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# Direct limit Lie groups and manifolds

By

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**Abstract**

We show that every countable strict directed system of finite-dimensional Lie groups has a direct limit in the category of smooth Lie groups modelled on sequentially complete, locally convex spaces. Similar results are obtained for countable directed systems of finite-dimensional manifolds, and for countable directed systems of finite-dimensional Lie groups and manifolds over totally disconnected local fields. An *uncountable* strict directed system of finite-dimensional Lie groups has a direct limit in the category of Lie groups in the sense of convenient differential calculus, provided certain technical hypotheses are satisfied.

## 1. Introduction

Let $M_1 \subseteq M_2 \subseteq \cdots$ be an ascending sequence of finite-dimensional topological manifolds, where $M_n$ is a closed submanifold of $M_{n+1}$ for all $n$, and $\dim M_n \to \infty$ as $n \to \infty$. Then the direct limit topological space $M := \varinjlim M_n = \bigcup_{n \in \mathbb{N}} M_n$ is a topological manifold modelled on $\mathbb{R}^\infty$, the real vector space of finite sequences, equipped with the finite topology (Hansen [12], 1971). Our main result is an analogue of this classical fact in the setting of smooth manifolds: if each $M_n$ is a smooth manifold and $M_n$ a closed $C^\infty$-submanifold of $M_{n+1}$ for all $n$, then $M$ can be given a smooth manifold structure modelled on $\mathbb{R}^\infty$ making it the direct limit of the sequence $(M_n)_{n\in\mathbb{N}}$ in the category of smooth manifolds (Theorem 4.3). The charts for the direct limit manifolds are limit maps of certain compatible families of charts of the finite-dimensional manifolds; to obtain these compatible families, we start with a suitable chart of $M_1$ and inductively use tubular neighbourhoods to extend the chart already constructed for $M_n$, restricted to a slightly smaller open set, to a chart of $M_{n+1}$. The finite-dimensional manifolds $M_n$ considered here need not be second countable, but we have to assume that each $M_n$ is paracompact.

In the special case where $M_n = G_n$ is a finite-dimensional Lie group, our construction allows us to turn the direct limit topological group $G := \varinjlim G_n$ into a smooth Lie group modelled on $\mathbb{R}^\infty$, which is the direct limit of the

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sequence $(G_n)_{n\in\mathbb{N}}$ in the category of smooth Lie groups. In contrast to earlier constructions of direct limits of Lie groups, we do not use the direct limit exponential function

$$\exp_G := \varinjlim \exp_{G_n} \colon \varinjlim \mathrm{L}(G_n) \to \varinjlim G_n$$

to define charts for $G$ (an approach followed by Natarajan et al. [27], 1991, [28], 1993, [29], 1994; Kriegl and Michor [22], 1997). Our method allows us to equip the direct limit topological group $G$ with a Lie group structure even if $\exp_G$ does not induce a local homeomorphism at 0 (as in Example 5.5): this was not possible before.*1 Direct limits of ascending sequences of manifolds or Lie groups over totally disconnected local fields can be constructed along similar lines (Section 8).

Now suppose that $((G_i)_{i\in I}, (\phi_{ij})_{i\geq j})$ is an uncountable directed system of finite-dimensional real Lie groups. Under certain technical assumptions (cf. Definition 6.2, Remark 6.3), it was shown by Natarajan et al. that the direct limit exponential map $\exp_G := \varinjlim \exp_{G_i} \colon \varinjlim \mathrm{L}(G_i) \to \varinjlim G_i =: G$ induces a local homeomorphism at 0, which can be used to define charts for $G$, whose transition maps are analytic on each finite-dimensional subspace ([27, Section 8]). Here, the direct limit group $G$ and direct limit Lie algebra $\mathfrak{g} := \varinjlim \mathrm{L}(G_i)$ are equipped with the respective topology of direct limit topological space. Examples show that $G$ need not be a topological group, and $\mathfrak{g}$ has discontinuous addition and Lie bracket in general (Theorem 7.1); it is therefore not obvious *a priori* in which sense $G$ can be considered as a Lie group. The authors of [27] were unaware of these problems, and gave incorrect proofs to the contrary in [28], Appendix (see [30], Appendix for corrections; the main problem has also been pointed out in Edamatsu [6]). We prove that $G$ is a Lie group in the sense of 'convenient differential calculus,' as defined in [21], [22] (a *convenient Lie group* for short). We show that the charts specified by Natarajan et al. make $G$ the direct limit convenient Lie group of the directed system $((G_i), (\phi_{ij}))$, if the direct limit Lie algebra $\mathfrak{g}$ is equipped with the *finest locally convex topology* instead of the direct limit topology (Theorem 6.4).*2 Another definition of Lie groups with separately analytic multiplication, modelled on topological Lie algebras, is proposed in ([30, Definition A.8]). However, this definition does not always apply in the situation we are interested in: neither the direct limit topology nor the finest locally convex topology make $\mathfrak{g}$ a topological Lie algebra in general (Theorem 7.1 (b)). We remark that the direct limit convenient Lie groups for certain *countable* strict directed systems of classical groups are already discussed in [22, Section 47], where it is shown that every Lie subalgebra of $\mathrm{gl}(\mathbb{N},\mathbb{R}) = \mathbb{R}^{(\mathbb{N}\times\mathbb{N})}$ is the Lie algebra of some smoothly arcwise connected Lie subgroup of $\mathrm{GL}(\mathbb{N},\mathbb{R}) \subseteq \mathbb{R}^{(\mathbb{N}\times\mathbb{N})} + \mathbf{1}$ (*loc. cit.* Theorem 47.9).

*1 Whenever the method of Natarajan et al. applies, the Lie group we construct is the smooth Lie group underlying the analytic Lie group provided by that method.

*2 It was already proposed to consider the finest locally convex topology on $\mathfrak{g}$ in [28], but our approach differs essentially since we do *not* transport the finest locally convex topology on $\mathfrak{g}$ to the group $G$, but only use it to make $\mathfrak{g}$ a convenient vector space on which the manifold is modelled in the sense of convenient differential calculus. Here, the $c^\infty$-refinement of the finest locally convex topology on $\mathfrak{g}$ is the finite topology (Lemma 6.1).

