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Extremal functions for plane quasiconformal mappings

By

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Abstract

For the family $\mathscr{F}(K)$ of K-quasiconformal mappings f from $\overline{\mathbb{C}} = \{|z| \leqslant +\infty\}$ onto $\overline{\mathbb{C}}$ such that $f(\mathbb{R}) = \mathbb{R}$ and $f(x) = x$ for $x = -1,\ 0,\ \infty$, the supremum $\lambda(K,t)$ and the infimum $\nu(K,t)$ of $f(t)$ for f ranging over $\mathscr{F}(K)$ with $t \in \mathbb{R}$ fixed are studied. They are expressed by the inverse μ^{-1} of the function $\mu(r)$, the modulus of the bounded, doubly-connected domain with the unit circle and the real interval $[0,r]$, $0 < r < 1$, as the boundary. Among a number of results obtained, asymptotic behaviors of $X(K,t)(X = \lambda,\ \nu)$ as $t \to \pm\infty$ for a fixed K and as $K \to +\infty$ for a fixed t are considered.

Introduction

Let $\mathscr{F}(K)$ be the family of K-quasiconformal mappings f from the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ onto $\overline{\mathbb{C}}$ such that $f(\mathbb{R}) = \mathbb{R}$ for the set $\mathbb{R}$ of real numbers and $f(x) = x$ for $x = -1,\ 0,\ \infty$. The contents of the present paper center around the extremal quantities

$$(0.1) \qquad \lambda(K,t) = \sup_{f \in \mathscr{F}(K)} f(t) \qquad \text{and} \qquad \nu(K,t) = \inf_{f \in \mathscr{F}(K)} f(t)$$

for $t \in \mathbb{R}$. Actually $\lambda(K,t)$ and $\nu(K,t)$ are attained by some members of $\mathscr{F}(K)$ because $\mathscr{F}(K)$ is a normal family by [L, p. 14, Theorem 2.1] and the Hurwitz-type theorem [L, p. 15, Theorem 2.2] is valid. In particular, they are finite.

If one defines $\lambda(K,t)$ for $t > 0$ directly by the right-hand side in the formula for λ in Theorem 1.1 (1) in the present paper, then, as will be seen, $\eta_K(t) = \lambda(K,t)$ for $\eta_K(t)$ in [QV] and [QVV].

Following the method of O. Lehto, K. I. Virtanen, and J. Väisälä [LVV] for the study of $\lambda(K,1)$ we determine the expression for $X(K,t)$, $X = \lambda,\ \nu$, in

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terms of the inverse function μ^{-1} of μ, and μ itself, where $\mu(r)$ is the modulus of the disk $\{|z| < 1\}$ slit along the closed, real interval $[0, r]$, $0 < r < 1$. Formulas for $X(K,t)$ and $t \in \mathbb{R}$ are summarized in Theorem 1.1 in Section 1. In particular, the set of values $f(t)$ for all $f \in \mathscr{F}(K)$ with a fixed $t \in \mathbb{R}$ is shown to be exactly the closed interval $[\nu(K,t), \lambda(K,t)]$.

The proof of Theorem 1.1 will be carried out in Sections 2 and 3 we exhibit various identities for $X(K,t)$, for example, $\lambda(K,t)\nu(K,1/t) = 1$ $(t \neq 0)$, the case $t = 1$ is earlier observed by Lehto, Virtanen, and Väisälä. See Theorem 3.1.

In Section 4 we shall consider the hyperbolic distance in the twice punctured complex plane $\mathbb{C}\setminus\{-1,0\}$, and prove that the hyperbolic distance between $t \in \mathbb{R}\setminus\{-1,0\}$ and $X(K,t)$ is exactly $\log\sqrt{K}$ for $X = \lambda,\ \nu$. This section is, in spirit, somewhat different from others, so that one can go directly from Section 3 to Section 5.

Section 5 is devoted to comparing $X(K,t)$ with $Y(K,s)$ for $X,\ Y = \lambda,\ \nu$ and $t, s \in \mathbb{R}$ in the form of inequalities; see Theorem 5.1.

In Section 6 we inquire into the orders of $X(K,t)$ for $X = \lambda,\ \nu$ as $t \to \pm\infty$ for a fixed K and those of $X(K,t)$ for $X = \lambda,\ \nu$ as $K \to +\infty$ for a fixed t. All the possible cases are summarized in Theorem 6.1, where the constants ± 16 and $\pm 1/16$ appear. A considerable part of our method depends again on Lehto, Virtanen, and Väisälä's [LVV], [LV1, p. 82], in which the behavior of the specified $\lambda(K,1)$ as $K \to +\infty$ is studied. For fixed $K > 1$ the graphs $s = X(K,t)$, $t \in \mathbb{R}$, in the ts-plane are also studied, where $X = \lambda, \nu$.

Section 7 is concerned with $\lim_{K\to 1} \partial^n X(K,t)/\partial K^n$ for $X = \lambda,\ \nu$; $n = 1,\ 2$, and $t \in \mathbb{R}$.

In Section 8 we consider some extensions of the family $\mathscr{F}(K)$.

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1. Extremal functions $\lambda(K,t)$ and $\nu(K,t)$

We rapidly review the definition of quasiconformality because the notation will sometimes appear.

A *quadrilateral* $Q = Q(z_1, z_2, z_3, z_4)$ in $\overline{\mathbb{C}}$ consists of a Jordan domain Q and a sequence of distinct points z_1, z_2, z_3, and z_4 on its boundary ∂Q, determining the positive orientation of ∂Q with respect to Q.

A meromorphic and univalent function f in a domain $A \subset \overline{\mathbb{C}}$ is called a *conformal mapping* from A onto $f(A)$. If the image $f(Q)$ of $Q = Q(z_1, z_2, z_3, z_4)$ by f conformal from Q onto $f(Q)$ is a Jordan domain, then the celebrated Carathéodory theorem ([C, p. 86, Theorem], [G, p. 41], and [D, p. 12]) says that f can be extended homeomorphically to the closure $\overline{Q}$ of Q; the extension is again denoted by f. Then $f(Q) = f(Q)(f(z_1), f(z_2), f(z_3), f(z_4))$ is a quadrilateral.

There exists a unique conformal mapping φ from $Q = Q(z_1, z_2, z_3, z_4)$ onto the rectangle $\{x+iy : 0<x<M, 0<y<1\}$ such that $\varphi(z_1)=0$, $\varphi(z_2)=M$, $\varphi(z_3)=M+i$ and $\varphi(z_4)=i$. Such a φ is called the *canonical mapping* of Q and the uniquely determined quantity $M=M(Q)=M(Q(z_1,z_2,z_3,z_4))$ is called the *modulus* of Q.

In the present paper the constant K always satisfies $1\leqslant K<+\infty$. A sense-preserving homeomorphism from a domain A in $\overline{\mathbb{C}}$ into $\overline{\mathbb{C}}$ is called a K-*quasiconformal mapping* from A onto $f(A)$ if $M(f(Q))\leqslant KM(Q)$ for each quadrilateral Q with $\overline{Q}\subset A$.

For the specified quadrilateral $H(t)\equiv H(0,t,\infty,-1)$ where $H=\{z : \operatorname{Im} z>0\}$ and $t>0$ we set $M(t)=M(H(t))$. We then have the well-known identity

$$M(t)=(2/\pi)\mu(1/\sqrt{1+t}) \qquad \text{for} \quad t>0, \tag{1.1}$$

where the function $\mu(r)$ of $0<r<1$ is defined in the next paragraph. See [L, p. 16].

For $0<r<1$ the disk $\{z : |z|<1\}$ slit along $[0,r]$ is mapped conformally onto the ring domain $\{z : 1<|z|<\rho\}$, where $\rho>1$ is uniquely determined by r. The function $\mu(r)=\log\rho$ for $0<r<1$ is then expressed by

$$\mu(r)=(\pi/2)\mathscr{K}(\sqrt{1-r^2})/\mathscr{K}(r), \tag{1.2}$$

where

$$\mathscr{K}(r)=\int_0^1\frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}=\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-r^2\sin^2\phi}}$$

is the *complete elliptic integral of the first kind* [WW, p. 499 and p. 518]; see [Hr, p. 316] in which the function $\nu(r)=\mu(r)/(2\pi)$ is considered. Hence μ is real-analytic. One can prove that $\mu(r)$ strictly decreases from $+\infty$ to 0 as r increases from 0 to 1. The inverse function μ^{-1} of μ is therefore defined in $(0,+\infty)$.

Note that

$$M(t)M(t^{-1})=1 \qquad \text{for} \quad t>0. \tag{1.3}$$

This is a consequence of

$$\mu(r)\mu(\sqrt{1-r^2})=\pi^2/4 \qquad \text{for} \quad 0<r<1, \tag{1.4}$$

which follows from (1.2); see [Hr, p. 316, (2)]. Setting $r=1/\sqrt{1+t}$ in (1.4) we immediately have (1.3). Again the identity $M(1)=1$ follows from (1.3).

Theorem 1.1. *For $t\in\mathbb{R}\setminus\{-1,0\}$ and $1\leqslant K<+\infty$,*

$$\{f(t) : f\in\mathscr{F}(K)\}=[\nu(K,t),\lambda(K,t)] \tag{1.5}$$

and $X(K,t)$, $X=\lambda$, ν, are expressed in terms of μ^{-1} and M in the following.

(1) *If* $t > 0$, *then*

$$\lambda(K,t) = \{\mu^{-1}(\pi K M(t)/2)\}^{-2} - 1 \quad \textit{and}$$
$$\nu(K,t) = \{\mu^{-1}(\pi M(t)/(2K))\}^{-2} - 1.$$

(2) *If* $t < -1$, *then*

$$\lambda(K,t) = -\{\mu^{-1}(\pi M(-1-t)/(2K))\}^{-2} \quad \textit{and}$$
$$\nu(K,t) = -\{\mu^{-1}(\pi K M(-1-t)/2)\}^{-2}.$$

(3) *If* $-1 < t < 0$, *then*

$$\lambda(K,t) = -\{\mu^{-1}(\pi K M(-t^{-1}-1)/2)\}^{2} \quad \textit{and}$$
$$\nu(K,t) = -\{\mu^{-1}(\pi M(-t^{-1}-1)/(2K))\}^{2}.$$

In particular, $\nu(K,t) > 0$ if $t > 0$ and $\lambda(K,t) < 0$ if $t < 0$. Furthermore, $\lambda(K,t) = \nu(K,t) = t$ for $t = -1,\ 0$, and $\nu(K,t) \leqslant t \leqslant \lambda(K,t)$ because $\mathscr{F}(1) = \{id\} \subset \mathscr{F}(K)$, where $id(z) \equiv z$. Obviously, $\nu(1,t) \equiv t \equiv \lambda(1,t)$.

It is known that $\lambda(K,1) = \{\mu^{-1}(\pi K/2)\}^{-2} - 1$; see [LV1, p. 81], [L, p. 16] and [LVV, p. 8]. This is the specified case of (1) for λ and $t = 1$.

The function $M(t)$ strictly increases from 0 to $+\infty$ as t increases from 0 to $+\infty$ and the inverse of M is $M^{-1}(t) = \{\mu^{-1}(\pi t/2)\}^{-2} - 1$ for $t > 0$, so that (1) reads $\lambda(K,t) = M^{-1}(KM(t))$ and $\nu(K,t) = M^{-1}(M(t)/K)$.

2. Proof of Theorem 1.1

The proof of Theorem 1.1 begins with (1.5) for $t > 0$ and (1).

