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# Rank one log del Pezzo surfaces of index two

By

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**Abstract**

Let $S$ be a rank one log del Pezzo surface of index two and $S^0$ the smooth part of $S$. In this paper we determine the singularity type of $S$, in a way different from Alekseev and Nikulin [1]. Moreover, we calculate the fundamental group of $S^0$ and prove that $S$ contains the affine plane as a Zariski open subset if and only if $\pi_1(S^0) = (1)$.

## 1. Introduction

Throughout the present article we work over an algebraically closed field $k$ of characteristic zero. Whenever we consider problems of topological nature, we assume that $k$ to be the complex number field $\mathbf{C}$. Let $S$ be a normal projective surface with only quotient singular points. The index of $S$ is the smallest positive integer $N$ such that $NK_S$ is a Cartier divisor. Since $S$ has only quotient singularities, the index of $S$ exists. Let $\pi : V \to S$ be a minimal resolution of singularities and $D$ the exceptional locus, which we identify with a reduced divisor with support $D$. We often denote $(V, D)$ and $S$ interchangeably.

**Definition 1.1.** Let $S$ be a normal projective surface with only quotient singular points. Then $S$ is called a *log del Pezzo surface* if the anticanonical divisor $-K_S$ is ample. A log del Pezzo surface $S$ is said to have rank one if the Picard number of $S$ is equal to one. In the present article we call a log del Pezzo surface of rank one an *LDP*1-*surface*.

In recent years, log del Pezzo surfaces have been studied by several authors. Gurjar and Zhang [8], [9] proved that the fundamental group of the smooth part of every log del Pezzo surface is finite. There are other proofs by Fujiki, Kobayashi and Lu [6] and by Keel and McKernan [12], independently. In [12], Keel and McKernan studied LDP1-surfaces and proved that the smooth part $S^0 := S - \operatorname{Sing} S$ of every LDP1-surface $S$ is log-uniruled, i.e., $S^0$ contains a non-empty Zariski open subset dominated by images of the affine line. LDP1-surfaces of index one (that is, Gorenstein LDP1-surfaces) have been studied by

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Brenton [2], Demazure [4], Furushima [7], Hidaka and Watanabe [10], Miyanishi and Zhang [19], etc. The classification of LDP1-surfaces of index two was announced by Alekseev and Nikulin [1, Theorem 7]. In [24], Zhang classified all LDP1-surfaces with only rational double points and unique rational triple point. Note that every LDP1-surface can have at most five singular points by [12, Section 9]. In [13], the author classified all LDP1-surfaces with unique singular point. The complete classification of LDP1-surfaces, however, is not yet fully explored.

In the present article, we shall study LDP1-surfaces of index two. In Section 3, by using Zhang's results on LDP1-surfaces (cf. [23] and [24]), we classify all LDP1-surfaces of index two. Our method is quite different from Alekseev and Nikulin [1]. In Section 4 we calculate the fundamental groups of the smooth parts of the LDP1-surfaces of index two. Our main result is the following theorem.

**Theorem 1.1.** *Let $S$ be an* LDP1*-surface of index two and let $\pi : (V, D) \to S$ be a minimal resolution of $S$, where $D$ is the reduced exceptional divisor. Let $S^0$ be the smooth part of $S$. Then the following assertions hold*:

(1) *There exist exactly* 18 *singularity types of* LDP1*-surfaces of index two, each of which is realizable and given in terms of the weighted dual graph of $D$ in Table* 1 (*see Appendix*).

(2) *Suppose that $(V, D)$ is not isomorphic to $(\Sigma_4, M_4)$. Then there exist a $(-1)$-curve $C \in \mathrm{MV}(V, D)$ (for the definition of $\mathrm{MV}(V, D)$, see Section* 2) *and a $\mathbf{P}^1$-fibration $\Phi : V \to \mathbf{P}^1$ such that $\varphi := \Phi|_{V-D} : V - D \to \mathbf{P}^1$ is an $\mathbf{A}^1$-fibration or an untwisted $\mathbf{A}^1_*$-fibration* (*for the definition, see* [17]). *Further, the configuration of $C + D$ as well as all singular fibers of $\Phi$ can be explicitly described. The configuration is given in Appendix, as the configuration $(n)$ for $2 \le n \le 18$.*

(3) *$\pi_1(S^0)$ is a finite group of order $\le 8$. The fundamental group $\pi_1(S^0)$ and the singularity type of the quasi-universal covering $U$ of $S$* (*see Section* 4) *are given in Table* 1 *together with other data.*

(4) *$S$ contains the affine plane as a Zariski open subset if and only if $\pi_1(S^0) = (1)$.*

A $(-n)$-curve is a smooth complete rational curve with self-intersection number $-n$. A connected reduced effective divisor $T$ on a smooth surface is a $(-2)$-rod (resp. a $(-2)$-fork) if $T$ consists entirely of $(-2)$-curves and $T$ can be contracted to a cyclic rational double point (resp. a non-cyclic rational double point). A $(-2)$-rod (resp. a $(-2)$-fork) corresponds to the exceptional locus of a minimal resolution of a rational double point of Dynkin type $A_n$ (resp. $D_n$ $(n \ge 4)$, $E_6$, $E_7$ or $E_8$). A reduced effective divisor $D$ is called an NC (resp. SNC) divisor if $D$ has only normal (resp. simple normal) crossings. We employ the following notation:

$K_X$: the canonical divisor on $X$.
$\rho(X)$: the Picard number of $X$.
$\Sigma_n (n \ge 0)$: a Hirzebruch surface of degree $n$.

$M_n (n \geq 0)$: a minimal section of $\Sigma_n$.
$\# D$: the number of all irreducible components in $\operatorname{Supp} D$.

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## 2. Preliminary results

**Definition 2.1.** (1) An SNC divisor $D$ on a smooth projective surface is said to be of type $K_n$ $(n \geq 1)$ if $D$ consists entirely of rational curves and has the weighted dual graph as shown in Figure 1.

(2) A quotient singular point $P$ on a normal surface $\overline{X}$ is said to be of type $K_n$ if the reduced exceptional divisor of a minimal resolution of $P \in \overline{X}$ is of type $K_n$. Note that if the index of $P$ is equal to two then $P$ is of type $K_n$ by [1, Proposition 2] (see also [25, Lemma 1.8]).

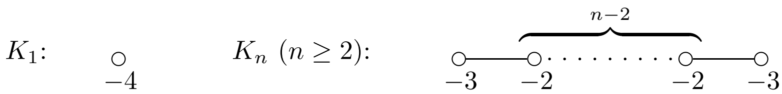



Figure 1

Let $T$ be a normal projective surface with only quotient singular points. If the index of $T$ is equal to two, then each singular point of $T$ is a rational double point or a quotient singular point of type $K_n$. As usual, rational double points are indicated by their Dynkin types $A_n$, $D_n$ $(n \geq 4)$, $E_6$, $E_7$, and $E_8$. When we say $T$ a surface of type $A_7 2K_1$ for example, this means that $T$ has three singular points, one of which is of type $A_7$ and other two are of type $K_1$. We indicate this by writing $S(A_7 2K_1)$.

Now, let $S$ be an LDP1-surface and let $\pi : V \to S$ be a minimal resolution of $S$. Let $D = \sum_i D_i$ be the reduced exceptional divisor with respect to $\pi$, where the $D_i$ are irreducible components of $D$. Since $S$ has only log-terminal singularities, there exists uniquely an effective $\mathbf{Q}$-divisor $D^{\#} = \sum_i \alpha_i D_i$ such that $0 \leq \alpha_i < 1$ for any $i$ and $D^{\#} + K_V$ is numerically equivalent to $\pi^* K_S$ (see [11], [18], [16], etc.). Hereafter in the present section, we retain this situation.

**Lemma 2.1.** (1) *$-(D^{\#} + K_V)$ is nef and big $\mathbf{Q}$-Cartier divisor. Moreover, for any irreducible curve $F$, $-(D^{\#} + K_V \cdot F) = 0$ if and only if $F$ is a component of $D$.*

(2) *Any $(-n)$-curve with $n \geq 2$ is a component of $D$.*

(3) *$V$ is a rational surface.*

*Proof.* See [24, Lemma 1.1].

**Lemma 2.2.** *There is no $(-1)$-curve $E$ such that, after contracting $E$ and consecutively (smoothly) contractible curves in $E + D$, the image of the divisor $E + D$ can be contracted to quotient singular points.*

*Proof.* See [23, Lemma 1.4].

By Lemma 2.1 (1), if $C$ is an irreducible curve not contained in $\operatorname{Supp} D$, then $-(C \cdot D^{\#} + K_V)$ takes value in $\{n/p|\ n \in \mathbf{N}\}$, where $p$ is the index of $S$. So we can find an irreducible curve $C$ such that $-(C \cdot D^{\#} + K_V)$ attains the smallest positive value. We denote the set of such irreducible curves by $\mathrm{MV}(V, D)$.

**Definition 2.2** (cf. [24, Definitions 1.2 and 3.2])**.** With the same notation as above, assume that $\rho(V) \geq 3$.

(1) $(V, D)$ is said to be of the first kind if there exits an irreducible curve $C \in \mathrm{MV}(V, D)$ such that $|C + D + K_V| \neq \emptyset$. $(V, D)$ is said to be of the second kind if $(V, D)$ is not of the first kind, i.e., $|C + D + K_V| = \emptyset$ for any curve $C \in \mathrm{MV}(V, D)$.

(2) Assume that $(V, D)$ is of the second kind. $(V, D)$ is said to be of type (IIa) if there exists a curve $C \in \mathrm{MV}(V, D)$ meeting at least two $(-2)$-curves in $\operatorname{Supp} D$. $(V, D)$ is said to be of type (IIb) if there exists a curve $C \in \mathrm{MV}(V, D)$ meeing only one component of $D$ but $(V, D)$ is not of type (IIa). $(V, D)$ is said to be of type (IIc) if $(V, D)$ is neither of type (IIa) nor of type (IIb).

We shall prove that if the index of $(V, D)$ is equal to two and $\rho(V) \geq 3$, then $(V, D)$ is of the second kind (see Theorem 3.1).

**Lemma 2.3.** *Assume that $(V, D)$ is of the second kind and that there exists a curve $C \in \mathrm{MV}(V, D)$ meeting at least three components $D_0$, $D_1$ and $D_2$ of $D$. Then either $G := 2C + D_0 + D_1 + D_2 + K_V \sim 0$ or there exists a $(-1)$-curve $\Gamma$ such that $G \sim \Gamma$ and $(C \cdot \Gamma) = (D_i \cdot \Gamma) = 0$ for $i = 0, 1, 2$.*

*Proof.* See [23, Lemma 2.3].