Our abstract results are illustrated by a discussion of the infinite matrix groups $\mathrm{GL}(I,\mathbb{R}) \subseteq \mathbb{R}^{(I\times I)} + \mathbf{1}$ and their Lie algebras (Section 7); all of the described pathologies occur even for these most natural examples of direct limit Lie groups.

For more information concerning direct limits of topological groups, the reader is referred to Tatsuuma et al. [34], 1998; discussions of specific examples of direct limits of Lie groups, considered as topological groups, can be found in Kolomytsev and Samoilenko [20], 1977, Ol'shanskiĭ [32], 1990, and Yamasaki [36], 1998. Information concerning universal complexifications of direct limit Lie groups can be found in [30] and [9].

## 2. Preliminaries and Notation

Let $(I,\leq)$ be a directed set and $\mathbb{A}$ a category. Recall that a *directed system* is a pair $\mathcal{S} := ((X_i)_{i\in I}, (\phi_{ji})_{j\geq i})$, where $X_i \in \mathrm{ob}\mathbb{A}$ and $\phi_{ji} \in \mathrm{Mor}(X_i, X_j)$ such that $\phi_{ii} = \mathrm{id}_{X_i}$ and $\phi_{kj}\circ\phi_{ji} = \phi_{ki}$, for all elements $k \geq j \geq i$ of $I$. A *cone over* $\mathcal{S}$ is a pair $(X, (\phi_i)_{i\in I})$, where $X \in \mathrm{ob}\mathbb{A}$ and $\phi_i\colon X_i \to X$ such that $\phi_j \circ \phi_{ji} = \phi_i$ whenever $j \geq i$. A cone $(X, (\phi_i)_{i\in I})$ is a *direct limit of* $\mathcal{S}$ (and we write $X = \varinjlim \mathcal{S}$ or $X = \varinjlim X_i$), if for every cone $(Y, (\psi_i)_{i\in I})$ over $\mathcal{S}$, there exists a unique morphism $\psi\colon X \to Y$ such that $\psi\circ\phi_i = \psi_i$ for all $i \in I$. If $\mathcal{T} = ((Y_i)_{i\in I}, (\psi_{ji})_{j\geq i})$ is another directed system over the same index set, $(Y, (\psi_i)_{i\in I})$ a cone over $\mathcal{T}$, and $(\eta_i)_{i\in I}$ a family of morphisms $\eta_i\colon X_i \to Y_i$ which is *compatible* in the sense that $\psi_{ji}\circ\eta_i = \eta_j\circ\phi_{ji}$ for all $j \geq i$, then $(Y, (\psi_i\circ\eta_i)_{i\in I})$ is a cone over $\mathcal{S}$. We write $\varinjlim \eta_i$ for the induced morphism $\psi\colon X \to Y$, determined by $\psi\circ\phi_i = \psi_i\circ\eta_i$. The directed systems $\mathcal{S}$ and $\mathcal{T}$ are called *equivalent* if there exists a compatible family $(\eta_i)_{i\in I}$ such that all morphisms $\eta_i$ are isomorphisms.

The existence of direct limits in many algebraic or topological categories can be proved by standard category-theoretical arguments. For the following, however, it is important that there are explicit realizations of the direct limits in the categories $\mathbb{SET}$ (sets and maps), $\mathbb{TOP}$ (not necessarily Hausdorff topological spaces, and continuous maps), $\mathbb{G}$ (groups and homomorphisms), and in the categories of vector spaces, Lie algebras, and semitopological groups (i.e., groups equipped with a topology which makes inversion continuous and the group multiplication separately continuous; morphisms are continuous group homomorphisms), cf. [24, Chapter IX.1], and [27]:

Suppose that $\mathcal{S} = ((X_i)_{i\in I}, (\phi_{ji})_{j\geq i})$ is a directed system of sets. Let $\Omega := \coprod_{i\in I} X_i \subseteq I \times \bigcup_{i\in I} X_i$ be the disjoint union of the sets $X_i$, together with the canonical inclusions $\lambda_i\colon X_i \to \Omega, x \mapsto (i,x)$. We define an equivalence relation on $\Omega$ via $\lambda_i(x) \sim \lambda_j(y)$ if there exists $k \geq i,j$ such that $\phi_{ki}(x) = \phi_{kj}(y)$. Set $X := \Omega/\sim$ and $\phi_i := q\circ\lambda_i$, where $q\colon \Omega \to \Omega/\sim$ is the canonical quotient map. Then $(X, (\phi_i))$ is easily seen to be the direct limit of $\mathcal{S}$ in $\mathbb{SET}$. Note that $X$ is the directed union of the sets $\mathrm{im}\phi_i$. If $((Y_i)_{i\in I}, (\psi_{ji})_{j\geq i})$ is another directed system in $\mathbb{SET}$ with the same index set, with direct limit $(Y, (\psi_i)_{i\in I})$, then clearly $(X\times Y, (\phi_i\times\psi_i)_{i\in I})$ is the direct limit of $((X_i\times Y_i)_{i\in I}, (\phi_{ji}\times\psi_{ji})_{j\geq i})$.

If $\mathcal{S} = ((X_i), (\phi_{ji}))$ is a directed system in $\mathbb{TOP}$, the direct limit $(X, (\phi_i))$

of $\mathcal{S}$ in $\mathbb{SET}$ becomes the direct limit in $\mathbb{TOP}$ if we give $X$ the final topology with respect to the family $(\phi_i)_{i\in I}$. Thus, by definition, a subset $U \subseteq X$ is open (resp., closed) if and only if $\phi_i^{-1}(U)$ is open (resp., closed) in $X_i$, for all $i \in I$. The directed system is called *strict* if all maps $\phi_{ji}$ are topological embeddings; then all maps $\phi_i$ are embeddings, see [28, Lemma A.5].

