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Minimal algebraic surfaces of general type with $c_1^2 = 3$, $p_g = 1$ and $q = 0$, which have non-trivial 3-torsion divisors

By

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Abstract

We shall give a concrete description of minimal algebraic surfaces X's defined over $\mathbb{C}$ of general type with the first chern number 3, the geometric genus 1 and the irregularity 0, which have non-trivial 3-torsion divisors. Namely, we shall show that the fundamental group is isomorphic to $\mathbb{Z}/3$, and that the canonical model of the universal cover is a complete intersection in $\mathbb{P}^4$ of type $(3, 3)$.

0. Introduction

In this paper, we shall give a concrete description of minimal algebraic surfaces X's defined over $\mathbb{C}$ of general type with $c_1^2 = 3$, $p_g = 1$, $q = 0$ and $\mathbb{Z}/3 \subset \mathrm{Tors}(X)$. Here, as usual, c_1, p_g, q and $\mathrm{Tors}(X)$ are the first chern class, the geometric genus, the irregularity and the torsion part of the Picard group of X, respectively.

In classical classification theories of the numerical Godeaux surfaces (i.e. minimal algebraic surfaces of general type with $c_1^2 = 1$, $p_g = 0$, $q = 0$), one fixes the torsion group or the fundamental group as an additional invariant, and finds concrete descriptions for each case (see for example [1] and [2]). For example, Miyaoka showed that if the torsion group $\mathrm{Tors}(X)$ is isomorphic to $\mathbb{Z}/5$, then the fundamental group π_1 is isomorphic to $\mathbb{Z}/5$ and the canonical model of the universal cover is a quintic surface in $\mathbb{P}^3$ (see [1]). It is well-known that the order $\sharp\mathrm{Tors}(X)$ is at most 5 for the numerical Godeaux surfaces.

Similar theories can be developed for other cases of numerical invariants, and there are many papers related to this direction. Minimal algebraic surfaces with $c_1^2 = 1$, $p_g = 1$ and $q = 0$ are completely understood ([10] and [12]), while minimal algebraic surfaces with $c_1^2 = 2$, $p_g = 1$ and $q = 0$ are classified in [13] and [14].

Consider the case $c_1^2 = 3$, $p_g = 1$ and $q = 0$. In this case, we see easily that the order of the torsion group $\mathrm{Tors}(X)$ is at most 6. Examples of surfaces

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with $c_1^2 = 3$, $p_g = 1$ and $q = 0$ can be found in Todorov's paper [11]. In the present paper, we consider the case $\mathbb{Z}/3 \subset \mathrm{Tors}(X)$, and give a concrete description of such surfaces. Namely, we shall show that the fundamental group is isomorphic to $\mathbb{Z}/3$ in this case, and that the canonical model of the universal cover is a complete intersection in $\mathbb{P}^4$ of type $(3,3)$ (Theorem 1). Using this result, we shall show that the number of moduli of X is 14 if a canonical divisor of X is ample. We shall also show that the case $\mathrm{Tors}(X) \simeq \mathbb{Z}/5$ is impossible (Proposition 1 or Remark 2).

In the present paper, following the method due to Miyaoka [1] and Reid [2], we take an unramified cover $Y \to X$ corresponding to a 3-torsion divisor, and study the canonical image of Y. Since we have $K_Y^2 = 9$ and $p_g(Y) = 5$, we can use the results and methods given by Konno in [3]. By a result due to Konno in [3], the degree of the canonical map Φ_{K_Y} of Y is either 1, 2, or 3 in our case. In Section 2, we shall consider the case $\deg \Phi_{K_Y} = 1$. In Section 3, we shall exclude the case $\deg \Phi_{K_Y} = 2$ for our surface Y. In Section 4, we shall exclude the case $\deg \Phi_{K_Y} = 3$ for our surface Y. Finally in Section 5, we shall compute the number of moduli of X with an ampleness canonical divisor. Note that only a little is known on surfaces with $c_1^2 = 3p_g - 6$ and $\deg \Phi_{K_Y} = 2$ (see [3]). Thus the exclusion of the case $\deg \Phi_{K_Y} = 2$ for our Y is the main part of the present paper in a sense. Throughout this paper, we work over the complex number field $\mathbb{C}$.

Notation. Let S be a compact complex manifold of dimension 2. We denote by $p_g(S)$, $q(S)$ and K_S, the geometric genus, the irregularity and a canonical divisor of S, respectively. The torsion group Tors(S) of S is the torsion part of the Picard group of S. For a coherent sheaf $\mathcal{F}$ on S, we denote by $h^i(\mathcal{F}) = \dim H^i(S, \mathcal{F})$ the dimension of the i-th cohomology group. As usual, $\mathbb{P}^n$ is the projective space of dimension n. We denote by $\Sigma_d \to \mathbb{P}^1$ the Hirzebruch surface of degree d. A curve Δ_0 is a section with self-intersection $-d$ of the Hirzebruch surface, and Γ is a fiber of $\Sigma_d \to \mathbb{P}^1$. The symbol $\sim$ means the linear equivalence of two divisors. For a finite set Σ, we denote by $\sharp\Sigma$ the number of elements of Σ. Moreover, we denote by $\varepsilon = \exp(2\pi\sqrt{-1}/3)$ a third root of unity.

