} d_K(p, q) &= \frac{1}{2}\log\frac{1 + k_{p,q}}{1 - k_{p,q}} = k_{p,q} + o(k_{p,q}) \\ &= \frac{(1 - |z_0|^2)d_H(p, q)}{2r} \sup_{\phi \in SA(R_p)} \left| \iint_{\Delta_r} \frac{\zeta - z_0}{|\zeta - z_0|}\phi \right| + o(d_H(p, q)). \end{aligned} \tag{3.8}$$

Thus

$$\lim_{q\to p}\frac{d_K(p,q)}{d_H(p,q)}=\frac{(1-|z_0|^2)}{2r}\sup_{\phi\in SA(R_p)}\left|\iint_{\Delta_r}\frac{\zeta-z_0}{|\zeta-z_0|}\phi\right|. \tag{3.9}$$

Particularly,

$$G_K(p,v_p)=\frac{(1-|z_0|^2)G_H(p,v_p)}{2r}\sup_{\phi\in SA(R_p)}\left|\iint_{\Delta_r}\frac{\zeta-z_0}{|\zeta-z_0|}\phi\right|. \tag{3.10}$$

Now

$$\iint_{\Delta_r}\frac{\zeta-z_0}{|\zeta-z_0|}\phi=\int_0^r sds\int_{|\zeta|=s}\frac{\zeta}{s}\phi(\zeta+z_0)\frac{d\zeta}{i\zeta}=2\pi r\operatorname{Res}_{\zeta=z_0}\phi(\zeta), \tag{3.11}$$

where and in what follows, $\operatorname{Res}_{\zeta=z_0}\phi(\zeta)$ denotes the residue of $\phi(\zeta)$ at z_0. It should be noted that the residue $\operatorname{Res}_{\zeta=z_0}\phi(\zeta)$ depends on the choice of the local parameter z_0. So

$$G_K(p,v_p)=\pi(1-|z_0|^2)G_H(p,v_p)\sup_{\phi\in SA(R_p)}|\operatorname{Res}_{\zeta=z_0}\phi(\zeta)|. \tag{3.12}$$

To get a lower bound of $G_K(p,v_p)$, as done in [GL2], we use the Poincaré theta series operator Θ_Γ (see [Ga]), which is defined as

$$\Theta_\Gamma\psi=\sum_{\gamma\in\Gamma}\psi\circ\gamma(\gamma')^2.$$

Letting

$$\psi(\zeta)=\frac{(1-|z_0|^2)^2}{(\zeta-z_0)(1-\overline{z_0}\zeta)^3},$$

and $\phi=\Theta_\Gamma\psi$. Then $\phi=\phi(\zeta)d\zeta^2\in A(R_p)$, and $\|\phi\|=\|\Theta_\Gamma\psi\|\le\|\psi\|=2\pi$. Now

$$\operatorname{Res}_{\zeta=z_0}\phi(\zeta)=\frac{1}{1-|z_0|^2}.$$

So $G_K(p,v_p)\ge G_H(p,v_p)/2$. Note that if the equality holds for a non-zero vector v_p, then $\|\phi\|=\|\Theta_\Gamma\psi\|=\|\psi\|=2\pi$, which implies that Γ is the trivial group, or equivalently, R is simply connected. As remarked in Section 2, the converse is also true.

We can summarize the above as

Theorem 1. *The infinitesimal form G_K of the distance d_K has the expression* (3.10) *and* (3.12). *When R is the punctured sphere $\overline{\mathbb{C}}_{pqr}$, $G_K=G_H$, when R is simply connected, $G_K=G_H/2$, in all other cases, $G_H/2<G_K<G_H$ except at the points on the zero section.*