If $\mathcal{S} = ((G_i), (\phi_{ji}))$ is a directed system of groups, let $(G, (\phi_i))$ be its direct limit in $\mathbb{SET}$. There is a unique group structure on $G$ which makes all maps $\phi_i$ homomorphisms; the multiplication is $\mu = \varinjlim \mu_i$, the inversion is $\kappa = \varinjlim \kappa_i$, where $\mu_i$ and $\kappa_i$ denote multiplication and inversion on $G_i$, respectively. Direct limits of vector spaces or Lie algebras can be treated similarly.

If $\mathcal{S} = ((G_i), (\phi_{ji}))$ is a directed system of semitopological groups, the direct limit $(G, (\phi_i))$ in $\mathbb{SET}$ becomes the direct limit of $\mathcal{S}$ in the category of semitopological groups if we equip it with the topology and group structure which make it the direct limit of $\mathcal{S}$ in $\mathbb{TOP}$ and $\mathbb{G}$, respectively. Following [28], if all semitopological groups involved are topological Hausdorff groups, we call the direct limit $G$ of $\mathcal{S}$ in the category of semitopological groups the *naïve direct limit* of $\mathcal{S}$; it need not be Hausdorff, nor a topological group. Naïve direct limits of topological vector spaces and topological Lie algebras are defined similarly, equipping the algebraic direct limit with the final topology.

## 3. Direct limits of topological spaces

In this section, we assemble some basic facts concerning direct limits of topological spaces for later use.

Let $\mathcal{S} = ((X_i)_{i\in I}, (\phi_{ji})_{j\geq i})$ be a strict directed system of topological spaces, with direct limit $(X, (\phi_i)_{i\in I})$. Then every map $\phi_i$ is a topological embedding by [28, Lemma A.5], whence $\mathcal{S}$ is equivalent to the directed system $\mathcal{S}' := ((Y_i)_{i\in I}, (\psi_{ji})_{j\geq i})$, where $Y_i := \mathrm{im}\phi_i$ and $\psi_{ji}\colon Y_i \hookrightarrow Y_j$ denotes inclusion; furthermore, $(X, (\psi_i)_{i\in I})$ is the direct limit of $\mathcal{S}'$, where $\psi_i\colon Y_i \hookrightarrow X$. Hence the investigation of strict directed systems of topological spaces can be reduced to the case that each $X_i$ is a subspace of the direct limit $X$, all maps $\phi_{ji}$ and $\phi_i$ being the respective inclusion maps. Then, a subset $U$ of $X$ is open if and only if $U \cap X_i$ is open in $X_i$ for all $i$, and a map $f\colon X \to Y$ into a topological space $Y$ is continuous if and only if all restrictions $f|_{X_i}$ are so. If $U$ is an open subset of $X$, a subset $V$ of $U$ is open in $U$ if and only if all intersections with the subspaces $X_i \cap U$ are open in $X_i \cap U$: hence $U$ is the direct limit of the subspaces $X_i \cap U$. We shall need a slight generalization of this simple observation:

**Lemma 3.1.** *Let $((X_i)_{i\in I}, (\phi_{ji})_{j\geq i})$ be a strict directed system of topological spaces and $U_i$ an open subset of $X_i$ for $i \in I$, where $\phi_{ji}(U_i) \subseteq U_j$ for all $i \leq j$. Then the maps $\psi_{ji} := \phi_{ji}|_{U_i}^{U_j}$ define a directed system $((U_i), (\psi_{ji}))$. If $(X, (\phi_i))$ and $(U, (\psi_i))$ denote the direct limits of the respective systems, the map $\lambda := \varinjlim \lambda_i\colon U \to X$ induced by the family of inclusions $\lambda_i\colon U_i \hookrightarrow X_i$ is a topological embedding onto an open subset of $X$.*

*Proof.* As $U = \bigcup_i \mathrm{im}\psi_i$ and $\lambda \circ \psi_i = \phi_i \circ \lambda_i$ is injective for all $i \in I$, we conclude that $\lambda$ is injective. $\lambda$ being continuous, it only remains to check that $\lambda$ is an open map. To this end, let $V$ be an open subset of $U$. Then, for every $i \in I$, we have $\phi_i^{-1}(\lambda(V)) = \bigcup_{j\geq i} \phi_i^{-1}(\lambda(\psi_j(\psi_j^{-1}(V))))$. Since $\lambda \circ \psi_j = \phi_j \circ \lambda_j$, we have $\lambda(\psi_j(\psi_j^{-1}(V))) = \phi_j(\psi_j^{-1}(V))$ for $j \geq i$. Furthermore, $W_{ij} := \phi_i^{-1}(\phi_j(\psi_j^{-1}(V))) = \phi_{ji}^{-1}(\psi_j^{-1}(V))$. Now $\psi_j^{-1}(V)$ is open in $U_j$, hence in $X_j$, and by continuity of $\phi_{ji}$, the subset $W_{ij}$ of $X_i$ is open. Hence so is $\phi_i^{-1}(\lambda(V)) = \bigcup_{j\geq i} W_{ij}$. □

Note that category-theoretical direct limits are unaffected by passage to cofinal subsystems of the directed system. If the directed set $I$ is countable, we easily construct a cofinal sequence $i_1 \leq i_2 \leq i_3 \leq \cdots$ and can therefore assume that $I = (\mathbb{N}, \leq)$ whenever this is convenient.

**Lemma 3.2.** *Let $((X_i)_{i\in I}, (\phi_{ji})_{j\geq i})$ and $((Y_i)_{i\in I}, (\psi_{ji})_{j\geq i})$ be strict directed systems of topological spaces, with direct limits $(X, (\phi_i))$ and $(Y, (\psi_i))$, respectively. Let $(P, (\pi_i))$ be the direct limit of $\mathcal{S} = ((X_i \times Y_i)_{i\in I}, (\phi_{ji} \times \psi_{ji})_{j\geq i})$. Then $(X \times Y, (\phi_i \times \psi_i)_{i\in I})$ is a cone over $\mathcal{S}$, and the induced map $\eta\colon P \to X \times Y$, determined by $\eta \circ \pi_i = \phi_i \times \psi_i$, is a continuous bijection.*

*Proof.* [5, Appendix 2, (1.9)(3)]. □

By Lemma 3.2, we can always identify $\varinjlim X_i \times Y_i$ with $\varinjlim X_i \times \varinjlim Y_i$, up to a possible refinement of the topology. Under suitable hypotheses, also the topologies will coincide:

**Proposition 3.3.** *If, in the situation of Lemma 3.2, the set $I$ is countable and all spaces $X_i$ and $Y_i$ are locally compact Hausdorff, then $\eta$ is a homeomorphism.*

*Proof.* We may assume without loss of generality that $I = (\mathbb{N}, \leq)$ and $X_1 \subseteq X_2 \subseteq \cdots \subseteq X$ and $Y_1 \subseteq Y_2 \subseteq \cdots \subseteq Y$, all maps $\phi_{ji}$, $\phi_i$, $\psi_{ji}$, and $\psi_i$ being the respective inclusion maps. Let $P = \varinjlim X_i \times Y_i$; as a set, we can identify $P$ with $X \times Y$ by the preceding. Then also the maps $\pi_i$ are the respective inclusion maps. Let $(x, y) \in P$ and suppose that $W$ is an open neighbourhood of $(x, y)$ in $P$. We show that $W$ is a neighbourhood of $(x, y)$ in $X \times Y$ as well. Passing to a cofinal subsystem, we may assume without loss of generality that $(x, y) \in X_1 \times Y_1$. For $i \in \mathbb{N}$, set $W_i := W \cap (X_i \times Y_i)$; then every $W_i$ is an open subset of $X_i \times Y_i$. Since $W_1$ is an open neighbourhood of $(x, y)$ in $X_1 \times Y_1$, there exist compact neighbourhoods $C_1$, $D_1$ of $x$ and $y$ in $X_1$ and $Y_1$, respectively, such that $C_1 \times D_1 \subseteq W_1$. Now $W_2$ is an open neighbourhood of $C_1 \times D_1$ in $X_2 \times Y_2$; therefore there exist compact subsets $C_2$ and $D_2$ of $X_2$ and $Y_2$, respectively, such that $C_2 \times D_2$ is a neighbourhood of $C_1 \times D_1$ in $X_2 \times Y_2$, and $C_2 \times D_2 \subseteq W_2$. Inductively, we find sequences of compact subsets $C_i$ and $D_i$ of $X_i$ and $Y_i$, respectively, such that $C_1 \times D_1$ is a neighbourhood of $(x, y)$ in $X_1 \times Y_1$, $C_i \times D_i \subseteq W_i$, and such that $C_{i+1} \times D_{i+1}$ is a neighbourhood of $C_i \times D_i$ in $X_{i+1} \times Y_{i+1}$, for all $i \in \mathbb{N}$. For $i \in \mathbb{N}$, let $U_i$ and $V_i$ denote the interior of $C_i$ and $D_i$ relative $X_i$ and $Y_i$, respectively. Set $U := \bigcup_{i\in\mathbb{N}} U_i$, $V := \bigcup_{i\in\mathbb{N}} V_i$. Since $U_1 \subseteq U_2 \subseteq \cdots$, Lemma 3.1 shows that $U$ is open in $X$; similarly, $V$ is open in $Y$. Now $U \times V \subseteq W$ is an open neighbourhood of $(x, y)$ in $X \times Y$. □

Proposition 3.3 has been found independently by Hirai et al. [14] and the author (as witnessed by [8]).

The following corollary is essential for the study of direct limit Lie groups, since it allows us to form limits of continuous maps other than homomorphisms.

**Corollary 3.4.** *Let* $\mathcal{S} = ((G_i)_{i\in I}, (\phi_{ji})_{j\geq i})$ *be a countable, strict directed system of locally compact Hausdorff groups* $G_i$*, with naïve direct limit* $(G, (\phi_i)_{i\in I})$*. Then* $G$ *is a topological Hausdorff group, and hence* $G$ *is the direct limit of* $\mathcal{S}$ *in* $\mathbb{TG}$*.*

*Proof.* (cf. [28, Corollary A.11 (a)]). For $i \in I$, let $\mu_i \colon G_i \times G_i \to G_i$ denote the respective multiplication map. Then $(\mu_i)_{i\in I}$ is a family of continuous maps compatible with the directed systems $\mathcal{T} = ((G_i \times G_i), (\phi_{ji} \times \phi_{ji}))$ and $\mathcal{S}$. By Proposition 3.3, $(G \times G, (\phi_i \times \phi_i))$ is the direct limit of $\mathcal{T}$ in the category of topological spaces. Multiplication on $G$ is the limit map $\varinjlim \mu_i$, and hence is continuous. By Proposition 3.6 below or [28, Corollary A.12], $G$ is Hausdorff. □

For an alternative proof of Corollary 3.4, we refer to [34, Theorem 2.7].

The hypotheses of local compactness of the groups and countability of the directed system in Proposition 3.3 are essential:

**Example 3.5.** Let $V$ be a real vector space, $I$ its set of finite-dimensional subspaces, with inclusion as the ordering. For $i \in I$, set $V_i := i$, and, for $j \geq i$, let $\phi_{ji}$ denote the inclusion map $V_i \hookrightarrow V_j$. We obtain a strict directed system of finite-dimensional vector spaces (hence of Lie groups), and $V$ is its naïve direct limit if we equip it with the final topology with respect to the inclusion maps $\phi_i \colon V_i \hookrightarrow V$. This topology is called the *finite topology* on $V$, or also the *topology of finitely open sets* [13]; by definition, a subset $U$ of $V$ is open in the finite topology if and only if all of its intersections with finite-dimensional vector subspaces of $V$ are open in these.

In addition to the finite topology on the real vector space $V$, certain particular vector space topologies will be relevant later on. There exists a finest locally convex (vector space) topology on $V$; the set of all balanced, absorbing, convex subsets of $V$ is a basis of 0-neighbourhoods for this topology (see, e.g., [16, Proposition 7.25, Definition 7.27]). There is also a finest vector space topology on $V$; to see its existence, form the product $P := \prod_{\tau\in\mathcal{T}}(V, \tau)$, where $\tau$ ranges through the set $\mathcal{T}$ of all vector space topologies on $V$, and give $V$ the topology making the diagonal map $V \to P$, $v \mapsto (v)_{\tau\in\mathcal{T}}$ a topological embedding. Clearly, we obtain a vector space topology on $V$ which is finer than any other vector space topology on $V$.

