J. Math. Kyoto Univ. (JMKYAZ)
43-1 (2003), 165–186

Recovery of the shape of an obstacle and the boundary impedance from the far-field pattern

By

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Abstract

In this paper, we consider an inverse scattering problem for an obstacle $D \subset \mathcal{R}^3$ with Robin boundary condition. A reconstruction procedure for identifying both the shape of the obstacle and the boundary impedance from the far-field pattern is proposed. Our method is to transform the far-field patterns into the near-field patterns, then construct an indicator function from which we can determine the boundary shape. Having known the shape of D, the boundary impedance is recovered by moment method for the known boundary.

1. Introduction

Let D be a simply connected domain in $\mathcal{R}^3$ with C^2 boundary ∂D. The scattering of time-harmonic acoustic plane waves for the obstacle D with impedance boundary condition is modeled by an exterior boundary value problem for the Helmholtz equation. That is, for given incident plane wave $u^i(x) = e^{ikx\cdot d}$, the total wave field $u = u^i + u^s \in H^1_{loc}(\mathcal{R}^3 \setminus \overline{D})$ satisfies

$$
\begin{cases}
\Delta u + k^2 u = 0, & \text{in} \quad R^3 \setminus \overline{D} \\
\dfrac{\partial u}{\partial \nu} + \lambda(x) u = 0, & \text{on} \quad \partial D \\
\dfrac{\partial u^s}{\partial r} - iku^s = O\left(\dfrac{1}{r}\right), & r = |x| \longrightarrow \infty,
\end{cases}
\tag{1.1}
$$

where ν is the unit normal vector of ∂D directed into the exterior of D.

We assume that $\lambda \in L^\infty(\partial D)$ and $\Im\lambda > 0$ a.e. on ∂D ($\Im\lambda$ denotes the imaginary part of λ). By the results in [4], we know that there exists a unique solution for the direct scattering problem.

For the incident field $u^i(x) = e^{ikx\cdot d}$, the far-field pattern $u^\infty(d,\theta)$ corresponding to the scattered wave $u^s(x)$ can be defined by

$$
u^s(x) = \frac{e^{ik|x|}}{|x|}\left\{u^\infty(d,\theta) + O\left(\frac{1}{|x|}\right)\right\}, \qquad |x| \longrightarrow \infty, \tag{1.2}
$$

Received August 15, 2001
Revised May 14, 2002

where $\theta, d \in S^2$ (S^2 is the unit sphere in $\mathcal{R}^3$).

Then our inverse problem is to determine $(\partial D, \lambda(x))$ from the corresponding far-field pattern $u^\infty(d, \theta)$, $\theta, d \in S^2$. Isakov conjectured that the uniqueness of this inverse problems is true. We will not only prove the uniqueness, but also give the reconstruction procedure for identifying ∂D and $\lambda(x)$.

For this inverse problems, the limiting cases $\lambda(x) = 0$ and $\lambda(x) = \infty$ a.e. on ∂D correspond to the Neumann boundary condition and the Dirichlet boundary condition respectively. In these cases the problem of recovering ∂D has been studied by many researchers ([1], [4], [8], [13], [15], [18]). The problem of reconstructing $\lambda(x)$ has also been studied by some researchers in the case that D is given ([2], [4], [16], [17]).

For the inverse scattering problem for determining both ∂D and boundary impedance, to the authors' knowledge, there are only few results. An approximate determination (or reconstruction) of the shape of D and boundary impedance was discussed in [20] by using the asymptotic behavior of the low frequency scattered waves associated with three different incident waves (or frequencies). In [11], one numerical method is proposed to determine both ∂D and impedance $\lambda(x)$. But it seems there is no published theoretic result for the uniqueness of this inverse problem. Also there is no result on reconstruction algorithm for ∂D and boundary impedance.

From the physical point of view, the formulation of the inverse problem with the Robin boundary condition is more reasonable. The obstacle is not completely sound-hard or sound-soft, and usually the boundary of the obstacle has some impedance. Moreover in many cases, the impedance is unknown. Therefore, the following two problems, from both the theoretical and the practical point of view, are interesting:

1. Is the identification of $(\partial D, \lambda(x))$ from $u^\infty(d, \theta)$ for all $\theta, d \in S^2$ unique?
2. If the uniqueness is true, can we give the exact reconstruction procedure for determining both ∂D and $\lambda(x)$?

In this paper, we want to propose a constructive method for the inverse problem of determining both the shape and impedance from the far-field patterns. In related with the constructive methods for the inverse problems, we refer to Ikehata ([7], [8], [9]), Ikehata and Nakamura ([10]) and Potthast ([19]) etc. But, to authors' knowledge, they assume that either the boundary condition is known or the shape of the obstacle is known. The problem of determining both the shape and the impedance has not been studied.

In this paper, by transforming the given far-field pattern into the Dirichlet-to-Neumann map, we can construct some indicator function to identify the obstacle without knowing its impedance. Although this method is a general method, some special care must be taken for analyzing the behaviour of the indicator function in applying it to each problem. After that, we reconstruct the impedance from the Dirichlet-to-Neumann map by the moment method.

These two reconstruction procedures also imply the uniqueness of identifying both the shape of obstacle and boundary impedance. Compared with the method proposed in [20], our method is theoretically exact.

Our main result is the following

Theorem 1. *For the inverse scattering problem, the far-field pattern $u^\infty(d,\theta)$, $\theta, d \in S^2$, can uniquely determine $(\partial D, \lambda(x))$. Moreover, the reconstructive algorithm can be realized by the steps at the end of Section* 3.

Remark 1.1. The uniqueness of identifying $(\partial D, \lambda(x))$ from $u^\infty(d,\theta)$ for all $d, \theta \in S^2$ becomes obvious from the reconstruction.

Our paper is organized as follows:

- Section 2: From far-field pattern to the Dirichlet-to-Neumann map
- Section 3: Probe method
- Section 4: Moment method for determining $\lambda(x)$
- Section 5: Some estimates
- Section 6: Appendix

2. From far-field pattern to the Dirichlet-to-Neumann map

In this section, we show how to obtain the Dirichlet-to-Neumann map $\Lambda_{\partial D,\lambda}$ from the far field pattern $u^\infty(d,\theta)$, $\theta, d \in S^2$. Our argument is the same as that of [8] except the part getting the Dirichlet-to-Neumann map from the Neumann derivatives to the outgoing Green function.

Without loss of generality, we assume that $\overline{D} \subset B(0, R/2)$ for some constant $R > 0$ and $\|\lambda\|_{L^\infty(\partial D)} \le \lambda_0$ for some constant $\lambda_0 > 0$. We also assume that 0 is not the Dirichlet eigenvalue of $\Delta + k^2$ in $B(0,R)$ for given $k > 0$.

Proposition 2.1. *The scattered solution $u^s(x,d)$ for $|x| > R/2$ can be determined uniquely from $u^\infty(d,\theta)$.*

Proof. Since $u^s(x)$ satisfies Helmholtz equation in $|x| > R/2$ and the Sommerfeld radiation condition, then $u^s(x)$ for $|x| > R/2$ can be expressed as

$$u^s(x) = k \sum_{n\ge 0} i^{n+1} \sum_{|m|\le n} b_n^m h_n^1(k|x|) Y_n^m\left(\frac{x}{|x|}\right), \tag{2.1}$$

where $h_n^1(k|x|)$ is the first spherical Hankel function of order n and $Y_n^m(x/|x|)$ is the spherical harmonic function.