Now we show that the Teichmüller density λ is precisely the metric density of the infinitesimal form G_K. Let

$$V(\zeta)=\left(1-\frac{|\zeta-z_0|}{r}\right)\chi_{\Delta_r}. \tag{3.13}$$

Clearly, $V(z_0) = 1$. For any $\phi \in A(R_p)$, noting that

$$\overline{\partial} V = -\frac{1}{2r}\frac{\zeta - z_0}{|\zeta - z_0|}\chi_{\Delta_r}, \tag{3.14}$$

we obtain from (3.11) that

$$V(\phi) = \iint_R \overline{\partial} V \phi = -\frac{1}{2r}\iint_{\Delta_r}\frac{\zeta - z_0}{|\zeta - z_0|}\phi = -\pi \operatorname{Res}_{\zeta = z_0} \phi.$$

So $V(\phi) = 0$ for all $\phi \in A(R)$, and

$$\lambda(z_0) = \frac{1}{2r}\sup_{\phi \in SA(R_p)}\left|\iint_{\Delta_r}\frac{\zeta - z_0}{|\zeta - z_0|}\phi\right| = \pi \sup_{\phi \in SA(R_p)} |\operatorname{Res}_{\zeta = z_0}\phi|. \tag{3.15}$$

By (3.12) and (3.15), noting that $\rho(z_0) = 1/(1 - |z_0|^2)$, we obtain

$$\frac{G_K(p, v_p)}{G_H(p, v_p)} = \frac{\lambda(z_0)}{\rho(z_0)}, \tag{3.16}$$

which implies that λ is the metric density of G_K. We have obtained

Theorem 2. *The Teichmüller density λ is the metric density of the infinitesimal form G_K and has the expression* (3.15). *When R is the punctured sphere $\overline{\mathbb{C}}_{pqr}$, $\lambda = \rho$, when R is simply connected, $\lambda = \rho/2$, in all other cases, $\rho/2 < \lambda < \rho$.*

4. Point shift differential and extremal differential

Since $\overline{\partial} V \in Belt(0, R)$, we conclude that $\beta(\overline{\partial} V) = 0$. By the Infinitesimal Frame Mapping Theorem, there exists unique ϕ in $SA(R_p)$, which we denote by ϕ_p, such that $\lambda(z_0)|\phi_p|/\phi_p \in Belt(\overline{\partial} V, R_p)$. For simplicity, we shall call ϕ_p an extremal differential (at the point p). Note that by definition,

$$\begin{aligned}\lambda(z_0) &= \sup_{\phi \in SA(R_p)} \operatorname{Re}\iint_R \overline{\partial} V \phi \\ &= \frac{1}{2r}\sup_{\phi \in SA(R_p)} \operatorname{Re}\iint_{\Delta_r}\frac{\zeta - z_0}{|\zeta - z_0|}\phi \\ &= \pi \sup_{\phi \in SA(R_p)} \operatorname{Re}(\operatorname{Res}_{\zeta = z_0}\phi).\end{aligned} \tag{4.1}$$

Then ϕ_p is the unique element in $SA(R_p)$ which attains the first supremum in (4.1), and every element $\phi \in A(R_p)$ satisfies

$$\operatorname{Res}_{\zeta = z_0}\phi = -\frac{\lambda(z_0)}{\pi}\iint_R \frac{|\phi_p|}{\phi_p}\phi. \tag{4.2}$$

Here it should be pointed out that the extremal differential ϕ_p depends on the choice of the local parameter z_0.

Remark 4. This is a convenient place to point out that the Teichmüller density $\lambda = \lambda_R$ is monotone, namely, $\lambda_{R_0} \geq \lambda_R$ holds for each Riemann surface and for each subdomain R_0 of R. In fact, if $p \in R_0$, and $\pi(z_0) = p$, then $\lambda_{R_0}(z_0) = \pi \sup_{\psi \in SA(R_{0p})} |\operatorname{Res}_{\zeta=z_0} \psi|$, $\lambda_R(z_0) = \pi \sup_{\phi \in SA(R_p)} |\operatorname{Res}_{\zeta=z_0} \phi|$. Noting that for any $\phi \in SA(R_p)$, $\psi = \phi/\|\phi\|_{R_0} \in SA(R_{0p})$, we conclude that $\lambda_{R_0}(z_0) \geq \lambda_R(z_0)$.