If $\dim V \leq \aleph_0$, then the finite topology on $V$, the finest locally convex topology, and the finest vector space topology coincide. If $\dim V > \aleph_0$, the finest vector space topology is properly finer than the finest locally convex topology ([16, Proposition A4.21]). Furthermore, in this case, the finite topology on $V$ is *not* a group topology, the addition map is not jointly continuous, see [17], [2]. Thus the naïve direct limit $V$ of the uncountable strict directed system of locally compact groups $V_i$ fails to be a topological group here, and we deduce that the mapping $\eta\colon \varinjlim(V_i \times V_i) \to \varinjlim V_i \times \varinjlim V_i$ defined in Lemma 3.2 is not a homeomorphism.

For an example of a countable strict directed system of non-locally compact topological groups whose naïve direct limit is not a topological group, see [34, Example 1.2]. For later use, we recall from [12, Lemma 2.4 and Proposition 4.1]:

**Proposition 3.6.** *Let $X$ be a topological space which is the direct limit of an ascending sequence $X_1 \subseteq X_2 \subseteq \cdots$ of topological subspaces. Then the following holds*:

(a) *If $X_n$ is locally compact for all $n \in \mathbb{N}$, then $X$ is Hausdorff.*

(b) *If $X_n$ is $T_1$ for all $n \in \mathbb{N}$, then every compact subset of $X$ is contained in some of the subspaces $X_n$.* □

## 4. Countable direct limits of manifolds

In this section, we construct the direct limit smooth manifolds of suitable countable directed systems of finite-dimensional smooth manifolds. The direct limit manifolds will be either finite-dimensional or modelled on $\mathbb{R}^\infty := \mathbb{R}^{(\mathbb{N})}$, equipped with the finite topology.

There are many different concepts of differentiability and differentiable manifolds in infinite dimensions (and indeed we shall use two different ones). In this section and the next, we consider infinite-dimensional manifolds and Lie groups in the sense of Milnor [25], modelled on sequentially complete, locally convex Hausdorff (s.c.l.c.) topological vector spaces, based on the concept of smooth mappings in the Michal-Bastiani sense (also known as Keller's $C^\infty_c$-maps [19]). In Section 6, we consider manifolds and Lie groups in the sense of convenient differential calculus.

Let $X$ and $Y$ be s.c.l.c. topological vector spaces, $U$ be an open subset of $X$, and $f\colon U \to Y$ be a continuous map. Given $x \in U$ and $h \in X$, the *derivative of $f$ at $x$ in the direction $h$* is defined as $df(x)(h) := \lim_{t\to 0} t^{-1}(f(x+th) - f(x))$, whenever the limit exists. We say that $f$ *is differentiable at $x$* if $df(x)(h)$ exists for all $h \in X$; it is $\mathcal{C}^1$ if it is differentiable at all $x$ in $U$ and $df\colon U \times X \to Y$, $(x, h) \mapsto df(x)(h)$ is continuous. Higher derivatives are defined recursively by means of the familiar formula $d^n f(x)(h_1, \ldots, h_n) := \lim_{t\to 0} t^{-1}(d^{n-1}f(x + th_n)(h_1, \ldots, h_{n-1}) - d^{n-1}f(x)(h_1, \ldots, h_{n-1}))$, provided that all limits involved exist. The function $f$ is said to be *of class $\mathcal{C}^n$* if $d^n f\colon U \times X^n \to Y$ is continuous; it is *of class $\mathcal{C}^\infty$* (or *smooth*) if it is of class $\mathcal{C}^n$ for all $n$. It can be shown that composites of $\mathcal{C}^p$-maps are of class $\mathcal{C}^p$ for $p \in \mathbb{N} \cup \{\infty\}$, whence $\mathcal{C}^p$-manifolds modelled on s.c.l.c. topological vector spaces (and $\mathcal{C}^p$-maps between these) can be defined in the usual way [25], [31] (cf. also [11]).

In the above situation, suppose that $X$ is a vector space of countable dimension, equipped with the finite topology, and suppose that $V_1 \le V_2 \le \cdots$ is a sequence of finite-dimensional subspaces such that $X = \bigcup_{i\in\mathbb{N}} V_i$; we set $U_i := U \cap V_i$. It is clear from the definitions that all derivatives (of a given order) of $f$ exist if and only if this holds for the derivatives of $f|_{U_i}$ for all $i$. If this is the case, for a given $n \in \mathbb{N}$ the function $d^n f$ is continuous if and only if all functions $d^n f\,|_{U_i \times V_i^n} = d^n\,(f|_{U_i})$ are so, by Lemma 3.1 and Proposition 3.3.

**Lemma 4.1.** *Let $\mathcal{S} = ((M_i)_{i\in\mathbb{N}}, (\phi_{ji})_{j\geq i})$ be a directed system of finite-dimensional paracompact $\mathcal{C}^p$-manifolds such that every map $\phi_{ji}$ is a $\mathcal{C}^p$-diffeomorphism onto a closed $\mathcal{C}^p$-submanifold of $M_j$, where $p \in \mathbb{N} \cup \{\infty\}$, $p \geq 3$. Let $(M, (\phi_i)_{i\in\mathbb{N}})$ denote the direct limit of $\mathcal{S}$ in $\mathbb{TOP}$. Set $d_i := \dim M_i$, and, for $j \geq i$, let $\lambda_{ji}$ denote the mapping $\mathbb{R}^{d_i} \to \mathbb{R}^{d_j} : v \mapsto (v, 0)$. Then, for every $x = \phi_n(y) \in M$, there exists an open neighbourhood $O_x$ of $x$ in $M$ such that, setting $U_i := \phi_i^{-1}(O_x)$ for $i \geq n$, there is a family $(h_i^{(x)})_{i\geq n}$ of $\mathcal{C}^{p-2}$-diffeomorphisms $h_i^{(x)} : \mathbb{R}^{d_i} \to U_i$ such that $h_j^{(x)} \circ \lambda_{ji} = \phi_{ji}|_{U_i}^{U_j} \circ h_i^{(x)}$ for all $j \geq i \geq n$, and $h_n^{(x)}(0) = y$.*