By the results in [4] (Theorems 2.15 and 2.16, p. 35–36), we know that b_n^m can be expressed in terms of $u^\infty(d,\theta)$:

$$b_n^m = \int_{S^2} u^\infty(d,\theta) Y_n^m(\theta) d\theta, \qquad n = 0, 1, 2, \ldots; \quad |m| \le n. \tag{2.2}$$

The proof is complete. □

In the following, we denote $\Omega = B(0,R)$. Let $G(x,y) = (e^{ik|x-y|})/(4\pi|x - y|)$ be the fundamental solution of the Helmholtz equation. For each $y \in \mathcal{R}^3\backslash\overline{D}$, define $E(\cdot,y)\in H^1_{loc}(\mathcal{R}^3\setminus\overline{D})$ as the solution to

$$
\begin{cases}
\Delta E+k^2E=0, & \text{in} \quad R^3\setminus\overline{D}\\
\dfrac{\partial E}{\partial\nu}+\lambda(x)E=-\dfrac{\partial}{\partial\nu}G-\lambda(x)G, & \text{on} \quad \partial D\\
\dfrac{\partial E}{\partial r}-ikE=O\left(\dfrac{1}{r}\right), & r=|x|\longrightarrow\infty.
\end{cases}
\tag{2.3}
$$

Then we have the following proposition for the outgoing Green function $G_D(x, y):=G(x,y)+E(x,y)$.

Proposition 2.2. *$G_D(x,y)$, as well as its normal derivatives $\partial/(\partial\nu(x))G_D(x,y)$ and $\partial/(\partial\nu(y))G_D(x,y)$ on $B(0,R)$, can be determined for $x,y\in\partial B(0,R)=\partial\Omega$ from $u^\infty(d,\theta)$ for all $d,\theta\in S^2$.*

Proof. Let $y\in\partial B(0,R_1)$ for any $R_1>R$.

From our assumption on k, we know that $\{e^{ikx.d}|d\in S^2\}$ is complete in $L^2(\partial B(0,R))$ (see [4], Theorem 5.5, p. 110). Therefore there exist $\alpha_j^n(y)$ and $d_j^n(y)$ such that

$$
\sum_{1\le j\le m_n(y)}\alpha_j^n(y)e^{ikx\cdot d_j^n(y)}\longrightarrow G(x,y)\qquad\text{in}\quad L^2(\partial B(0,R))
\tag{2.4}
$$

as $n\longrightarrow\infty$.

On the other hand, since $\sum_{1\le j\le m_n(y)}\alpha_j^n(y)e^{ikx.d_j^n(y)}$ and $G(x,y)$ satisfy the Helmholtz equation in $B(0,R)$, by the results in [4] (Theorem 5.4, p. 109), we know that (2.4) implies

$$
\sum_{1\le j\le m_n(y)}\alpha_j^n(y)e^{ikx.d_j^n(y)}\longrightarrow G(x,y)
$$

uniformly on any compact subset of $B(0,R)$ (together with all their derivatives).

Therefore we have

$$
\sum_{1\le j\le m_n(y)}\alpha_j^n(y)e^{ikx.d_j^n(y)}\longrightarrow G(x,y)\qquad\text{in}\quad L^2(\partial D),
$$

$$
\frac{\partial}{\partial\nu}\left(\sum_{1\le j\le m_n(y)}\alpha_j^n(y)e^{ikx.d_j^n(y)}\right)\longrightarrow\frac{\partial}{\partial\nu}G(x,y)\qquad\text{in}\quad L^2(\partial D)
$$

as $n\longrightarrow\infty$.

This fact tells us

$$
\left(\frac{\partial}{\partial\nu}+\lambda(x)\right)\left(\sum_{1\le j\le m_n(y)}\alpha_j^n(y)e^{ikx.d_j^n(y)}\right)\longrightarrow\left(\frac{\partial}{\partial\nu}+\lambda(x)\right)G(x,y)
\tag{2.5}
$$

in $L^2(\partial D)$ as $n\longrightarrow\infty$.

Since $u^s(x,d) \in H^1(\mathcal{R}^3 \setminus \overline{D})$ satisfies

$$
(2.6) \qquad \begin{cases} \Delta u^s + k^2 u^s = 0, & \text{in} \quad R^3 \setminus \overline{D} \\ \dfrac{\partial u^s}{\partial \nu} + \lambda(x) u^s = -\left(\dfrac{\partial}{\partial \nu} + \lambda(x)\right) e^{ikd.x}, & \text{on} \quad \partial D \\ \dfrac{\partial u^s}{\partial r} - iku^s = O\left(\dfrac{1}{r}\right), & r = |x| \longrightarrow \infty, \end{cases}
$$

and $E(x,y)$ satisfies (2.3), by the standard results in scattering theory, we have

$$
\sum_{1 \le j \le m_n(y)} \alpha_j^n(y) u^s(x, d_j^n(y)) \longrightarrow E(x,y) \qquad \text{uniformly on} \quad R/2 < |x| < 2R
$$

for $y \in \partial B(0,R_1)$. From Proposition 2.1, we know that $u^s(x, d_j^n(y))$, $|x| > R/2$ can be determined from the far-field pattern $u^\infty(d,\theta)$. Therefore, for $R/2 < |x| < 2R$, $E(x,y)$ and its derivatives $\partial/(\partial\nu(x))E(x,y)$ and $\partial/(\partial\nu(y))E(x,y)$ for $y \in \partial B(0,R_1)$ can also be determined from the far-field pattern $u^\infty(d,\theta)$, since R_1 is arbitrary.

Finally, by letting $R_1 \longrightarrow R$, we complete the proof. $\square$

Let $u(x) \in H^1(\mathcal{R}^3 \setminus \overline{D})$ satisfy the mixed-boundary value problem

$$
(2.7) \qquad \begin{cases} \Delta u + k^2 u = 0, & \text{in} \quad \Omega \setminus \overline{D} \\ \dfrac{\partial u}{\partial \nu} + \lambda(x) u = 0, & \text{on} \quad \partial D \\ u(x) = f, & \text{on} \quad \partial \Omega \end{cases}
$$

for $f \in H^{1/2}(\partial\Omega)$.

Lemma 2.1. *There exists a unique solution of the problem* (2.7) *for any* $f \in H^{1/2}(\partial\Omega)$.

Proof. Firstly, we prove the uniqueness. It is enough to prove $f = 0$ implies $u = 0$ in $\Omega \setminus \overline{D}$. For $f = 0$, it is easy to see from (2.7) that

$$
0 = \int_{\Omega\setminus\overline{D}} (\Delta u + k^2 u)\overline{u}dx = \int_{\partial D} \lambda u \overline{u} ds - \int_{\Omega\setminus\overline{D}} (\bigtriangledown u. \bigtriangledown \overline{u} - k^2 u\overline{u})dx,
$$
$$
0 = \int_{\Omega\setminus\overline{D}} (\Delta \overline{u} + k^2 \overline{u})udx + \int_{\partial D} \overline{\lambda}\overline{u} u ds - \int_{\Omega\setminus\overline{D}} (\bigtriangledown \overline{u}. \bigtriangledown u - k^2 \overline{u}u)dx.
$$

Subtracting these two equalities generates

$$
\int_{\partial D} \Im\lambda(x)|u|^2 ds = 0,
$$

which leads to $u = 0$ on ∂D from the assumption on $\lambda(x)$.

Therefore we get $\partial u/(\partial\nu(x)) = 0$ on ∂D. Now the uniqueness of the Cauchy problem for the Helmholtz equations implies $u = 0$ in $\Omega \setminus \overline{D}$. By the integral equation method for the scattering problem ([3], [4]), we know that this problem can be transformed into a Fredholm integral equation of the second kind. Therefore the uniqueness implies the existence. □

By this unique solution $u \in H^1(\Omega)$ of (2.7) for $f \in H^{1/2}(\partial\Omega)$, we can define the Dirichlet-to-Neumann map

$$\Lambda_{\partial D,\lambda} : f \longrightarrow \frac{\partial u}{\partial \nu}|_{\partial\Omega} \in H^{-1/2}(\partial\Omega).$$

In next Lemma, we show the relations between the far field patterns and the Dirichlet-to-Neumann map.