**Lemma 2.4.** *Assume that $(V, D)$ is of the second kind. Then every curve $C \in \mathrm{MV}(V, D)$ is a $(-1)$-curve.*

*Proof.* See [23, Lemma 2.2] and [8, Proposition 3.6]. See also [13, Lemma 1.5].

**Lemma 2.5.** *Let $\Phi : V \to \mathbf{P}^1$ be a $\mathbf{P}^1$-fibration. Then the following assertions hold*:

(1) *$\#\{$irreducible components of $D$ not in any fiber of $\Phi\} = 1 + \sum(\#\{(-1)$-curves in $F\} - 1)$, where $F$ moves over all singular fibers of $\Phi$.*

(2) *If a singular fiber $F$ consists only of $(-1)$-curves and $(-2)$-curves then $F$ has one of the configurations* (i), (ii) *and* (iii) *in Figure* 2. *In Figure* 2, *the integer over a curve is the self-intersection number of the corresponding curve.*

(3) *Suppose that there exists a singular fiber $F$ such that $F$ is of type* (i) *or* (ii) *in Figure* 2. *Let $C$ be the unique $(-1)$-curve in* $\operatorname{Supp} F$. *Suppose further that $C \in \mathrm{MV}(V, D)$. Then each singular fiber consists of $(-2)$-curves*

*and* $(-1)$*-curves, say* $E_1$ *and* $E_2$ *(possibly* $E_1 = E_2$*), and* $E_i \in \mathrm{MV}(V, D)$ *for* $i = 1, 2$.

*Proof.* See [23, Lemmas 1.5 and 1.6].

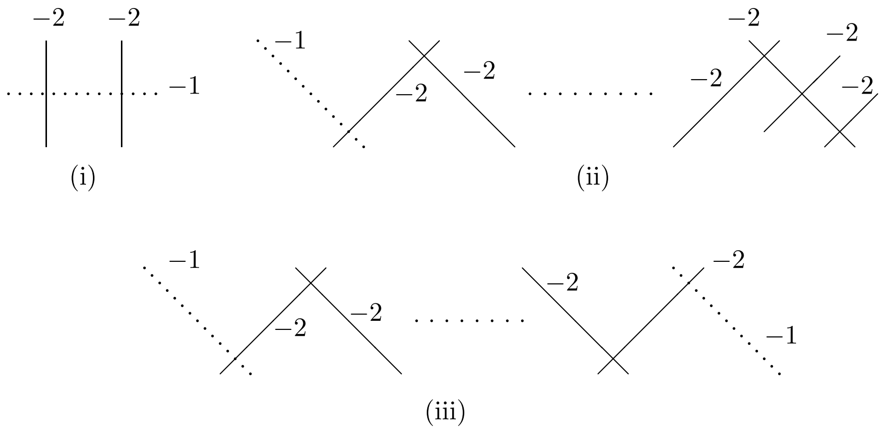



Figure 2

## 3. Classification

Let $(V, D)$ be an LDP1-surface of index two. If $\rho(V) \le 2$, then $(V, D) \cong (\Sigma_4, M_4)$ (see No. 1 in Table 1). We assume that $\rho(V) \ge 3$. Let $D = \sum_{i=1}^{r} D^{(i)}$ be the decomposition of $D$ into connected components. Assume that $D^{(i)}$ $(1 \le i \le s)$ is of type $K_n$ and $D^{(j)}$ $(j > s)$ is a $(-2)$-rod or a $(-2)$-fork. It is then clear that $s \ge 1$ and $D^{\#} = (1/2)\sum_{i=1}^{s} D^{(i)}$ (see Section 2 for the definition of $D^{\#}$). Further, for any curve $E$ not in $\operatorname{Supp} D$, $-(E \cdot D^{\#} + K_V) \ge 1/2$.

We prove the following result.

**Theorem 3.1.** *Let* $(V, D)$ *be an* LDP1*-surface of index two. Assume that* $\rho(V) \ge 3$*. Then* $(V, D)$ *is of the second kind, i.e.,* $|C + D + K_V| = \emptyset$ *for any curve* $C \in \mathrm{MV}(V, D)$.

*Proof.* Suppose to the contrary that $(V, D)$ is of the first kind, i.e., there exists a curve $C \in \mathrm{MV}(V, D)$ such that $|C + D + K_V| \ne \emptyset$. By [23, Lemma 2.1], there exists uniquely a decomposition of $D$ as a sum of effective integral divisors $D = D' + D''$ such that:

(i) $(C \cdot D_i) = (D'' \cdot D_i) = (K_V \cdot D_i) = 0$ for any component $D_i$ of $D'$.

(ii) $C + D'' + K_V \sim 0$.

Namely, the pair $(V, C + D)$ is a quasi-Iitaka surface (for the definition, see [23, Section 3]). Since $(V, D)$ has index two and each connected component of $D'$ is a $(-2)$-rod or a $(-2)$-fork, $D''$ is a connected component of $D$ and of type $K_n$. In particular, $D^{\#} = (1/2)D''$.

Since $|C+K_V| = |-D''| = \emptyset$ by (ii), $C \cong \mathbf{P}^1$. So $(C \cdot D'') = -(C \cdot C + K_V) = 2$. Since $D^{\#} = (1/2)D''$, we have

$$0 > (C \cdot D^{\#} + K_V) = \frac{1}{2}(D'' \cdot C) + (C \cdot K_V) = 1 + (C \cdot K_V) = -1 - (C^2).$$

Hence $(C^2) \geq 0$ and $-(C \cdot D^{\#} + K_V) \geq 1$.

Since $\rho(V) \geq 3$, there exists a $(-1)$-curve $E$ on $V$. Then

$$\left(\frac{1}{2} \leq\right) - (E \cdot D^{\#} + K_V) = 1 - \frac{1}{2}(E \cdot D'') \leq 1.$$

Since $C \in \mathrm{MV}(V, D)$, we know that $(C^2) = 0$ and $(E \cdot D'') = 0$ for any $(-1)$-curve $E$ on $V$. Then $\mathcal{O}_C(C) \cong \mathcal{O}_{\mathbf{P}^1}$. Consider the following exact sequence:

$$0 \to \mathcal{O}_V \to \mathcal{O}_V(C) \to \mathcal{O}_{\mathbf{P}^1} \to 0.$$

Since $V$ is a rational surface, the induced cohomology exact sequence implies that $h^0(V, \mathcal{O}_V(C)) = 2$ and a complete linear system $|C|$ is free. So $|C|$ defines a $\mathbf{P}^1$-fibration $\Phi := \Phi_{|C|} : V \to \mathbf{P}^1$. Let $F$ be a singular fiber of $\Phi$, where we note that $V$ is not relatively minimal. If $F$ contains some components of $D''$ then, by Lemma 2.1 (2), $F$ has a $(-1)$-curve meeting $D''$. This is a contradiction. If $F$ contains no components of $D''$, then $F$ has a $(-1)$-curve $G$ meeting $D''$ because some components of $D''$ meet $C$. This is also a contradiction. □

We consider LDP1-surfaces of index two and type (IIa) in the following theorem.

**Theorem 3.2.** *Let $(V, D)$ be an* LDP1-*surface of index two and type* (II*a*). *Let $C \in \mathrm{MV}(V, D)$ be a curve meeting at least two $(-2)$-curves in* Supp $D$. *Then the following assertions hold.*

(1) *The singularity type of $(V, D)$ is one of $2A_1D_6K_1$ and $A_1A_5K_3$ (see No.* 2 *and No.* 3 *in Table* 1*).*

(2) *There exist a $\mathbf{P}^1$-fibration $\Psi : V \to \mathbf{P}^1$ and a component $H$ of $D$ such that $H$ is a section of $\Psi$ and the other components of $D$ are contained in singular fibers of $\Psi$. In particular, $V - D$ is affine-ruled, i.e., $V - D$ contains a non-empty Zariski open subset isomorphic to $U \times \mathbf{A}^1$, where $U$ is a smooth algebraic curve.*

(3) *The configuration of $C + D$ and all singular fibers of $\Psi$ is given in the configuration $(n)$ for $n = 2$ or* 3 *in Appendix.*

(4) *All the cases are realizable.*

*Proof.* By Lemma 2.4, $C$ is a $(-1)$-curve. Let $D_1$ and $D_2$ be two $(-2)$-curves in Supp $D$ which $C$ meets. Since $|C+D+K_V| = \emptyset$, $(C \cdot D_1) = (C \cdot D_2) = 1$. So a divisor $F_0 := 2C + D_1 + D_2$ defines a $\mathbf{P}^1$-fibration $\Phi = \Phi_{|F_0|} : V \to \mathbf{P}^1$. By Lemma 2.5 (3), each singular fiber of $\Phi$ consists only of $(-1)$-curves and $(-2)$-curves.

(I) *The case where $C$ meets a component $D_0$ of $D-(D_1+D_2)$.* By Lemma 2.3, either $G := 2C + D_0 + D_1 + D_2 + K_V = F_0 + D_0 + K_V \sim 0$ or there exists a $(-1)$-curve $\Gamma$ such that $G \sim \Gamma$ and $(C \cdot \Gamma) = (D_i \cdot \Gamma) = 0$ for $i = 0, 1, 2$. We consider the following two cases I-1 and I-2 separately.

*Case* I-1. $G \sim 0$. Then $D_0$ is a 2-section of $\Phi$ because $(D_0 \cdot F_0) = -(D_0 \cdot D_0 + K_V) = 2$. Since the dual graph of $C + D$ is a tree by [15, Lemma I.2.1.3], $(D_0 \cdot D_1) = (D_0 \cdot D_2) = (D_1 \cdot D_2) = 0$. If $D_i$ is a component of $D - (D_0 + D_1 + D_2)$, then

$$0 \leq (D_i \cdot F_0) = (D_i \cdot -D_0 - K_V) \leq 0.$$

So $(D_i \cdot F_0) = (D_i \cdot D_0) = (D_i \cdot K_V) = 0$. Hence $(D_j \cdot D - D_j) = 0$ for $j = 0, 1, 2$ and each connected component of $D - D_0$ is a $(-2)$-rod or a $(-2)$-fork. Since the index of $(V, D)$ is equal to two, $(D_0^2) = -4$.