Remark 5. It is known that the Poincaré density is holomorphically contractive, namely, $f^*\rho_S \leq \rho_R$ for any holomorphic mapping f between two Riemann surfaces R and S, and it is invariant under covering projections, that is, $f^*\rho_S = \rho_R$ when $f : R \to S$ is a covering. Neither is true for the Teichmüller density λ, however. In fact, let $\pi : \Delta \to R$ be a universal covering for a non-simply connected Riemann surface R, then $\pi^*\lambda_R > \pi^*(\rho_R/2) = \rho_\Delta/2 = \lambda_\Delta$, except at the points on the zero section. More generally, for any covering mapping $\pi : R \to S$ between two Riemann surfaces, it is easy to see that $\pi^*\lambda_S \geq \lambda_R$.

We recall that $k_{p,q}|\phi_{p,q}|/\phi_{p,q}$ is the Beltrami coefficient of the Teichmüller shift mapping $T_{p,q}$. Now we prove

Theorem 3. *Under the notations before* ($\pi(z_0) = p$, $\pi(z) = q$, $d_\rho(z_0, z) = d_\rho(p, q)$),
(1) $\lim_{q \to p} \|(z - z_0)/(|z - z_0|)\phi_{p,q} - \phi_p\| = 0$;
(2) $\lim_{q \to p} \|(z - z_0)/(|z - z_0|)\phi_{q,p} + \phi_p\| = 0$.

Proof. We apply Proposition C to μ (which is the Beltrami coefficient of the mapping f by (3.2)) and $k_{p,q}|\phi_{p,q}|/\phi_{p,q}$ and obtain

$$\frac{1 + k_{p,q}}{1 - k_{p,q}} \leq \iint_R |\phi_{p,q}| \frac{|1 + \mu \frac{\phi_{p,q}}{|\phi_{p,q}|}|^2}{1 - |\mu|^2}. \tag{4.3}$$

By (3.5) and (3.15) we get

$$k_{p,q} = |z - z_0|\lambda(z_0) + o(|z - z_0|)$$

and so

$$\frac{1 + k_{p,q}}{1 - k_{p,q}} = 1 + 2|z - z_0|\lambda(z_0) + o(|z - z_0|). \tag{4.4}$$

By (3.4) and (3.14) we have

$$\mu(\zeta) = (z - z_0)\overline{\partial} V(\zeta) + o(|z - z_0|)$$

and so

$$\iint_R |\phi_{p,q}| \frac{|1 + \mu \frac{\phi_{p,q}}{|\phi_{p,q}|}|^2}{1 - |\mu|^2} = 1 + 2 \operatorname{Re} \iint_R (z - z_0)\overline{\partial} V \phi_{p,q} + o(|z - z_0|). \tag{4.5}$$

Since $\lambda(z_0)|\phi_p|/\phi_p \in Belt(\overline{\partial}V, R_p)$, we get from (4.3)–(4.5) that

$$\lim_{q\to p} \mathrm{Re} \iint_R \frac{|\phi_p|}{\phi_p} \frac{z-z_0}{|z-z_0|}\phi_{p,q} = 1,$$

which implies that $((z-z_0)/|z-z_0|\phi_{p,q})$ is a Hamilton sequence for the infinitesimal extremal Beltrami differential $\lambda(z_0)|\phi_p|/\phi_p$ as $q \to p$. Since $\lambda(z_0)|\phi_p|/\phi_p$ represents an infinitesimal Reich-Strebel point, $((z-z_0)/|z-z_0|\phi_{p,q})$ must converge in norm to ϕ_p as $q \to p$. This finishes the proof of the first conclusion.