For $f \in \mathscr{F}(K)$, the real-valued function $f(t)$ of $t \in \mathbb{R}$ is strictly increasing, so that $f(t) > f(0) = 0$ for $t > 0$. Since $f(H(t)) = H(f(t))$, it then follows that $M(t)/K \leqslant M(f(t)) \leqslant KM(t)$, or equivalently,

$$\{\mu^{-1}(\pi M(t)/(2K))\}^{-2} - 1 \leqslant f(t) \leqslant \{\mu^{-1}(\pi K M(t)/2)\}^{-2} - 1$$

for $t > 0$. The left-most term is strictly positive because $\mu^{-1}(q) < 1$ for $q > 0$.

Consequently, in order to prove (1.5) for $t > 0$ and (1) at the same time, it suffices to show that for $s > 0$ satisfying

$$M(t)/K \leqslant M(s) \leqslant KM(t) \tag{2.1}$$

there always exists $f \in \mathscr{F}(K)$ such that $f(t) = s$.

Let φ_t and φ_s be the canonical mappings of $H(t)$ and $H(s)$, respectively, and set

$$h_\Lambda(z) = \Lambda \operatorname{Re} z + i \operatorname{Im} z = 2^{-1}(\Lambda+1)z + 2^{-1}(\Lambda-1)\overline{z} \qquad \text{for} \quad z \in \mathbb{C},$$

where $\Lambda = M(s)/M(t)$; and $h_\Lambda(\infty) = \infty$ by definition. Then the affine mapping h_Λ is $K(\Lambda)$-quasiconformal from $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$, where $K(\Lambda) = \max(\Lambda, \Lambda^{-1})$.

Set $\psi = {\varphi_s}^{-1} \circ h_\Lambda \circ \varphi_t$. Then F defined by $F(z) = \psi(z)$ for $\operatorname{Im} z \geqslant 0$ and $F(z) = \overline{\psi(\overline{z})}$ for z with $\overline{z} \in H$, is a $K(\Lambda)$-quasiconformal mapping from $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$ such that $F(\mathbb{R}) = \mathbb{R}$, $F(\zeta) = \zeta$ for $\zeta = -1$, 0, and ∞. Hence $F \in \mathscr{F}(K)$ is the requested mapping because $F(t) = s$ and $1 \leqslant K(\Lambda) \leqslant K$ by $1/K \leqslant \Lambda \leqslant K$, a consequence of (2.1).

For the remainder of the proof we consider

$$\Theta_k(f) = {S_k}^{-1} \circ f \circ S_k \tag{2.2}$$

for $k = 2,\ 3$ and for $f \in \mathscr{F}(K)$, where

$$S_2(z) = -1 - z \qquad \text{and} \qquad S_3(z) = -z^{-1} - 1 \tag{2.3}$$

are Möbius transformations. Then Θ_k maps $\mathscr{F}(K)$ one-to-one onto $\mathscr{F}(K)$ for $k = 2,\ 3$.

We therefore have

$$\lambda(t) = \sup_{f \in \mathscr{F}(K)} \Theta_k(f)(t) \tag{2.4}$$

and

$$\nu(t) = \inf_{f \in \mathscr{F}(K)} \Theta_k(f)(t). \tag{2.5}$$

Here and hereafter, we sometimes write $X(t) = X(K, t)$ for $X = \lambda,\ \nu$, whenever the meaning is clear from the context.

Suppose that $t < -1$. Then $-1 - t > 0$ and the right-hand sides of (2.4) and (2.5) for $k = 2$ are $-\nu(-1-t) - 1$ and $-\lambda(-1-t) - 1$, respectively. Hence the formulas in (2) follow from those in (1).

Suppose that $-1 < t < 0$. Then $-(1+t)/t > 0$ and the right-hand sides of (2.4) and (2.5) for $k = 3$ are $-1/\{\lambda(-(1+t)/t)+1\}$ and $-1/\{\nu(-(1+t)/t)+1\}$, respectively. We thus have the formulas in (3) in view of those in (1). □

Remark. Although we mentioned in the introduction that the supremum $\lambda(K, t)$ and the infimum $\nu(K, t)$ in (0.1) are attained by functions of $\mathscr{F}(K)$, we have actually proved these facts without appealing to the normal family property of $\mathscr{F}(K)$.

3. Formulas; Corollaries and Remarks

To deal with our forthcoming problems in a uniform way we begin with

Theorem 3.1. *Let $K \geqslant 1$ and $t \in \mathbb{R}$. Then*

(1) $\lambda(K, t)\nu(K, t^{-1}) = 1 \qquad$ *for* $\quad t \neq 0$;

(2) $\lambda(K, t) + \nu(K, -1 - t) = -1 \qquad$ *for all* $\quad t$;

(3) $X(K, t) = -1/(X(K, -t^{-1} - 1) + 1) \qquad$ *for* $\quad t \neq 0$,

where $X = \lambda,\ \nu$;

(4) $$X(K,t) = -1/X(K, -(1+t)^{-1}) - 1 \qquad \textit{for} \quad t \neq -1,$$

where $X = \lambda,\ \nu$;

(5) $$X(K,t) = -Y(K, -t/(1+t))/(Y(K, -t/(1+t)) + 1) \qquad \textit{for} \quad t \neq -1,$$

where $(X, Y) = (\lambda, \nu)$ *or* $(X, Y) = (\nu, \lambda)$.

Proof. Significant Möbius transformations other than *id*, which map the set $\{-1, 0, \infty\}$ onto itself are

(3.1) $$S_1(z) = 1/z,\ S_4(z) = S_1 \circ S_2(z) = -1/(1+z),$$
$$S_5(z) = S_1 \circ S_2 \circ S_1(z) = -z/(1+z),$$

and, furthermore, S_2 and $S_3 = S_2 \circ S_1$ of (2.3). Then each Θ_k of (2.2) for $1 \leqslant k \leqslant 5$, this time, maps the family $\mathscr{F}(K)$ one-to-one onto itself. Hence

$$\lambda(t) = \max_{f \in \mathscr{F}(K)} \Theta_k(f)(t) \qquad \text{and} \qquad \nu(t) = \min_{f \in \mathscr{F}(K)} \Theta_k(f)(t).$$

Since $S_1 :\ \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$, $S_2 :\ \mathbb{R} \to \mathbb{R}$, and $S_5 :\ \mathbb{R} \setminus \{-1\} \to \mathbb{R} \setminus \{-1\}$ all are decreasing on each subinterval, whereas $S_3 :\ \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{-1\}$ and $S_4 :\ \mathbb{R} \setminus \{-1\} \to \mathbb{R} \setminus \{0\}$ are increasing on each subinterval, so that (1), (2), and (5) follow from the former and (3) and (4) follow from the latter monotone property of S_k.

For example, $S_k^{-1} = S_k$ for $k = 1,\ 2$, and 5, so that

$$\lambda(t) = \max \Theta_k(f)(t) = S_k(\min f(S_k(t))) = S_k(\nu(S_k(t)))$$

shows the case $(X, Y) = (\lambda, \nu)$ in (1), (2), and (5). Note that $S_3^{-1} = S_4$, and hence $S_4^{-1} = S_3$. □

One can also prove (3)–(5) directly with the combination of (1) and (2).

To avoid the restriction $t \neq 0$ or $t \neq -1$ in Theorem 3.1 one could define $X(K, +\infty) = +\infty$ and $X(K, -\infty) = -\infty$ for $X = \lambda,\ \nu$. For example, let $t \to +0$ in (3). Then, since $-(1+t)/t < -1$ for $t > 0$, the right-hand sides tend to 0. Another natural device is that $X(K, \infty) = \infty$ for the point at infinity ∞.

Two corollaries emanate from Theorem 1.1. First, as a consequence of Theorem 1.1 we naturally have relations between λ and ν which are "transcendental" in contrast with those in Theorem 3.1.

Corollary 3.2. *For* $t > 0$

(3.2) $$M(\lambda(K,t)) = K^2 M(\nu(K,t));$$

for $t < -1$,

(3.3) $$M \circ S_4(\lambda(K,t)) = K^2 M \circ S_4(\nu(K,t));$$

and for $-1 < t < 0$,

(3.4) $$M \circ S_3(\lambda(K,t)) = K^2 M \circ S_3(\nu(K,t)).$$

Recall that $S_3(z) = -1/z - 1$ and $S_4(z) = -1/(1+z)$, so that $S_3^{-1} = S_4$.

For the proof we begin with the case $t > 0$. It follows from (1.1) and (1) of Theorem 1.1 that

$$M(\lambda(K,t)) = KM(t) \quad \text{and} \quad M(\nu(K,t)) = K^{-1}M(t). \tag{3.5}$$

Hence (3.2). In case $t < -1$, we invoke (4) in Theorem 3.1 to have $X(-1/(1+t)) = S_4(X(t))$ for $X = \lambda,\ \nu$. Since $-1/(1+t) > 0$ for $t < -1$, the identity (3.3) is a consequence of (3.2). In case $-1 < t < 0$, we recall (3) in Theorem 3.1 to have $X(-(1+t)/t) = S_3(X(t))$ for $X = \lambda,\ \nu$. Since $-(1+t)/t > 0$ for $-1 < t < 0$, the requested (3.4) follows.

Corollary 3.3. *Suppose that $t > 0$. Then $X(2K,t)$ is expressed in terms of $X(K,t)$ as follows.*

$$\lambda(2K,t) = (\sqrt{1+\lambda(K,t)} + \sqrt{\lambda(K,t)})^4 - 1. \tag{3.6}$$

$$\nu(2K,t) = (\sqrt{1+\nu(K,t)} - 1)^2/(4\cdot\sqrt{1+\nu(K,t)}). \tag{3.7}$$

Equivalences of (3.6) and (3.7) are

$$\lambda(K/2,t) = (\sqrt{1+\lambda(K,t)} - 1)^2/(4\cdot\sqrt{1+\lambda(K,t)}) \tag{3.8}$$

and

$$\nu(K/2,t) = (\sqrt{1+\nu(K,t)} + \sqrt{\nu(K,t)})^4 - 1 \tag{3.9}$$

for $t > 0$ and $K \geqslant 2$.

The formulas in the case $t < 0$ follow from (3.6), (3.7) (and (3.8), (3.9)) and Theorem 3.1. For example, if $t < -1$, we combine (2) in Theorem 3.1 and (3.7) for $-1-t > 0$ to have $\lambda(2K,t) = -(\sqrt{-\lambda(K,t)} + 1)^2/(4\cdot\sqrt{-\lambda(K,t)})$. The formulas (3.6)–(3.9) produce recursion ones, so that we are able to have the formulas for $X(2^nK,t)$ and $X(2^{-n}K,t)$ for $n = 2, 3, \dots$.

For the proof of Corollary 3.3 we recall two identities for μ due to J. Hersch [Hr, p. 316, (3) and (3′)] which read $2\mu(r) = \mu((1-\sqrt{1-r^2})^2r^{-2})$ and $\mu(r) = 2\mu(2\sqrt{r}/(1+r))$ for $0 < r < 1$. Somewhat laborious calculation with $r = \mu^{-1}(\rho)$ and

$$\Upsilon(\rho) \equiv \{\mu^{-1}(\rho)\}^{-2} - 1, \quad \rho > 0, \tag{3.10}$$

shows that

$$\Upsilon(\rho) = (\sqrt{1+\Upsilon(2\rho)} - 1)^2/(4\cdot\sqrt{1+\Upsilon(2\rho)}) \tag{3.11}$$

and

$$\Upsilon(\rho) = (\sqrt{1+\Upsilon(\rho/2)} + \sqrt{\Upsilon(\rho/2)})^4 - 1. \tag{3.12}$$

Setting $\rho = \pi K M(t)$ in (3.12) and using Theorem 1.1 (1), one has (3.6), whereas setting $\rho = \pi M(t)/(4K)$ in (3.11) one has (3.7).