1. Statement of the main results

We begin with a bound of the order of the torsion group. By Deligne's well-known argument [4, Theorem 14] and the unbranched covering trick, we have the following:

Lemma 1.1. *Let X be a minimal algebraic surface of general type with $c_1^2 = 3$, $p_g = 1$ and $q = 0$. Let $\pi : Y \to X$ be an unramified cover of finite degree m. Then $m \leq 6$ and $q(Y) = 0$.*

Proof. Apply Noether's inequality to the surface Y. □

Corollary 1.1. *Let X be as in Lemma 1.1. Then $\sharp\mathrm{Tors}(X) \leq 6$.*

In this paper, we consider the case $\mathbb{Z}/3 \subset \mathrm{Tors}(X)$, and find a concrete description of such surfaces. More precisely, we shall show the following:

Theorem 1. *Let X be a minimal algebraic surface of general type with $c_1^2 = 3$, $p_g = 1$, $q = 0$ and $\mathbb{Z}/3 \subset \mathrm{Tors}(X)$. Then both the fundamental group $\pi_1(X)$ and the torsion group $\mathrm{Tors}(X)$ are isomorphic to $\mathbb{Z}/3$. The canonical model Z of the universal cover Y of X is a complete intersection in $\mathbb{P}^4$ of type $(3,3)$ defined by the following equations*:

$$\begin{aligned} F_i = {} & a_0^{(i)} X_0^3 + a_1^{(i)} X_0X_1X_3 + a_2^{(i)} X_0X_1X_4 + a_3^{(i)} X_0X_2X_3 + a_4^{(i)} X_0X_2X_4 \\ & + a_5^{(i)} X_1^3 + a_6^{(i)} X_1^2X_2 + a_7^{(i)} X_1X_2^2 + a_8^{(i)} X_2^3 \\ & + a_9^{(i)} X_3^3 + a_{10}^{(i)} X_3^2X_4 + a_{11}^{(i)} X_3X_4^2 + a_{12}^{(i)} X_4^3 = 0 \end{aligned} \tag{1}$$

for $i = 1, 2$, where $(X_0 : \cdots : X_4)$ are homogeneous coordinates of the projective space $\mathbb{P}^4$.

Remark 1. Here the induced action on Z of the Galois group $\mathrm{Gal}(Y/X) = G = \langle \tau_0 \rangle$ is given by

$$\tau_0 : (X_0 : X_1 : X_2 : X_3 : X_4) \mapsto (X_0 : \varepsilon X_1 : \varepsilon X_2 : \varepsilon^{-1} X_3 : \varepsilon^{-1} X_4),$$

where $\varepsilon = \exp(2\pi\sqrt{-1}/3)$. This action on Z has no fixed points, since any automorphism of a fundamental cycle has fixed points. This imposes certain conditions on the coefficients $a_j^{(i)}$'s of the defining polynomials F_i's. Conversely, if a complete intersection Z in $\mathbb{P}^4$ of the form given in this theorem has at most rational double points, and if, moreover, it has no fixed points by the action on $\mathbb{P}^4$ defined above, then the minimal desingularization X of Z/G is a minimal algebraic surface of general type with the invariants as in Theorem 1. For example, put

$$\begin{aligned} F_1 &= X_0^3 + X_1^3 + X_2^3 + X_3^3 + X_4^3, \\ F_2 &= \alpha_0 X_0^3 + \alpha_1 X_1^3 + \alpha_2 X_2^3 + \alpha_3 X_3^3 + \alpha_4 X_4^3, \end{aligned}$$

where $\alpha_0, \ldots, \alpha_4$ are five distinct non-zero constants.

Theorem 2. *Let X be a surface as in Theorem 1. Let Θ_X be the sheaf of germs of holomorphic vector field on X. Assume that a canonical divisor K_X is ample. Then $h^1(\Theta_X) = 14$ and $h^2(\Theta_X) = 0$. Thus the number of moduli of X is 14.*

By Remark 2 given in the final section, we have the following:

Proposition 1. *There are no minimal algebraic surfaces X's with $c_1^2 = 3$, $p_g = 1$, $q = 0$ and $\mathrm{Tors}(X) \simeq \mathbb{Z}/5$.*

Theorem 1 together with Proposition 1 sharpens Corollary 1.1 as follows:

Proposition 2. *Let X be as in Lemma* 1.1. *Then* $\sharp\mathrm{Tors}(X) \le 4$.

In what follows, X is a minimal algebraic surface of general type with $c_1^2 = 3$, $p_g = 1$, $q = 0$ and $\mathbb{Z}/3 \subset \mathrm{Tors}(X)$. We denote by $\pi : Y \to X$ the unramified Galois triple covering associated with a 3-torsion divisor.