Noting that $T_{q,p} = T_{p,q}^{-1}$, we conclude that

$$\phi_{p,q} = -(1-k_{p,q}^2)\phi_{q,p} \circ T_{p,q}(\partial T_{p,q})^2. \tag{4.6}$$

So the second conclusion follows from the first one. □

Example 1. When $R = \Delta$, $\pi = id$, and the computation in Section 3 shows that

$$\phi_{z_0} = -\frac{1}{2\pi}\frac{(1-|z_0|^2)^2}{(1-\overline{z_0}\zeta)^3(\zeta-z_0)}.$$

So in this case we have

$$\lim_{z\to z_0}\left\| \frac{z-z_0}{|z-z_0|}\phi_{z_0,z} + \frac{1}{2\pi}\frac{(1-|z_0|^2)^2}{(1-\overline{z_0}\zeta)^3(\zeta-z_0)} \right\| = 0. \tag{4.7}$$

Particularly, for the classical Teichmüller shift mapping, $\|\phi_\delta - 1/2\pi\zeta\| \to 0$ as $\delta \to 0+$. On the other hand, by (4.2) we obtain that for any $z_0 \in \Delta$ and any $\phi \in A(\Delta_{z_0})$, it holds that

$$\frac{1}{2\pi(1-|z_0|^2)}\iint_\Delta \frac{(1-\overline{z_0}\zeta)^3(\zeta-z_0)}{|(1-\overline{z_0}\zeta)^3(\zeta-z_0)|}\phi = \mathrm{Res}_{\zeta=z_0}\,\phi. \tag{4.8}$$

So we have the following reproducing formula for functions $\phi \in A(\Delta)$,

$$\phi(z) = \frac{1}{2\pi(1-|z|^2)}\iint_\Delta \frac{(1-\bar{z}\zeta)^3}{|(1-\bar{z}\zeta)^3(\zeta-z)|}\phi. \tag{4.9}$$

5. Smoothness of the distance d_K and the Teichmüller density λ

We choose $p_0 \in R$ and consider the map $F: R \to T(R_{p_0})$ defined as

$$F(p) = \Phi(\mu_{p_0,p}), \tag{5.1}$$

where $\mu_{p_0,p}$ is the Beltrami coefficient of the Teichmüller shift mapping $T_{p_0,p}$. By a result of Bers [Be] (see also [Na2]), F is a holomorphic mapping. By definition,

$$d_K(p_1,p_2) = d_T(F(p_1), F(p_2)), \tag{5.2}$$

so $G_K = F^*G_T$, or more precisely, $G_K(p, v_p) = G_T(F(p), d_pF(v_p))$, that is, the infinitesimal form G_K is the pull-back of the Teichmüller metric G_T under the holomorphic mapping F. On the other hand, it is well known that G_T is continuous on the whole tangent bundle, and it is even continuously differentiable except at the points on the zero section when R is of conformally finite type (see [Ga]). Since the map F is holomorphic, we obtain the following theorem.

Theorem 4. *The infinitesimal form G_K is continuous on the whole bundle TR. Consequently, the Teichmüller density λ is continuous on the whole surface R. Furthermore, when R is of conformally finite type, G_K is continuously differentiable except at the points on the zero section, so λ is continuously differentiable on the whole surface R.*

Remark 5. It is known that the holomorphic curvature of the Teichmüller metric G_T is identically equal to -4 (see [AP] for more details). So the holomorphic curvature of G_K is bounded above by -4.

Remark 6. Let d_K^i denote the inner distance induced by d_K, or equivalently, it is the integral of the infinitesimal form G_K. Since $F(R) \subset T(R_{p_0})$ does not contain any Teichmüller geodesic segment (see [EL2], Theorem 2), we conclude that $d_K < d_K^i$ on the whole surface unless $R = \overline{\mathbb{C}}_{pqr}$, so d_K is not an inner distance. It is known that both the hyperbolic distance d_H and the Teichmüller distance d_T are inner.