4. Hyperbolic distance

The extremal functions $X(K,t)$ for $X = \lambda,\ \nu$ will be studied in more detail in conjunction with the hyperbolic distance. One must not neglect the result of O. Teichmüller [T2, p. 364] described below; see [LVV, p. 6] also. Let $P(z)$ be the *hyperbolic density* at a point z of the domain $\mathbb{C}^* = \mathbb{C} \setminus \{-1, 0\}$, so that $\Delta \log P = 4P^2$ everywhere in $\mathbb{C}^*$, in other words, the Gaussian curvature of the metric $P(z)\,|dz|$ is the constant -4. More precisely, $1/P(z) = (1 - |w|^2)|\psi'(w)|$ at $z = \psi(w) \in \mathbb{C}^*$ for a universal covering projection ψ from the open unit disk onto $\mathbb{C}^*$. The *hyperbolic distance* $\sigma(z,w)$ between z and w in $\mathbb{C}^*$ is then

$$\sigma(z,w) = \int P(\zeta)|d\zeta|,$$

where the integral is taken along a geodesic joining z with w in $\mathbb{C}^*$.

Let $\mathscr{G}(K)$ be the family of all the K-quasiconformal mappings f from $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$ such that $f(\zeta) = \zeta$ for $\zeta = -1,\ 0,\ \infty$, so that $\mathscr{F}(K)$ is a proper subset of $\mathscr{G}(K)$. The celebrated Teichmüller result cited above reads that $\{f(z) :\ f \in \mathscr{G}(K)\} = U(z,K)$ for every $z \in \mathbb{C}^*$, where $U(z,K) = \{w \in \mathbb{C}^* :\ \sigma(w,z) \leqslant \log\sqrt{K}\}$ is the closed hyperbolic disk of center z and radius $\log\sqrt{K} \geqslant 0$. Hence

$$[\nu(K,t), \lambda(K,t)] = \{f(t) :\ f \in \mathscr{F}(K)\} \subset U(t,K) \cap \mathbb{R} \tag{4.1}$$

for all $t \in \mathbb{R} \setminus \{-1, 0\}$.

It follows from Theorem 1.1 that $X(K,t)$ for a fixed $K \geqslant 1$ is a strictly increasing function of $t \in \mathbb{R}$, where $X = \lambda,\ \nu$. Set $I_1 = (0, +\infty)$, $I_2 = (-\infty, -1)$, and $I_3 = (-1, 0)$. We can then prove that $[\nu(t), \lambda(t)] \subset I_j$ for $t \in I_j$ and for $j \in \{1,\ 2,\ 3\}$. This is obvious for $j = 1$ because $\nu(t) > 0$ for $t > 0$. Since $\lambda(t) < \lambda(-1) = -1$ for $t \in I_2$, we obtain the inclusion formula for $j = 2$. Finally, for $t \in I_3$, $-1 = \nu(-1) < \nu(t) \leqslant \lambda(t) < \lambda(0) = 0$ implies the inclusion formula. Consequently, for $t \in I_j$,

$$[\nu(K,t), \lambda(K,t)] \subset U(t,K) \cap I_j. \tag{4.2}$$

In fact equality holds in (4.2), as we now show.

Theorem 4.1. *Suppose that $t \in I_j$ for some $j \in \{1,\ 2,\ 3\}$. Then*

$$[\nu(K,t), \lambda(K,t)] = U(t,K) \cap I_j. \tag{4.3}$$

This theorem is obvious for $K = 1$ because $\nu(1,t) = \lambda(1,t) = t$ and $U(t,1) = \{t\}$. Fix $t \in I_j$ for $j \in \{1,\ 2,\ 3\}$. Then $\mathbb{C}^* = \bigcup_{K \geqslant 1} U(t,K)$, so that $U(t,K) \cap I_j \subsetneqq U(t,K) \cap \mathbb{R}$ for $t \in I_j$ and for $K > 1$ depending on t; in fact, for $s \in I_k,\ k \neq j$, there exists $K > 1$ such that $\log\sqrt{K} \geqslant \sigma(s,t)$, so that $s \in U(t,K) \cap \mathbb{R}$.

The proof of Theorem 4.1 is postponed.

One of the universal covering projections from the unit disk onto $\mathbb{C}^*$ is the elliptic modular function ("the bat" or "the umbrella") omitting -1, 0, and ∞, so that if $[a,b] \subset \mathbb{C}^*$ for $a,\ b \in \mathbb{R}$, then $[a,b]$ itself is the geodesic between a and $b > a$. Consequently one has

$$\int_{\nu(K,t)}^{t} P(x)\,dx = \int_{t}^{\lambda(K,t)} P(x)\,dx = \log\sqrt{K} \tag{4.4}$$

for $t \in \mathbb{R}\setminus\{-1,0\}$, where $x \in \mathbb{R}$. Differentiating the first and the second equations in (4.4) with respect to $t \in \mathbb{R}\setminus\{-1,0\}$, one immediately has $P(\lambda(K,t))d\lambda(K,t)/dt = P(t) = P(\nu(K,t))d\nu(K,t)/dt$. This shows that $P(t)dt$ is invariant, $P(X(K,t))\,dX(K,t) = P(t)\,dt$ for the diffeomorphism $X(K,t)$ of $\mathbb{R}\cap\mathbb{C}^*$ onto itself for $X = \lambda,\ \nu$ and for a fixed K, where $dX(K,t) = (d/dt)X(K,t)\,dt$. In case $t > 0$, the identities in (1) in Theorem 1.1 yield

$$\frac{P(\lambda(K,t))}{P(\nu(K,t))} = \frac{d\nu(K,t)/dt}{d\lambda(K,t)/dt} = \frac{\Upsilon'(\pi M(t)/(2K))}{K^2\Upsilon'(\pi K M(t)/2)},$$

where Υ is given in (3.10), $\lambda(K,t) = \Upsilon(\pi K M(t)/2)$, and $\nu(K,t) = \Upsilon(\pi M(t)/(2K))$.

Actually Theorem 4.1 rests on

Theorem 4.2.

$$\{|f(t)|:\ f\in\mathscr{G}(K)\} = [\nu(K,t),\lambda(K,t)] \qquad \textit{for} \quad t>0; \tag{4.5}$$

$$\begin{aligned}&\{|1+f(t)|:\ f\in\mathscr{G}(K)\}\\ &\qquad = [\nu(K,-1-t),\lambda(K,-1-t)] \qquad \textit{for} \quad t<-1;\end{aligned} \tag{4.6}$$

$$\begin{aligned}&\{|f(t)/(1+f(t))|:\ f\in\mathscr{G}(K)\}\\ &\qquad = [\nu(K,-t/(1+t)),\lambda(K,-t/(1+t))] \qquad \textit{for} \quad -1<t<0.\end{aligned} \tag{4.7}$$

Proof. Since

$$[\nu(t),\lambda(t)] = \{f(t):\ f\in\mathscr{F}(K)\} \subset \{|f(t)|:\ f\in\mathscr{G}(K)\}$$

for $t > 0$, the identity (4.5) will follow if we establish the estimates $\nu(t) \leqslant |f(t)| \leqslant \lambda(t)$ for all $f \in \mathscr{G}(K)$.

For a doubly-connected domain $B \subset \overline{\mathbb{C}}$ which can be conformally mapped onto the annulus $\{1 < |z| < R\}$, $1 < R < +\infty$, the quantity $M(B) = \log R$ is well-defined and is called the *modulus of the ring domain* B. For example, for $r_1 > 0$ and $r_2 > 0$ let $B(r_1,r_2)$ be $\mathbb{C}$ minus the real intervals $[-r_1,0]$ and $[r_2,+\infty)$. O. Teichmüller proved that

$$M(B(r_1,r_2)) = \log\rho = 2\mu(\sqrt{r_1/(r_1+r_2)});$$

see [T1, pp. 222–223] where $\rho = \Psi(r_2/r_1)$ in Teichmüller's notation; see [LV1, p. 55] and [L, p. 11] also. We return to general B. If two components of $\overline{\mathbb{C}}\setminus B$

contain pairs of points 0, z and ∞, w, respectively, where $z \in \mathbb{C} \setminus \{0\}$ and $w \in \mathbb{C} \setminus \{0\}$, then the celebrated Teichmüller modulus theorem [T1, p. 222] (see also [L, p. 11] and [LV1, p. 56]) reads that

$$M(B) \leqslant M(B(|z|,|w|)) = 2\mu(\sqrt{|z|/(|z|+|w|)}). \tag{4.8}$$

For $f \in \mathscr{G}(K)$ and for $t > 0$,

$$\pi M(t) = M(B(1,t)) \leqslant KM(f(B(1,t))) \tag{4.9}$$

by the ring-domain-modulus criterion; see [L, p. 13] and [LV1, p. 41]. On the other hand, it follows from (4.8) that $M(f(B(1,t))) \leqslant M(B(1,|f(t)|)) = 2\mu(1/\sqrt{1+|f(t)|})$ because $f(-1) = -1$. Combining this with (4.9), one has $\nu(t) \leqslant |f(t)|$. Next, consider $g(z) = -f(-z)$, $z \in \overline{\mathbb{C}}$. Then, this time,

$$\pi M(t^{-1}) = M(B(t,1)) \leqslant KM(g(B(t,1))) \leqslant KM(B(|g(-t)|,1)),$$

and $|g(-t)| = |f(t)|$, so that

$$\pi M(t^{-1}) \leqslant 2K\mu(\sqrt{|f(t)|/(|f(t)|+1)}),$$

whence $\nu(1/t) \leqslant 1/|f(t)|$. Consequently, $|f(t)| \leqslant 1/\nu(1/t) = \lambda(t)$.

Before proceeding further we note that $\Theta_k(f)$ for $1 \leqslant k \leqslant 5$ can be defined also for $f \in \mathscr{G}(K)$, so that $\Theta_k(f) \in \mathscr{G}(K)$, $1 \leqslant k \leqslant 5$. Actually, Θ_k is a one-to-one mapping from $\mathscr{G}(K)$ onto $\mathscr{G}(K)$, $1 \leqslant k \leqslant 5$.

For the proof of (4.6) we first remark that $-1-t > 0$ for $t < -1$. We may apply (4.5) to $-1-t > 0$ instead of t to observe that the set $A \equiv \{|f(-1-t)| : f \in \mathscr{G}(K)\}$ is equal to the interval $[\nu(-1-t), \lambda(-1-t)]$. On the other hand, since Θ_2 is one-to-one and onto,

$$A = \{|\Theta_2(f)(-1-t)| : \; f \in \mathscr{G}(K)\} = \{|1+f(t)| : \; f \in \mathscr{G}(K)\}.$$

This shows (4.6). For the proof of (4.7) we apply (4.5) to $-t/(1+t) > 0$ for $-1 < t < 0$ to observe that the set $\{|f(-t/(1+t))| : \; f \in \mathscr{G}(K)\}$ is exactly the interval $[\nu(-t/(1+t)), \lambda(-t/(1+t))]$. Since $|\Theta_5(f)(-t/(1+t))| = |f(t)|/|1+f(t)|$, the identity (4.7) immediately follows. □

Proof of Theorem 4.1. Suppose $K > 1$ and suppose first that $t \in I_1$. Since (4.5) claims that $U(t,K) = \{f(t) : \; f \in \mathscr{G}(K)\}$ lies in the closed ring $\{z : \; \nu(t) \leqslant |z| \leqslant \lambda(t)\}$ it follows that $U(t,K) \cap I_1 \subset [\nu(t), \lambda(t)]$. Combining this inclusion formula with that in (4.2) for $j = 1$ we have (4.3) for $j = 1$.

To prove the remaining cases we first remark that S_k for $1 \leqslant k \leqslant 5$ are conformal from $\mathbb{C}^*$ onto $\mathbb{C}^*$ so that $\sigma(z,w) = \sigma(S_k(z), S_k(w))$ for z, $w \in \mathbb{C}^*$.