Lemma 2.2. *Suppose that u satisfies* (2.7). *Then $(\partial u/\partial\nu)|_{\partial\Omega}$ can be obtained from $u^\infty(d,\theta)$, $\theta, d \in S^2$.*

Proof. It is enough to prove for $f \in C^\infty(\partial\Omega)$. Let $x_0 \in \partial B(0, R_0)$ for $R/2 < R_0 < R$. By the Green's formula, we have

$$\begin{aligned}
u(x_0) &= \int_{\Omega\setminus\overline{D}} (G_D \Delta u - u\Delta G_D)\, dx \\
&= \int_{\partial\Omega} \left(G_D \frac{\partial u}{\partial \nu_1} - u \frac{\partial G_D}{\partial \nu_1} \right) ds + \int_{\partial D} \left(G_D \frac{\partial u}{\partial \nu_1} - u \frac{\partial G_D}{\partial \nu_1} \right) ds \\
&= \int_{\partial\Omega} \left(G_D(x,x_0) \frac{\partial u(x)}{\partial \nu_1} - f(x) \frac{\partial G_D(x,x_0)}{\partial \nu_1} \right) ds \\
&\quad + \int_{\partial D} \left(G_D \lambda + \frac{\partial G_D}{\partial \nu} \right) u ds \\
&= \int_{\partial\Omega} \left(G_D(x,x_0) \frac{\partial u(x)}{\partial \nu_1(x)} - f(x) \frac{\partial G_D(x,x_0)}{\partial \nu_1(x)} \right) ds
\end{aligned}$$

where ν_1 is the outward normal to the boundary of domain $\Omega \setminus \overline{D}$.

Taking the normal derivatives of u on $\partial B(0,R_0)$ in above expression and letting $R_0 \longrightarrow R$, by the properties of the single layer potential and double layer potential ([4]), we have

(2.8)
$$\frac{1}{2}\frac{\partial u(x_0)}{\partial \nu_1(x_0)} = \int_{\partial\Omega} \frac{\partial G_D(x,x_0)}{\partial \nu_1(x_0)} \frac{\partial u(x)}{\partial \nu_1(x)} ds - \frac{\partial}{\partial \nu_1(x_0)} \int_{\partial\Omega} f(x) \frac{\partial G_D(x,x_0)}{\partial \nu_1(x)} ds.$$

The equation (2.8) is a Fredholm integral equation of the second kind with respect to $(\partial u(x))/(\partial\nu(x))|_{\partial\Omega}$. There exists a unique solution due to the unique solvability of (2.7) (Lemma 2.1).

By Proposition 2.2, we know that, for $x, y \in \partial\Omega$, $\nabla_x G_D(x,y)$ and $\nabla_y G_D(x, y)$ can be obtained from $u^\infty(d,\theta)$, $\theta, d \in S^2$. Therefore $(\partial u/\partial\nu)|_{\partial\Omega}$ can be obtained from $u^\infty(d,\theta)$, $\theta, d \in S^2$.

The proof is complete. □

From this lemma, we can see that the original inverse problem can be stated as that reconstructing the shape of the obstacle and the boundary impedance from the Dirichlet-to-Neumann map $\Lambda_{\partial D,\lambda}$.

Remark 2.1. The Dirichlet-to-Neumann map $(\partial u/\partial \nu)|_{\partial\Omega} = \Lambda_{\partial D,\lambda} f$ can be defined by the following weak form: a bounded linear functional on $H^{1/2}(\partial\Omega)$ such that

$$\langle \Lambda_{\partial D,\lambda} f, g\rangle = \int_{\Omega\setminus\overline{D}} (\bigtriangledown u \bigtriangledown v - k^2 uv)dx - \int_{\partial D} \lambda uvds$$

for any $g \in H^{1/2}(\partial\Omega)$, where v is any element in $H^1(\Omega\setminus\overline{D})$ such that $v\,|_{\partial\Omega} = g$.

Corresponding to $D = \emptyset$, we can introduce a Dirichlet-to-Neumann map $\Lambda_{0,0} : H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega)$ by

$$\Lambda_{0,0} : f \longrightarrow \frac{\partial u_1}{\partial \nu}|_{\partial\Omega}$$

where $u_1(x) \in H^1(\Omega)$ satisfies

$$\begin{cases} \Delta u_1 + k^2 u_1 = 0, & \text{in} \quad \Omega \\ u_1(x) = f, & \text{on} \quad \partial\Omega. \end{cases} \tag{2.9}$$

Since we assume that 0 is not the Dirichlet eigenvalue of the operator $\Delta + k^2$ in Ω, $\Lambda_{0,0}$ can be well defined and is independent of $(\partial D, \lambda(x))$.

Lemma 2.3. *Let $u \in H^1(\Omega\setminus\overline{D})$ and $u_1 \in H^1(\Omega)$ be the solutions to* (2.7) *and* (2.9)*, respectively. There exists a constant $C = C(k, R, \lambda_0)$ such that*

$$\|u - u_1\|_{H^1(\Omega\setminus\overline{D})} \leq C\,\|u_1\|_{H^1(D)}$$

holds for all $f \in H^{1/2}(\partial\Omega)$.

In the cases of $\lambda = 0$ and $\lambda = \infty$, this result may be found in [9]. The proof of this Lemma is given in Section 5.

3. Probe method

Definition 1. For any continuous curve $c = \{c(t)|\ 0 \leq t \leq 1\}$, if it satisfies

(1) $c(0), c(1) \in \partial\Omega$,

(2) $c(t) \in \Omega (0 < t < 1)$,

then we call c a needle in Ω.

Definition 2. For any needle c in Ω, we call

$$t(c, D) = \sup\{0 < t < 1\ ;\ c(s) \in \Omega\setminus\overline{D} \text{ for all } 0 < s < t\}$$

geometric impact parameter (GIP). It is obvious that $t(c, D) = 1$ if c does not touch any point on ∂D.

From this definition, we know if a needle c touches $\overline{D}$, then $t(c,D) < 1$ and $t(c,D)$ is the first hitting time, i.e., $c(t(c,D)) \in \partial D$ and $c(t) \in \Omega \setminus \overline{D}$ for $0 < t < t(c,D)$.

Since $\Omega \setminus \overline{D}$ is connected, it is easy to get a reconstruction algorithm for ∂D in terms of the geometric impact parameter and the needle, i.e.,

$$\partial D = \{c(t) \ ; \ t = t(c,D), c \text{ is a needle and } t(c,D) < 1\}. \tag{3.1}$$

In order to reconstruct ∂D, it suffices to consider the problem of calculating the GIP for each needle from the Dirichlet-to-Neumann map.

Lemma 3.1. *Suppose that Γ is an arbitrary open set of $\partial\Omega$. For each $t > 0$, there exists a sequence $\{v_n\}_{n=1,2,\ldots}$ in $H^1(\Omega)$, which satisfy $\Delta v_n + k^2 v_n = 0$, such that* $\mathrm{supp}(v_n|_{\partial\Omega}) \subset \Gamma$ *and*

$$v_n \longrightarrow G(\cdot - c(t)) \qquad \textit{in} \quad H^1_{loc}(\Omega \setminus \{c(t')|0 < t' \le t\}).$$

This result comes from the Runge approximation theorem (see [7]).

Remark 3.1. Usually the Runge approximation is not constructive, because its proof is done by using the unique continuation and Hahn-Banach theorem. However for the Helmholtz equation, it is possible to make the Runge approximation constructive by using the translation theory (see [6]) and has been done in [10] for the case $k = 0$.

It is obvious that $v_n|_{\partial\Omega}$ depends on $c(t)$. We denote it by $v_n|_{\partial\Omega} = f_n(\cdot, c(t))$, where $f_n(\cdot, c(t)) \in H^{1/2}(\partial\Omega)$ and $\mathrm{supp}(f_n(\cdot, c(t)) \subset \Gamma$.

For a given needle c in Ω and $0 < t < 1$, we can construct a function

$$I(t,c) = \lim_{n \longrightarrow \infty} \langle \overline{(\Lambda_{\partial D,\lambda} - \Lambda_{0,0}) f_n(\cdot, c(t))}, f_n(\cdot, c(t)) \rangle \tag{3.2}$$

where $\langle \cdot, \cdot \rangle$ is the pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$.

Next we show that $\Re I(t,c)$ ($\Re$ denotes the real part) can be used to calculate GIP.