By using $\rho(V) = \#D + 1$ and Lemma 2.5 (1), we know that every singular fiber has the configuration (i) or (ii) in Figure 2. Applying the Hurwitz formula to $\Phi|_{D_0} : D_0 \to \mathbf{P}^1$, we see that $\Phi$ has at most two singular fibers. Let $u : V \to \Sigma_n$ be a contraction of all $(-1)$-curves and consecutively (smoothly) contractible curves in the fibers of $\Phi$. By Lemma 2.1 (2), $n = 0$ or 1. We put $u_*(D_0) \sim 2M_n + \alpha\ell$, where $\ell$ is a fiber of $\Phi_1 = \Phi \circ u^{-1} : \Sigma_n \to \mathbf{P}^1$. Since $u_*(D_0)$ is a smooth rational curve, we have $\alpha = n + 1$ and $(u_*(D_0)^2) = (2M_n + (n+1)\ell)^2 = 4$. Then we know that $\Phi$ has just two singular fibers $F_0$ and $F_1$ and that $\#F_1 = 1 + (8 - 2) = 7$. Hence the configuration of $F_1$ looks like that of (ii) in Figure 2. The singularity type of $(V, D)$ is then $2A_1 D_6 K_1$.

The configuration of $C + D + E_1$ looks like that of Figure 3, where $E_1$ is the unique $(-1)$-curve in $\mathrm{Supp}(F_1)$. Put $G_0 := 4E_1 + 3D_3 + 2D_5 + D_0 + D_5$. Then $G_0$ defines a $\mathbf{P}^1$-fibration $\Psi := \Phi_{|G_0|} : V \to \mathbf{P}^1$, $C$ and $D_6$ are sections of $\Psi$ and $D - D_6$ is contained in singular fibers of $\Psi$. Let $G_i$ $(i = 1, 2)$ be the singular fiber of $\Psi$ containing $D_i$. By considering $\rho(V) = \#D + 1 = 10$ and Lemma 2.5 (1), we can easily see that the configuration of $C + D$ and all singular fibers of $\Psi$ is given in the configuration (2) in Appendix. In particular, $V - D$ is affine-ruled.

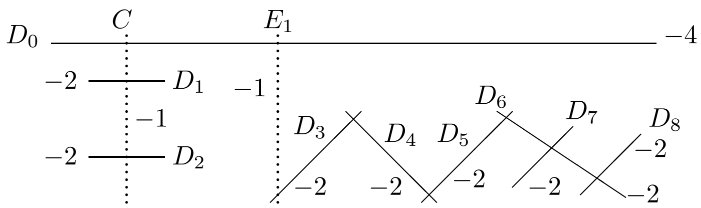



Figure 3

*Case* I-2. There exists a $(-1)$-curve $\Gamma$ such that $G \sim \Gamma$ and $(C \cdot \Gamma) = (D_i \cdot \Gamma) = 0$ for $i = 0, 1, 2$. Since $G = F_0 + D_0 + K_V \sim \Gamma$ and $(D_0 \cdot \Gamma) = 0$, $(F_0 \cdot D_0) = -(D_0 + K_V \cdot D_0) = 2$, i.e., $D_0$ is a 2-section of $\Phi$. Since $(\Gamma \cdot C) = (\Gamma \cdot D_i) = 0$ $(i = 0, 1, 2)$, $\Gamma$ is contained in a fiber $F_1$ of $\Phi$. By Lemma 2.5 (3), the configuration of $F_1$ looks like that of (i), (ii) or (iii) in Figure 2. If $F_1$

has the configuration (i) or (iii) in Figure 2, then there exists a $(-1)$-curve $E$ (possibly $\Gamma$) and a reduced effective divisor $\Delta(\leq D)$ such that $|E+\Delta+K_V| \neq \emptyset$ because $(\Gamma \cdot D_0) = 0$. By Lemma 2.5 (3), $E \in \mathrm{MV}(V, D)$. Then $(V, D)$ is of the first kind, a contradiction. So the configuration of $F_1$ looks like that of (ii) in Figure 2. Since each connected component of $D$ can be contracted to a quotient singular point, $D_0$ meets $F_1$ as follows (Figure 4):

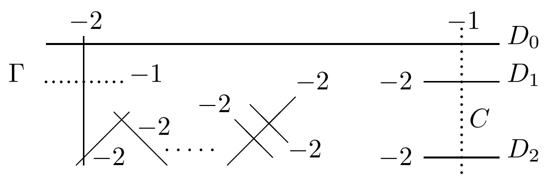



Figure 4

Since $(V, D)$ has index two, $(D_0^2) = -2$. We claim that $D - D_0$ is contained in fibers of $\Phi$. Indeed, suppose that $D_i \leq D - D_0$ is not in any fiber of $\Phi$. Then $(D_i \cdot \Gamma) = (D_i \cdot F_0 + D_0 + K_V) \geq (D_i \cdot F_0) \geq 1$. On the other hand,

$$((D_i \cdot \Gamma) \geq)(D_i \cdot F_0) = (D_i \cdot F_1) \geq (D_i \cdot 2\Gamma) > (D_i \cdot \Gamma).$$

This is absurd. So each connected component of $D$ is a $(-2)$-rod or a $(-2)$-fork. This contradicts that the index of $(V, D)$ is equal to two. Therefore, Case I-2 does not take place.

(II) *The case where $C$ does not meet any component of $D - (D_1 + D_2)$.* We claim that there exist no $(-4)$-curves in $\mathrm{Supp}\, D$. Indeed, if $D_i$ is a $(-4)$-curve in $\mathrm{Supp}\, D$, then $(D_i \cdot D - D_i) = 0$. Since $(C \cdot D_i) = 0$, $D_i$ is contained in a singular fiber of $\Phi$. This is a contradiction because each singular fiber of $\Phi$ consists only of $(-1)$-curves and $(-2)$-curves.

Since $(V, D)$ has index two and $D$ contains no $(-4)$-curves, there exists a $(-3)$-curve $D_0$ in $\mathrm{Supp}\, D$. Then $(D_0 \cdot D_j) = 1$, where $j = 1$ or $2$, because $D_0$ is not contained in any fiber of $\Phi$. Assume that $j = 1$. Let $D^{(i)}$ $(i = 1, 2)$ be the connected component of $D$ containing $D_i$. Then $D^{(1)}$ is of type $K_n$ $(n \geq 3)$ and $D^{(2)}$ is a $(-2)$-rod or a $(-2)$-fork because $-(C \cdot D^\# + K_V) \geq 1/2$. Let $D_4$ be the $(-3)$-curve in $\mathrm{Supp}(D^{(1)})$ other than $D_0$. Then $D_4$ also meets $D_1$. So $D^{(1)}$ is of type $K_3$. Since $(D - D_1 \cdot D_1) = 2$, by using the arguments as in the proof of [23, Lemma 5.3], we know that $(D - D_2 \cdot D_2) = 0$.

Let $F_0, \ldots, F_r$ $(r \geq 0)$ be all singular fibers of $\Phi$. We claim that:

Claim 1. $r = 1$ and the configuration of $F_1$ looks like that of (iii) in Figure 2.

*Proof.* If $r = 0$, then $\rho(V) = 2 + (\#F_0 - 1) = 4$. On the other hand, $\rho(V) = \#D + 1 \geq \#D^{(1)} + \#D^{(2)} + 1 = 5$, which is a contradiction. So $r \geq 1$. Since $(D - D_2 \cdot D_2) = 0$, $D - D^{(1)}$ is contained in singular fibers of $\Phi$. By using $\rho(V) = \#D + 1$ and Lemma 2.5 (1), we know that $r = 1$. If the configuration of $F_1$ looks like that of (i) or (ii) in Figure 2, then the unique $(-1)$-curve $E_1$ in

$\mathrm{Supp}(F_1)$ meets both of $D_0$ and $D_4$ which are sections of $\Phi$. Then

$$-(E_1 \cdot D^{\#} + K_V) \le 1 - \frac{1}{2}(E_1 \cdot D_0 + D_4) \le 0,$$

which is a contradiction. This proves Claim 1.

Let $E_1$ and $E_1'$ be the two $(-1)$-curves in $\mathrm{Supp}(F_1)$. Since $D_0$ and $D_4$ are sections of $\Phi$ and $D - D^{(1)}$ is contained in singular fibers of $\Phi$, we may assume that $(E_1 \cdot D_0) = (E_1' \cdot D_4) = 1$. Note that $(F_1)_{\mathrm{red}} - (E_1 + E_1') \neq 0$ by $\rho(V) = \#D + 1$ and Lemma 2.2. Let $\mu : V \to \Sigma_3$ be the contraction of all $(-1)$-curves and consecutively (smoothly) contractible curves in fibers of $\Phi$ except for those meeting $D_0$. Then $M_3 = \mu_*(D_0)$, $(\mu_*(D_0) \cdot \mu_*(D_4)) = 0$ and $(\mu_*(D_4)^2) = 3$. By Claim 1, we can easily see that $\rho(V) = 2 + (\#F_0 - 1) + (\#F_1 - 1) = 2 + (\#F_0 - 1) + ((\mu_*(D_4)^2) - (D_4^2)) = 10$. Hence the singularity type of $(V, D)$ is $A_1A_5K_3$ and the configuration of $C + D + E_1 + E_1'$ is given in the configuration (3) in Appendix.

The assertions (1)–(3) are thus verified. The assertion (4) is clear. □

We consider LDP1-surfaces of index two and type (IIb) in the following theorem.

**Theorem 3.3.** *Let $(V, D)$ be an* LDP1*-surface of index two and type* (IIb)*. Let $C \in \mathrm{MV}(V, D)$ be a curve meeting only one component of $D$. Then the following assertions hold.*

(1) *The singularity type of $(V, D)$ is one of $K_5$, $K_9$, $A_2K_6$ and $A_4K_5$ (see No. $n$ $(4 \le n \le 7)$ in Table 1).*

(2) *There exists a $\mathbf{P}^1$-fibration $\Phi : V \to \mathbf{P}^1$ such that the configuration of $C + D$ and all singular fibers of $\Phi$ is given in the configuration $(n)$ for $4 \le n \le 7$ in Appendix. In particular, all components of $D$, except one section or two disjoint sections, are contained in singular fibers of $\Phi$.*

(3) *$V - D$ is affine-ruled.*

(4) *All the cases are realizable.*

*Proof.* By Lemma 2.4, $C$ is a $(-1)$-curve. Let $D_i$ be the unique component of $D$ meeting $C$ and let $D'$ be the connected component of $D$ containing $D_i$.