An immediate consequence of Theorem 4 is the continuity of extremal differentials. For any point $p \in R$, under the local parameter $z = \pi^{-1}$, $\lambda(z) = -\pi \operatorname{Res}_{\zeta=z} \phi_p$. Now let $p_n \to p$. By the continuity of λ, $\operatorname{Res}_{\zeta=z_n} \phi_{p_n} \to \operatorname{Res}_{\zeta=z} \phi_p$. On the other hand, since (ϕ_{p_n}) is a normal family, we conclude without loss of generality that (ϕ_{p_n}) converges to some $\phi \in A(R_p)$ locally uniformly in R_p. By Fatou's Lemma, $\|\phi\| \leq 1$. Since $\operatorname{Res}_{\zeta=z_n} \phi_{p_n} \to \operatorname{Res}_{\zeta=z} \phi$, $\operatorname{Res}_{\zeta=z} \phi = \operatorname{Res}_{\zeta=z} \phi_p$, so we must have $\|\phi\| = 1$ and so $\phi = \phi_p$. Consequently, by Lebesgue's dominated convergence theorem we have $\|\phi_{p_n} - \phi_p\| \to 0$. We have proved

Theorem 5. $\lim_{q \to p} \|\phi_q - \phi_p\| = 0$.

We also have some corresponding properties of the distance function d_K. We consider the continuous function $G : R \to \mathbb{R}$ defined as

$$G(p) = d_K(p_0, p) = d_T(F(p_0), F(p)) = d_T(\Phi(0), \Phi(\mu_{p_0,p})). \tag{5.3}$$

When R is of conformally finite type, it is known that d_T is continuously differentiable except at the points on the diagonal (see [Ea] or [Ga]). Since F is holomorphic, G is continuously differentiable whenever $p \neq p_0$. When R is of conformally infinite type, by a result of Lakic [La], the function $\Phi(\nu) \to d_T(\Phi(0), \Phi(\nu))$ is continuously differentiable at $\Phi(\mu)$ if it is a Strebel point. Clearly, when $p \neq p_0$, $F(p)$ is a Strebel point in $T(R_{p_0})$. So G is again continuously differentiable whenever $p \neq p_0$. So we have

Theorem 6. *The function $G : R \to \mathbb{R}$ is continuously differentiable except at the point p_0. Consequently, the distance function $d_K : R \times R \to \mathbb{R}$ is continuously differentiable off the diagonal.*

Remark 7. Recall that

$$k_{p_0,p} = \frac{e^{2d_K(p_0,p)} - 1}{e^{2d_K(p_0,p)} + 1} = \frac{e^{2G(p)} - 1}{e^{2G(p)} + 1},$$

so $k_{p_0,p}$ is also continuously differentiable except at the point p_0. It is also logarithmically pluri-subharmonic on the whole surface R. In fact, let g_T denote the Green function of a Teichmüller space, then (see [Kru2])

$$g_T = \log \frac{e^{2d_T} - 1}{e^{2d_T} + 1}, \tag{5.4}$$

and so

$$g_T(F(p_0), F(p)) = \log \frac{e^{2d_K(p_0,p)} - 1}{e^{2d_K(p_0,p)} + 1} = \log \frac{e^{2G(p)} - 1}{e^{2G(p)} + 1} = \log k_{p_0,p}, \tag{5.5}$$

which implies that $k_{p_0,p}$ is logarithmically pluri-subharmonic on the whole surface R.

In the rest of the section, we shall give the derivatives of the mapping $F : R \to T(R_{p_0})$ and the function $G : R \to \mathbb{R}$, which is of independent interest.