Lemma 1.2. *The surface Y satisfies $p_g(Y) = 5$, $q(Y) = 0$ and $K_Y^2 = 9$.*

Following the methods in [1] and [2], we study the canonical map Φ_{K_Y} of Y by using the canonical ring of Y. Let T_0 be the non-trivial 3-torsion divisor of X. We have a natural isomorphism

$$\alpha_m : H^0(Y, \mathcal{O}_Y(mK_Y)) \simeq \bigoplus_{l=0,1,-1} H^0(X, \mathcal{O}_X(mK_X - lT_0)) \tag{2}$$

for $m \ge 1$. Let us choose a generator τ_0 of the Galois group $G = \mathrm{Gal}(Y/X)$ in such a way that the spaces $H^0(X, \mathcal{O}_X(mK_X))$, $H^0(X, \mathcal{O}_X(mK_X - T_0))$ and $H^0(X, \mathcal{O}_X(mK_X + T_0))$ correspond to the eigenspaces of τ_0^* of eigenvalue 1, ε and ε^{-1}, respectively, where the action of G on $H^0(Y, \mathcal{O}_Y(mK_Y))$ is induced by the one on Y. We have $h^0(\mathcal{O}_X(K_X)) = 1$, while by the Riemann-Roch theorem, we have $h^0(\mathcal{O}_X(K_X - lT_0)) = 2$ for $l = -1, 1$. So we can take a base $x_0, \dots, x_4$ of $H^0(Y, \mathcal{O}_Y(K_Y))$ such that $x_0 \in \alpha_1^{-1} H^0(\mathcal{O}_X(K_X))$, $x_i \in \alpha_1^{-1} H^0(\mathcal{O}_X(K_X - T_0))$ for $i = 1, 2$, and $x_i \in \alpha_1^{-1} H^0(\mathcal{O}_X(K_X + T_0))$ for $i = 3, 4$. The canonical map is given by

$$\Phi_{K_Y} : p \mapsto (x_0(p) : x_1(p) : x_2(p) : x_3(p) : x_4(p)). \tag{3}$$

Note that we have $K_Y^2 = 3p_g(Y) - 6$. We frequently use results and methods given in [3]. See [3, Lemma 1.3] for a proof of the following proposition.

Proposition 3 (Konno)**.** *Let Y be a minimal algebraic surface of general type with $c_1^2 = 3p_g - 6$ and $p_g \ge 5$. Let $\Phi_{K_Y} : Y \to Z \subset \mathbb{P}^{p_g - 1}$ be the canonical map of Y. Then $1 \le \deg \Phi_{K_Y} \le 3$. Moreover, if $\deg \Phi_{K_Y} = 1$ or 3, then the canonical linear system $|K_Y|$ has no base points.*

By this proposition, we have $1 \le \deg \Phi_{K_Y} \le 3$ for our triple cover Y of the surface X.

2. The case $\deg \Phi_{K_Y} = 1$

In this section, we shall consider the case $\deg \Phi_{K_Y} = 1$. In this case the canonical map Φ_{K_Y} is holomorphic by Proposition 3. There are 13 monomials of $x_0, \dots, x_4$ in $\alpha_3^{-1} H^0(\mathcal{O}_X(3K_X))$, while we have $h^0(\mathcal{O}_X(3K_X)) = 11$. Thus we have at least two non-trivial linear relations, say $F_1(x) = 0$ and $F_2(x) = 0$, among these 13 monomials. Here $F_1(X)$ and $F_2(X)$ are homogeneous polynomials of degree 3 of the form given in Theorem 1. Put $V_i = \{F_i = 0\} \subset \mathbb{P}^4$ for $i = 1, 2$. Then we have $Z = \Phi_{K_Y}(Y) \subset V_1 \cap V_2$. We have two cases:

Case 1. V_1 and V_2 have no common irreducible components, *Case* 2. V_1 and V_2 have a common irreducible component W_0.

Claim 2.1. If V_1 and V_2 have no common irreducible components, then $Z = V_1 \cap V_2$. Moreover Z is the canonical model of Y.

See [3, Theorem 4.2] for a proof of this claim. The outline is as follows. The first assertion follows from $\deg Z = 9$. In this case, both V_1 and V_2 are irreducible. Let H be a hyperplane in $\mathbb{P}^4$. We have a natural inclusion $H^0(\mathcal{O}_{V_1 \cap V_2}(mH)) \subset H^0(\mathcal{O}_Y(mK_Y))$. By an easy computation we have

$$h^0(\mathcal{O}_{V_1 \cap V_2}(mH)) = \frac{9}{2}m(m-1) + 6 = h^0(\mathcal{O}_Y(mK_Y))$$

for any $m \geq 2$. This implies that the five elements $x_0, \ldots, x_4 \in H^0(Y, \mathcal{O}_Y(K_Y))$ span the canonical ring of the surface Y. Thus, the surface Z is the canonical model of the surface Y. We have already seen in Remark 1 that Case 1 in fact occurs.

Next, we consider Case 2. In this case, the common irreducible component W_0 is a quadric hypersurface in $\mathbb{P}^4$. Note that W_0 is the only quadric hypersurface containing Z, since we have $\deg Z = 9 > 2 \cdot 2$. This implies that W_0 is invariant under the action of $G = \mathrm{Gal}(Y/X)$ on $\mathbb{P}^4$, where the action of G on $\mathbb{P}^4$ is induced from the one on Z. Since W_0 is a quadric hypersurface, the isomorphism class is determined by its rank. Konno showed that W_0 is singular ([3, Section 3]). As regards the action of the Galois group G, we can show the following using the canonical ring of Y.