Suppose that $D'$ is a $(-2)$-rod or a $(-2)$-fork. By Lemma 2.2, there exists an effective divisor $\Delta_0$ with $\mathrm{Supp}\,\Delta_0 \subset \mathrm{Supp}\,D'$ such that $2C + \Delta_0$ defines a $\mathbf{P}^1$-fibration $\Phi_0 := \Phi_{|2C+\Delta|} : V \to \mathbf{P}^1$. Since the index of $(V, D)$ is equal to two, there exists a connected component $D''$ of $D$ such that $D''$ is of type $K_n$. Then $D''$ is contained in a singular fiber $G$ of $\Phi_0$ and there exists a $(-1)$-curve $E$ in $\mathrm{Supp}\,G$ meeting $D''$. Then we have

$$-(E \cdot D^{\#} + K_V) \le \frac{1}{2} < -(C \cdot D^{\#} + K_V) = 1.$$

This is absurd. Hence $D'$ is of type $K_n$. Lemma 2.2 implies that $n \ge 5$ and $D_i$ is not a terminal component of $D'$.

Let $D' = D'_1 + \cdots + D'_n$ be the decomposition of $D'$ into irreducible components, where we assume that $D_i = D'_i$ and $(D'_j \cdot D'_{j+1}) = 1$ for $j = 1, \ldots, n-1$. By Lemma 2.2, there exist an effective divisor $\Delta$ supported on $D'$ and an integer $e > 0$ such that $F_0 := eC + \Delta$ defines a $\mathbf{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbf{P}^1$. The dual graph of $C + (\Delta)_{\mathrm{red}}$ looks like that of (1) or (2) in Figure 5. Note that we may assume that $i = 2$ in the configuration (1) in Figure 5.

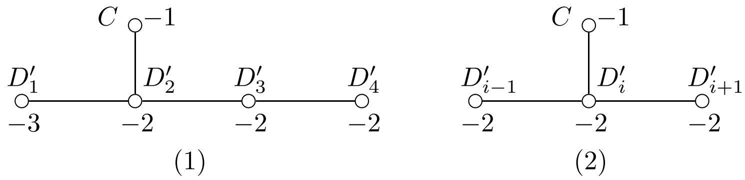


Figure 5

*Case* (1). Then $F_0 = 3(C + D'_2) + 2D'_3 + D'_1 + D'_4$. Moreover, $D'_5$ is a section of $\Phi$ and $D - D'_5$ is contained in singular fibers of $\Phi$. Let $F_0, F_1, \ldots, F_r$ $(r \geq 0)$ be all singular fibers of $\Phi$. By using $\rho(V) = \#D + 1$ and Lemma 2.5 (1), we know that $F_i$ $(1 \leq i \leq r)$ has only one $(-1)$-curve, say $E_i$. So $r \leq 1$ and the equality holds if and only if $n \geq 6$. If $r = 0$, then the singularity type of $(V, D)$ is $K_5$ and the configuration of $C + D$ is given in the configuration (4) in Appendix.

Assume that $r = 1$. If $(F_1)_{\mathrm{red}} - E_1$ is connected, then we can easily see that the singularity type of $(V, D)$ is $K_9$ and the configuration of $C + D + E_1$ is given in the configuration (5) in Appendix. Assume that $(F_1)_{\mathrm{red}} - E_1$ is not connected. Put $D'' := D - D'$. Since $E_1$ is the unique $(-1)$-curve in $\mathrm{Supp}(F_1)$ and $(0 <) - (E_1 \cdot D^{\#} + K_V) \leq 1 - (1/2)(E_1 \cdot D') \leq 1/2$, $D''$ is a $(-2)$-rod or a $(-2)$-fork. Note that $(E_1 \cdot D') = (E_1 \cdot D'_n) = 1$ because the intersection matrix of $(F_1)_{\mathrm{red}} - D'_n = D'' + E_1 + D'_6 + \cdots + D'_{n-1}$ is negative definite. By using [23, Lemma 1.6 (1)], we know that $n = 6$ and $\#D'' = 2$. Hence the singularity type of $(V, D)$ is $A_2K_6$ and the configuration of $C + D$ and $F_1$ is given in the configuration (6) in Appendix.

*Case* (2). Then $F_0 = 2(C + D'_i) + D'_{i-1} + D'_{i+1}$. Moreover, $D'_{i-2}$ and $D'_{i+2}$ are sections of $\Phi$ and $D - (D'_{i-2} + D'_{i+2})$ is contained in singular fibers of $\Phi$.

We consider the case where $D'_{i-2}$ and $D'_{i+2}$ are $(-2)$-curves. Then $n \geq 7$ and $3 < i < n - 2$. Let $F_1$ (resp. $F_2$) be the singular fiber of $\Phi$ containing $D'_1 + \cdots + D'_{i-3}$ (resp. $D'_{i+3} + \cdots + D'_n$). By using $\rho(V) = \#D + 1$ and Lemma 2.5 (1), we know that $F_1 = F_2$, $F_1$ has just two $(-1)$-curves $E_1$ and $E'_1$, and that $\Phi$ has no singular fibers other than $F_0$ and $F_1$. Let $m$ be the number of connected components of $(F_1)_{\mathrm{red}} - (E_1 + E'_1)$. Then $m = 2$ or 3. If $m = 2$, then we may assume that $E_1$ meets both of $D'_1 + \cdots + D'_{i-3}$ and $D'_{i+3} + \cdots + D'_n$. Then $-(E_1 \cdot D^{\#} + K_V) \leq 0$, which is a contradiction. So $m = 3$. Since $-(E_1 \cdot D^{\#} + K_V)$ and $-(E'_1 \cdot D^{\#} + K_V)$ are positive, we know that $(E_1 \cdot D') = (E'_1 \cdot D') = 1$ and $D'' := (F_1)_{\mathrm{red}} - (E_1 + E'_1 + D'_1 + \cdots + D'_{i-3} + D'_{i+3} + \cdots + D'_n)$ is a $(-2)$-rod or a $(-2)$-fork. Then $E_1$ and $E'_1$ meets $D''$. This is a contradiction because the intersection matrix of $E_1 + E_1' + D''$ is then not negative definite. Thus we know that $D'_{i-2}$ or $D'_{i+2}$ is a $(-3)$-curve. We may assume that $D'_{i-2}$ is a $(-3)$-curve, i.e., $i = 3$.

Assume that $D'_{i+2} = D'_5$ is a $(-2)$-curve, i.e., $n \geq 6$. Let $F_1$ be the singular fiber of $\Phi$ containing $D'_6, \ldots, D'_n$. Then $F_1$ has at least two $(-1)$-curves because $F_1$ must have a $(-1)$-curve meeting $D'_1$ which is a section of $\Phi$. By using $\rho(V) = \#D + 1$ and Lemma 2.5 (1), we know that $F_1$ has just two $(-1)$-curves $E_1$ and $E'_1$ and that there exist no singular fibers of $\Phi$ other than $F_0$ and $F_1$. We may assume that $(E_1 \cdot D'_1) = 1$. Then $(E_1 \cdot D'_6 + \cdots + D'_n) = 0$ since $-(E_1 \cdot D^{\#} + K_V) > 0$. So $D'' := (F_1)_{\mathrm{red}} - (E_1 + E'_1 + D'_6 + \cdots + D'_n) \neq 0$. Further, $D''$ is a $(-2)$-rod or a $(-2)$-fork because $E_1$ must meet $D''$. Since $E'_1$ also meets $D''$, the intersection matrix of $E_1 + E'_1 + D'$ is not negative definite. This is a contradiction. Thus we know that $D_5^{(1)}$ is a $(-3)$-curve, i.e., $n = 5$.

Let $F_1, \ldots, F_r$ be all singular fibers of $\Phi$ other than $F_0$. Since $\rho(V) = \#D + 1 = 2 + \sum_{i=0}^{r}(\#F_i - 1) = 5 + \sum_{i=1}^{r}(\#F_i - 1) \geq 6$, we have $r \geq 1$. By using an argument similar to the case (II) as in the proof of Theorem 3.2, we know that $r = 1$, $F_1$ contains just two $(-1)$-curves $E_1$ and $E'_1$ and that $\#F_1 = 6$. Hence the singularity type of $(V, D)$ is $A_4K_5$ and the configuration of $C + D$ and $F_1$ is given in the configuration (7) in Appendix.

The assertions (1) and (2) are thus verified. The assertion (4) is clear. Since $(C \cdot D) = (C \cdot D') = 1$ and the dual graph of $D'$ is linear, the assertion (3) follows from [23, Lemma 6.2]. □

We consider LDP1-surfaces of index two and type (IIc) in the following theorem.

**Theorem 3.4.** *Let $(V, D)$ be an* LDP1*-surface of index two and type* (IIc)*. Then the following assertions hold.*

(1) *The singularity type of $(V, D)$ is one of $A_2K_2$, $2A_2K_3$, $2A_3K_2$, $A_7K_2$, $A_3D_5K_1$, $2D_4K_1$, $D_8K_1$, $A_4K_1$, $A_7K_1$, $A_1A_5K_1$ and $A_72K_1$ (see No. $n$ $(8 \leq n \leq 18)$ in Table* 1*).*

(2) *There exist a curve $C \in \mathrm{MV}(V, D)$ and a $\mathbf{P}^1$-fibration $\Phi : V \to \mathbf{P}^1$ such that the configuration of $C + D$ and all singular fibers of $\Phi$ is given in the configuration $(n)$ for $8 \leq n \leq 18$ in Appendix. In particular, all components, except one section or two disjoint sections, are contained in singular fibers of $\Phi$.*

(3) *$V - D$ is affine-ruled if $n \neq 12$.*

(4) *All the cases are realizable.*

*Proof.* We take a curve $C \in \mathrm{MV}(V, D)$. Then $C$ meets a $(-n)$-curve $D_0$ ($n = 3$ or $4$) and a $(-2)$-curve $D_1$ by the hypothesis. Further, $C$ is a $(-1)$-curve by Lemma 2.4. Let $D^{(i)}$ $(i = 0, 1)$ be the connected component of $D$ containing $D_i$. Since $-(C \cdot D^{\#} + K_V) > 0$, $D^{(1)}$ is a $(-2)$-rod or a $(-2)$-fork and $-(C \cdot D^{\#} + K_V) = 1/2$. Let $D^{(i)} = \sum_{j=1}^{r_i} D_j^{(i)}$ $(i = 0, 1)$ be the irreducible decomposition of $D^{(i)}$, where we put $D_1^{(i)} = D_i$.

(I) *The case $n = 3$.* Then $D^{(0)}$ is of type $K_{r_0}$ $(r_0 \geq 2)$ and $D_1^{(0)}$ is a terminal component of $D^{(0)}$. We claim that:

Claim 1. $D^{(1)}$ is a $(-2)$-rod and $D_1^{(1)}$ is a terminal component of $D^{(1)}$.