Proposition 7. *Under the local parameter $z = \pi^{-1}$, for any $z \in \Delta$ and non-zero $v \in \mathbb{C}$,*

$$\begin{aligned} &d_z(F \circ \pi)(v) \\ &\quad = d_{\mu_{p_0,\pi(z)}} \Phi \left((1 - k^2_{p_0,\pi(z)}) G_K(\pi(z), d_z\pi(v)) \frac{v}{|v|} \frac{|\phi_{\pi(z)}|}{\phi_{\pi(z)}} \circ T_{p_0,\pi(z)} \frac{\overline{\partial T_{p_0,\pi(z)}}}{\partial T_{p_0,\pi(z)}} \right). \end{aligned}$$

Proof. Since we have already known that F is holomorphic, we only need to calculate

$$\begin{aligned} &d_z(F \circ \pi)(v) \\ &\quad = \lim_{t \to 0+} \frac{F(\pi(z + tv)) - F(\pi(z))}{t} = \lim_{t \to 0+} \frac{\Phi(\mu_{p_0,\pi(z+tv)}) - \Phi(\mu_{p_0,\pi(z)})}{t}. \end{aligned} \tag{5.6}$$

Let μ_t denote the Beltrami coefficient of the mapping $T_{\pi(z),\pi(z+tv)} \circ T_{p_0,\pi(z)}$, then $\Phi(\mu_t) = \Phi(\mu_{p_0,\pi(z+tv)}) = F(\pi(z + tv))$ and it is a holomorphic mapping. Now

$$\mu_t = \frac{\mu_{p_0,\pi(z)} + \mu_{\pi(z),\pi(z+tv)} \circ T_{p_0,\pi(z)} \frac{\overline{\partial T_{p_0,\pi(z)}}}{\partial T_{p_0,\pi(z)}}}{1 + \overline{\mu}_{p_0,\pi(z)} \mu_{\pi(z),\pi(z+tv)} \circ T_{p_0,\pi(z)} \frac{\overline{\partial T_{p_0,\pi(z)}}}{\partial T_{p_0,\pi(z)}}}. \tag{5.7}$$

Since

$$\begin{aligned} &\mu_{\pi(z),\pi(z+tv)} \\ &\quad = k_{\pi(z),\pi(z+tv)} \frac{|\phi_{\pi(z),\pi(z+tv)}|}{\phi_{\pi(z),\pi(z+tv)}} = (tG_K(\pi(z), d_z\pi(v)) + o(t)) \frac{|\phi_{\pi(z),\pi(z+tv)}|}{\phi_{\pi(z),\pi(z+tv)}}, \end{aligned}$$

and by Theorem 3, $\|v\phi_{\pi(z),\pi(z+tv)} - |v|\phi_{\pi(z)}\| \to 0$ as $t \to 0+$, we conclude by (5.7) that

$$\mu'(0+) = (1-k^2_{p_0,\pi(z)})G_K(\pi(z), d_z\pi(v))\frac{v}{|v|}\frac{|\phi_{\pi(z)}|}{\phi_{\pi(z)}} \circ T_{p_0,\pi(z)}\frac{\overline{\partial T_{p_0,\pi(z)}}}{\partial T_{p_0,\pi(z)}}. \tag{5.8}$$

Consequently, by (5.6) and (5.8) we obtain as required that

$$\begin{aligned} d_z(F\circ\pi)(v) &= \frac{d}{dt}\Phi(\mu_t)|_{t=0+} = d_{\mu_0}\Phi(\mu'(0+)) \\ &= d_{\mu_{p_0,\pi(z)}}\Phi\left((1-k^2_{p_0,\pi(z)})G_K(\pi(z), d_z\pi(v))\frac{v}{|v|}\frac{|\phi_{\pi(z)}|}{\phi_{\pi(z)}} \circ T_{p_0,\pi(z)}\frac{\overline{\partial T_{p_0,\pi(z)}}}{\partial T_{p_0,\pi(z)}}\right) \end{aligned}$$