Since $S_2(t) \in I_1$ for $t \in I_2$, it follows that $[\nu(S_2(t)), \lambda(S_2(t))]$ is the intersection of $U(S_2(t),K)$ with I_1. Since the left interval is just $S_2([\nu(t),\lambda(t)])$ by (2) in Theorem 3.1, and since $U(S_2(t),K) = S_2(U(t,K))$, together with $I_1 = S_2(I_2)$, the identity (4.3) for $j = 2$ follows from $S_2([\nu(t),\lambda(t)]) = S_2(U(t,K) \cap I_2)$.

The identity (4.3) for $j = 3$ may be reduced to the case $j = 2$ with the assistance of (1) in Theorem 3.1 and S_1. □

Making use of the identities in (4.4) one can prove

Corollary 4.3.

(4.10) $$\nu(K,t) \leqslant t/\sqrt{K} \leqslant t \leqslant \sqrt{K}t \leqslant \lambda(K,t) \qquad \textit{for} \quad t > 0;$$

(4.11)
$$\nu(K,t) \leqslant \sqrt{K}t + \sqrt{K} - 1 \leqslant t \leqslant t/\sqrt{K} + 1/\sqrt{K} - 1 \leqslant \lambda(K,t) \qquad \textit{for} \quad t < -1;$$

(4.12) $$\nu(K,t) \leqslant \frac{\sqrt{K}t}{1-(\sqrt{K}-1)t} \leqslant t \leqslant \frac{t}{(\sqrt{K}-1)t+\sqrt{K}} \leqslant \lambda(K,t) \qquad \textit{for} \quad -1 < t < 0.$$

In all chains of inequalities (4.10)–(4.12) the equality holds in the first and the last, respectively, if and only if $K = 1$.

It is well known that $1/P(z)$ is not less than the distance between $z \in \mathbb{C}^*$ and $\{-1, 0\}$, namely,

$$1/P(z) > \min\{|z|, |1+z|\}, \quad z \in \mathbb{C}^*;$$

the strict inequality holds everywhere in $\mathbb{C}^*$; see [Y1, p. 116, (7.4)]. For a rapid and self-contained proof we let $z \in \mathbb{C}^*$ and let $\delta(z) = \min\{|z|, |1+z|\}$. On the other hand, there exists a universal covering projection ψ from the disk $\Delta = \{|w| < 1\}$ onto $\mathbb{C}^*$ such that $z = \psi(0)$. Let φ be the inverse of ψ in $\mathscr{D} \equiv \{\zeta : \ |\zeta - z| < \delta(z)\}$ such that $\varphi(z) = 0$, so that $\gamma(\zeta) = \varphi(\delta(z)\zeta + z)$ maps Δ into Δ with $\gamma(0) = 0$. Hence by the Schwarz lemma,

$$\delta(z)P(z) = \delta(z)/|\psi'(0)| = \delta(z)|\varphi'(z)| = |\gamma'(0)| \leqslant 1.$$

Suppose that $\delta(z)P(z) = 1$. Then $\Delta = \gamma(\Delta) = \varphi(\mathscr{D})$. Hence $\mathscr{D} = \psi(\Delta) = \mathbb{C}^*$. This is absurd. Therefore $\delta(z)P(z) < 1$ everywhere in $\mathbb{C}^*$.

Hence, for $t > 0$,

$$\log\sqrt{K} = \int_{\nu(t)}^{t} P(x)\,dx \leqslant \int_{\nu(t)}^{t} \frac{dx}{x} = \log\frac{t}{\nu(t)}$$

and similarly $\log\sqrt{K} \leqslant \log\{\lambda(t)/t\}$, from which (4.10) follows. Suppose that the equality holds in the first or in the last in (4.10) for $K > 1$. Then $P(x) = 1/x$ for all $x \in [\nu(t), t]$ or all $x \in [t, \lambda(t)]$, respectively. This is a contradiction. Replacing t by $-1-t$ and $-t/(t+1)$, respectively, in (4.10), and then applying (2) and (5) in Theorem 3.1, respectively, one obtains (4.11) and (4.12).

Set $c_H = \Gamma(1/4)^4/(4\pi^2) = 4.376879\cdots$, where Γ means Euler's gamma function. Note that $c_H = (4/\pi)\mathscr{K}(1/\sqrt{2})^2$ by the known formula $\mathscr{K}(1/\sqrt{2}) = \Gamma(1/4)^2/(4\sqrt{\pi})$ (see [BB, p. 25, Theorem 1.7]) and $\Gamma(1/4) = 3.625609\cdots$. Set further,

$$\omega_K = \exp\{(K-1)c_H\} \qquad \text{and} \qquad A_K(t) = \frac{(\log t - \log\omega_K)\log\omega_K}{\log\omega_K + (K-1)\log t}$$

for $0 < t \neq (\omega_K)^{1/(1-K)}$.

Corollary 4.4.

(4.13) $$\lambda(K,t) \leqslant \omega_K t^K \qquad \textit{for} \quad t \geqslant 1$$

and

(4.14a) $$\nu(K,t) \geqslant (\omega_K^{-1} t)^{1/K} \qquad \textit{if} \quad t \geqslant 1 \textit{ and } \nu(K,t) > 1;$$

(4.14b) $$\nu(K,t) \geqslant \exp A_K(t) \qquad \textit{if} \quad t \geqslant 1 \textit{ and } \nu(K,t) \leqslant 1.$$

(4.15a) $$\lambda(K,t) \leqslant (\omega_K t)^{1/K} \qquad \textit{if} \quad 0 < t < 1 \textit{ and } \lambda(K,t) < 1;$$

(4.15b) $$\lambda(K,t) \leqslant \exp\{-A_K(t^{-1})\} \qquad \textit{if} \quad 0 < t < 1 \textit{ and } \lambda(K,t) \geqslant 1;$$

and

(4.16) $$\nu(K,t) \geqslant \omega_K^{-1} t^K \qquad \textit{for} \quad 0 < t < 1.$$

(4.17) $$\lambda(K,t) \leqslant S_5(\omega_K^{-1} S_5(t)^K) \qquad \textit{for} \ -1/2 < t < 0$$

and

(4.18a)
$$\nu(K,t) \geqslant S_5((\omega_K S_5(t))^{1/K}) \qquad \textit{if} \quad -1/2 < t < 0 \textit{ and } \nu(K,t) > -1/2;$$

(4.18b)
$$\nu(K,t) \geqslant S_5(\exp\{-A_K(-t^{-1}-1)\}) \quad \textit{if} \ -1/2 < t < 0 \textit{ and } \nu(K,t) \leqslant -1/2.$$

(4.19a)
$$\lambda(K,t) \leqslant S_5((\omega_K^{-1} S_5(t))^{1/K}) \qquad \textit{if} \quad -1 < t \leqslant -1/2 \textit{ and } \lambda(K,t) < -1/2;$$

(4.19b)
$$\lambda(K,t) \leqslant S_5(\exp A_K(S_5(t))) \qquad \textit{if} \quad -1 < t \leqslant -1/2 \textit{ and } \lambda(K,t) \geqslant -1/2;$$

and

(4.20) $$\nu(K,t) \geqslant S_5(\omega_K S_5(t)^K) \qquad \textit{for} \quad -1 < t \leqslant -1/2.$$

(4.21) $$\lambda(K,t) \leqslant S_2(\omega_K^{-1} S_2(t)^K) \qquad \textit{for} \quad -2 < t < -1$$

and

(4.22a) $$\nu(K,t) \geqslant S_2((\omega_K S_2(t))^{1/K}) \qquad \textit{if} \quad -2 < t < -1 \textit{ and } \nu(K,t) > -2;$$

(4.22b)
$$\nu(K,t) \geqslant S_2(\exp\{-A_K(-1/(1+t))\}) \quad \textit{if} \ -2 < t < -1 \textit{ and } \nu(K,t) \leqslant -2.$$

(4.22c)

(4.23a) $$\lambda(K,t) \leqslant S_2((\omega_K^{-1} S_2(t))^{1/K}) \qquad \textit{if} \quad t \leqslant -2 \textit{ and } \lambda(K,t) < -2;$$

(4.23b) $$\lambda(K,t) \leqslant S_2(\exp A_K(S_2(t))) \qquad \textit{if} \quad t \leqslant -2 \textit{ and } \lambda(K,t) \geqslant -2;$$

and

(4.24) $$\nu(K,t) \geqslant S_2(\omega_K S_2(t)^K) \qquad \textit{for} \quad t \leqslant -2.$$

For the proof of Corollary 4.4, we recall here the result of J. Hempel [Hm, p. 443, (4.1)] for the hyperbolic density for $\mathbb{C} \setminus \{1, 0\}$, which can be reduced to the inequality

$$1/P(z) \leqslant 2|z|(|\log|z|| + c_H), \quad z \in \mathbb{C}^*,$$

by the map $z \mapsto -z$ from $\mathbb{C}^*$ to $\mathbb{C} \setminus \{1, 0\}$; note that $c_H = 1/\{2P(1)\}$; see [Y1, p. 118, (8.2)] also.

Suppose that $t > 1$. Then

$$\log \sqrt{K} = \int_t^{\lambda(t)} P(x)\, dx \geqslant \int_t^{\lambda(t)} \frac{dx}{2x(c_H + \log x)} = \frac{1}{2} \log \frac{c_H + \log \lambda(t)}{c_H + \log t},$$

whence (4.13). Suppose further that $\nu(K, t) > 1$. Then

$$\log \sqrt{K} = \int_{\nu(t)}^t P(x)\, dx \geqslant \int_{\nu(t)}^t \frac{dx}{2x(c_H + \log x)} = \frac{1}{2} \log \frac{c_H + \log t}{c_H + \log \nu(t)},$$

whence (4.14a). Next consider the case $\nu(K, t) \leqslant 1$. Then

$$\begin{aligned} \log \sqrt{K} &= \int_{\nu(t)}^1 P(x)\, dx + \int_1^t P(x)\, dx \\ &\geqslant \int_{\nu(t)}^1 \frac{dx}{2x(c_H - \log x)} + \int_1^t \frac{dx}{2x(c_H + \log x)} \\ &= \frac{1}{2} \log \frac{c_H - \log \nu(t)}{c_H} + \frac{1}{2} \log \frac{c_H + \log t}{c_H}, \end{aligned}$$

whence (4.14b).

Suppose that $0 < t < 1$. Then $\nu(t) = 1/\lambda(t^{-1})$ and $t^{-1} > 1$, so that (4.16) is a consequence of (4.13) for t^{-1}. If $\lambda(t) < 1$, then $\nu(t^{-1}) > 1$, so that (4.15a) follows from (4.14a). Similarly, (4.15b) is a consequence of (4.14b).

The remaining cases are consequences of (4.13)–(4.16) by our standard reasoning. Implication formulas are as follows.

$$-1/2 < t < 0 \implies 0 < S_5(t) < 1 \implies \begin{cases} (4.16) & \implies \quad (4.17) \\ (4.15\text{a}) & \implies \quad (4.18\text{a}) \\ (4.15\text{b}) & \implies \quad (4.18\text{b}) \end{cases}$$

$$-1 < t \leqslant -1/2 \implies 1 \leqslant S_5(t) \implies \begin{cases} (4.14\text{a}) & \implies \quad (4.19\text{a}) \\ (4.14\text{b}) & \implies \quad (4.19\text{b}) \\ (4.13) & \implies \quad (4.20) \end{cases}$$

Here $X(t) = S_5 \circ Y \circ S_5(t)$ for $Y = \lambda,\ \nu$ by (5) in Theorem 3.1.

$$-2 < t < -1 \implies 0 < S_2(t) < 1 \implies \begin{cases} (4.16) & \implies & (4.21) \\ (4.15\text{a}) & \implies & (4.22\text{a}) \\ (4.15\text{b}) & \implies & (4.22\text{b}) \end{cases}$$

$$t \leqslant -2 \implies 1 \leqslant S_2(t) \implies \begin{cases} (4.14\text{a}) & \implies & (4.23\text{a}) \\ (4.14\text{b}) & \implies & (4.23\text{b}) \\ (4.13) & \implies & (4.24) \end{cases}$$

Here $X(t) = S_2 \circ Y \circ S_2(t)$ for $Y = \lambda,\ \nu$, by (2) in Theorem 3.1.