*Proof.* Suppose that $D^{(1)}$ is a $(-2)$-fork or $D_1^{(1)}$ is not a terminal component of $D^{(1)}$. Then there exists an effective divisor $\Delta$ with $\operatorname{Supp}\Delta \subset \operatorname{Supp}(D^{(1)})$ such that $G_0 := 2C + \Delta$ defines a $\mathbf{P}^1$-fibration $\Phi_0 := \Phi_{|G_0|} : V \to \mathbf{P}^1$. Then $D_1^{(0)}$ is a 2-section of $\Phi_0$ and the configuration of $G_0$ looks like that of (i) or (ii) in Figure 2. Further, another $(-3)$-curve in $\operatorname{Supp}(D^{(0)})$ is contained in a fiber of $\Phi_0$. On the other hand, by Lemma 2.5 (3), every singular fiber of $\Phi_0$ consists only of $(-1)$-curves and $(-2)$-curves. This is a contradiction. This proves Claim 1.

By Claim 1, we may assume that $(D_j^{(1)} \cdot D_{j+1}^{(1)}) = 1$ for $j = 1, \ldots, r_1 - 1$. Lemma 2.2 implies that $r_1 \geq 2$. So $F_0 := 3C + 2D_1^{(1)} + D_2^{(1)} + D_1^{(0)}$ defines a $\mathbf{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbf{P}^1$, $D_2^{(0)}$ (which exists) and $D_3^{(1)}$ (which exists if $r_1 \geq 3$) are sections of $\Phi$ and $D - (D_2^{(0)} + D_3^{(1)})$ is contained in singular fibers of $\Phi$.

We consider the following five cases I-1–I-5 separately.

*Case* I-1. $r_1 = 2$. If $r_0 = 2$, then, by virtue of Lemma 2.5 (1), we know that $F_0$ is the unique singular fiber of $\Phi$. Hence the singularity type of $(V, D)$ is $A_2K_2$ and the configuration of $C + D$ is given in the configuration (8) in Appendix. We assume that $r_0 \geq 3$. Let $F_1$ be the fiber of $\Phi$ containing $D_3^{(0)}, \ldots, D_{r_0}^{(0)}$. By using $\rho(V) = \#D + 1$ and Lemma 2.5 (1), we know that $\Phi$ has no singular fibers other than $F_0$ and $F_1$ and that $F_1$ has a unique $(-1)$-curve $E_1$. If $(F_1)_{\mathrm{red}} - E_1$ is connected, then $(F_1)_{\mathrm{red}} - E_1 = D_3^{(0)} + \cdots + D_{r_0}^{(0)}$, $(E_1 \cdot D) = 1$ and

$$-(E_1 \cdot D^{\#} + K_V) = \frac{1}{2} = -(C \cdot D^{\#} + K_V).$$

This is a contradiction because $(V, D)$ is then of type (IIb). Hence $D' := (F_1)_{\mathrm{red}} - (E_1 + D_3^{(0)} + \cdots + D_{r_0}^{(0)}) \neq 0$. It is then easy to see that $D'$ is a $(-2)$-rod or a $(-2)$-fork. Since $F_1$ consists of a $(-1)$-curve $E_1$, a $(-3)$-curve $D_{r_0}^{(0)}$ and $(-2)$-curves, we know that $r_0 = 3$ and $\#D' = 2$ by [23, Lemma 1.6 (1)]. Hence the singularity type of $(V, D)$ is $2A_2K_3$ and the configuration of $C + D + E_1$ is given in the configuration (9) in Appendix.

*Case* I-2. $r_0 = 2$ and $r_1 = 3$. Then $D_2^{(0)}$ and $D_3^{(1)}$ are sections of $\Phi$. Let $F_0, F_1, \ldots, F_t$ $(t \geq 0)$ be all singular fibers of $\Phi$. By using an argument similar to the case (II) as in the proof of Theorem 3.2, we know that $t = 1$, the configuration of $F_1$ looks like that of (ii) in Figure 2 and that $\#F_1 = 5$. Hence the singularity type of $(V, D)$ is $2A_3K_2$ and the configuration of $C + D$ and $F_1$ is given in the configuration (10) in Appendix.

*Case* I-3. $r_0 = 2$ and $r_1 \geq 4$. Then $D_2^{(0)}$ and $D_3^{(1)}$ are sections of $\Phi$. Let $F_1$ be the fiber of $\Phi$ containing $D_4^{(1)}, \ldots, D_{r_1}^{(1)}$. Then $F_1$ has at least two

$(-1)$-curves. By using $\rho(V) = \#D + 1$ and Lemma 2.5 (1), we know that $F_1$ has just two $(-1)$-curves $E_1$ and $E_1'$ and $\Phi$ has no singular fibers other than $F_0$ and $F_1$. Assume that $(E_1 \cdot D_2^{(0)}) = 1$. Let $\mu : V \to W$ be a contraction of a $(-1)$-curve $E_1'$ and consecutively (smoothly) contractible curves in the fiber $F_1$ except for those meeting $D_2^{(0)}$ or $D_3^{(1)}$ such that $\mu_*(D_4^{(1)})$ becomes a $(-1)$-curves. Then $\mu_*(F_1)$ has just two $(-1)$-curves $\mu_*(E_1)$ and $\mu_*(D_4^{(1)})$. Further, the multiplicities of $\mu_*(E_1)$ and $\mu_*(D_4^{(1)})$ in $\mu_*(F_1)$ are equal to one. So the configuration of $\mu_*(F_1)$ looks like that of (iii) in Figure 2. The configuration of $\mu_*(C + D + E_1 + E_1')$ looks like that of Figure 6. Note that $\mu \neq \text{id}$. Since $F_1$ has just two $(-1)$-curves, the birational morphism $\mu$ starts with a blowing-up at a center $P$ on $\mu_*(D_4^{(1)}) - \{\mu_*(D_4^{(1)}) \cap \mu_*(D_3^{(1)})\}$. If $P \in \mu_*(D_A) \cap \mu_*(D_4^{(1)})$, then $E_1$ must meet two components of $D$ whose coefficients in $D^\#$ are equal to 1/2. Then $-(E_1 \cdot D^\# + K_V) = 0$, which is a contradiction. So $P \notin \mu_*(D_A)$. Then $D^{(1)} \geq \mu'(\mu_*(D_1^{(1)} + D_2^{(1)} + D_3^{(1)} + D_4^{(1)} + D_A + D_B + D_C))$. Since $D^{(1)}$ is a $(-2)$-rod by Claim 1, we konw that $\mu$ is the blowing-up with center $P$. Hence the singularity type of $(V, D)$ is $A_7K_2$ and the configuration of $C+D+E_1+E_1'$ is given in the configuration (11) in Appendix.

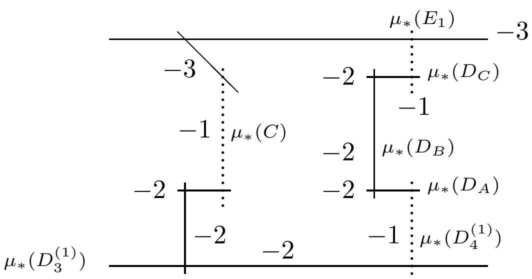



Figure 6

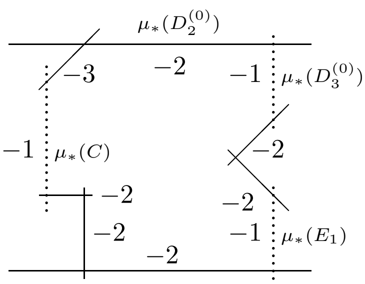



Figure 7

*Case* I-4. $r_0 \geq 3$ and $r_1 = 3$. Let $F_1$ be the fiber of $\Phi$ containing $D_3^{(0)}, \ldots, D_{r_0}^{(0)}$. Then, by using an argument similar to Case I-3, we know that $F_1$ has just two $(-1)$-curves $E_1$ and $E_1'$ and $\Phi$ has no singular fibers other than

$F_0$ and $F_1$. Since either $E_1$ or $E_1'$ meets $D_3^{(1)}$, which is a section of $\Phi$, we may assume that $(E_1 \cdot D_3^{(1)}) = 1$. Let $\mu : V \to W$ be a contraction of $(-1)$-curve $E_1'$ and consecutively (smoothly) contractible curves in the fiber $F_1$ except for those meeting $D_2^{(0)}$ or $D_3^{(1)}$ such that $\mu_*(D_3^{(0)})$ becomes a $(-1)$-curve. Then, by usnig an argument similar to Case I-3, we know that the configuration of $\mu_*(C + E_1 + E_1' + D)$ looks like that of Figure 7. Note that the fundamental points of $\mu$ lie on $\mu_*(D_3^{(0)}) - \{\mu_*(D_3^{(0)}) \cap \mu_*(D_2^{(0)})\}$. We can easily see that $D$ contains a connected componetnt $D'$ which can be contracted to a quotient singular point of index $\geq 3$. This is a contradiction. Therefore, Case I-4 does not take place.

*Case* I-5. $r_0 \geq 3$ and $r_1 \geq 4$. Let $F_1$ (resp. $F_2$) be the fiber of $\Phi$ containing $D_3^{(0)}, \ldots, D_{r_0}^{(0)}$ (resp. $D_4^{(1)}, \ldots, D_{r_1}^{(1)}$). If $F_1 \neq F_2$, then $F_1$ and $F_2$ have at least two $(-1)$-curves. This contradicts Lemma 2.5 (1). So $F_1 = F_2$. Further, Lemma 2.5 (1) implies that $F_1$ has at most two $(-1)$-curves.

Suppose that $F_1$ has a unique $(-1)$-curve $E_1$. Then $(F_1)_{\mathrm{red}} = E_1 + D_3^{(0)} + \cdots + D_{r_0}^{(0)} + D_4^{(1)} + \cdots + D_{r_1}^{(1)}$. By using [23, Lemma 1.6 (1)] (see Case I-1), we know that $r_0 = 3$, $r_2 = 5$ and that $E_1$ meets $D_3^{(0)}$ and $D_5^{(1)}$. It follows from $\rho(V) = \#D + 1$ and Lemma 2.5 (1) that $\Phi$ has another singular fiber $F_2$ and $F_2$ has just two $(-1)$-curves $E_2$ and $E_2'$. We may assume that $(E_2 \cdot D_2^{(0)}) = 1$. Then $-(E_2 \cdot D^{\#} + K_V) \leq 1/2 = -(C \cdot D^{\#} + K_V)$. So $E_2$ meets $D$ in only $(-2)$-curves. This contradicts the assumption that $(V, D)$ is of type (IIc). Hence $F_1$ has just two $(-1)$-curves.