Theorem 2. *For a given needle $c(t)$ in Ω, $t(c,D)$ is given by*

$$\begin{aligned} t(c,D) = \sup\{0 < t < 1; &\textit{the limit in } (3.2) \textit{ exists for all } t' \ (0 < t' < t), \\ &\inf_{0<t'<t} \Re(I(t',c)) > -\infty\}. \end{aligned} \tag{3.3}$$

Proof. We do some preliminary considertion. For a given needle $c(t)$, by Lemma 3.1, we know that there exists a sequences $\{v_n(x)\} \subset H^1(\Omega)$ which satisfies

$$\begin{cases} \Delta v_n + k^2 v_n = 0, & \text{in} \quad \Omega \\ v_n = f_n(\cdot, c(t)), & \text{on} \quad \partial\Omega, \qquad \mathrm{supp}\, f_n(\cdot, c(t)) \subset \Gamma, \end{cases} \tag{3.4}$$

and

$$v_n \longrightarrow G(\cdot - c(t)) \quad \text{in} \quad H^1_{loc}(\Omega \setminus \{c(t')|0 < t' \le t\}) \qquad (n \longrightarrow \infty). \tag{3.5}$$

Let $u_n(x) \in H^1(\Omega)$ satisfy

$$\begin{cases} \Delta u_n + k^2 u_n = 0, & \text{in} \quad \Omega \setminus \overline{D} \\ \dfrac{\partial u_n}{\partial \nu} + \lambda(x) u_n = 0, & \text{on} \quad \partial D \\ u_n(x) = f_n, & \text{on} \quad \partial\Omega, \end{cases} \tag{3.6}$$

then $w_n = u_n - v_n|_{\Omega\setminus\overline{D}} \in H^1(\Omega \setminus \overline{D})$ satisfies

$$\begin{cases} \Delta w_n + k^2 w_n = 0, & \text{in} \quad \Omega \setminus \overline{D} \\ \dfrac{\partial w_n}{\partial \nu} + \lambda(x) w_n = -\left(\dfrac{\partial v_n}{\partial \nu} + \lambda(x) v_n\right), & \text{on} \quad \partial D \\ w_n(x) = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{3.7}$$

By the calculation in Section 6, we have

$$\begin{aligned} &- \overline{\langle (\Lambda_{\partial D,\lambda} - \Lambda_{0,0}) f_n(\cdot, c(t)), f_n(\cdot, c(t)) \rangle} \\ &\quad = \int_{\Omega\setminus\overline{D}} \{|\nabla w_n|^2 - k^2 |w_n|^2\} dx + \int_D \{|\nabla v_n|^2 - k^2 |v_n|^2\} dx \\ &\quad + \int_{\partial D} \{\overline{\lambda}|v_n|^2 - \lambda |w_n|^2\} ds - \int_{\partial D} (\lambda - \overline{\lambda}) \overline{w}_n v_n ds. \end{aligned} \tag{3.8}$$

By Lemmas 2.3 and 3.1, we know that, for $0 < t < t(c, D)$, it holds that

$$w_n \longrightarrow w \quad \text{in} \quad H^1(\Omega \setminus \overline{D}), \qquad n \longrightarrow \infty, \tag{3.9}$$

where w satisfies

$$\begin{cases} \Delta w + k^2 w = 0, & \text{in} \quad \Omega \setminus \overline{D} \\ \dfrac{\partial w}{\partial \nu} + \lambda(x) w = -\left(\dfrac{\partial G(\cdot - c(t))}{\partial \nu} + \lambda(x) G(\cdot - c(t))\right), & \text{on} \quad \partial D \\ w(x) = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{3.10}$$

Let $0 < t < t(c, D)$ and n tend to infinity in (3.8). Then by (3.9), we have

$$\begin{aligned} -I(t, c) &= \int_D \{|\nabla G(\cdot - c(t))|^2 - k^2 |G(\cdot - c(t))|^2\} dx \\ &\quad + \int_{\Omega\setminus\overline{D}} \{|\nabla w|^2 - k^2 |w|^2\} dx \\ &\quad + \int_{\partial D} \{\overline{\lambda}|G(\cdot - c(t))|^2 - \lambda |w|^2\} ds + \int_{\partial D} (\lambda - \overline{\lambda}) \overline{w} G ds. \end{aligned} \tag{3.11}$$

Now we prove (3.3). If $t(c,D)=1$, this is clear. So let $t(c,D)<1$. For any t $(0<t<t(c,D))$, it is easy to see the limit of (3.2) exists for all t' $(0<t'<t)$ and $\inf_{0<t'<t}\Re(I(t',c))>-\infty$. Hence $t(c,D)\le\tilde{t}$, where $\tilde{t}$ is the right handside of (3.3). Suppose $t(c,D)<\tilde{t}$. For $0<t<t(c,D)$, we have

(3.12)

$$
\begin{aligned}
-\Re I(t,c) = &\int_D [|\bigtriangledown G(\cdot - c(t))|^2 - k^2|G(\cdot - c(t))|^2]dx \\
&+\int_{\Omega\setminus\overline{D}} (|\bigtriangledown w|^2 - k^2|w|^2)dx + \int_{\partial D}\Re\lambda[|G(\cdot - c(t))|^2 - |w^2|]ds \\
&-2\int_{\partial D}\Im\lambda\Im(\overline{w}G(\cdot - c(t)))ds \\
\ge &\int_D |\bigtriangledown G(\cdot - c(t))|^2dx - k^2\int_D |G(\cdot - c(t))|^2dx - k^2\int_{\Omega\setminus\overline{D}}|w|^2dx \\
&+\int_{\partial D}\Re\lambda[|G(\cdot - c(t))|^2 - |w|^2]ds - 2\int_{\partial D}\Im\lambda\Im[\overline{w}G(\cdot - c(t))]ds.
\end{aligned}
$$

According to the singularity analysis about $w(x,x_0)$ and $G(x-x_0)$ for $x_0\in\partial D$ (see Theorems 3 and 4 below) and from (3.12), we have

$$\lim_{t\uparrow t(c,D)}\Re(I(t,c))=-\infty.$$

This contradicts to $t(c,D)<\tilde{t}$. □

The **reconstruction algorithm** for the shape of the obstacle can be realized by the following steps:

- Calculate the Dirichlet-to-Neumann map $\Lambda_{\lambda,D}$ from the far field patterns $u^\infty(d,\theta)$, $d,\theta\in S^2$.
- For any given needle $c(t)$, calculate the sequences v_n and $f_n(\cdot,c)$.
- Calculate $\langle\overline{(\Lambda_{\partial D,\lambda}-\Lambda_{0,0})f_n(\cdot,c(t))},f_n(\cdot,c(t))\rangle$.
- Calculate $I(c,t)$ and $t(c,D)$ by (3.2) and (3.3), respectively.
- Calculate ∂D by (3.1).

What we still have to prove for Theorem 1 is to reconstruct boundary impedance, which will be given in the next section.

Remark 3.2. The "reconstruction" in this paper means that there exist some formulae such that the unknown functions can be calculated directly from these formulae. This is different from the "reconstruction" in the numerical simulations. Of course, there is the close relation between the "mathematical reconstruction" and "numerical reconstruction". In this paper, we will not discuss the "numerical reconstruction".

4. Moment method for determining $\lambda(x)$

In this section, we reconstruct the boundary impedance $\lambda(x)$. Since in the previous section, we have reconstructed ∂D from the far field patterns $u^\infty(d,\theta)$, $d,\theta\in S^2$, therefore in this section we assume that ∂D is known.