□

Using the formula of dF, we can examine directly that $G_K = F^*G_T$. In fact, under the local parameter $z = \pi^{-1}$, by (2.19),

$$\begin{aligned} &G_T(F(\pi(z)), d_z(F\circ\pi)(v)) \\ &\quad = G_T(\Phi(\mu_{p_0,\pi(z)}), d_{\mu_{p_0,\pi(z)}}\Phi(\mu'(0+))) \\ &\quad = \sup_{\phi\in SA(R_{\pi(z)})}\left|\iint_R \left(\frac{\mu'(0+)}{1-k^2_{p_0,\pi(z)}}\frac{\overline{\partial T_{p_0,\pi(z)}}}{\overline{\partial T_{p_0,\pi(z)}}}\right)\circ (T_{p_0,\pi(z)})^{-1}\phi\right| \\ &\quad = \sup_{\phi\in SA(R_{\pi(z)})}\left|\iint_R G_K(\pi(z), d_z\pi(v))\frac{v}{|v|}\frac{|\phi_{\pi(z)}|}{\phi_{\pi(z)}}\phi\right| = G_K(\pi(z), d_z\pi(v)). \end{aligned}$$

Now we give the derivative of the function G. Noting that

$$\begin{aligned} G(\pi(z+tv)) &= d_T(\Phi(0), F(\pi(z+tv))) \\ &= d_T(\Phi(0), F(\pi(z)) + t d_z(F\circ\pi)(v) + o(t)), \end{aligned} \tag{5.9}$$

when $z \in \Delta - \pi^{-1}(p_0)$, $F(\pi(z))$ is a Strebel point in $T(R_{p_0})$, by Lemma 2 in [La], we can obtain

(5.10)

$$\begin{aligned} d_z(G\circ\pi)(v) &= \frac{1}{1-k^2_{p_0,\pi(z)}}\operatorname{Re}\iint_R \mu'(0+)\phi_{p_0,\pi(z)} \\ &= \operatorname{Re}\iint_R G_K(\pi(z), d_z\pi(v))\frac{v}{|v|}\frac{|\phi_{\pi(z)}|}{\phi_{\pi(z)}}\circ T_{p_0,\pi(z)}\frac{\overline{\partial T_{p_0,\pi(z)}}}{\partial T_{p_0,\pi(z)}}\phi_{p_0,\pi(z)} \\ &= -\operatorname{Re}\iint_R G_K(\pi(z), d_z\pi(v))\frac{v}{|v|}\frac{|\phi_{\pi(z)}|}{\phi_{\pi(z)}}\phi_{\pi(z),p_0} \\ &= \pi \operatorname{Re}(v \operatorname{Res}_{\zeta=z}\phi_{\pi(z),p_0}). \end{aligned}$$

When $z \in \pi^{-1}(p_0)$, by Proposition D and (5.9) we obtain

$$
\begin{aligned}
G(\pi(z+tv)) &= d_T(\Phi(0), td_z(F\circ\pi)(v) + o(t)) \\
&= |t| \sup_{\phi\in SA(R_{p_0})} \left| \iint_R G_K(\pi(z), d_z\pi(v)) \frac{v}{|v|} \frac{|\phi_{\pi(z)}|}{\phi_{\pi(z)}} \phi \right| + o(t) \\
&= |t| G_K(\pi(z), d_z\pi(v)) + o(t) = -|t|\pi|v| \operatorname{Res}_{\zeta=z} \phi_{p_0} + o(t) \\
&= |tv|\lambda(z) + o(t).
\end{aligned} \tag{5.11}
$$

Clearly, G is not differentiable at p_0.

We state the above discussion as a proposition, which can be served as a supplement to Theorem 6.