Claim 2.2. Assume that V_1 and V_2 have a common irreducible component W_0. Let τ_0, T_0, $x_0, \ldots, x_4$ be as in Section 1. Then we can take τ_0, T_0, $x_0, \ldots, x_4$, in such a way that we have one of the following two cases:
Case 2-1. $W_0 = \{X_0X_1 - X_3X_4 = 0\} \subset \mathbb{P}^4$,
Case 2-2. $W_0 = \{X_0X_1 - X_3^2 = 0\} \subset \mathbb{P}^4$.

Case 2 in fact occurs for certain surfaces Y's with $p_g = 5$, $q = 0$ and $c_1^2 = 9$, when we do not restrict our Y to the triple cover of our X. Such surfaces are called surfaces of type I-0 in [3]. See [3] for such surfaces. We shall exclude both Case 2-1 and Case 2-2, using the fact that our Y is an unramified Galois triple cover of X.

2.1. Exclusion of Case 2-1

First, let us exclude Case 2-1 in Claim 2.2. In Case 2-1, the hypersurface W_0 is a cone over the Hirzebruch surface $\Sigma_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Here Σ_0 is a non-singular quadric hypersurface in $\mathbb{P}^3 \subset \mathbb{P}^4$. We denote by p_0 the vertex of W_0. Let Λ_0 be the linear system consisting of the pull-back by Φ_{K_Y} of all hyperplanes passing through the vertex p_0. We denote by Λ and F, the variable part of Λ_0 and the fixed part of Λ_0, respectively. We let $p : \tilde{Y} \to Y$ be a composition of quadric transformations such that the variable part $|M|$ of $p^*\Lambda$ is free from base points. We take the shortest one among those with this property. Then we have

$$p^*K_Y \sim M + \tilde{E} + p^*F,$$

where $\tilde{E}$ is an exceptional divisor. We have a morphism

$$\tilde{\mu} = \Phi_M : \tilde{Y} \to \Sigma_0 \subset \mathbb{P}^3$$

determined by the linear system $|M|$. This holomorphic map $\tilde{\mu}$ is just the composition $g \circ \Phi_{K_Y}$, where the rational map g is the projection from the vertex p_0.

Let us compute intersection numbers among these divisors, and derive a contradiction. First note that the vertex p_0 is invariant under the action of $G = \mathrm{Gal}(Y/X) = \langle \tau_0 \rangle$ on W_0. This together with $(\tau_0|_{W_0})^3 = \mathrm{id}_{W_0}$ implies that the linear system Λ_0 is spanned by the pull-back of divisors on X. Indeed, since we have $p_0 = (0:0:1:0:0)$, the linear system Λ_0 is spanned by (x_0), (x_1), (x_3) and (x_4), where $x_0, \dots, x_4$ are global sections as in Claim 2.2 and (x_i)'s are the effective divisors determined by x_i's. Thus both Λ and F are spanned by the pull-back of divisors on X, hence $\Lambda^2 \equiv \Lambda F \equiv F^2 \equiv 0 \bmod 3$. Moreover, we may assume that the action of G on Y lifts to the one on $\tilde{Y}$, since Λ is spanned by the pull-back of divisors on X. Then we have $\tilde{E}^2 \equiv 0 \bmod 3$. Since $\Phi_{K_Y}(F)$ is contained in $\{p_0\}$, we have $\Lambda_0 F = 0$. Thus we have

$$9 = \Lambda_0^2 = M^2 + M\tilde{E} + Mp^*F, \tag{4}$$

where each term of the right hand side is a non-negative integer. We have

$$M\tilde{E} = -\tilde{E}^2 \equiv 0 \mod 3,$$
$$Mp^*F = \Lambda F \equiv 0 \mod 3,$$
$$M^2 = 2 \deg \tilde{\mu}.$$

Moreover, we have

$$\Lambda F = F^2 + FK_Y - 2F^2 \equiv 0 \mod 2$$

by the Riemann-Roch theorem. Thus we have

$$M^2 \equiv Mp^*F \equiv 0 \mod 6.$$

Since Y is not birational to a ruled surface, we have $M^2 > 0$. Thus by (4) and Hodge's index theorem, we have the following:

$$M^2 = 6, \quad M\tilde{E} = 3, \quad \tilde{E}^2 = -3, \quad \deg \tilde{\mu} = 3, \quad F = 0. \tag{5}$$

Then the linear system Λ has exactly 3 base points, and the set of base points, say $\{p_1, p_2, p_3\}$, forms an orbit of the action of G on Y. Let $\tilde{E} = \sum_{i=1}^3 \tilde{E}_i$ be the decomposition of $\tilde{E}$ into the sums of components lying over each base point p_i. Then we have $M\tilde{E}_i = 1$, $\tilde{E}_i^2 = -1$ for each $1 \le i \le 3$, and $\tilde{E}_i\tilde{E}_j = 0$ for $i \neq j$.