*Proof.* By the remarks in Section 3, we may assume w.l.o.g. that $M_1 \subseteq M_2 \subseteq \cdots \subseteq M$, all maps $\phi_{ji}$ and $\phi_i$ being the respective inclusion maps. Then $x = y$. Passing to a cofinal subsystem, we may assume that $x \in M_1$. Choose $r_1 > r_2 > \cdots > 1$. There is a $\mathcal{C}^p$-diffeomorphism $H_1 : \,]{-r_1}, r_1[^{d_1} \to W_1$ onto an open neighbourhood $W_1$ of $x$ in $M_1$, such that $H_1(0) = x$. By [23], Corollary II 3.8 and Theorem IV 5.1, there exists a tubular neighbourhood of $M_1$ in $M_2$, of class $\mathcal{C}^{p-2}$. That is, there is a $\mathcal{C}^{p-2}$-vector bundle $\pi : E \to M_1$ over $M_1$, an open neighbourhood $Z$ of the zero section $\eta$ in $E$, and a $\mathcal{C}^{p-2}$-diffeomorphism $f : Z \to V$ onto an open neighbourhood $V$ of $M_1$ in $M_2$ such that $f \circ \eta|^Z$ is the inclusion map $M_1 \hookrightarrow M_2$. Set $F := \pi^{-1}(W_1)$, $Z' := F \cap Z$, and $q := \pi|_F^{W_1}$. Then $q : F \to W_1$ is a vector bundle of class $\mathcal{C}^{p-2}$. Being homeomorphic to $]{-r_1}, r_1[^{d_1}$, the topological space $W_1$ is paracompact and contractible. By [15], Corollary 2.5, $F$ is a trivial bundle, i.e., we find a fiber-preserving $\mathcal{C}^{p-2}$-diffeomorphism $g : W_1 \times \mathbb{R}^s \to F$, where $s + d_1 = d_2$. Now $g^{-1}(Z')$ is an open neighbourhood of the compact subset $\overline{W_1' \times \{0\}}$ in $W_1 \times \mathbb{R}^s$, where $W_1' := H_1(]{-r_2}, r_2[^{d_1})$, and after re-parametrization in the $\mathbb{R}^s$-directions, we may assume that $W_1' \times J$ is contained in this neighbourhood, where $J := ]{-r_2}, r_2[^s$. We abbreviate $W_2 := f(g(W_1' \times J))$; then the map $H_2 := f|_{Z'}^{W_2} \circ g|_{W_1' \times J}^{Z'} \circ (H_1 \times \mathrm{id}_J)|_{]-r_2, r_2[^{d_2}}^{W_1' \times J}$ is a $\mathcal{C}^{p-2}$-diffeomorphisms.

Proceeding in this fashion, we obtain open neighbourhoods $W_i$ of $x$ in $M_i$ and $\mathcal{C}^{p-2}$-diffeomorphisms $H_i : \,]{-r_i}, r_i[^{d_i} \to W_i$ such that, for all $i \in \mathbb{N}$,

$$W_{i+1} \cap M_i = H_i(]{-r_{i+1}}, r_{i+1}[^{d_i}) = H_{i+1}(]{-r_{i+1}}, r_{i+1}[^{d_i} \times \{0\})$$

and

$$H_i|_{]-r_{i+1}, r_{i+1}[^{d_i}}^{W_{i+1} \cap M_i} = H_{i+1}|_{]-r_{i+1}, r_{i+1}[^{d_i} \times \{0\}}^{W_{i+1} \cap M_i} \circ \theta_i,$$

where $\theta_i : \,]{-r_{i+1}}, r_{i+1}[^{d_i} \hookrightarrow \,]{-r_{i+1}}, r_{i+1}[^{d_i} \times \{0\}$. Let $U_i := H_i(]{-1}, 1[^{d_i})$ and $h_i^{(x)} := H_i|_{]-1,1[^{d_i}}^{U_i} \circ u^{d_i}$, where $u : \mathbb{R} \to ]{-1}, 1[$ is a $\mathcal{C}^\infty$-diffeomorphism such that $u(0) = 0$. Then $O_x := \bigcup_{i\in\mathbb{N}} U_i$ has the required properties. □

For the remainder of this section, we introduce the following notation: we suppose that $M_1 \subseteq M_2 \subseteq \cdots$ is a directed system of $\mathcal{C}^p$-manifolds, as described in Lemma 4.1 and its proof, with direct limit topological space $M = \bigcup_{i \in \mathbb{N}} M_i$. We abbreviate $V := \varinjlim \mathbb{R}^{d_i}$. Given $x \in M$, $x \in M_{n(x)}$, say, we let $(h_i^{(x)})_{i \geq n(x)}$ be a family of $\mathcal{C}^{p-2}$-diffeomorphisms $h_i^{(x)} \colon \mathbb{R}^{d_i} \to U_i^{(x)}$, as constructed in Lemma 4.1, and define $O_x := \bigcup_{i \geq n(x)} U_i^{(x)}$. We let $h_x := \varinjlim h_i^{(x)} \colon V \to O_x$ denote the homeomorphism whose restriction to $\mathbb{R}^{d_i}$ is $h_i^{(x)}$ for all $i \geq n(x)$, and we set $g_x := h_x^{-1}$.