Proposition 8. *The function G is not differentiable at p_0. Under the local parameter $z = \pi^{-1}$, for any $z \in \Delta - \pi^{-1}(p_0)$ and non-zero $v \in \mathbb{C}$, it holds that*

$$
\begin{aligned}
d_z(G\circ\pi)(v) &= -\operatorname{Re} \iint_R G_K(\pi(z), d_z\pi(v)) \frac{v}{|v|} \frac{|\phi_{\pi(z)}|}{\phi_{\pi(z)}} \phi_{\pi(z),p_0} \\
&= \pi \operatorname{Re}(v \operatorname{Res}_{\zeta=z} \phi_{\pi(z),p_0}).
\end{aligned}
$$

There is a generalization of the concept of Teichmüller shift mapping, which is called a point shift mapping. More precisely, let $Q : S \to R$ be a quasiconformal mapping, and $q_0 \in S$ is a fixed point. An extremal mapping $f_{Q,p}$ which sends q_0 to a point $p \in R$ and is homotopic to Q modulo ∂S is called a point shift mapping. Note that $f_{Q,p}$ need not be uniquely extremal. Let $E(Q)$ denote the set of extremal mappings in the class of quasiconformal mappings which are homotopic to Q modulo ∂S and set

$$
V_Q = \{p = f(q_0) : f \in E(Q)\}. \tag{5.12}
$$

Then V_Q is called the variablity set of Q with respect to q_0 (see [St2], [EL1], [EL2]). Strebel [St2], Earle and Lakic [EL1-2] proved that V_Q is compact, connected and simply connected. When $p \in R - V_Q$, $f_{Q,p}$ is uniquely extremal, it is a Teichmüller mapping with Beltrami coefficient $\mu_{Q,p} = k_{Q,p}|\psi_{Q,p}|/\psi_{Q,p}$, where $k_{Q,p} > 0$, $\psi_{Q,p} \in SA(S_{q_0})$ has a simple pole at q_0. The inverse mapping $f_{Q,p}^{-1}$ is also a Teichmüller mapping with Beltrami coefficient $k_{Q,p}|\phi_{Q,p}|/\phi_{Q,p}$, where $\phi_{Q,p} \in SA(R_p)$ has a simple pole at p and $\psi_{Q,p} = -(1-k_{Q,p}^2)\phi_{Q,p} \circ f_{Q,p}(\partial f_{Q,p})^2$.

Consider $F_Q : R \to T(S_{q_0})$ and $G_Q : R \to \mathbb{R}$ as

$$
F_Q(p) = \Phi(\mu_{Q,p}), \tag{5.13}
$$

$$
G_Q(p) = d_T(\Phi(0), F_Q(p)) = d_T(\Phi(0), \Phi(\mu_{Q,p})). \tag{5.14}
$$

Then F_Q is holomorphic on R and G_Q is continuously differentiable in $R - V_Q$. Note that we still have $d_K(p_1,p_2) = d_T(F_Q(p_1), F_Q(p_2))$ and so $G_K = F_Q^* G_T$. By the same reasoning as above, we can obtain

Proposition 9. *Under the local parameter $z = \pi^{-1}$, for any $z \in \Delta$ and non-zero $v \in \mathbb{C}$,*

$$d_z(F_Q \circ \pi)(v)$$
$$= d_{\mu_{Q,\pi(z)}} \Phi \left(G_K(\pi(z), d_z\pi(v)) \frac{v}{|v|} (1 - |\mu_{Q,\pi(z)}|^2) \frac{|\phi_{\pi(z)}|}{\phi_{\pi(z)}} \circ f_{Q,\pi(z)} \overline{\frac{\partial f_{Q,\pi(z)}}{\partial f_{Q,\pi(z)}}} \right).$$

Proposition 10. *Under the local parameter $z = \pi^{-1}$, for any $z \in \Delta - \pi^{-1}(V_Q)$ and non-zero $v \in \mathbb{C}$, it holds that*

$$d_z(G_Q \circ \pi)(v) = -\operatorname{Re} \iint_R G_K(\pi(z), d_z\pi(v)) \frac{v}{|v|} \frac{|\phi_{\pi(z)}|}{\phi_{\pi(z)}} \phi_{Q,\pi(z)}$$
$$= \pi \operatorname{Re}(v \operatorname{Res}_{\zeta = z} \phi_{Q,\pi(z)}).$$

For further properties of the function G_Q, see Theorem 3 in [EL2].

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