Remark. S. Agard [A, p. 10, (3.1)] proved a remarkable result that

$$\lambda(K,t) = \sup_{f \in \mathscr{G}(K)} \max_{|z|=t} |f(z)| \qquad \text{for} \quad t \geqslant 1;$$

he makes use of the notation $P_2(t, K)$ for the right-hand side in the above when $t \geqslant 1$. G. J. Martin solved an extremal problem in [M, Theorem 1.1]. Namely, for $t > 0$ let $\mathcal{A}(t)$ be the family of holomorphic functions $f : \{|z| < 1\} \to \mathbb{C} \setminus \{0,1\}$ with $|f(0)| = t$. Then

$$\lambda(K,t) = \sup_{f \in \mathcal{A}(t)} \max_{|z|=(K-1)/(K+1)} |f(z)|.$$

See the forthcoming paper [Y2] for the details.

5. Comparison of $X(K,s)$ with $X(K,t)$ for $X = \lambda,\ \nu$

Our main result in this section is

Theorem 5.1. *Let t and s be real numbers.*
(1) *If $s > 0$ and $t > 0$, then*

$$-\lambda(K,-s/t)\nu(K,t) \leqslant \nu(K,s) \leqslant \lambda(K,s) \leqslant -\nu(K,-s/t)\lambda(K,t).$$

(2) *If $s < 0$ and $t < 0$, then*

$$-\nu(K,-s/t)\nu(K,t) \leqslant \nu(K,s) \leqslant \lambda(K,s) \leqslant -\lambda(K,-s/t)\lambda(K,t).$$

(3) *If $s < 0$ and $t > 0$, then*

$$-\lambda(K,-s/t)\lambda(K,t) \leqslant \nu(K,s) \leqslant \lambda(K,s) \leqslant -\nu(K,-s/t)\nu(K,t).$$

(4) *If $s > 0$ and $t < 0$, then*

$$-\nu(K,-s/t)\lambda(K,t) \leqslant \nu(K,s) \leqslant \lambda(K,s) \leqslant -\lambda(K,-s/t)\nu(K,t).$$

Equalities hold in (1) and (2) if $t = s \neq 0$.

Let $\mathscr{E}(K)$ be the family of all the K-quasiconformal mappings f from $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$ such that $f(\mathbb{R}) = \mathbb{R}$, and $f(\infty) = \infty$. Hence $\mathscr{F}(K)$ is a proper subset of $\mathscr{E}(K)$. Fix $a \in \mathbb{R}$ and $b \in \mathbb{R}$ with $a \neq b$. We then associate with a function $f \in \mathscr{E}(K)$ a new function

$$\Theta_{a,b}(f)(z) = \frac{f((b-a)z+b) - f(b)}{f(b) - f(a)}, \quad z \in \overline{\mathbb{C}}.$$

Then $\Theta_{a,b}$ is a mapping from $\mathscr{E}(K)$ onto $\mathscr{F}(K)$. To prove the "onto" property let $g \in \mathscr{F}(K)$ and set $f(z) = g((z-b)/(b-a))$, $z \in \overline{\mathbb{C}}$. Then $f \in \mathscr{E}(K)$, $f(a) = -1$, and $f(b) = 0$. Hence $\Theta_{a,b}(f) = g$.

We thus have, for a, b, and $t \in \mathbb{R}$ with $a \neq b$,

$$\min_{f \in \mathscr{E}(K)} \Theta_{a,b}(f)(t) = \nu(K,t) \quad \text{and} \tag{5.1}$$

$$\max_{f \in \mathscr{E}(K)} \Theta_{a,b}(f)(t) = \lambda(K,t). \tag{5.2}$$

Proof of Theorem 5.1. Let s, t, a, and b all be in $\mathbb{R}$ and suppose that $st \neq 0 \neq a - b$. Set $c = (b-a)t + b$. Then $c \neq b$ and

$$-\nu(-s/t) = \max_{f \in \mathscr{E}(K)} \{-\Theta_{c,b}(f)(-s/t)\} \quad \text{and} \tag{5.3}$$

$$-\lambda(-s/t) = \min_{f \in \mathscr{E}(K)} \{-\Theta_{c,b}(f)(-s/t)\}, \tag{5.4}$$

where one observes that

$$-\Theta_{c,b}(f)(-s/t) = \frac{f((b-a)s+b) - f(b)}{f((b-a)t+b) - f(b)} = \frac{\Theta_{a,b}(f)(s)}{\Theta_{a,b}(f)(t)}, \tag{5.5}$$

so that

$$\Theta_{a,b}(f)(s) = -\Theta_{c,b}(f)(-s/t)\Theta_{a,b}(f)(t). \tag{5.6}$$

Set $A = -\nu(-s/t)$ and $B = -\lambda(-s/t)$. Suppose that $st > 0$ so that $0 < B \leqslant A$. If $t > 0$, then $s > 0$ and

$$\nu(s) = \min_{f \in \mathscr{E}(K)} \Theta_{a,b}(f)(s) \geqslant B \min_{f \in \mathscr{E}(K)} \Theta_{a,b}(f)(t) = B\nu(t),$$
$$\lambda(s) = \max_{f \in \mathscr{E}(K)} \Theta_{a,b}(f)(s) \leqslant A \max_{f \in \mathscr{E}(K)} \Theta_{a,b}(f)(t) = A\lambda(t).$$

Hence (1) is established. The rest of the proof is now obvious. □

Remark 1. Set $c(K) = \lambda(K,1) = 1/\nu(K,1)$. Set $t = 1$ in (1) in Theorem 5.1 and consider (2) in Theorem 3.1. Then we have

$$c(K)^{-1}(\nu(K,s-1)+1) \leqslant \nu(K,s) \leqslant \lambda(K,s) \leqslant c(K)(\lambda(K,s-1)+1) \tag{5.7}$$

for $s > 0$. Set $t = 1$ in (3) in Theorem 5.1 and consider (2) in Theorem 3.1 again. Then we have

$$(5.8) \quad c(K)(\nu(K, s-1)+1) \leqslant \nu(K,s) \leqslant \lambda(K,s) \leqslant c(K)^{-1}(\lambda(K,s-1)+1)$$

for $s < 0$. It should be mentioned that (5.7) and (5.8) can be used recursively to produce new inequalities. For example, if $s > 1$, then

$$\begin{aligned} c(K)^{-2}(\nu(K,s-2)+1+c(K)) \leqslant \nu(K,s) &\leqslant c(K)\nu(K,s+1)-1 \\ &\leqslant c(K)^2\nu(K,s+2)-c(K)-1. \end{aligned}$$

Since $\mu(1/\sqrt{2}) = \pi/2$ it follows that $c(K) \geqslant 1$ and $c(K) = 1$ if and only if $K = 1$.

Remark 2. Let f be a K-quasiconformal mapping from the upper half-plane H onto H such that $f(\infty) = \infty$. Actually f can be extended K-quasiconformally to $\overline{\mathbb{C}}$ by the reflection, so that the resulting function, again denoted by f, is in $\mathscr{E}(K)$. For $x,\ y \in \mathbb{R}$ with $y \neq 0$ set $a = x - y$ and $b = x$ in (5.1) and (5.2). We then have

$$\lambda(K,t^{-1})^{-1} = \nu(K,t) \leqslant \{f(x+yt)-f(x)\}/\{f(x)-f(x-y)\} \leqslant \lambda(K,t)$$

for $t \in \mathbb{R}\setminus\{0\}$. In the specified case $t = 1$ this is simply the necessary condition of A. Beurling and L. V. Ahlfors [BA]; see [LV1, p. 81, Theorem 6.2].

6. Asymptotic behavior of $X(K,t)$, $X = \lambda,\ \nu$

As obvious consequences of Theorem 1.1 one observes that, for a fixed $K \geqslant 1$,

$$\begin{aligned} \lim_{t\to+\infty} \lambda(K,t) &= \lim_{t\to+\infty} \nu(K,t) = +\infty \quad \text{and} \\ \lim_{t\to-\infty} \lambda(K,t) &= \lim_{t\to-\infty} \nu(K,t) = -\infty. \end{aligned}$$

Furthermore, for a fixed $t > 0$,

$$\lim_{K\to+\infty} \lambda(K,t) = +\infty \qquad \text{and} \qquad \lim_{K\to+\infty} \nu(K,t) = 0;$$

for a fixed $t < -1$,

$$\lim_{K\to+\infty} \lambda(K,t) = -1 \qquad \text{and} \qquad \lim_{K\to+\infty} \nu(K,t) = -\infty;$$

and for a fixed $t,\ -1 < t < 0$,

$$\lim_{K\to+\infty} \lambda(K,t) = 0 \qquad \text{and} \qquad \lim_{K\to+\infty} \nu(K,t) = -1.$$

The following theorem provides information on orders of all the described limits, so that, is significant.

Theorem 6.1. *First fix $K \geqslant 1$. Then*

$$\lim_{t\to+\infty} t^{-a}X(K,t) = 16^{a-1}, \tag{6.1}$$

where $a = K$ for $X = \lambda$ and $a = 1/K$ for $X = \nu$;

$$\lim_{t\to-\infty} (-t)^{-a}X(K,t) = -16^{a-1}, \tag{6.2}$$

where $a = 1/K$ for $X = \lambda$ and $a = K$ for $X = \nu$. Next, fix $t > 0$. Then

$$\begin{aligned} &\lim_{K\to+\infty} \lambda(K,t)\exp\{-\pi K M(t)\} = 1/16 \quad \text{and} \\ &\lim_{K\to+\infty} \nu(K,t)\exp\{\pi K M(t^{-1})\} = 16. \end{aligned} \tag{6.3}$$

Fix $t < -1$. Then

$$\begin{aligned} &\lim_{K\to+\infty} (\lambda(K,t)+1)\exp\{\pi K M(-1/(1+t))\} = -16 \quad \text{and} \\ &\lim_{K\to+\infty} \nu(K,t)\exp\{-\pi K M(-1-t)\} = -1/16. \end{aligned} \tag{6.4}$$

Finally fix $-1 < t < 0$. Then

$$\begin{aligned} &\lim_{K\to+\infty} \lambda(K,t)\exp\{\pi K M(-t^{-1}-1)\} = -16 \quad \text{and} \\ &\lim_{K\to+\infty} (\nu(K,t)+1)\exp\{\pi K M(-t/(1+t))\} = 16. \end{aligned} \tag{6.5}$$

The proof of Theorem 6.1 is postponed. A somewhat more general discussion is possible; we describe it here.