By the definition of the Dirichlet-to-Neumann map $\Lambda_{\partial D,\lambda}$, we know that, if $u(x) \in H^1(\Omega \setminus \overline{D})$ satisfies

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in} \quad \Omega \setminus \overline{D} \\ \dfrac{\partial u}{\partial \nu} + \lambda(x) u = 0, & \text{on} \quad \partial D \\ u(x) = f, & \text{on} \quad \partial\Omega \end{cases} \tag{4.1}$$

for a given $f(x) \in H^{1/2}(\partial\Omega)$, then its Neumann boundary datum can be obtained formally by

$$\frac{\partial u(x)}{\partial \nu} |_{\partial\Omega} = \Lambda_{\partial D,\lambda} f(x) \in H^{-1/2}(\partial\Omega).$$

Lemma 4.1. *Suppose that* $u_j(x)$, $j = 1, 2, \dots$ *satisfy* (4.1) *with* $f = f_j$. *Put* $\phi_j(x) = u_j(x)|_{\partial D}$. *If*

$$\overline{\operatorname{span}\{f_j(x)\}} = H^{1/2}(\partial\Omega), \tag{4.2}$$

then we have

$$\overline{\operatorname{span}\{\phi_j(x)\}} = H^{1/2}(\partial D).$$

Proof. Assume that $f(x) \in H^{-1/2}(\partial D)$ which satisfies

$$\int_{\partial D} \phi_j \overline{f} ds = 0, \qquad j = 1, 2, \dots, \tag{4.3}$$

we want to prove that $f(x) = 0$. Here $\int_{\partial D} \phi_j \overline{f} ds$ is applied to denote the pairing $\langle \overline{f}, \phi_j \rangle$ between $H^{1/2}(\partial D)$ and $H^{-1/2}(\partial D)$. Later on we use the same convention for the integral on ∂D.

Consider the following mixed boundary value problem

$$\begin{cases} \Delta v + k^2 v = 0, & \text{in} \quad \Omega \setminus \overline{D} \\ \dfrac{\partial v}{\partial \nu} + \lambda(x) v = \overline{f}, & \text{on} \quad \partial D \\ v = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{4.4}$$

Since $\Im\lambda > 0$, similarly to the treatment of proving Lemma 2.1, we know there exists a unique solution for (4.4).

By the Green's formula, we know that

$$\begin{aligned} 0 &= \int_{\Omega\setminus\overline{D}} (v\Delta u_j - u_j \Delta v) dx \\ &= \int_{\partial\Omega} \left(\frac{\partial u_j}{\partial \nu_1} v - \frac{\partial v}{\partial \nu_1} u_j \right) + \int_{\partial D} \left(\frac{\partial u_j}{\partial \nu_1} v - \frac{\partial v}{\partial \nu_1} u_j \right), \end{aligned} \tag{4.5}$$

where ν_1 is the outward normal of domain $\Omega \setminus \overline{D}$.

Noticing $\nu_1 = -\nu$ on ∂D and $v|_{\partial\Omega} = 0$, we have

$$\int_{\partial\Omega} \frac{\partial v}{\partial \nu} u_j ds = \int_{\partial D} \left(\lambda v u_j + \frac{\partial v}{\partial \nu} u_j \right) ds.$$

Therefore, it holds that

$$\int_{\partial\Omega} f_j \frac{\partial v}{\partial \nu} ds = \int_{\partial D} \phi_j \overline{f} ds = 0, \qquad j = 1, 2, \ldots.$$

Since $\overline{\mathrm{span}\{f_j(x)\}} = H^{1/2}(\partial\Omega)$, we obtain

$$\frac{\partial v}{\partial \nu}|_{\partial\Omega} = 0.$$

By the uniqueness of the Cauchy problem for the Helmholtz equations in domain $\Omega \setminus \overline{D}$, we have $v(x) = 0$ in $\Omega \setminus \overline{D}$. Then by (4.4), we know $f(x) = 0$.

The proof is complete. $\square$

On the other hand, we can obtain $u_j|_{\partial D}$ and $(\partial u_j / \partial \nu)|_{\partial D}$ by solving the following Cauchy problem

$$\begin{cases} \Delta u_j + k^2 u_j = 0, & \text{in} \quad \Omega \setminus \overline{D} \\ u_j = f_j, \qquad \dfrac{\partial u_j}{\partial \nu} = \Lambda_{\partial D, \lambda} f_j & \text{on} \quad \partial\Omega \end{cases} \tag{4.6}$$

for a given $f_j(x)$.

Taking the integral in the impedance boundary condition, we have that the impedance $\lambda(x)$ satisfies

$$\int_{\partial D} \lambda(x) u_j(x) ds = -\int_{\partial D} \frac{\partial u_j}{\partial \nu} ds, \qquad j = 1, 2, \ldots \tag{4.7}$$

Here note that $\mathrm{span}\{u_j|_{\partial D}\}$ is dense in $H^{1/2}(\partial D)$ by Lemma 4.1, hence $\lambda(x)$ can be solved uniquely from this moment problem.

Now the recovery of the impedance λ can be realized by the following steps:

- Choose f_j, $j = 1, 2, \ldots$ such that $\overline{\mathrm{span}\{f_j\}_{j=1}^{\infty}} = H^{1/2}(\partial\Omega)$.
- For every f_j, solve the Cauchy problem (4.6) and obtain $u_j|_{\partial D}$ and $(\partial u_j / \partial \nu)|_{\partial D}$.
- Solve the moment problem (4.7) to get λ.

5. Some estimates

In this section we give the proof of Lemma 2.3 and estimate of $\|w\|_{L^2(\Omega \setminus \overline{D})}$. Although the boundary condition is different, the proofs are the same as those of [9] except for some estimates given in Theorem 4 below, we give the detailed proofs for the readers' convenience.

Proof of Lemma 2.3. Put $p(x) = u(x) - u_1(x)|_{\Omega\setminus\overline{D}}$. It is easy to verify that $p(x)$ satisfies

$$(5.1) \qquad \begin{cases} \Delta p + k^2 p = 0, & \text{in} \quad \Omega \setminus \overline{D} \\ \dfrac{\partial p}{\partial \nu} + \lambda(x) p = -g, & \text{on} \quad \partial D \\ p(x) = 0, & \text{on} \quad \partial \Omega \end{cases}$$

where $g(x) = ((\partial u_1/\partial \nu) + \lambda(x) u_1)|_{\partial D}$.

By the Green's formula, we have

$$\int_{\Omega\setminus\overline{D}} (\bigtriangledown p \bigtriangledown \phi - k^2 p\phi) dx - \int_{\partial D} \lambda p \phi ds = \int_{\partial D} g\phi ds = \langle g, \phi|_{\partial D}\rangle$$

for all $\phi \in V = \{\phi \in H^1(\Omega \setminus \overline{D}), \phi|_{\partial\Omega} = 0\}$.

Therefore

$$\|p\|_{H^1(\Omega\setminus\overline{D})} \le C \|g\|_{H^{-1/2}(\partial D)} = C \left\| \frac{\partial u_1}{\partial \nu} + \lambda(x) u_1 \right\|_{H^{-1/2}(\partial D)}.$$

On the other hand, if we restrict u_1 in D, then applying the trace theorem to u_1, which satisfies

$$(5.2) \qquad \begin{cases} \Delta u_1 + k^2 u_1 = 0, & \text{in} \quad D \\ \dfrac{\partial u_1}{\partial \nu} + \lambda(x) u_1 = g_1, & \text{on} \quad \partial D, \end{cases}$$

we have $\|g_1\|_{H^{-1/2}(\partial D)} \le C \|u_1\|_{H^1(D)}$.