**Proposition 4.2.** *$M$ is a Hausdorff space, and $\mathcal{A} := \{g_x \colon x \in M\}$ is an atlas for $M$ which makes $M$ a $\mathcal{C}^{p-2}$-manifold. For every $i \in \mathbb{N}$, the inclusion map $\phi_i \colon M_i \hookrightarrow M$ is an embedding of $\mathcal{C}^{p-2}$-manifolds. A map $f \colon M \to N$ into a $\mathcal{C}^{p-2}$-manifold $N$ is of class $\mathcal{C}^{p-2}$ if and only if $f \circ \phi_i$ is of class $\mathcal{C}^{p-2}$ for all $i$, whence $M$ is the direct limit of the above system in the category of $\mathcal{C}^{p-2}$-manifolds.*

*Proof.* For simplicity of notation, we regard each $\mathbb{R}^{d_i}$ (and $V = \bigcup_{i \in \mathbb{N}} \mathbb{R}^{d_i}$) as a subspace of $\mathbb{R}^\infty$ (via $t \mapsto (t, 0)$). Note that, for every $x \in M$ and $i \geq n(x)$, the bijection $g_x$ maps $O_x \cap M_i$ onto $\mathbb{R}^{d_i}$. Now given $x, y \in M$, let $n := \max\{n(x), n(y)\}$. Then $x, y \in M_n$. Set $\tau := g_y|_{O_x \cap O_y} \circ g_x^{-1}|_Q^{O_x \cap O_y}$, where $Q := g_x(O_x \cap O_y)$. Let $(h_i^{(x)})_{i \geq n}$ and $(h_i^{(y)})_{i \geq n}$ denote the families of $\mathcal{C}^{p-2}$-diffeomorphisms used to define $h_x = g_x^{-1}$ and $h_y = g_y^{-1}$, respectively. Then $\tau$ is of class $\mathcal{C}^{p-2}$, since, by construction of the maps $h_i^{(x)}$ and $h_i^{(y)}$, for every $i \geq n$ we have

$$\tau|_{Q \cap \mathbb{R}^{d_i}} = \lambda_i \circ (h_i^{(y)})^{-1}|_{U_i^{(x)} \cap U_i^{(y)}} \circ h_i^{(x)}|_{Q \cap \mathbb{R}^{d_i}}^{U_i^{(x)} \cap U_i^{(y)}},$$

where $h_i^{(x)}$ and $h_i^{(y)}$ are $\mathcal{C}^{p-2}$-diffeomorphisms onto the open submanifolds $U_i^{(x)}$ and $U_i^{(y)}$ of $M_i$, respectively, and $\lambda_i \colon \mathbb{R}^{d_i} \hookrightarrow V$ denotes inclusion. The transition functions being of class $\mathcal{C}^{p-2}$, $\mathcal{A}$ is a $\mathcal{C}^{p-2}$-atlas for $M$. Since $M$ is Hausdorff by Proposition 3.6 (a), we obtain a manifold of class $\mathcal{C}^{p-2}$.

Now suppose that $f \colon M \to N$ is a map into a $\mathcal{C}^{p-2}$-manifold $N$ such that all maps $f_i := f|_{M_i}$ are of class $\mathcal{C}^{p-2}$. Then $f$ is continuous since the maps $f_i$ are continuous, $M$ being the direct limit of its subspaces $M_i$ as a topological space. Given $x \in M$, let $g_x \colon O_x \to V$ be the chart as above. Furthermore, let $\phi \colon W \to U$ be a chart around $f(x)$ in $N$, where $U$ is an open subset of the vector space on which $N$ is modelled. Then there is an open neighbourhood $P \subseteq O_x$ of $x$ in $M$ such that $f(P) \subseteq W$, since $f$ is continuous. Thus $F := \phi \circ f|_P^W \circ g_x^{-1}|_Q^P$ is defined, where $Q := g_x(P)$. Let $E$ be a finite-dimensional subspace of $V$; without loss of generality $E = \mathbb{R}^{d_i}$ for some $i \geq n(x)$. Now $g_x^{-1}|_E$ is a $\mathcal{C}^{p-2}$-diffeomorphism of $E$ onto an open submanifold $S$ of $M_i$, by the construction of $g_x^{-1}$. Since $f_i$ is of class $\mathcal{C}^{p-2}$ by assumption, the formula $F|_{Q \cap E} = (\phi \circ f_i|_{P \cap S}^W) \circ g_x^{-1}|_{Q \cap E}^{P \cap S}$ shows that $F$ is of class $\mathcal{C}^{p-2}$. Hence $f$ is of class $\mathcal{C}^{p-2}$. The remainder is obvious. □

As a special case, we obtain:

**Theorem 4.3.** *Let $M_1 \subseteq M_2 \subseteq \cdots$ be an ascending sequence of finite-dimensional paracompact smooth manifolds, where $M_n$ is a closed $C^\infty$-submanifold of $M_{n+1}$ for all $n$. Then there exists a unique smooth manifold structure on the direct limit topological space $M := \varinjlim M_n = \bigcup_{n\in\mathbb{N}} M_n$ which makes $M$ the direct limit of its submanifolds $M_n$ in the category of smooth manifolds.*

We conclude this section with further technical information.

**Proposition 4.4.** *Let $M_1 \subseteq M_2 \subseteq \cdots \subseteq M$ be as in Lemma 4.1 above, and $x \in M_n$. Then the path component $P$ of $x$ in $M$ is open, coincides with the connected component $C$ of $x$ in $M$, and $C = \varinjlim_{i\geq n} C_i$, where $C_i$ is the connected component of $x$ in $M_i$ for $i \geq n$.*

*Proof.* For $i \geq n$, let $U_i$ denote the path component of $x$ in $M_i$. Then $U_i$ is open in $M_i$ and coincides with the connected component of $x$ in $M_i$. The family $(U_i)_{i\geq n}$ satisfies the requirements of Lemma 3.1; thus $U := \bigcup_{i\geq n} U_i$ is open in $M$, is path connected, and contains $x$. If $\gamma\colon [0,1] \to M$ is any path starting at $x$, its image is contained in some $M_i$ by Proposition 3.6 (b), whence $\gamma(1) \in U_i \subseteq U$. Thus $U$ is the path component of $x$ in $M$. Since all path components of $M$ are open by the preceding, they coincide with the connected components. □

Here is an analogue of Proposition 3.3 for smooth manifolds.