Theorem 6.2. *There exists a real, continuous function Δ of real variable $x > 0$ such that*

(6.6)
$$0 < \Delta(x) < 8 \quad \text{for } x \geqslant \log 2 \quad \text{and} \quad -5/2 < \Delta(x) < 5/2 \quad \text{for } 0 < x < \log 2,$$

for which the following formulas are valid, where

$$Q(x) = 4^{-1}e^{x} - e^{-x} \qquad \text{for} \quad x > 0.$$

For $t > 0$,

$$\lambda(K,t) = Q(\pi K M(t)/2)^2 + \Delta(\pi K M(t)/2)\exp\{-\pi K M(t)\}, \tag{6.7}$$

$$\nu(K,t) = Q(\pi M(t)/(2K))^2 + \Delta(\pi M(t)/(2K))\exp\{-\pi M(t)/K\}, \tag{6.8}$$

and

$$1/\nu(K,t) = Q(\pi K M(t^{-1})/2)^2 + \Delta(\pi K M(t^{-1})/2)\exp\{-\pi K M(t^{-1})\}. \tag{6.9}$$

For $t < -1$,

(6.10) $$\lambda(K,t) = -Q(\pi M(-1-t)/(2K))^2 - 1 - \Delta(\pi M(-1-t)/(2K))\exp\{-\pi M(-1-t)/K\},$$

(6.11) $$\nu(K,t) = -Q(\pi KM(-1-t)/2)^2 - 1 - \Delta(\pi KM(-1-t)/2)\exp\{-\pi KM(-1-t)\},$$

and

(6.12) $$1/(\lambda(K,t)+1) = -Q(\pi KM(-1/(1+t))/2)^2 - \Delta(\pi KM(-1/(1+t))/2)\exp\{-\pi KM(-1/(1+t))\}.$$

For $-1 < t < 0$,

(6.13) $$1/\lambda(K,t) = -Q(\pi KM(-t^{-1}-1)/2)^2 - 1 - \Delta(\pi KM(-t^{-1}-1)/2)\exp\{-\pi KM(-t^{-1}-1)\},$$

and

(6.14) $$1/(\nu(K,t)+1) = Q(\pi KM(-t/(1+t))/2)^2 + 1 + \Delta(\pi KM(-t/(1+t))/2)\exp\{-\pi KM(-t/(1+t))\}.$$

Note that $Q(x)^2 = e^{2x}/16 - 1/2 + e^{-2x}$. More detailed properties of $\Delta(x)$ will be observed. For example, $\limsup_{x\to+\infty}\Delta(x) \leqslant 1/2$. Furthermore, if $1/2 < A < 8$, then there exists $\alpha > 0$ such that $\Delta(x) < A$ for $x \geqslant \log 2 + \alpha$; see the forthcoming Remark 4.

We can now prove (6.3)–(6.5) in Theorem 6.1 in the following procedure.

(6.7) $\Longrightarrow$ λ-part in (6.3).	(6.9) $\Longrightarrow$ ν-part in (6.3).
(6.11) $\Longrightarrow$ ν-part in (6.4).	(6.12) $\Longrightarrow$ λ-part in (6.4).
(6.13) $\Longrightarrow$ λ-part in (6.5).	(6.14) $\Longrightarrow$ ν-part in (6.5).

From (1.1) and Theorem 1.1 (3) it is obvious that $X(K,t)$ is bounded for $-1 < t < 0$ if K is fixed. See [LV1, p. 82, (6.10)] (see [LVV] also) for $t = 1$ in (6.7). The cited asymptotic expansion reads $\lambda(K,1) = 16^{-1}e^{\pi K} - 2^{-1} + O(e^{-\pi K})$; see also [LVV, Theorem 3] and [AVV1, p. 7, Theorem 2.13].

Proof of Theorem 6.2. We define $\Delta(x)$ by the formula

(6.15) $$\Upsilon(x) = Q(x)^2 + \Delta(x)e^{-2x},$$

where $\Upsilon(x)$ is the function of (3.10), namely, $\Upsilon(x) = \{\mu^{-1}(x)\}^{-2} - 1 > 0$ for $x > 0$. First of all we prove that Δ satisfies (6.6).

Setting $r = \mu^{-1}(x)$ in the following inequality [LV1, p. 62],

$$(6.16) \qquad 0 < \frac{2(1+\sqrt{1-r^2})}{r} - e^{\mu(r)} < r^3, \quad 0 < r < 1,$$

one has

$$(6.17) \qquad 0 < \delta(x) < \mu^{-1}(x)^3,$$

where $\delta(x) = 2(\sqrt{1+\Upsilon(x)} + \sqrt{\Upsilon(x)}) - e^x,\ x > 0$. Here, in terms of the function $J(Y) = 4^{-1}Y + Y^{-1},\ Y > 0$, one may express $\Upsilon(x)$ as

$$(6.18) \qquad \Upsilon(x) = J(e^x + \delta(x))^2 - 1,$$

so that (6.17) may be rewritten as

$$(6.19) \qquad 0 < \delta(x) < J(e^x + \delta(x))^{-3}, \quad x > 0.$$

Since $Q(x)^2 = J(e^x)^2 - 1$, it follows from (6.18) that

$$(6.20) \quad \Delta(x) \equiv (\Upsilon(x) - Q(x)^2)e^{2x} = (J(e^x + \delta(x))^2 - J(e^x)^2)e^{2x}, \quad x > 0.$$

Suppose that $x \geqslant \log 2$. Then, by the mean-value theorem,

$$\Delta(x) = \delta(x)J'(e^x + \theta(x)\delta(x))(J(e^x + \delta(x)) + J(e^x))e^{2x},$$

where $0 < \theta(x) < 1$, which, together with the three estimates,

$$\begin{aligned} &0 < J'(e^x + \theta(x)\delta(x)) < 4^{-1}, \\ &2 < J(e^x + \delta(x)) + J(e^x) < 2J(e^x + \delta(x)), \quad \text{and} \\ &e^{2x} < 4^2 J(e^x)^2 < 4^2 J(e^x + \delta(x))^2, \end{aligned}$$

shows that $0 < \Delta(x) < 8\delta(x)J(e^x + \delta(x))^3$. It then follows from (6.19) that $\Delta(x) < 8$.

In the case where $0 < x < \log 2$, we have $1 < e^x + \delta(x) < 3$, so that

$$\begin{gathered} 2 < J(e^x + \delta(x)) + J(e^x) < 5/2,\ 1 < e^{2x} < 4, \quad \text{and} \\ -4^{-1} < J(e^x + \delta(x)) - J(e^x) < 4^{-1}. \end{gathered}$$

Hence (6.20) yields that $-5/2 < \Delta(x) < 5/2$.

We have (6.7) and (6.8) on setting $x = \pi K M(t)/2$ and $x = \pi M(t)/(2K)$ in (6.15), respectively.

Since $\nu(K, t) = 1/\lambda(K, 1/t)$ we have (6.9) by (6.7). For $t < -1$ we recall (2) in Theorem 3.1 to have (6.10) and (6.11) from (6.8) and (6.7), respectively. The formula (4) of Theorem 3.1 and (6.7) give (6.12). For $-1 < t < 0$ we have $1/\lambda(K, t) = -1 - \lambda(K, -t^{-1} - 1)$ by (3) for λ in Theorem 3.1, so that (6.13) is a consequence of (6.7). Finally, (6.14) follows from Theorem 3.1 (5) and (6.7) with t replaced by $-t/(1+t) > 0$. □

Proofs of (6.1) *and* (6.2) *in Theorem* 6.1. First of all, a consequence of Hersch's inequality [Hr, p. 318, (9)]

$$2\log\frac{1+\sqrt{1-r}}{\sqrt{r}} \leqslant \mu(r) \leqslant 2\log\frac{1+\sqrt{1+r}}{\sqrt{r}}, \quad 0<r<1,$$

is that

$$\lim_{r\to 0}(\mu(r)-\log(4/r)) = 0. \tag{6.21}$$

This also follows from

$$\lim_{r\to 0}(\mathscr{K}(r)-\pi/2) = \lim_{r\to 0}(\mathscr{K}(\sqrt{1-r^2})-\log(4/r)) = 0;$$

see [WW, p. 521].

For the proof of (6.1) one begins with

$$\lim_{t\to+\infty} X(K,t)\exp\{-\pi a M(t)\} = 16^{-1}, \tag{6.22}$$

which results from (6.7) (for $X=\lambda$) and (6.8) (for $X=\nu$). Set $r=1/\sqrt{1+t}$ for $t>0$. Then $-\pi a M(t) = -2a\mu(r)$, so that (6.1) follows from (6.21) and (6.22).

For the proof of (6.2) one finds

$$\lim_{t\to-\infty} X(K,t)\exp\{-\pi a M(-1-t)\} = -16^{-1}; \tag{6.23}$$

this follows from (6.10) (for λ) and (6.11) (for ν). Set $r=1/\sqrt{-t}$ for $t<0$. Then, this time, $-\pi a M(-1-t) = -2a\mu(r)$, which, combined with (6.21) and (6.23), proves (6.2). □

Remark 1. Since $M^{-1}(s)=\Upsilon(\pi s/2)$ for $s>0$, it follows from (6.15) that

$$\lim_{s\to+\infty} e^{-\pi s}M^{-1}(s) = 16^{-1}.$$

Remark 2. Since $\Upsilon(x)\to 0$ and $Q(x)\to -3/4$ as $x\to 0$, it follows that $\Delta(x)\to -9/16$ as $x\to 0$, so that $\Delta(x)<0$ for x near 0.

Remark 3. It follows from Theorem 1.1 that

$$\lim_{t\to 0} X(K,t) = \lim_{t\to -1}(X(K,t)+1) = 0$$

for $X=\lambda,\ \nu$. We actually obtain much more:

$$\lim_{t\to+0} t^{-a}X(K,t) = 16^{1-a}; \tag{6.24}$$

$$\lim_{t\to-0} (-t)^{-a}X(K,t) = -16^{1-a}; \tag{6.25}$$

$$\lim_{t\to-1+0} (1+t)^{-a}(1+X(K,t)) = 16^{1-a}; \tag{6.26}$$

$$\lim_{t\to-1-0} (-1-t)^{-a}(1+X(K,t)) = -16^{1-a}, \tag{6.27}$$

where $a=1/K$ for $X=\lambda$ and $a=K$ for $X=\nu$ in (6.24) and (6.26), while $a=K$ for $X=\lambda$ and $a=1/K$ for $X=\nu$ in (6.25) and (6.27).

If $K > 1$, the graph $s = X(K,t)$, $t \in \mathbb{R}$, in the ts-plane, is not smooth at $t = -1,\ 0$, for $X = \lambda,\ \nu$. The following are consequences of (6.24)–(6.27).

$$\begin{aligned}\lim_{t\to+0}\frac{\lambda(K,t)}{t} &= \lim_{t\to-0}\frac{\nu(K,t)}{t} = \lim_{t\to-1+0}\frac{\lambda(K,t)+1}{t+1}\\ &= \lim_{t\to-1-0}\frac{\nu(K,t)+1}{t+1} = +\infty;\\ \lim_{t\to+0}\frac{\nu(K,t)}{t} &= \lim_{t\to-0}\frac{\lambda(K,t)}{t} = \lim_{t\to-1+0}\frac{\nu(K,t)+1}{t+1}\\ &= \lim_{t\to-1-0}\frac{\lambda(K,t)+1}{t+1} = 0.\end{aligned}$$

For the proof of (6.24) set $s = 1/t$, $t > 0$, so that $X(K,t) = 1/Y(K,s)$, for $(X,Y) = (\lambda,\nu)$ or (ν,λ). Then $t^{-a}X(K,t) = s^{a}Y(K,s)^{-1}$, so that (6.24) is a consequence of (6.1). For the proof of (6.25) set $s = 1/t,\ t < 0$. Then $(-t)^{-a}X(K,t) = (-s)^{a}Y(K,s)^{-1}$, which, together with (6.2), gives (6.25). For the proof of (6.26), set $s = -1/(1+t)$ for $t > -1$. Then (4) in Theorem 3.1 yields that $(1+t)^{-a}(1+X(K,t)) = -(-s)^{a}X(K,s)^{-1}$, which, combined with (6.2), gives (6.26). Finally, setting $s = -1/(1+t)$ for $t < -1$, and making use of (4) in Theorem 3.1 one has $(-1-t)^{-a}(1+X(K,t)) = -s^{a}X(K,s)^{-1}$, which, combined with (6.1), gives (6.27).