The proof is complete. □

Theorem 3. *There exists constant C which is independent of D such that*

$$\|w\|_{L^2(\Omega\setminus\overline{D})} \le C.$$

Proof. First we define a function $v(x)$ by

$$(5.3) \qquad \begin{cases} \Delta v + k^2 v = w, & \text{in} \quad \Omega \setminus \overline{D} \\ \dfrac{\partial v}{\partial \nu} + \overline{\lambda}(x) v = 0, & \text{on} \quad \partial D \\ v(x) = 0, & \text{on} \quad \partial \Omega. \end{cases}$$

Then we have

$$(5.4) \qquad \|v\|_{H^2(\Omega\setminus\overline{D})} \le C \|w\|_{L^2(\Omega\setminus\overline{D})}.$$

Since $\Omega \setminus \overline{D}$ is a domain in $\mathcal{R}^3$, by the Sobolev embedding theorems, we know that $H^2(\Omega\setminus\overline{D})$ can be embedded into $B^{1/2}(\Omega\setminus\overline{D})$ (Hölder space with exponent 1/2) and

$$\|v\|_{B^{1/2}} \le C \|v\|_{H^2}.$$

Therefore we have

$$\|v\|_{B^{1/2}} \leq C \|w\|_{L^2}.$$

From this inequality, we know that

$$\begin{cases} |v(x) - v(y)| \leq C|x - y|^{1/2} \|w\|_{L^2(\Omega \setminus \overline{D})}, & x, y \in \Omega \setminus \overline{D} \\ \|v\|_{L^\infty(\Omega \setminus \overline{D})} \leq C \|w\|_{L^2(\Omega \setminus \overline{D})}. \end{cases} \tag{5.5}$$

Now remind the definition of the weak solutions w and v to (3.10) and (5.3) respectively, we have

$$\begin{aligned} \int_{\Omega \setminus \overline{D}} |w(x)|^2 dx &= \int_{\Omega \setminus \overline{D}} (\Delta + k^2) v(x) \overline{w}(x) dx \\ &= \int_{\partial D} \overline{\lambda}(x) \overline{w}(x) v(x) ds - \int_{\Omega \setminus \overline{D}} \left(\bigtriangledown \overline{w}(x) \bigtriangledown v(x) - k^2 \overline{w}(x) v(x) \right) dx \\ &= - \int_{\partial D} v(x) \left(\frac{\partial}{\partial \nu_1} + \overline{\lambda} \right) \overline{G}(x - c(t)) ds \\ &= - \int_{\partial D} v(x) \overline{\lambda G}(x - c(t)) ds - \int_{\partial D} (v(x) - v(c(t)) \frac{\partial}{\partial \nu_1} \overline{G}(x - c(t)) ds \\ &\quad - v(c(t)) \int_{\partial D} \frac{\partial}{\partial \nu_1} \overline{G}(x - c(t)) ds. \end{aligned} \tag{5.6}$$

On the other hand, if $y \notin \overline{D}$, then

$$- \int_{\partial D} \frac{\partial}{\partial \nu_1} \overline{G}(x - y) ds + k^2 \int_D \overline{G}(x - y) dx = \int_D (\Delta + k^2) \overline{G}(x - y) dx = 0.$$

Hence (5.6) leads to

$$\begin{aligned} \|w\|^2_{L^2(\Omega \setminus \overline{D})} = &- \int_{\partial D} v \overline{\lambda G}(\cdot - c(t)) ds - k^2 v(c(t)) \int_D \overline{G}(\cdot - c(t)) dx \\ &- \int_{\partial D} (v - v(c(t))) \frac{\partial}{\partial \nu_1} \overline{G}(\cdot - c(t)) ds. \end{aligned} \tag{5.7}$$

Here note that the integrals

$$\int_{\partial D} |\overline{G}(x - c(t))| ds, \quad \int_{\partial D} |x - c(t)|^{1/2} |\frac{\partial}{\partial \nu} \overline{G}(x - c(t))| ds, \quad \int_D |\overline{G}(x - c(t))| dx$$

are bounded as $c(t) \longrightarrow \partial \Omega$, then by (5.5) and (5.7), we have

$$\|w\|^2_{L^2(\Omega \setminus \overline{D})} \leq C \|w\|_{L^2(\Omega \setminus \overline{D})}.$$

The proof is complete. □

Theorem 4. *Assume $x_0 \in \partial D$ and $c(t) \in (\Omega \setminus \overline{D}) \cap \partial B(x_0, \delta)$ for some $\delta > 0$, where $B(x_0, \delta)$ is an open ball centered at x_0 with radius δ, then there* *exists some constant $C > 0$ such that for δ small enough the following estimates hold*:

$$\int_D |\bigtriangledown G(x - c(t))|^2 dx \geq \frac{C}{\delta}, \qquad \int_D |G(x - c(t))|^2 dx \leq C,$$

$$\int_{\partial D} |G(x - c(t))|^2 ds \leq C |\ln \delta|, \quad \int_{\partial D} |w(x, c(t))|^2 ds \leq C \int_{\partial D} |G(x - c(t))|^2 ds,$$

here C may be different.

Proof. Denote the tangent plane of ∂D at point x_0 by $T(x_0, \partial D)$. From the expressions of Green's function, we have

(5.8)
$$|\bigtriangledown G(x - c(t))|^2 = O\left(\frac{1}{|x - c(t)|^4}\right), \qquad |G(x - c(t))|^2 = O\left(\frac{1}{|x - c(t)|^2}\right).$$

If $\delta > 0$ is small enough, $\partial D \cap B(x_0, \delta)$ approximates $T(x_0, \partial D) \cap B(x_0, \delta)$. So we have for δ small enough that

(5.9)
$$\begin{aligned}\int_D \frac{1}{|x - c(t)|^4} dx &\geq \int_{D \cap B(x_0,\delta)} \frac{1}{|x - c(t)|^4} dx \geq \int_{D \cap B(x_0,\delta)} \frac{1}{(2\delta)^4} dx \\ &= \frac{1}{(2\delta)^4} \int_{D \cap B(x_0,\delta)} dx \geq \frac{1}{(2\delta)^4} \frac{1}{4} \int_{B(x_0,\delta)} dx = \frac{C}{\delta}.\end{aligned}$$

Hence we have obtained the first estimate. The second estimate is obvious.

On the other hand, let $c(t') \in \Omega \setminus \overline{D}$ satisfy

$$c(t') \in \partial B(x_0, \delta), \qquad c(t') - x_0 \text{ is perpendicular to } T(x_0, \partial D).$$

Then there exists a constant $C > 0$ such that $|x - c(t)| \geq C|x - c(t')|$ for $\delta > 0$ small enough. So

$$\int_{\partial D} |G(x - c(t))|^2 dx \leq C \left(\int_{\partial D_1} + \int_{\partial D_2}\right) \frac{1}{|x - c(t')|^2} dx,$$

where we define

$$\partial D_1 = \partial D \cap \left\{x \colon |x - c(t')| \geq \frac{1}{|\ln \delta|}\right\}, \quad \partial D_2 = \partial D \cap \left\{x \colon |x - c(t')| \leq \frac{1}{|\ln \delta|}\right\}.$$

The first integral leads to

(5.10)
$$\int_{\partial D_1} \frac{1}{|x - c(t')|^2} ds \leq 4 |\ln \delta|^2 \int_{\partial D_1} ds \leq C |\ln \delta|^2.$$

For the second integral, since

$$\partial D_2' = \left\{x : x \in T(x_0, \partial D), |x - x_0|^2 \leq \frac{1}{|\ln \delta|^2} - \delta^2\right\}$$

approximates ∂D_2 for small $\delta > 0$, we have

$$
\begin{aligned}
\int_{\partial D_2} \frac{1}{|x - c(t')|^2} ds \le 2 \int_{\partial D_2'} \frac{1}{|x - c(t')|^2} ds &= \int_{\partial D_2'} \frac{1}{|x - x_0|^2 + \delta^2} ds \\
&= 2 \int_0^{2\pi} \int_0^{\sqrt{|\ln\delta|^{-2} - \delta^2}} \frac{r dr d\theta}{r^2 + \delta^2} \\
&= 4\pi(|\ln\delta| - \ln(|\ln\delta|) \le C|\ln\delta|
\end{aligned} \tag{5.11}
$$

for $\delta > 0$ small enough. Then the third estimate follows from (5.10)–(5.11).