Let $E_1$ and $E_1'$ be two $(-1)$-curves in $\mathrm{Supp}(F_1)$. We may assume that $(E_1 \cdot D_3^{(0)} + \cdots + D_{r_0}^{(0)}) > 0$. Then $E_1 \in \mathrm{MV}(V, D)$. Since $(V, D)$ is of type (IIc), $E_1$ meets $D' := (F_1)_{\mathrm{red}} - (E_1 + E_1' + D_3^{(0)} + \cdots + D_{r_0}^{(0)})$. Note that $D'$ consists only of $(-2)$-curves. Suppose that $(E_1 \cdot D' - (D_4^{(1)} + \cdots + D_{r_1}^{(1)})) = 1$. Then $(E_1' \cdot D_4^{(1)} + \cdots + D_{r_1}^{(1)}) = 1$. Since the intersection matrix of $(F_1)_{\mathrm{red}} - E_1'$ is negative definite, $D' - (D_4^{(1)} + \cdots + D_{r_1}^{(1)})$ is an irreducible $(-2)$-curve and $(E_1 \cdot D_3^{(0)} + \cdots + D_{r_0}^{(0)}) = (E_1 \cdot D_{r_0}^{(0)}) = 1$. Further, $(E_1' \cdot D_3^{(0)} + \cdots + D_{r_0}^{(0)}) = (E_1' \cdot D_{r_0}^{(0)}) = 1$. The intersection matrix of $E_1 + E_1' + D_{r_0}^{(0)} + D' - (D_4^{(1)} + \cdots + D_{r_1}^{(1)})$ is then not negative definite, which is a contradiction. Suppose that $(E_1 \cdot D_4^{(1)} + \cdots + D_{r_1}^{(1)}) = 1$. Since the intersection matrix of $E_1 + D_3^{(0)} + \cdots + D_{r_0}^{(0)} + D_4^{(1)} + \cdots + D_{r_1}^{(1)}$ is negative definite, $(E_1 \cdot D_3^{(0)} + \cdots + D_{r_0}^{(0)}) = (E_1 \cdot D_{r_0}^{(0)}) = 1$ and $r_1 = 4$. Further, $r_0 \geq 4$ and $(E_1' \cdot D_3^{(0)} + \cdots + D_{r_0 - 1}^{(0)}) = 1$. Since $(1/2 \leq) - (E_1' \cdot D^{\#} + K_V) \leq 1/2 = -(C \cdot D^{\#} + K_V)$, it follows that $E_1' \in \mathrm{MV}(V, D)$. Then $(V, D)$ is of type (IIa) or (IIb). This is also a contradiction. Therefore, Case I-5 does not take place.

(II) *The case* $n = 4$. In this case we may assume that every curve $E \in \mathrm{MV}(V, D)$ meets a $(-4)$-curve in $\mathrm{Supp}\, D$. Since $(D_0^2) = -4$, $D_0$ is a connected component of $D$. Let $D^{(1)}$ be the connected component of $D$ containing $D_1$. Then $D^{(1)}$ is a $(-2)$-rod or a $(-2)$-fork.

(II-1) *The case where* $D_1$ *is not a terminal component of* $D^{(1)}$ *or* $D^{(1)}$ *is a* $(-2)$*-fork*. Then there exists an effective divisor $\Delta$ with $\mathrm{Supp}\,\Delta \subset \mathrm{Supp}(D^{(1)})$ such that $F_0 := 2C + \Delta$ gives rise to a $\mathbf{P}^1$-fibration $\Phi = \Phi_{|F_0|} : V \to \mathbf{P}^1$. Then $D_0$ is a 2-section of $\Phi$ and the configuration of $F_0$ looks like that of (i) or (ii) in Figure 2. It follows from Lemma 2.5 (3) that every singular fiber of $\Phi$ consists of $(-1)$-curves and $(-2)$-curves. Hence each connected component of $D$ other than $D_0$ is a $(-2)$-rod or a $(-2)$-fork because $D - (D_0 + D^{(1)})$ is contained in fibers of $\Phi$.

Let $\sigma : V \to W$ be the contraction of $C$ and put $B := \sigma_*(D - D_1)$. Then the pair $(W, B)$ is an LDP1-surface by [23, Lemma 4.3]. In particular, $(W, B)$ is a dP3-surface (for the definition, see [24, Introduction]). Put $B_0 := \sigma_*(D_0)$ and $D_1' := \sigma_*(D_1)$. Then $B_0$ is a $(-3)$-curve and $(B_0 \cdot B - B_0) = 0$. Further, $B^\# = (1/3)B_0$. Note that $(W, B) \neq (\Sigma_3, M_3)$ since $\#D^{(1)} \geq 3$.

Claim 2. $(W, B)$ is of type (IIa) or (IIc).

*Proof.* Suppose that $(W, B)$ is of the first kind. Then there exists a curve $E \in \mathrm{MV}(W, B)$ such that $|E + B + K_W| \neq \emptyset$. By using the same argument as in the proof of Theorem 3.1, we know that $E$ is a $(-1)$-curve, $(E \cdot B) = (E \cdot B_0) = 2$ and $E + B_0 + K_W \sim 0$. Since $\sigma'(E)$ is not a component of $D$, $\sigma'(E)$ is a $(-1)$-curve by Lemma 2.1. Then $-(\sigma'(E) \cdot D^\# + K_V) = 1 - (1/2)(\sigma'(E) \cdot D_0) = 0$, which is a contradiction. Hence $(W, B)$ is of the second kind. By [24, Theorem 4.1], $(W, B)$ is not of type (IIb). This proves Claim 2.

We consider the following two Cases II-1-1 and II-1-2 separately.

*Case* II-1-1. $D_1$ *is not a terminal component of* $D^{(1)}$. Note that $-(D_1' \cdot B^\# + K_W) = 2/3$. For any curve $E \in \mathrm{MV}(W, B)$, $E$ is a $(-1)$-curve and $-(E \cdot B^\# + K_W) \geq 2/3$ by Lemma 2.4 and Claim 2. So $D_1' \in \mathrm{MV}(W, B)$. By the hypothesis that $(D^{(1)} - D_1 \cdot D_1) \geq 2$, $(W, B)$ is of type (IIa). It then follows from [24, Theorem 3.3] that the configuration of $D_1' + B$ looks like that of Figure 8, where a solid line stands for a component of $B$; a line with $*$ on it is a section of the vertical $\mathbf{P}^1$-fibration $\phi : W \to \mathbf{P}^1$. Hence the singularity type of $(V, D)$ is $A_3D_5K_1$ and the configuration of $C + D$ and all singular fibers of a $\mathbf{P}^1$-fibration $\Psi := \phi \circ \sigma : V \to \mathbf{P}^1$ is given in the configuration (12) in Appendix.

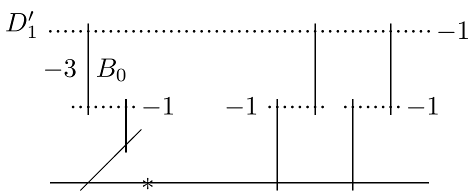



Figure 8

*Case* II-1-2. *$D_1$ is a terminal component of $D^{(1)}$.* By the hypothesis in Case II-1, $D^{(1)}$ is a $(-2)$-fork. By Claim 2 and [24, Theorems 3.3 and 5.2], the dual graph of $B$ is one of those given in the cases No. $m$ ($m = 28$, 66, 67, 68, 84, 97) in [24, Appendix]. If $m = 28$, then there exists a curve $E \in \mathrm{MV}(W, B)$ such that $E$ meets the $(-3)$-curve $B_0$ and two $(-2)$-curves in $\mathrm{Supp}\, B$ by [24, Theorem 3.3]. Then $\sigma'(E) \in \mathrm{MV}(V, D)$ and $\sigma'(E)$ meets two $(-2)$-curves and $D_0$. So $(V, D)$ is of type (IIa), which is a contradiction. Hence $(W, B)$ is of type (IIc). We consider the following three subcases II-1-2-1 through II-1-2-3 separately.

*Subcase* II-1-2-1. $m = 66$, 67 or 68. Note that $D_1' \in \mathrm{MV}(W, B)$. By [24, Theorem 5.2], $D_1'$ must meet $B_0$ and a terminal component of $\sigma_*(D^{(1)} - D_1)$. Then $D^{(1)}$ is a $(-2)$-rod, which contradicts the hypothesis in Case II-1-2. Therefore, this subcase does not take place.

*Subcase* II-1-2-2. $m = 84$. The configuration of $D_1' + B$ then looks like that of Figure 9, where $D_1' = B_1'$ or $B_2'$ (cf. [24, Appendix]). If $D_1' = B_2'$, then we can easily see that $\sigma'(B_1') \in \mathrm{MV}(V, D)$ and $\sigma'(B_1')$ satisfies the hypothesis in Case II-1-1. So we are reduced to the situation treated in Case II-1-1. If $D_1' = B_1'$, then the singularity type of $(V, D)$ is $2D_4K_1$ and the configuration of $C + C' + D$, where $C' = \sigma'(B_2')$, looks like that of Figure 10, where $(D_i^2) = -2$ for $1 \le i \le 8$. Put $G_0 := D_0 + D_1 + D_5 + 2(C + C')$. Then $G_0$ defines a $\mathbf{P}^1$-fibration $\Psi := \Phi_{|G_0|} : V \to \mathbf{P}^1$, $D_2$ and $D_6$ are sections of $\Psi$ and $D - (D_2 + D_6)$ is contained in singular fibers of $\Psi$. By using $\rho(V) = \# D + 1 = 10$ and Lemma 2.5 (1), we can easily see that the configuration of $C + D$ and all singular fibers of $\Phi$ is given in the configuration (13) in Appendix.