By an argument similar to the one in the proof of Theorem 1 in [5], we show that each $\tilde{E}_i$ is an exceptional curve of the first kind for $1 \le i \le 3$. Put

$$K_{\tilde{Y}} \sim p^*(K_Y) + E,$$

where E is an exceptional divisor. Let $E = \sum_{i=1}^{3} E_i$ be the decomposition of E into the sums of components lying over each base point p_i. We have $\tilde{E}_i \geq E_i$ and $\text{supp}(\tilde{E}_i) = \text{supp}(E_i)$, since the morphism p is the shortest one. Since $M\tilde{E}_i = 1$, we have $ME_i = 1$. Thus there exists an exceptional curve $E_i^{(0)}$ of the first kind such that

$$E_i = E_i^{(0)} + E_i', \qquad \tilde{E}_i = E_i^{(0)} + \tilde{E}_i',$$

where E_i' and $\tilde{E}_i'$ are effective divisors and $ME_i^{(0)} = 1$, $ME_i' = M\tilde{E}_i' = 0$. Thus we have

$$K_{\tilde{Y}} \sim M + \sum_{i=1}^{3}(2E_i^{(0)} + E_i' + \tilde{E}_i'). \tag{6}$$

Note that neither E_i' nor $\tilde{E}_i'$ contain $E_i^{(0)}$ as a component. We have $K_{\tilde{Y}}E_i^{(0)} = -1$. Thus by (6), we obtain $E_i^{(0)}E_i' = E_i^{(0)}\tilde{E}_i' = 0$. From these equalities and the assumption that p is the shortest one, we infer that $E_i' = \tilde{E}_i' = 0$. Thus $\tilde{E}_i = E_i^{(0)}$ is an exceptional curve of the first kind.

Finally we derive a contradiction as follows: By the argument as above, we have $K_{\tilde{Y}} \sim M + 2\tilde{E}$. We denote by Γ and Δ_0, a fiber and a section of the Hirzebruch surface $\Sigma_0 \to \mathbb{P}^1$ as in Section 0. Let D be the pull-back $\Phi_M^*(\Gamma)$. Since $M \sim \Phi_M^*(\Delta_0 + \Gamma)$, we have

$$D^2 + DK_{\tilde{Y}} = 3 + 2D\tilde{E}.$$

This contradicts the Riemann-Roch theorem, since the right hand side is odd. Thus Case 2-1 in Claim 2.2 is impossible.

2.2. Exclusion of Case 2-2

Next, we exclude Case 2-2 in Claim 2.2. In Case 2-2, the hypersurface W_0 is a generalized cone over a rational curve $C \simeq \mathbb{P}^1$. This rational curve C is a conic in $\mathbb{P}^2 \subset \mathbb{P}^4$. The singular locus of W_0 is given by $X_0 = X_1 = X_3 = 0$ in $\mathbb{P}^4$. We call this line the ridge of W_0. Let Λ_0 be a linear system consisting of the pull-back of all hyperplanes containing the ridge. We denote by Λ and F, the variable part and the fixed part of Λ_0, respectively. Again Λ_0 is spanned by the pull-back of divisors on X, namely by (x_0), (x_1) and (x_3). Thus Λ and F are also spanned by the pull-back of divisors on X. In particular, we have $\Lambda^2 \equiv \Lambda F \equiv F^2 \equiv 0 \bmod 3$. Let F' be the maximal common component of divisors (x_0) and (x_1). Then we have $(x_i) = D_i' + F'$ for an effective divisor D_i' for $i = 1, 2$, where D_1' and D_2' have no common components. By the equality $(x_0x_1) = (x_3^2)$, we have $D_i' = 2D_i$ for an effective divisor D_i for $i = 1, 2$. Then we have $\Lambda_0 \sim 2D + F$ for an effective divisor $D \sim D_1$. The linear system $|D_1| = |D_2|$ is a linear pencil without fixed components. We have

$$9 = \Lambda_0^2 = 4D^2 + 2DF + K_YF, \tag{7}$$

where each term of the right hand side is non-negative integer. Since $4D^2 = \Lambda^2 \equiv 0 \bmod 3$, we get $D^2 = 0$. Then by the Riemann-Roch theorem, we have

$DF = D^2 + DK_Y \equiv 0 \bmod 2$, hence $2DF = \Lambda F \equiv 0 \bmod 12$. Thus we obtain $D^2 = DF = 0$ and $F^2 = 9$. By Hodge's index theorem, we infer $D = 0$. This contradicts the equality $h^0(\mathcal{O}_Y(D)) = 2$. Thus Case 2-2 is excluded.

3. The case $\deg \Phi_{K_Y} = 2$

In this section, we exclude the possibility of the case $\deg \Phi_{K_Y} = 2$. Surfaces Y's of this case are called surfaces of type II in [3]. There exist many surfaces of type II, and they are not classified completely even in [3]. However for our case of triple covering, we can exclude the possibility of type II using the action of the Galois group $G = \mathrm{Gal}(Y/X)$. Since only a little is known on surfaces of type II, the exclusion of the possibility of type II for our Y is the main part of the present paper in a sense.