Note that the graphs $s = \lambda(K,t)$ and $s = \nu(K,t)$ in case $K > 1$ for $t \in \mathbb{R}$ are actually mirror images of each other with respect to the straight line $s = t$. In other words, the function $\lambda(t) = \lambda(K,t)$ of $t \in \mathbb{R}$ is the inverse function of $\nu(t) = \nu(K,t)$ of $t \in \mathbb{R}$, or equivalently, $\lambda(\nu(t)) = t$ for all $t \in \mathbb{R}$. This is trivial for $t = -1$ and 0. If $t > 0$, then $\lambda(\nu(t)) = t$ follows from direct computation with the aid of Theorem 1.1 (1); see also (1.1) and (3.5). Hence $\nu(\lambda(t)) = t$ for $t > 0$ also follows. If $t < -1$, then $-1-t > 0$, so that Theorem 3.1 (2), together with $\nu(\lambda(-1-t)) = -1-t$ shows that $\lambda(\nu(t)) = t$. Hence $\nu(\lambda(t)) = t$ is also true for $t < -1$. If $-1 < t < 0$, then $1/t < -1$ so that $\nu(\lambda(1/t)) = 1/t$. Then making use of Theorem 3.1 (1), twice, one has $\lambda(\nu(t)) = \lambda(1/\lambda(1/t)) = 1/\nu(\lambda(1/t)) = t$.

Remark 4. We can further prove that $\limsup_{x\to+\infty}\Delta(x) \leqslant 1/2$. For this purpose we quote a better estimate

$$0 < \frac{2(1+\sqrt{1-r^2})}{r} - e^{\mu(r)} < \phi(r), \quad 0 < r < 1,$$

than (6.16), where

$$\phi(r) = r^3(1+\sqrt[4]{1-r^2})^{-2}(1+\sqrt{1-r^2})^{-2};$$

see [LV1, p. 62]. On setting $\rho = 1/J(e^x + \delta(x))$, the estimate (6.19) is improved as $0 < \delta(x) < \phi(\rho)$ for $x > 0$. Hence, as in the proof of Theorem 6.2,

$$\Delta(x) < 8\rho^{-3}\phi(\rho) \qquad \text{for} \quad x \geqslant \log 2. \tag{6.28}$$

Since $\rho \to 0$ as $x \to +\infty$, we have the desired estimate.

Let us consider (6.28) in detail. We prove that for each $A \in (1/2, 8)$ there exists $\alpha > 0$ such that $\Delta(x) < A$ for $x \geqslant \log 2 + \alpha$. The function $\psi(r) = 8\phi(r)/r^3$ increases from $1/2$ to 8 as r increases from 0 to 1. Hence if $A \in (1/2, 8)$, then there exists $\alpha > 0$ such that

$$\psi(1/\cosh\alpha) < A. \tag{6.29}$$

Then for $x \geqslant \log 2 + \alpha$, we have $1/\rho = J(e^x + \delta(x)) > J(2e^\alpha) = \cosh\alpha$. Hence $\Delta(x) < A$.

7. The limit of $\partial^n X(K,t)/\partial K^n$ as $K \to 1$, $n = 1,\ 2$

As a consequence of Theorem 1.1, the limit $\partial X(1,t)/\partial K \equiv \lim_{K\to 1} \partial X(K, t)/\partial K$ exists for each $t \in \mathbb{R}$ and for $X = \lambda,\ \nu$. For example, calculation shows that $\partial\nu(1,t)/\partial K = -\partial\lambda(1,t)/\partial K$ and $\partial\lambda(1,t)/\partial K = \Phi(\pi M(t)/2)$ for $t > 0$, where

$$\Phi(x) = x(d/dx)\{\mu^{-1}(x)\}^{-2} = x(d/dx)\Upsilon(x), \quad x > 0.$$

Consequently, it follows from de l'Hôpital's rule, together with $X(1,t) = t$ for $X = \lambda,\ \nu$, that

$$\lim_{K\to 1}(\lambda(K,t) - t)/(K-1) = \partial\lambda(1,t)/\partial K \equiv \mathscr{S}(t) \quad \text{and}$$
$$\lim_{K\to 1}(\nu(K,t) - t)/(K-1) = \partial\nu(1,t)/\partial K$$

for all $t \in \mathbb{R}$; in particular, $\mathscr{S}(t) = 0$ for $t = -1,\ 0$.

Theorem 7.1. *The following identities hold.*

$$\partial\nu(1,t)/\partial K = -\mathscr{S}(t) \qquad \textit{for all} \quad t \in \mathbb{R}. \tag{7.1}$$

$$\mathscr{S}(t^{-1}) = \mathscr{S}(t)t^{-2} \qquad \textit{for all} \quad t \in \mathbb{R}\setminus\{0\}. \tag{7.2}$$

$$\mathscr{S}(-1-t) = \mathscr{S}(t) \qquad \textit{for all} \quad t \in \mathbb{R}. \tag{7.3}$$

The following formulas follow at once from (7.2) and (7.3).

$$\mathscr{S}(-t^{-1}-1) = \mathscr{S}(t)t^{-2} \qquad \text{for all} \quad t \in \mathbb{R}\setminus\{0\}. \tag{7.4}$$

$$\mathscr{S}(-t/(1+t)) = \mathscr{S}(t)(1+t)^{-2} \qquad \text{for all} \quad t \in \mathbb{R}\setminus\{-1\}. \tag{7.5}$$

Proof of Theorem 7.1. We have already observed (7.1) for $t > 0$. For $t < -1$, it follows from (2) in Theorem 3.1 that $\partial\nu(1,t)/\partial K = -\mathscr{S}(-1-t)$ and $\partial\nu(1,-1-t)/\partial K = -\mathscr{S}(t)$. Since (7.1) is true for $-1-t > 0$ instead of t, we have (7.1) for $t < -1$. Suppose next that $-1 < t < 0$. It then follows from (5) in Theorem 3.1 that $\partial\nu(1,t)/\partial K$ and $\mathscr{S}(t)$ are equal to $-(1+t)^2\mathscr{S}(-t/(1+t))$ and $-(1+t)^2\partial\nu(1,-t/(1+t))/\partial K$, respectively. Since (7.1) is true for $-t/(1+t) > 0$, we have (7.1) for $-1 < t < 0$.

It follows from (1) in Theorem 3.1 and (7.1) that $\mathscr{S}(t) = -t^2\partial\nu(1, t^{-1})/\partial K = t^2\mathscr{S}(t^{-1})$ for $t \neq 0$; this is (7.2). Similarly we have (7.3) with the aid of (2) in Theorem 3.1 and (7.1). □

One can consider the "second derivative".

First of all, the limit $\partial^2 X(1,t)/\partial K^2 \equiv \lim_{K\to 1} \partial^2 X(K,t)/\partial K^2$ exists for $X = \lambda$, ν and for $t \in \mathbb{R}$. For example, calculation for $t > 0$ shows that $\partial^2\lambda(1,t)/\partial K^2 = \Psi(\pi M(t)/2)$, where

$$\begin{aligned}\Psi(x) &= x^2(d^2/dx^2)\{\mu^{-1}(x)\}^{-2}\\ &= x^2(d^2/dx^2)\Upsilon(x)\\ &= x\Phi'(x) - \Phi(x), \quad x > 0.\end{aligned}$$

Furthermore,

$$\partial^2\nu(1,t)/\partial K^2 = \Psi(\pi M(t)/2) + 2\Phi(\pi M(t)/2), \quad t > 0. \tag{7.6}$$

Returning to general $t \in \mathbb{R}$, we observe that

$$\lim_{K\to 1}(K-1)^{-1}(\partial\lambda(K,t)/\partial K - \mathscr{S}(t)) = (\partial^2/\partial K^2)\lambda(1,t) \equiv \mathscr{U}(t) \quad \text{and}$$
$$\lim_{K\to 1}(K-1)^{-1}(\partial\nu(K,t)/\partial K + \mathscr{S}(t)) = (\partial^2/\partial K^2)\nu(1,t);$$

in particular, $\mathscr{U}(-1) = \mathscr{U}(0) = 0$.

Theorem 7.2.

$$(\partial^2/\partial K^2)\nu(1,t) = -\mathscr{U}(-1-t) = \mathscr{U}(t) + 2\mathscr{S}(t) \qquad \textit{for all} \quad t \in \mathbb{R}. \tag{7.7}$$

$$\mathscr{U}(t^{-1}) = -\mathscr{U}(t)t^{-2} - 2\mathscr{S}(t)t^{-2} + 2\mathscr{S}(t)^2 t^{-3} \qquad \textit{for all} \quad t \in \mathbb{R}\setminus\{0\}. \tag{7.8}$$

The following formulas promptly follow from (7.2), (7.4), (7.7), and (7.8).

$$\mathscr{U}(-t^{-1} - 1) = \mathscr{U}(t)t^{-2} - 2\mathscr{S}(t)^2 t^{-3} \qquad \text{for all} \quad t \in \mathbb{R}\setminus\{0\}. \tag{7.9}$$

$$\mathscr{U}(-t/(1+t)) = -\mathscr{U}(t)(1+t)^{-2} - 2\mathscr{S}(t)(1+t)^{-2} + 2\mathscr{S}(t)^2(1+t)^{-3} \tag{7.10}$$

for all $t \in \mathbb{R}\setminus\{-1\}$. For example, by (3) in Theorem 3.1 (7.2), and (7.8) we obtain $\mathscr{U}(-t^{-1}-1) = -\mathscr{U}(t^{-1}) - 2\mathscr{S}(t^{-1})$, which, combined with (7.7), (7.8), and (7.2) shows (7.9).

Proof of Theorem 7.2. First of all, it follows from (2) in Theorem 3.1 that

$$\partial^2\nu(1,t)/\partial K^2 = -\mathscr{U}(-1-t) \qquad \text{and} \qquad \mathscr{U}(t) = -\partial^2\nu(1,-1-t)/\partial K^2$$

for all $t \in \mathbb{R}$. Hence, to establish (7.7) it remains to prove that

$$\partial^2\nu(1,t)/\partial K^2 = \mathscr{U}(t) + 2\mathscr{S}(t) \tag{7.11}$$

for all $t \in \mathbb{R}$. This is a direct consequence of (7.6) in case $t > 0$. Suppose that $t < -1$. We may then replace t with $-1 - t > 0$ in (7.11) to have

$$\begin{aligned}\mathscr{U}(t) &= -\partial^2\nu(1, -1-t)/\partial K^2\\ &= -\mathscr{U}(-1-t) - 2\mathscr{S}(-1-t)\\ &= \partial^2\nu(1,t)/\partial K^2 - 2\mathscr{S}(t);\end{aligned}$$

the last equality follows from (7.3). Hence we have (7.11) for $t < -1$. Supposing $-1 < t < 0$ we may replace t with $1/t < -1$ in (7.11) to have

$$\partial^2\nu(1,t^{-1})/\partial K^2 = \mathscr{U}(t^{-1}) + 2\mathscr{S}(t^{-1}) = \mathscr{U}(t^{-1}) + 2\mathscr{S}(t)t^{-2} \tag{7.12}$$

by (7.2). On the other hand, it follows from (1) in Theorem 3.1 that

$$\partial^2\nu(1,t^{-1})/\partial K^2 = -\mathscr{U}(t)t^{-2} + 2\mathscr{S}(t)^2t^{-3} \quad \text{and} \tag{7.13}$$

$$\partial^2\nu(1,t)/\partial K^2 = -t^2\mathscr{U}(t^{-1}) + 2t^3\mathscr{S}(t^{-1})^2 = -t^2\mathscr{U}(t^{-1}) + 2\mathscr{S}(t)^2t^{-1} \tag{7.14}$$

by (7.2). Eliminating $\partial^2\nu(1,1/t)/\partial K^2$ and $\mathscr{U}(1/t)$ from (7.12)–(7.14) one has (7.11) for $-1 < t < 0$.