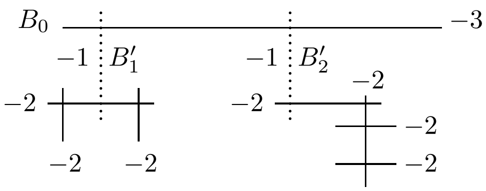



Figure 9

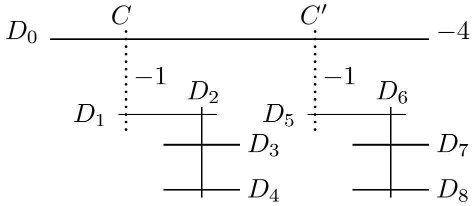



Figure 10

*Subcase* II-1-2-3. $m = 97$. The configuration of $D_1' + B$ then looks like that of Figure 11 (cf. [24, Appendix]). So the singularity type of $(V, D)$ is

$D_8K_1$. Put $D_{i+1} := \sigma'(B_i)$, $i = 1, \ldots, 7$ and $G_0 := D_0 + 4C + 3D_1 + 2D_2 + D_3$. Then $G_0$ defines a $\mathbf{P}^1$-fibration $\Psi := \Phi_{|G_0|} : V \to \mathbf{P}^1$, $D_4$ is a section of $\Psi$ and $D - D_4$ is contained in singular fibers of $\Psi$. We can easily see that the configuration of $C + D$ and all singular fibers of $\Psi$ is given in the configuration (14) in Appendix.

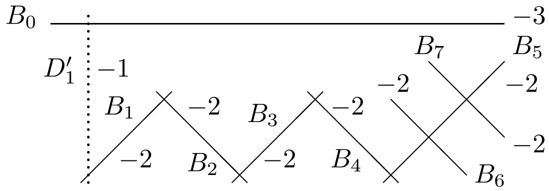



Figure 11

(II-2) *The case where $D^{(1)}$ is a $(-2)$-rod and $D_1$ is a terminal component of $D^{(1)}$.* Let $D^{(1)} = D_1^{(1)} + \cdots + D_r^{(1)}$ be the decomposition of $D^{(1)}$ into irreducible components, where $D_1^{(1)} = D_1$ and $(D_i^{(1)} \cdot D_{i+1}^{(1)}) = 1$ for $i = 1, \ldots, r-1$. By Lemma 2.2 and $\rho(V) = \#D + 1$, $r \geq 4$. A divisor $F_0 := 4C + 3D_1^{(1)} + 2D_2^{(1)} + D_0 + D_3^{(1)}$ defines a $\mathbf{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbf{P}^1$. Then $D_4^{(1)}$ is a section of $\Phi$ and $D - D_4^{(1)}$ is contained in singular fibers of $\Phi$. If $r = 4$, then, by using $\rho(V) = \#D + 1$ and Lemma 2.5 (1), we know that $\Phi$ has no singular fibers other than $F_0$. Hence the singularity type of $(V, D)$ is $A_4K_1$ and the configuration of $C + D$ is given in the configuration (15) in Appendix.

We assume that $r \geq 5$. Let $F_1$ be the singular fiber of $\Phi$ containing $D_5^{(1)}, \ldots, D_r^{(1)}$. It then follows form $\rho(V) = \#D + 1$ and Lemma 2.5 (1) that $F_1$ has a unique $(-1)$-curve $E_1$ and $\Phi$ has no singular fibers other than $F_0$ and $F_1$. If $(F_1)_{\text{red}} - E_1$ is connected, then $(F_1)_{\text{red}} = E_1 + D_5^{(1)} + \cdots + D_r^{(1)}$ and the configuration of $F_1$ looks like that of (ii) in Figure 2. Hence the singularity type of $(V, D)$ is $A_7K_1$ and the configuration of $C + D$ and $F_1$ is given in the configuration (16) in Appendix.

We assume further that $(F_1)_{\text{red}} - E_1$ is not connected. We note that $D' := (F_1)_{\text{red}} - (E_1 + D_5^{(1)} + \cdots + D_r^{(1)})$ is connected because $F_1$ has no $(-1)$-curves other than $E_1$. Since the intersection matrix of $E_1 + D_5^{(1)} + \cdots + D_r^{(1)}$ is negative definite, $(E_1 \cdot D_j^{(1)}) = 1$, where $j = 5$ or $r$. If $D'$ is a $(-2)$-rod or a $(-2)$-fork, then the configuration of $F_1$ looks like that of (i) in Figure 2 by Lemma 2.5 (2). Hence the singularity type of $(V, D)$ is $A_1A_5K_1$ and the configuration of $C + D$ and $F_1$ is given in the configuration (17) in Appendix. Assume that $D'$ is of type $K_n$. Then $-(E_1 \cdot D^{\#} + K_V) = 1/2$ and hence $E_1 \in \mathrm{MV}(V, D)$. By the hypothesis in (II), $D'$ is of type $K_1$. If $E_1$ does not meet $D_r^{(1)}$, then we are reduced to the situation treated in (II-1). So we may assume that $(E_1 \cdot D^{(1)}) = (E_1 \cdot D_r^{(1)}) = 1$. Since $(F_1)_{\text{red}}$ is a linear chain and $D'$ is a $(-4)$-curve, $r = 7$. Hence the singularity type of $(V, D)$ is $A_72K_1$ and the configuration of $C+D$ and all singular fibers of $\Phi$ is given in the configuration (18) in Appendix.

The assertions (1) and (2) are thus verified. The assertion (4) is clear. The assertion (3) can be verified by using [23, Lemma 3.3]. □

The assertions (1) and (2) of Theorem 1.1 follows from Theorems 3.1 through 3.4.

## 4. Quasi-universal coverings

Let $S$ (or $(V,D)$) be an LDP1-surface of index two and let $U^0$ be the universal covering of $S^0 = S - \mathrm{Sing}\, S = V - D$, which is an algebraic surface because $\pi_1(S^0)$ is finite by [8], [9] (see also [6] and [12]). Let $U$ be the normalization of $S$ in the function field of $U^0$. We call $U$ the quasi-universal covering of $S$ (cf. [19] and [24]). It then follows from [24, Proposition 6.1] that $U$ is a log del Pezzo surface. In this section, to complete the proof of Theorem 1.1, we look into the fundamental group $\pi_1(S^0)$ of $S^0$ and the quasi-universal covering $U$ of $S$. To exhibit our arguments, we treat only three cases $S = S(A_4K_5)$, $S = S(2A_1D_6K_1)$ and $S = S(2D_4K_1)$.

*Case* $S = S(A_4K_5)$. The configuration of $D$ is given in the configuration (7) in Appendix, where a linear pencil $|E_1+D_6+D_7+D_8+D_9+E_2|$ defines the vertical $\mathbf{P}^1$-fibration $\Phi : V \to \mathbf{P}^1$. Let $u : V \to \Sigma_3$ be the contraction of $C$, $D_3$, $D_4$, $E_2$, $D_9$, $D_8$, $D_7$ and $D_6$. Let $F$ be a fiber of $\Phi$. Then $F \sim 2(C+D_3)+D_2+D_4 \sim E_1+D_6+D_7+D_8+D_9+E_2$ and $D_5 \sim D_1+3F-(D_6+2D_7+3D_8+4D_9+5E_2)-(D_3+D_4+C)$. Put $G := C+D_1+D_2+D_3+D_4-E_2$ and $\Delta := 4D_1+2D_2+3D_4+D_5+D_6+2D_7+3D_8+4D_9$. Then $5G \sim \Delta$. Note that $\mathrm{Pic}(V)$ is a free abelian group of rank ten with a free basis $\{D_1, D_3, D_4, D_6, D_7, D_8, D_9, F, C-E_2, E_2\}$. In $\mathrm{Pic}(V-D)$, which is $\mathrm{Pic}(V)$ modulo the subgroup generated by the components of $D$, we have $F = 2C = E_1+E_2$ and $0 = D_5 = 3F-5E_2-C$. So, in $\mathrm{Pic}(V-D)$, $5(C-E_2) = 0$. Hence $\mathrm{Pic}(V-D) \cong \mathbf{Z} \oplus \mathbf{Z}/(5)$. By the universal coefficient theorem, we have $H_1(V-D;\mathbf{Z}) \cong \mathbf{Z}/(5)$ (see [5, Section 8] and [14, Proof of Proposition 4.13]).

Let $g_1 : T_1 \to V$ be the composite of the following morphisms in the given order: the $\mathbf{Z}/(5)$-covering defined by the relation $5G \sim \Delta$, the normalization of the covering surface and the minimal resolution of the isolated singularities on the normalized surface. The configuration of $g_1^{-1}(D)$ looks like that of Figure 12, where a solid line stands for a component of $g_1^{-1}(D)$ and $g_1^{-1}(C) = \sum_{i=1}^{5} \tilde{C}_i$. The $\mathbf{P}^1$-fibration $\Phi$ induces a $\mathbf{P}^1$-fibration $\Phi_1 : T_1 \to \mathbf{P}^1$ of which all singular fibers are those two given in Figure 12. Note that $T_1$ is a rational surface and $\rho(T_1) = 23$.

Let $g_2 : T_1 \to T$ be the contraction of $g_1^{-1}(D-D_3)$. Put $B := g_{2*}(g_1^{-1}(D_3))$ and $C_i := g_2(\tilde{C}_i)$, $i = 1, \ldots, 5$. Let $h : T \to U$ be the contraction of $B$. Then the singularity type of $U$ is $K_1$ and $\rho(U) = 5$. Note that $g_1$ induces a finite

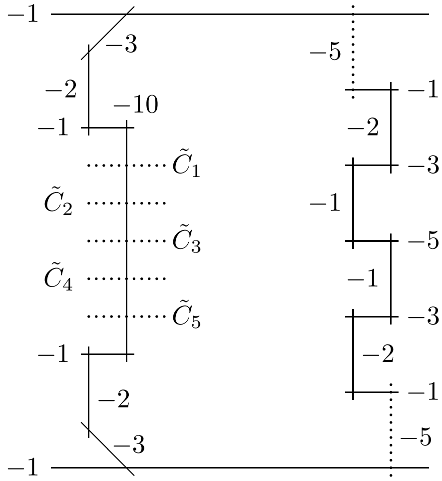



Figure 12

morphism $\overline{g}_1 : U \to S$, which is étale outside $\mathrm{Sing}\, S$, and $U$ is a log del Pezzo surface by [24, Corollary 6.2]. A divisor $H := C_2 + C_3 + C_4 + C_5 + B$ on $T$ defines a $\mathbf{P}^1$-fibration $\Psi : T \to \mathbf{P}^1$ and $C_1$ is a section of $\Psi$. So $T - H$ contains the affine plane $\mathbf{C}^2$ and hence $U - \mathrm{Sing}\, U$ is simply connected. Therefore, $U$ is the quasi-universal covering of $S$ and $\pi_1(S^0) \cong \mathbf{Z}/(5)$.