First we study the base points of the linear system $|K_Y|$. Let $|L|$ and F be the variable part and the fixed part of the linear system $|K_Y|$, respectively. Again we denote by $p : \tilde{Y} \to Y$ a composition of quadric transformations which is the shortest among the ones with the property that the variable part of $|p^*L|$ has no base point. We take p in such a way that the action of the Galois group $G = \langle \tau_0 \rangle$ lifts to one on $\tilde{Y}$. This is possible, since $|K_Y|$ is spanned by the pull-back of divisors on X. We have

$$p^*K_Y \sim M + \tilde{E} + p^*F,$$

where M and $\tilde{E}$ are the variable part and the fixed part of p^*L, respectively. From this we infer

$$9 = K_Y^2 = M^2 + M\tilde{E} + Mp^*F + K_YF, \tag{8}$$

where each term of the right hand side is a non-negative integer. We have

$$M\tilde{E} = -\tilde{E}^2 \equiv 0 \mod 3, \quad Mp^*F = LF \equiv 0 \mod 3,$$
$$K_YF \equiv 0 \mod 3,$$

hence $M^2 \equiv 0 \bmod 3$. Moreover we have

$$M^2 = 2\deg \Phi_M(\tilde{Y}) \equiv 0 \mod 2,$$
$$Mp^*F = LF = L^2 + LK_Y - 2L^2 \equiv 0 \mod 2.$$

Thus from (8), we infer $M^2 = 6$ and $Mp^*F = LF = 0$. So by (8), the inequalities $K_Y^2 \geq L^2 = M^2 + M\tilde{E} \geq M^2$ and Hodge's index theorem, we obtain

$$M^2 = 2\deg Z = 6, \quad M\tilde{E} = -\tilde{E}^2 = 3, \quad F = 0, \tag{9}$$

where $Z = \Phi_M(\tilde{Y})$ is the canonical image of Y. Similarly to the proof of exclusion of Case 2-1 of Claim 2.2, we can show that $|L|$ has exactly 3 base points and that each base point is resolved by a single quadric transformation.

The set P of these three base points forms an orbit of the action of G. Let $q : \tilde{X} \to X$ be a quadric transformation with the center $\pi(P)$. We have the following commutative diagram:

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{p} & Y \\ {\scriptstyle \tilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ \tilde{X} & \xrightarrow{q} & X, \end{array} \tag{10}$$

where $\tilde{\pi}$ is an unramified Galois triple cover of $\tilde{X}$. Note that $\mathrm{Gal}(\tilde{Y}/\tilde{X}) \simeq \mathrm{Gal}(Y/X)$. By (9), the canonical image Z of Y is a surface of minimal degree in $\mathbb{P}^4$. By a classification of surfaces of minimal degree (see for example [6, Lemma 1.2] or [9]), we have the following:

Lemma 3.1. *Let $Z = \Phi_{K_Y}(Y)$ be the canonical image of Y, then Z is one of the following*:

Case 3-1. *an image of the Hirzebruch surface Σ_3 under the morphism determined by the linear system $|\Delta_0 + 3\Gamma|$,*

Case 3-2. *the Hirzebruch surface Σ_1 embedded by $|\Delta_0 + 2\Gamma|$.*

3.1. Exclusion of Case 3-1

We exclude Cases 3-1 and 3-2. First, we exclude Case 3-1. In this case, the canonical image Z is a cone over a twisted cubic curve $C \subset \mathbb{P}^3$. We denote by p_0 the vertex of the cone Z. Let Λ_0 be the linear system consisting of the pull-back by Φ_M of hyperplanes passing through p_0. We denote by Λ and F', the variable part and the fixed part of Λ_0, respectively. We have a natural isomorphism $\beta : H^0(\mathcal{O}_{\tilde{Y}}(M)) \simeq \mathbb{C}[X_0, \ldots, X_4]_1$, where $\mathbb{C}[X_0, \ldots, X_4]_1$ is the homogeneous part of degree 1 of the homogeneous coordinate ring of $\mathbb{P}^4$. We have $\Lambda_0 = \mathbb{P}(\beta^{-1}(V))$ for a linear subspace $V \subset \mathbb{C}[X_0, \ldots, X_4]_1$. Since the vertex p_0 is invariant under the action of $G = \mathrm{Gal}(\tilde{Y}/\tilde{X}) = \langle \tau_0 \rangle$ on $\mathbb{P}^4$, the subspace V is stable under the action of G on $\mathbb{C}[X_0, \ldots, X_4]_1$. This together with $(\tau_0^*)^3 = \mathrm{id}$ implies that V is spanned by eigenvectors of τ_0^*. Thus Λ and F' are both spanned by the pull-back of divisors on $\tilde{X}$. Since $(\tau_0^*)^3 = \mathrm{id}$, we have $\mathbb{C}[X_0, \ldots, X_4]_1 = V \oplus W$, where W is a 1-dimensional linear subspace invariant under the action of G. We take a base $Y_0, \ldots, Y_4$ of $\mathbb{C}[X_0, \ldots, X_4]_1$ such that $Y_i \in V$ for $0 \le i \le 3$ and $Y_4 \in W$. Let H_0 be a hyperplane in $\mathbb{P}^4$ defined by $Y_4 = 0$. Then Z is a cone over the twisted cubic $C = Z \cap H_0$. Note that C and H_0 are both invariant under the action of G on $\mathbb{P}^4$. See [6, Lemma 1.5] for a proof of the following lemma.