The fourth estimate will be given in Appendix. □

6. Appendix

6.1. Expression of $\Lambda_{\partial D,\lambda}$

Here we lead to the expression of (3.8). Let $v(x) \in H^1(\Omega \setminus \overline{D})$. From the definition of the weak solution of u_n to (3.6), we have

$$
\begin{aligned}
0 &= \int_{\Omega\setminus\overline{D}} v \Delta u_n dx + \int_{\Omega\setminus\overline{D}} k^2 v u_n dx \\
&= \int_{\partial\Omega} v \frac{\partial u_n}{\partial n} ds - \int_{\partial D} v \frac{\partial u_n}{\partial \nu} ds - \int_{\Omega\setminus\overline{D}} (\bigtriangledown u_n \bigtriangledown v - k^2 v u_n) dx.
\end{aligned} \tag{6.1}
$$

Hence, reminding the boundary condition of u_n, we have

$$
\int_{\partial\Omega} \frac{\partial u_n}{\partial n} v ds = \int_{\Omega\setminus\overline{D}} (\bigtriangledown u_n \bigtriangledown v - k^2 u_n v) dx - \int_{\partial D} \lambda u_n v ds.
$$

Taking $v = \overline{v}_n$ in this expression, we have

$$
\langle \Lambda_{\partial D,\lambda} f_n, \overline{f_n} \rangle = \int_{\Omega\setminus\overline{D}} (\bigtriangledown u_n \bigtriangledown \overline{v}_n - k^2 u_n \overline{v}_n) dx - \int_{\partial D} \lambda u_n \overline{v}_n ds. \tag{6.2}
$$

Analogously, we have

$$
\langle \Lambda_{0,0} f_n, \overline{f_n} \rangle = \int_{\Omega} (\bigtriangledown v_n \bigtriangledown \overline{v}_n - k^2 v_n \overline{v}_n) dx. \tag{6.3}
$$

Remind $w_n = u_n - v_n$ and consider the expression

$$
\begin{aligned}
&\int_{\Omega\setminus\overline{D}} [|\bigtriangledown w_n|^2 - k^2 |w_n|^2] dx - \int_{\partial D} \lambda |w_n|^2 ds \\
&\qquad = \int_{\Omega\setminus\overline{D}} (\bigtriangledown w_n \bigtriangledown \overline{w}_n - k^2 w_n \overline{w}_n) dx - \int_{\partial D} \lambda w_n \overline{w}_n ds.
\end{aligned} \tag{6.4}
$$

Since $(\overline{u}_n - \overline{v}_n)|_{\partial\Omega} = 0$, we have

$$
\int_{\Omega\setminus\overline{D}} (\bigtriangledown u_n \bigtriangledown (\overline{u}_n - \overline{v}_n) - k^2 u_n (\overline{u}_n - \overline{v}_n)) dx - \int_{\partial D} \lambda u_n (\overline{u}_n - \overline{v}_n) ds = 0.
$$

Hence, it follows from (6.4) that

$$
\begin{aligned}
&\int_{\Omega\setminus\overline{D}}[|\bigtriangledown w_n|^2 - k^2|w_n|^2]dx - \int_{\partial D}\lambda|w_n|^2 ds \\
&= -\left(\int_{\Omega\setminus\overline{D}}(\bigtriangledown v_n \bigtriangledown \overline{u}_n - k^2 v_n\overline{u}_n)dx - \int_{\partial D}\lambda v_n\overline{u}_n ds\right) \\
&\quad + \int_{\Omega\setminus\overline{D}}(\bigtriangledown v_n \bigtriangledown \overline{v}_n - k^2 v_n\overline{v}_n)dx - \int_{\partial D}\lambda v_n\overline{v}_n ds \\
&= -\int_{\Omega\setminus\overline{D}}\overline{(\bigtriangledown \overline{v}_n \bigtriangledown u_n - k^2\overline{v}_n u_n)}dx + \int_{\partial D}\overline{\lambda u_n\overline{v}_n}ds \\
&\quad - \int_{\partial D}\overline{\lambda u_n\overline{v}_n}ds + \int_{\partial D}\lambda v_n\overline{u}_n ds + \int_{\Omega}\overline{(\bigtriangledown v_n \bigtriangledown \overline{v}_n - k^2 v_n\overline{v}_n)}dx \\
&\quad - \int_D(|\bigtriangledown v_n|^2 - k^2|v_n|^2)dx - \int_{\partial D}\lambda|v_n|^2 ds \\
&= -\overline{\langle\Lambda_{\partial D,\lambda}f_n, \overline{f}_n\rangle} + \overline{\langle\Lambda_{0,0}f_n, \overline{f}_n\rangle} + \int_{\partial D}(\lambda - \overline{\lambda})v_n\overline{u}_n ds \\
&\quad - \int_D(|\bigtriangledown v_n|^2 - k^2|v_n|^2)dx - \int_{\partial D}\lambda|v_n|^2 ds.
\end{aligned}
\tag{6.5}
$$

Combining (6.4) and (6.5) and noticing $u_n = w_n + v_n$, we have (3.8).

6.2. Estimate for w

Here we prove the fourth estimate in Theorem 4.

For given needle $c \in \Omega\setminus\overline{D}$, put $x_0 = c(t) \in \Omega\setminus\overline{D}$ and let $a \in \partial D$ be the point at which the needle c first hits ∂D. Suppose x_0 is very near to a. Consider two families of functions $\{w(\cdot, x_0)\}$, $\{z(\cdot, x_0)\}$ depending on x_0 in some function space X. We donote by $w(\cdot, x_0) \sim z(\cdot, x_0)$ in X if $\{w(\cdot, x_0) - z(\cdot, x_0)\}$ is a bounded set in X.

Let $G_0(x - x_0) = 1/(4\pi|x - x_0|)$. Then it is easy to see that

$$(\partial_\nu + \lambda)G(x - x_0) \sim (\partial_\nu + \lambda)G_0(x - x_0)$$

in $L^2(\partial D)$, hence

$$w(\cdot, x_0) \sim w_0(\cdot, x_0) \qquad \text{in} \quad H^1(\Omega\setminus\overline{D}), \tag{6.6}$$

where $w = w(\cdot, x_0) \in H^1(\Omega\setminus\overline{D})$ is the solution to (3.10) and $w_0 = w_0(\cdot, x_0) \in H^1(\Omega\setminus\overline{D})$ is the solution to

$$
\begin{cases}
\Delta w_0 + k^2 w_0 = 0, & \text{in} \quad \Omega\setminus\overline{D} \\
\left(\dfrac{\partial}{\partial\nu} + \lambda\right) w_0 = -\left(\dfrac{\partial}{\partial\nu} + \lambda\right) G_0(\cdot - x_0), & \text{on} \quad \partial D \\
w_0 = 0, & \text{on} \quad \partial\Omega.
\end{cases}
\tag{6.7}
$$

By the Sobolev embedding $H^{1/2}(\partial D) \hookrightarrow L^r(\partial D)$ with $2 \le r \le 4$ and the Holder inequality, for any $q(4/3 \le q \le 2)$, there exists a constant $C > 0$ such that

$$
\begin{aligned}
(6.8) \qquad \left|\int_{\partial D} \lambda(x) G_0(x-x_0)\phi ds\right| &\le \|\lambda G_0(\cdot - x_0)\|_{L^q(\partial D)} \|\phi\|_{L^r(\partial D)} \\
&\le C\, \|\lambda G_0(\cdot - x_0)\|_{L^q(\partial D)} \|\phi\|_{H^{1/2}(\partial D)}
\end{aligned}
$$

for $\phi \in H^{1/2}(\partial D)$, where $1/r = 1 - 1/q$.

Hence $\lambda G_0(\cdot - x_0) \sim 0$ in $H^{-1/2}(\partial D)$, and by the well poseness of our boundary value problem, this implies

$$
(6.9) \qquad w_0(\cdot, x_0) \sim w_1(\cdot, x_0) \qquad \text{in} \quad H^1(\Omega \setminus \overline{D}),
$$

where $w_1 = w_1(\cdot, x_0) \in H^1(\Omega \setminus \overline{D})$ is the solution to

$$
(6.10) \qquad \begin{cases}
\Delta w_1 + k^2 w_1 = 0, & \text{in} \quad \Omega \setminus \overline{D} \\
\left(\dfrac{\partial}{\partial \nu} + \lambda(x)\right) w_1 = -\dfrac{\partial}{\partial \nu} G_0(\cdot - x_0), & \text{on} \quad \partial D \\
w_1 = 0, & \text{on} \quad \partial \Omega.
\end{cases}
$$

Now consider the solution $w_2 = w_2(\cdot, x_0) \in H^1(\Omega \setminus \overline{D})$ to

$$
(6.11) \qquad \begin{cases}
\Delta w_2 = 0, & \text{in} \quad \Omega \setminus \overline{D} \\
\left(\dfrac{\partial}{\partial \nu} + \lambda(x)\right) w_2 = -\dfrac{\partial}{\partial \nu} G_0(\cdot - x_0), & \text{on} \quad \partial D \\
w_2 = 0, & \text{on} \quad \partial \Omega.
\end{cases}
$$

For this problem, we have

Claim 1. $\lambda(x) w_2(x, x_0) \sim 0$ in $H^{-1/2}(\partial D)$, $w_2(x, x_0) \sim 0$ in $H^{-1}(\Omega \setminus \overline{D})$, then $w_1(\cdot, x_0) \sim w_2(\cdot, x_0)$ in $H^1(\Omega \setminus \overline{D})$.