**Remark 1.** In the Case $S = S(A_4K_5)$, we can easily see that $\pi_1(S^0)$ is cyclic by using [20, Lemma 1.5].

*Case* $S = S(2A_1D_6K_1)$. By using a similar argument to the case $S = S(A_4K_5)$, we know that $H_1(S^0; \mathbf{Z}) = \mathbf{Z}/(2) \oplus \mathbf{Z}/(2)$, $\rho(U) = 1$ and $U$ is the surface obtained by contracting the minimal section on $\Sigma_2$. We calculate the fundamental group of $S^0$. The configuration of $D$ is given in the configuration (2) in Appendix. Let $\Phi : V \to \mathbf{P}^1$ be the vertical $\mathbf{P}^1$-fibration. Then $\varphi := \Phi|_{V-D} : V - D \to \mathbf{P}^1$ is an $\mathbf{A}^1$-fibration onto $\mathbf{P}^1$. It is then clear that every fiber of $\varphi$ is irreducible and $\varphi$ has three multiple fibers $m_i\Gamma_i$ $(i = 1, 2, 3)$ with $\{m_1, m_2, m_3\} = \{2, 2, 4\}$. By [5, Proposition (4.19)], $\pi_1(V - D)$ $(= \pi_1(S^0))$ is generated by $\sigma_1$, $\sigma_2$ and $\sigma_3$ with the relation $\sigma_1\sigma_2\sigma_3 = \sigma_1^2 = \sigma_2^2 = \sigma_3^4 = 1$. Hence $\pi_1(S^0)$ is the binary dihedral group of order 8.

*Case* $S = S(2D_4K_1)$. By using a similar argument to the case $S = S(A_4K_5)$, we know that $H_1(S^0; \mathbf{Z}) = \mathbf{Z}/(2) \oplus \mathbf{Z}/(2)$ and $U = \mathbf{P}^1 \times \mathbf{P}^1$. Moreover, we know that the degree of the quasi-universal covering morphism of $S$ is equal to eight. Hence $\pi_1(S^0)$ is a non-abelian group of order 8, i.e., the binary dihedral group of order 8 or the quaternion group of order 8.

Thus, we can verify the assertion (3) of Theorem 1.1.

*Proof of the assertion* (4) *of Theorem* 1.1. Let $(V, D)$ be an LDP1-surface of index two. If $V - D$ contains the affine plane $\mathbf{C}^2$ as a Zariski open subset, then $V - D$ is simply connected. Assume that $V - D$ is simply connected. By the assertion (3) of Theorem 1.1, the singularity type of $(V, D)$ is one of $K_1$, $K_5$, $A_2K_2$ and $A_4K_1$. Then $(V, D)$ is a surface corresponding to the configuration $(n)$ for $n = 1$, 4, 8 or 15. It is then clear that $V - D$ contains the affine plane $\mathbf{C}^2$ as a Zariski open subset.

The proof of Theorem 1.1 is thus completed.

## Appendix. Table and list of configurations

In Table 1, we employ the following notation for finite groups.

$D_2$: the binary dihedral group of order 8.

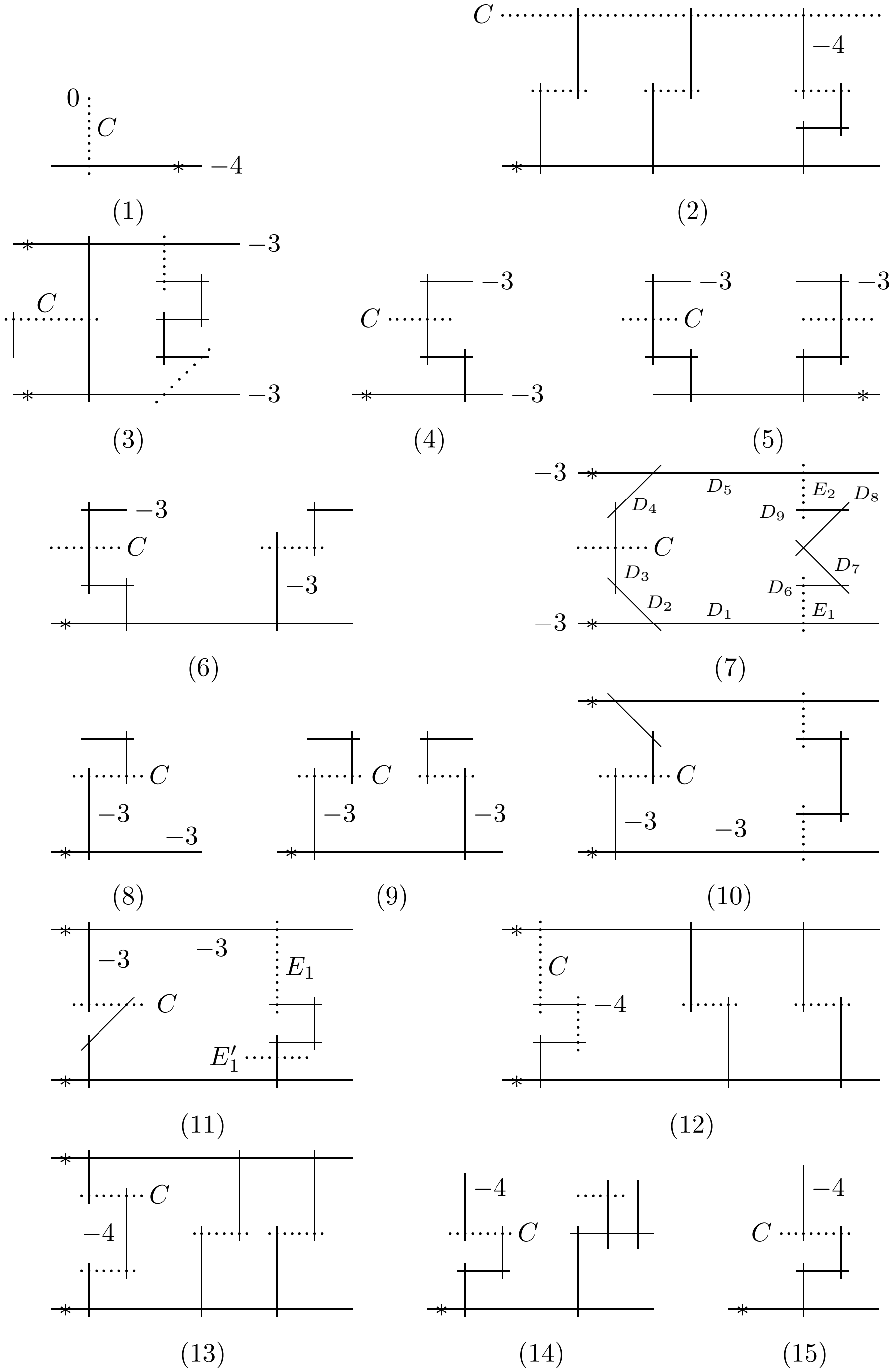
$Q_3$: the quaternion group of order 8.

In No. 13, we do not know yet which of $D_2$ and $Q_3$ the fundamental group $\pi_1(S^0)$ takes.

| No. | Sing $S$ | $H_1(S^0; \mathbf{Z})$ | $\pi_1(S^0)$ | $\rho(U)$ | Sing $U$ |
|---|---|---|---|---|---|
| 1 | $K_1$ | (0) | (1) | 1 | $S = U$ |
| 2 | $2A_1D_6K_1$ | $\mathbf{Z}/(2) \oplus \mathbf{Z}/(2)$ | $D_2$ | 1 | $A_1$ |
| 3 | $A_1A_5K_3$ | $\mathbf{Z}/(6)$ | $\mathbf{Z}/(6)$ | 3 | $A_1$ |
| 4 | $K_5$ | (0) | (1) | 1 | $U = S$ |
| 5 | $K_9$ | $\mathbf{Z}/(3)$ | $\mathbf{Z}/(3)$ | 5 | $K_3$ |
| 6 | $A_2K_6$ | $\mathbf{Z}/(3)$ | $\mathbf{Z}/(3)$ | 3 | $K_2$ |
| 7 | $A_4K_5$ | $\mathbf{Z}/(5)$ | $\mathbf{Z}/(5)$ | 5 | $K_1$ |
| 8 | $A_2K_2$ | (0) | (1) | 1 | $U = S$ |
| 9 | $2A_2K_3$ | $\mathbf{Z}/(3)$ | $\mathbf{Z}/(3)$ | 1 | $K_1$ |
| 10 | $2A_3K_2$ | $\mathbf{Z}/(4)$ | $\mathbf{Z}/(4)$ | 1 | $A_1$ |
| 11 | $A_7K_2$ | $\mathbf{Z}/(4)$ | $\mathbf{Z}/(4)$ | 4 | $2A_1$ |
| 12 | $A_3D_5K_1$ | $\mathbf{Z}/(4)$ | $\mathbf{Z}/(4)$ | 2 | $2A_1A_2$ |
| 13 | $2D_4K_1$ | $\mathbf{Z}/(2) \oplus \mathbf{Z}/(2)$ | $D_2$ or $Q_3$ | 2 | $U = \mathbf{P}^1 \times \mathbf{P}^1$ |
| 14 | $D_8K_1$ | $\mathbf{Z}/(2)$ | $\mathbf{Z}/(2)$ | 2 | $A_1D_5$ |
| 15 | $A_4K_1$ | (0) | (1) | 1 | $U = S$ |
| 16 | $A_7K_1$ | $\mathbf{Z}/(2)$ | $\mathbf{Z}/(2)$ | 2 | $A_1A_3$ |
| 17 | $A_1A_5K_1$ | $\mathbf{Z}/(2)$ | $\mathbf{Z}/(2)$ | 1 | $A_1A_2$ |
| 18 | $A_72K_1$ | $\mathbf{Z}/(4)$ | $\mathbf{Z}/(4)$ | 1 | $A_1$ |

Table 1

In the following list of configurations, the numbers in brackets coincide with the classifying numbers in Table 1; a solid line stands for a component of $D$; the self-intersection number of a $(-2)$-curve in $\operatorname{Supp} D$ is omitted; a dotted line in the configuration $(n)$ for $n \geq 2$ is a $(-1)$-curve; a line with $*$ on it is not contained in any fiber of the vertical $\mathbf{P}^1$-fibration on $V$.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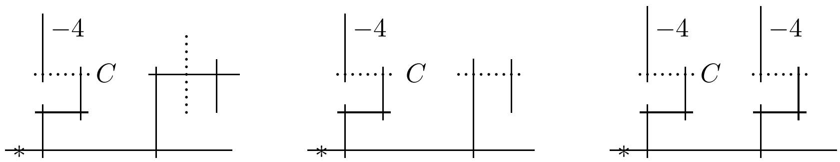


(16) (17) (18)

Figure 13

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