Lemma 3.2. *There exists a linear pencil $|D|$ on $\tilde{Y}$ without fixed components such that $\Lambda \sim 3D$.*

We have $M \sim 3D + F'$. We derive a contradiction by computing intersection numbers among these divisors. First note that $MF' = 0$, since $\Phi_M(F') = p_0$. Thus we have $6 = M^2 = 9D^2 + 3DF'$, hence

$$D^2 = 0, \qquad DF' = 2. \tag{11}$$

Thus the linear system Λ has no base points. We have a holomorphic map Φ_Λ determined by the linear system Λ. This Φ_Λ is just the extension of the rational map $p \mapsto (y_0(p) : \cdots : y_3(p))$, where y_i's are the same as in the proof of Lemma 3.2. It follows that D is a pull-back $\Phi_M^*(q_0)$ by Φ_M, where q_0 is an effective divisor of degree 1 on C. The curve C has an action of G compatible to the one on $\tilde{Y}$, since C is stable under the action on H_0 of the Galois group $G = \langle \tau_0 \rangle$. The isomorphism $\tau_0|_C$ has at least 2 fixed points, say q_1 and q_2, since we have $\tau_0^3 = \mathrm{id}$ and the curve C is isomorphic to $\mathbb{P}^1$. Put $D_i'' = \Phi_M^*(q_i)$ for $i = 1, 2$. Then D_i'' is a member of $|D|$ stable under the action of G, hence a pull-back of a divisor on $\tilde{X}$, for $i = 1, 2$. Then both $|D|$ and F' are spanned by pull-back of divisors on $\tilde{X}$. Thus the intersection number DF' must be a multiple of 3, which contradicts the equality (11). This proves that Case 3-1 is impossible.

3.2. Exclusion of Case 3-2

Next, we exclude Case 3-2 in Lemma 3.1. In this case, the canonical image Z of Y is the Hirzebruch surface Σ_1 embedded by $|\Delta_0 + 2\Gamma|$. The curve Δ_0 is a line in $\mathbb{P}^4$. Let Λ_0 be a linear system consisting of the pull-back by Φ_M of all hyperplanes containing Δ_0 in $\mathbb{P}^4$. We denote by F the fixed part of Λ_0. The curve Δ_0 is the unique (-1)-curve on Z, since Σ_1 is obtained by a single quadric transformation of $\mathbb{P}^2$. Thus Δ_0 is invariant under the action of G on $\mathbb{P}^4$. Then, as in the proof of exclusion of Case 3-1, we see that Λ_0 is spanned by the pull-back of divisors on $\tilde{X}$, and that so is F. So the intersection number F^2 has to be a multiple of 3. However, we have $F = \Phi_M^*(\Delta_0)$, hence $F^2 = -2$. This is a contradiction. This proves that Case 3-2 is impossible.

4. The case $\deg \Phi_{K_Y} = 3$

In this section, we exclude the case $\deg \Phi_{K_Y} = 3$. This case corresponds to surfaces of type III in [3]. We exclude this case by using the action of the Galois group $\mathrm{Gal}(Y/X)$.

First, note that the canonical system $|K_Y|$ is free from base points by Proposition 3. The canonical image $Z = \Phi_{K_Y}(Y)$ is a surface of minimal degree in $\mathbb{P}^4$. Thus, as in the previous section, the surface Z is either an image of the Hirzebruch surface Σ_3 by $|\Delta_0 + 3\Gamma|$, or the Hirzebruch surface Σ_1 embedded in $\mathbb{P}^4$ by $|\Delta_0 + 2\Gamma|$. For a proof of the follwing lemma, see [3, Lemma 2.2].

Lemma 4.1 (Konno). *The canonical image Z of Y is an image of the Hirzebruch surface Σ_3 by $|\Delta_0 + 3\Gamma|$.*

Thus we have only to exclude the case in which Z is an image of the Hirzebruch surface Σ_3 by $|\Delta_0 + 3\Gamma|$. In this case, Z is a cone over a twisted cubic curve. We denote by p_0 the vertex of the cone Z. Let Λ_0 be a linear system consisting of the pull-back by Φ_{K_Y} of all hyperplanes passing through p_0. We denote by Λ and F the variable part and the fixed part of Λ_0, respectively. As in the proof of exclusion of Case 3-1, we see that Λ and F are both spanned by the pull-back of divisors on X. Moreover, by the proof of Lemma 3.2, we see that there exists a linear pencil $|D| = |D_1| = |D_2|$ without fixed components such that Λ is spanned by four divisors $3D_1$, $2D_1 + D_2$, $D_1 + 2D_2$ and $3D_2$. We denote by b the number of base points of $|D|$. Note that the linear system Λ also has exactly b base points. By the proof of [3, Lemma 2.2], we have $D^2 = 1$. We obtain $b = 1$ by this equality. However, since Λ is spanned by the pull-back of divisors on X, the number of the base points of Λ must be a multiple of 3, which contradicts the equality $b = 1$. Thus the case $\deg \Phi_{K_Y} = 3$ is impossible. This completes the proof of Theorem 1. □

5. The number of moduli

Let X be a surface as in Theorem 1 such that a canonical divisor K_X is ample. We give a proof of Theorem 2 in this section. Namely we show that $h^1(\Theta_X) = 14$ and $h^2(\Theta_X) = 0$, where Θ_X is the sheaf of germs of holomorphic vector field on X. This means that the number of moduli of X is 14. In what follows, we assume ampleness of a canonical divisor K_X.