Since (7.12) and (7.13) both are true for $t \neq 0$, we have (7.8). □

Remark 1. One can prove that

$$\begin{aligned}&\mathscr{S}'(t) = (M'(t)/M(t))(\mathscr{S}(t) + \mathscr{U}(t)) \qquad \text{for} \quad t > 0,\\ &\mathscr{S}'(t) = (M'(-1-t)/M(-1-t))(\mathscr{S}(t) + \mathscr{U}(t)) \qquad \text{for} \quad t < -1, \quad \text{and}\\ &\mathscr{S}'(t) = 2\mathscr{S}(t)\,t^{-1} - \{M'(-t^{-1}-1)/M(-t^{-1}-1)\}\{\mathscr{S}(t^{-1}) + \mathscr{U}(t^{-1})\}\end{aligned}$$

for $-1 < t < 0$. With the assistance of (7.2) and (7.8) one can express $\mathscr{S}(1/t) + \mathscr{U}(1/t)$ by t, $\mathscr{S}(t)$, and $\mathscr{U}(t)$ in the last formula for $-1 < t < 0$.

Remark 2. In the expression of $\Phi(x)$ and $\Psi(x)$ one needs the derivatives $(\mu^{-1})'(x) = 1/\mu'(r)$ and $(\mu^{-1})''(x) = -\mu''(r)/\mu'(r)^3$ for $x = \mu(r)$, $0 < r < 1$. Recall the *complete elliptic integral of the second kind* [WW, pp. 517–518], that is,

$$E(r) = \int_0^1 \sqrt{(1-r^2x^2)/(1-x^2)}dx, \quad 0 < r < 1.$$

Then

$$\mu'(r) = -\frac{\pi^2}{4}\cdot\frac{1}{r(1-r^2)\mathscr{K}(r)^2}, \quad 0 < r < 1,$$

and

$$\mu''(r) = -\frac{\pi^2}{4}\cdot\frac{(1+r^2)\mathscr{K}(r) - 2E(r)}{r^2(1-r^2)^2\mathscr{K}(r)^3}, \quad 0 < r < 1;$$

see [AVV3, p. 82, (5.9)] and [BB, p. 137, (4.6.3*a*)] for $\mu'(r)$. We thus have

$$\Phi(x) = (4/\pi)r^{-2}(1-r^2)\mathscr{K}(\sqrt{1-r^2})\mathscr{K}(r) \quad \text{and}$$
$$\Psi(x) = (16/\pi^2)r^{-2}(1-r^2)\mathscr{K}(\sqrt{1-r^2})^2\mathscr{K}(r)[(2-r^2)\mathscr{K}(r)-E(r)]$$

for $x = \mu(r)$, $0 < r < 1$.

A remarkable result among others in [AVV2] is that $\mu(1/s)$ is a concave function of $s > 1$ in the sense that $d\mu(1/s)/ds$ is a decreasing function of $s > 1$; see [AVV2, p. 545, Theorem 4.5], whereas $\mu(r)$ for $0 < r < 1$ is neither convex nor concave.

8. Generalizations

Hitherto our study depends on the fundamental fact that $-1,\ 0,\ \infty$ are on the great circle $\mathbb{R} \cup \{\infty\}$ on $\overline{\mathbb{C}}$. Hence it is natural to consider the following. Let $C(a,b,c)$ be the circle, and not necessarily a great circle, on $\overline{\mathbb{C}}$ passing through three distinct points $a,\ b \in \mathbb{C}$ and $c \in \overline{\mathbb{C}}$, and let $\mathscr{F}(K,a,b,c)$ be the family of all the K-quasiconformal mappings from $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$ such that $f(\zeta) = \zeta$ for $\zeta = a,\ b,\ c$, and moreover, $f(C(a,b,c)) = C(a,b,c)$. Set $V(\zeta) = V_{K,a,b,c}(\zeta) = \{f(\zeta) :\ f \in \mathscr{F}(K,a,b,c)\}$ for $\zeta \in C(a,b,c)$, so that $V(\zeta) = \{\zeta\}$ for $\zeta = a,\ b$, and c. Define a Möbius transformation $T \equiv T_{a,b,c}$ by

$$T_{a,b,c}(z) = \frac{c(a-b)z + b(a-c)}{(a-b)z + a - c}, \quad z \in \overline{\mathbb{C}},$$

if $c \neq \infty$, and $T_{a,b,c}(z) = (b-a)z + b$ if $c = \infty$, so that $T(-1) = a$, $T(0) = b$, and $T(\infty) = c$. Then $V(\zeta)$ for $\zeta \in C(a,b,c) \setminus \{a,b,c\}$ is a closed subarc of $C(a,b,c)$ with $V(\zeta) = T([\nu(K,t),\lambda(K,t)])$, where $T(t) = \zeta$. Actually, $f \longmapsto T^{-1} \circ f \circ T$ is a one-to-one mapping from $\mathscr{F}(K,a,b,c)$ onto $\mathscr{F}(K)$.

As a specified case we fix $\eta \in \mathbb{C} \setminus \{0\}$, and set $a = 0,\ b = \eta^* = -1/\overline{\eta} \in \mathbb{C} \setminus \{0\}$, the antipodal point of η, and $c = \infty$. Then $T(z) = \eta^*(z+1)$ and $T(-\zeta\overline{\eta} - 1) = \zeta$ for $\zeta \in C(a,b,c) \setminus \{a,b,c\}$, so that

$$V(\zeta) = T([\nu(K,-\zeta\overline{\eta}-1),\lambda(K,-\zeta\overline{\eta}-1)]) = \{s/\overline{\eta} :\ s \in [\nu(K,\zeta\overline{\eta}),\lambda(K,\zeta\overline{\eta})]\}$$

by (2) in Theorem 3.1.

Under the additional restriction that $\eta = u$ is a nonzero real number in the preceding paragraph, we have $V(s) = \{t/u :\ t \in [\nu(K,su),\lambda(K,su)]\}$ for $s \in \mathbb{R} \setminus \{0,-1/u\}$.

Another generalization of $\mathscr{F}(K)$ is the family

$$\mathscr{F}(K,u) = \{f \in \mathscr{E}(K) :\ f(\zeta) = \zeta,\ \zeta = 0,\ -u\ \}$$

defined for $u \in \mathbb{R} \setminus \{0\}$. Then $\mathscr{F}(K) = \mathscr{F}(K,1) = \mathscr{F}(K,0,-1,\infty)$. Define

$$\Omega(f)(z) = f(uz)/u, \quad z \in \overline{\mathbb{C}},$$

for $f \in \mathscr{F}(K,u)$ to observe that Ω is a one-to-one mapping from $\mathscr{F}(K,u)$ onto $\mathscr{F}(K)$, so that

$$\lambda_u(K,t) = \max_{f\in\mathscr{F}(K,u)} f(t) \qquad \text{and} \qquad \nu_u(K,t) = \min_{f\in\mathscr{F}(K,u)} f(t)$$

both exist for $t \in \mathbb{R}$. Exactly,

$$X_u(K,t) = uX(K,t/u) \qquad \text{for} \quad X = \lambda,\ \nu$$

if $u > 0$, whereas

$$X_u(K,t) = uY(K,t/u) \qquad \text{for} \quad (X,Y) = (\lambda,\nu) \text{ or } (\nu,\lambda)$$

if $u < 0$.

We can extend Theorem 3.1 from $X(t) = X(K,t)$ to $X_u(t) = X_u(K,t)$. More precisely, the following hold. Here $X = \lambda,\ \nu$ and $(X,Y) = (\lambda,\nu)$ or (ν,λ) as usual.

$$(1_u) \qquad X_u(K,t)Y_u(K,u^2/t) = u^2 \qquad \text{for} \quad t \neq 0;$$

$$(1_+) \qquad X_u(K,t)Y_t(K,u) = ut \qquad \text{for} \quad tu > 0;$$

$$(1_-) \qquad X_u(K,t)X_t(K,u) = ut \qquad \text{for} \quad tu < 0;$$

$$(2_u) \quad X_u(K,t) + Y_u(K,-u-t) = -u \qquad \text{for all} \quad t;$$

$$(3_u) \qquad X_u(K,t) = \frac{-u^2}{X_u(K,-u(u+t)/t)+u} \qquad \text{for} \quad t \neq 0;$$

$$(3_+) \qquad X_u(K,t) = \frac{-ut}{X_t(K,-u-t)+t} \qquad \text{for} \quad tu > 0;$$

$$(3_-) \qquad X_u(K,t) = \frac{-ut}{Y_t(K,-u-t)+t} \qquad \text{for} \quad tu < 0;$$

$$(4_u) \qquad X_u(K,t) = \frac{-u^2}{X_u(K,-u^2/(u+t))} - u \qquad \text{for} \quad t \neq -u;$$

$$(4_+) \qquad X_u(K,t) = \frac{-ut}{X_t(K,-ut/(u+t))} - u \qquad \text{for} \quad tu > 0;$$

$$(4_-) \qquad X_u(K,t) = \frac{-ut}{Y_t(K,-ut/(u+t))} - u \qquad \text{for} \quad tu < 0;$$

$$(5_u) \qquad X_u(K,t) = \frac{-uY_u(K,-ut/(u+t))}{Y_u(K,-ut/(u+t))+u} \qquad \text{for} \quad t \neq -u;$$

$$(5_+) \qquad X_u(K,t) = \frac{-uY_t(K,-t^2/(u+t))}{Y_t(K,-t^2/(u+t))+t} \qquad \text{for} \quad tu > 0;$$

$$(5_-) \qquad X_u(K,t) = \frac{-uX_t(K,-t^2/(u+t))}{X_t(K,-t^2/(u+t))+t} \qquad \text{for} \quad tu < 0.$$

The proofs are of one pattern. It follows from (1) in Theorem 3.1 that

$$uX(t/u)uY(S_1(t/u)) = u^2,$$

which, combined with $uS_1(t/u) = u^2/t$, shows (1_u). Also,

$$uX(t/u)tY(S_1(t/u)) = ut,$$

together with $tS_1(t/u) = u$, shows (1_+) and (1_-).

The remaining cases are proved by the following deductions.

$$(1) \implies uX(t/u) + uY(S_2(t/u)) = -u \quad \text{and} \quad uS_2(t/u) = -u - t \implies (2_u);$$

$$(3) \implies X(t/u) = \frac{-1}{X(S_3(t/u)) + 1}$$

$$\implies \begin{cases} uX(t/u) = \dfrac{-u^2}{uX(S_3(t/u)) + u} \implies (3_u), \\ uX(t/u) = \dfrac{-ut}{tX(S_3(t/u)) + t} \implies \begin{cases} (3_+), \\ (3_-); \end{cases} \end{cases}$$

$$(4) \implies X(t/u) = \frac{-1}{X(S_4(t/u))} - 1$$

$$\implies \begin{cases} uX(t/u) = \dfrac{-u^2}{uX(S_4(t/u))} - u \implies (4_u), \\ uX(t/u) = \dfrac{-ut}{tX(S_4(t/u))} - u \implies \begin{cases} (4_+), \\ (4_-); \end{cases} \end{cases}$$

$$(5) \implies X(t/u) = \frac{-Y(S_5(t/u))}{Y(S_5(t/u)) + 1}$$

$$\implies \begin{cases} uX(t/u) = \dfrac{-u^2 Y(S_5(t/u))}{uY(S_5(t/u)) + u} \implies (5_u), \\ uX(t/u) = \dfrac{-utY(S_5(t/u))}{tY(S_5(t/u)) + t} \implies \begin{cases} (5_+), \\ (5_-). \end{cases} \end{cases}$$

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