Proof. The proof given here also gives a more precise estimate for w_2, which will be used in the sequel.

Let $y = (y_1, y_2, y_3) = (y_1(x, x_0), y_2(x, x_0), y_3(x, x_0))$ be a boundary normal coordinates near point a such that

$$
y(a) = 0, \qquad J(x) := \frac{\partial(y(x, x_0))}{\partial x} = I \text{ (identity matrix)}
$$

at $x = x_0$ and $D_0 = \{y_1 < 0\}$ locally near point a. Also, let

$$
\begin{aligned}
A(x) := |J(x)|^{-1} J(x) (J(x))^T, \qquad & x(y(x, x_0); x_0) = x, \\
\tilde{A}(y) := A(x(y; x_0)), \qquad & \tilde{u}(y) := u(x(y; x_0)).
\end{aligned}
$$

Then it is easy to see

(1) $\tilde{A}(y) \in C^1$ near $y = 0$;

(2) $\Delta u = 0$ near point $a \iff \nabla \cdot \tilde{A} \nabla \cdot \tilde{u} = 0$ near 0;

(3) $\delta(x(y;x_0)-x_0)=\delta(y-y_0)$;
(4) $\partial_\nu=\partial_{y_1}$.

In order to simplify the description of our argument, from now on we extend $x(y;x_0)$ and $\tilde{A}(y)$ to an open ball $V\subset R^3$ centered at $y=0$ without destroying their regularities and positivity of $\tilde{A}(y)$. By a direct estimate, we can easily see

$$\tilde{G}_0(y;y_0)\sim G_0(y-y_0)\qquad\text{in}\quad H^1(V),$$

where we have adopted the convention $y_0=y(x_0;x_0)$.

Now consider the solution $\tilde{w}_2^0\in H^1(R^3_+)$ to

$$\begin{cases}\Delta\tilde{w}_2^0=0, & \text{in}\quad y_1>0\\ \partial_{y_1}\tilde{w}_2^0=-\partial_{y_1}G_0(y-y_0), & \text{on}\quad y_1=0.\end{cases}\tag{6.12}$$

and put $\tilde{w}_2(y):=w_2(x(y,x_0))$. If we can prove

Claim 2. $\nabla\cdot((\tilde{A}(y)-\tilde{A}(y_0))\bigtriangledown\tilde{w}_2^0)\sim 0$ in $(H_0^1(V\cap R^3_+))^*$,
then we have

$$\tilde{w}_2\sim\tilde{w}_2^0\qquad\text{in}\quad H^1(V\cap R^3_+)\tag{6.13}$$

by observing
(6.14)

$$\begin{cases}\bigtriangledown\cdot(\tilde{A}\bigtriangledown(\tilde{w}_2-\tilde{w}_2^0))=-\bigtriangledown\cdot((\tilde{A}(y)-\tilde{A}(y_0))\bigtriangledown\tilde{w}_2^0), & \text{in}\quad V\cap R^3_+\\ \partial_{y_1}(\tilde{w}_2-\tilde{w}_2^0)=-\partial_{y_1}\tilde{G}_0(y,y_0)+\partial_{y_1}G_0(y-y_0), & \text{on}\quad y_1=0.\end{cases}$$

Proof for Claim 2 will be given in Subsection 6.3. Therein we also yield a prescise expression for $\tilde{w}_2^0(y)$, which completes the proof of Claim 1. $\square$

Now we can see that

$$w_1(\cdot,x_0)\sim w_2(\cdot,x_0)\qquad\text{in}\quad H^1(\Omega\setminus\overline{D}),\tag{6.15}$$

from Claim 1 and the well posedness of our boundary value problem.

Now summing up (6.6), (6.9), (6.15) and (6.13), as well as the expression of $\tilde{w}_2^0(y)$ in the sequel altogether leads to

$$\int_{\partial D}|w(x,x_0)|^2ds\le C\left(\int_{\partial D}|G(x-c(t))|^2ds+1\right),$$

which completes the proof of the fourth estimate in Theorem 4.

6.3. Proof for Claim 2

Let $y_0=(y_{01},y_{02},y_{03})=(y_{01},y_0')$. Then it is well known that $H(y)=H(y;y_0)=G_0(y-y_0)$ can be given by

$$H(y)=\begin{cases}H_+(y)=H_+(y;y_0), & \text{in}\quad y_1>y_{01}\\ H_-(y)=H_-(y;y_0), & \text{in}\quad y_1<y_{01}\end{cases}\tag{6.16}$$

with the solution $H_{\pm}(y)$ to

$$
(6.17) \qquad \begin{cases} \Delta H_{\pm}(y) = 0, & \text{in} \quad \pm(y_1 - y_{01}) > 0 \\ H_{+}(y)|_{y_1 = y_{01}+0} = H_{-}(y)|_{y_1=y_{01}-0}, \\ \partial_{y_1} H_{+}(y)|_{y_1=y_{01}+0} - \partial_{y_1} H_{-}(y)|_{y_1=y_{01}-0} = -\delta(y' - y_0'). \end{cases}
$$

Denote by $\Gamma_{\pm}(y_1, \eta')$ and $w(y_1, \eta')$ the Fourier transforms of $H_{\pm}(y)$ and $\tilde{w}_2^0(y)$ with respect to y', respectively. Then, $\Gamma'_{\pm} := e^{iy_0'\cdot\eta'}\Gamma_{\pm}$ and $w' := e^{iy_0'\cdot\eta'}w$ satisfy

$$
(6.18) \qquad \begin{cases} (\partial_{y_1}^2 - |\eta'|^2)\Gamma'_{\pm} = 0, & \text{in} \quad \pm(y_1 - y_{01}) > 0 \\ \Gamma'_{+}|_{y_1=y_{01}+0} = \Gamma'_{-}|_{y_1=y_{01}-0}, \\ \partial_{y_1}\Gamma'_{+}|_{y_1=y_{01}+0} - \partial_{y_1}\Gamma'_{-}|_{y_1=y_{01}-0} = -1. \end{cases}
$$

and

$$
(6.19) \qquad \begin{cases} (\partial_{y_1}^2 - |\eta'|^2)w' = 0, & \text{in} \quad y_1 > 0 \\ \partial_{y_1} w' = -\partial_{y_1}\Gamma'_{-}, & \text{on} \quad y_1 = 0. \end{cases}
$$

respectively. $\Gamma'_{\pm} = \Gamma'_{\pm}(y_1)$ is given by

$$
\Gamma'_{\pm}(y_1) = 2^{-1}|\eta'|^{-1}e^{\mp(y_1-y_{01})|\eta'|},
$$

Hence $w' = w'(y_1) = 2^{-1}|\eta'|^{-1}e^{-(y_1+y_{01})|\eta'|}$. Comparing these two formula, we have

$$
\tilde{w}_2^0(y) = H_{+}(y_1, y'; -y_{01}, y_0') = \frac{1}{4\pi\sqrt{(y_1+y_{01})^2 + |y'-y_0'|^2}}.
$$

This immediately proves Claim 2.

Acknowledgements. The authors thank the referees for their careful reading and suggestions. The first named author is partly supported by NSF of China (No. 10271032). He would like to thank Prof. M. Yamamoto in the University of Tokyo for his kind hospitality and discussions during his long stay in Japan. The second named author is partly supported by the Science Foundation of Southeast University, China (No. 9207011148). He is grateful to Dr. M. Ikehata for useful discussions during his visit in Gunma University.

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