Let $\pi : Y \to X$ be the universal cover of the surface X. By the Riemann-Roch-Hirzebruch theorem, we have

$$h^1(\Theta_X) = 10\chi(\mathcal{O}_X) - 2c_1^2(X) + h^2(\Theta_X) = 14 + h^2(\Theta_X).$$

The equality $h^0(\Theta_X) = 0$ holds, since X is of general type. On the other hand, we have

$$h^2(\Theta_X) = h^0(\Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{O}_X(K_X)) \leq h^0(\Omega_Y^1 \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(K_Y)) = h^2(\Theta_Y),$$

where Ω_X^1 and Ω_Y^1 are the sheaves of germs of holomorphic 1-forms on X and Y, respectively. Thus in order to prove Theorem 2, we have only to show that $h^2(\Theta_Y) = 0$.

Lemma 5.1. *The surface Y satisfies $h^2(\Theta_Y) = 0$ on the assumption given in Theorem 2.*

Proof. The morphism π is of degree three. Since a canonical divisor K_X is ample, the universal cover Y has no (-2)-curves. Thus Y is a smooth complete intersection in $\mathbb{P}^4$ of type $(3,3)$ by Theorem 1. Let

$$\iota : Y \to W = \mathbb{P}^4$$

be the inclusion morphism as in Theorem 1. We denote by $\mathcal{J}$ the sheaf of ideals on W defining Y. We have natural exact sequences

$$0 \to \Theta_Y \to \iota^*\Theta_W \to \mathcal{O}_Y(3H)^{\oplus 2} \to 0,$$
$$0 \to \mathcal{J} \otimes_{\mathcal{O}_W} \Theta_W \to \Theta_W \to \iota^*\Theta_W \to 0$$

of sheaves, where H is a hyperplane in $\mathbb{P}^4$. By these exact sequences of sheaves, we obtain isomorphisms

$$H^2(\Theta_Y) \simeq H^2(\iota^*\Theta_W) \simeq H^3(\mathcal{J} \otimes_{\mathcal{O}_W} \Theta_W). \tag{12}$$

Thus we have only to prove that $H^3(\mathcal{J} \otimes_{\mathcal{O}_W} \Theta_W) = 0$. Meanwhile by short exact sequences of sheaves

$$0 \to \mathcal{O}_W(-6H) \to \mathcal{O}_W(-3H)^{\oplus 2} \to \mathcal{J} \to 0,$$
$$0 \to \mathcal{O}_W(-6H) \otimes_{\mathcal{O}_W} \Theta_W \to \mathcal{O}_W(-3H)^{\oplus 2} \otimes_{\mathcal{O}_W} \Theta_W \to \mathcal{J} \otimes_{\mathcal{O}_W} \Theta_W \to 0,$$

we obtain an exact sequence of cohomology groups

$$H^3(\mathcal{O}_W(-3H) \otimes_{\mathcal{O}_W} \Theta_W)^{\oplus 2} \to H^3(\mathcal{J} \otimes_{\mathcal{O}_W} \Theta_W) \to H^4(\mathcal{O}_W(-6H) \otimes_{\mathcal{O}_W} \Theta_W).$$

By the Riemann-Roch theorem we have

$$h^3(\mathcal{O}_W(-3H) \otimes_{\mathcal{O}_W} \Theta_W) = h^1(\Omega^1_W \otimes_{\mathcal{O}_W} \mathcal{O}_W(-2H)),$$
$$h^4(\mathcal{O}_W(-6H) \otimes_{\mathcal{O}_W} \Theta_W) = h^0(\Omega^1_W \otimes_{\mathcal{O}_W} \mathcal{O}_W(H)).$$

Thus the equality $H^3(\mathcal{J} \otimes_{\mathcal{O}_W} \Theta_W) = 0$ follows from the well-known theorem given below (Theorem 3). This equality together with isomorphisms (12) gives the assertion, which completes the proof of Theorem 2. □

Theorem 3 (Bott [15])**.** *Let Ω^p be the sheaf of germs of holomorphic p-forms on the projective space $\mathbb{P}^n$. Then the dimension $h^q(\mathbb{P}^n, \Omega^p)$ is zero except in the following three cases*: i) $p = q$ *and* $d = 0$, ii) $q = 0$ *and* $p < d$, iii) $q = n$ *and* $d < p - n$.

Remark 2. We remark that there are no minimal algebraic surfaces X's with $c_1^2 = 3$, $p_g = 1$, $q = 0$ and $\mathrm{Tors}(X) \simeq \mathbb{Z}/5$. Assume that we had a minimal algebraic surface X with such invariants. Then we would have an unramified Galois cover $Y \to X$ of degree 5 corresponding to the torsion group. Then Y is a minimal algebraic surface with $K_Y^2 = 2p_g(Y) - 3$, $p_g(Y) = 9$. However, we have the following theorem:

Theorem 4 (Horikawa [7], Section 1)**.** *Let Y be a minimal algebraic surface of general type with $K_Y^2 = 2p_g(Y) - 3$. If $p_g(Y) \geq 5$, then the canonical linear system $|K_Y|$ has a unique base point.*

By this theorem, we see that the canonical system $|K_Y|$ of our surface Y has a unique base point, and that this base point is a fixed point of any automorphisms of Y. This contradicts the assumption that $Y \to X$ is an unramified Galois cover of degree 5. Thus there are no minimal algebraic surfaces X's with $c_1^2 = 3$, $p_g = 1$, $q = 0$ and $\mathrm{Tors}(X) \simeq \mathbb{Z}/5$.

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