**Proposition 4.5.** *Let $((M_i)_{i\in\mathbb{N}}, (\phi_{ji})_{j\geq i})$ and $((N_i)_{i\in\mathbb{N}}, (\psi_{ji})_{j\geq i})$ be strict directed systems of finite-dimensional $\mathcal{C}^p$-manifolds, as in Lemma 4.1, with direct limit $\mathcal{C}^{p-2}$-manifolds $M$ and $N$, respectively. Then $\varinjlim M_i \times N_i = M \times N$ in the category of $\mathcal{C}^{p-2}$-manifolds.*

*Proof.* If $(x,y) \in M \times N$ and $g_x$, $g_y$ are the above-defined charts of $N$ and $M$ around $x$ and $y$, respectively, with respective domains of definition $O_x$ and $O_y$, then $O_x \times O_y$ is open in the direct limit manifold $S := \varinjlim M_i \times N_i$, and clearly $g_x \times g_y$ is a chart of $S$ as constructed in Lemma 4.1. □

## 5. Countable direct limits of Lie groups

A *smooth Lie group* is a group, equipped with a smooth manifold structure modelled on some s.c.l.c. topological vector space, such that the group operations are smooth maps. $\mathbb{LIE}_\infty$ denotes the category of smooth Lie groups and smooth homomorphisms. As a consequence of Theorem 4.3, we deduce in this section that every countable strict directed system of finite-dimensional Lie groups has a direct limit in the category $\mathbb{LIE}_\infty$ (Theorem 5.1). We then investigate continuous homomorphisms between direct limit Lie groups (Proposition 5.2), provide an alternative description of the Lie algebras of direct limit Lie groups (Proposition 5.4), and describe a direct limit Lie group whose exponential function does not induce a local homeomorphism at 0 (Example 5.5).

**Theorem 5.1.** *Let $\mathcal{S} := ((G_i)_{i\in I}, (\phi_{ji})_{j\geq i})$ be a countable strict directed system of finite-dimensional Lie groups, with topological group direct limit $(G, (\phi_i))$. Then there is a unique smooth manifold structure on $G$ which makes $G$ the direct limit of $\mathcal{S}$ in the category $\mathbb{LIE}_\infty$. The maps $\phi_i$ are embeddings onto $\mathcal{C}^\infty$-submanifolds of $G$.*

*Proof.* The identity component $K$ of a Lie group $L$ is a $\sigma$-compact locally compact space and therefore paracompact. Hence so is $L$, being the topological coproduct of the open closed cosets of $K$. Theorem 4.3 yields a smooth manifold structure on $G$ which makes it a direct limit in the category of smooth manifolds modelled on s.c.l.c. spaces. Since $G$ is, at the same time, a direct limit in the sense of abstract groups, and in the sense of sets, cones of smooth homomorphisms induce smooth homomorphisms. The remainder is plain. □

**Proposition 5.2.** *Let $G$ and $L$ be the direct limits of countable strict directed systems of finite-dimensional Lie groups $G_i \leq G$ and $L_i \leq L$, respectively, and assume that $H$ is a finite-dimensional Lie group. Then*

(a) *every continuous homomorphism $f\colon H \to G$ is smooth*;

(b) *every continuous homomorphism $f\colon G \to L$ is smooth.*

*Proof.* (a) We may assume w.l.o.g. that $H$ is connected, since translations in $H$ and $G$ are smooth and $H$ has an open identity component. Let $C$ be a compact identity neighbourhood in $H$; then $f(C) \subseteq G_i$ for some $i$ by Proposition 3.6 (b). Hence $f(H) \subseteq G_i$, because $C$ generates $H$. Since $G_i$ is a submanifold of $G$ and the continuous homomorphism $f|^{G_i}$ between finite-dimensional Lie groups is smooth, so is $f$.

(b) $f$ is induced by the cone $(L, (f|_{G_i})_{i\in I})$, where each continuous homomorphism $f|_{G_i}$ is smooth by Part (a). □

**5.3.** Suppose that $\mathcal{S}$ and $G$ are as in Theorem 5.1. Let $(\mathfrak{g}, (\psi_i)_{i\in I})$ be the direct limit of $\mathcal{T} := ((\mathrm{L}(G_i))_{i\in I}, (\mathrm{L}(\phi_{ji}))_{j\geq i})$ in the category of topological Lie algebras, where $\mathrm{L}(G_i) = \mathrm{Hom}(\mathbb{R}, G_i)$ and $\mathrm{L}(\phi_{ji}) = \mathrm{Hom}(\mathbb{R}, \phi_{ji})$. The set underlying $\mathfrak{g}$ being the direct limit of the sets $\mathrm{L}(G_i)$, the cone $(\mathrm{Hom}(\mathbb{R}, G),$ $(\mathrm{Hom}(\mathbb{R}, \phi_i))_{i\in I})$ over $\mathcal{T}$ in $\mathbb{SET}$ induces a mapping $\eta\colon \mathfrak{g} \to \mathrm{Hom}(\mathbb{R}, G)$. Let us check that $\eta$ is bijective: we may assume $I = (\mathbb{N}, \leq)$ and $G_1 \subseteq G_2 \subseteq \cdots \subseteq G$, all maps $\phi_i$ and $\phi_{ji}$ being the respective inclusion maps. Suppose $X \in \mathrm{Hom}(\mathbb{R}, G)$. By Proposition 3.6 (b), we have $X([-1, 1]) \subseteq G_i$ for some $i \in \mathbb{N}$, whence $\mathrm{im} X \leq G_i$ indeed since $[-1, 1]$ generates $\mathbb{R}$. We have proved that every one-parameter subgroup of $G$ is a one-parameter subgroup of some $G_i$. It follows from this that $\eta$ is surjective. All maps $\mathrm{Hom}(\mathbb{R}, \phi_i)$ being injective, so is $\eta$. We use the bijection $\eta$ to transport the topological Lie algebra structure of $\mathfrak{g}$ to $\mathrm{Hom}(\mathbb{R}, G)$.

Note that $\mathfrak{g}$ is isomorphic to the Lie algebra of $G$ as defined in [25].

**Proposition 5.4.** *In the above situation, the following holds*:

(a) *Addition and Lie bracket on* $\mathrm{Hom}(\mathbb{R}, G)$ *are given by the Trotter Product and Commutator Formulas, respectively* (*which do converge*). *Thus, given*

$X, Y \in \mathrm{Hom}(\mathbb{R}, G)$*, we have, for all* $t \in \mathbb{R}$*,*

$$(X+Y)(t) = \lim_{n\to\infty} \left( X\left(\frac{t}{n}\right) Y\left(\frac{t}{n}\right)\right)^n$$

*and*

$$[X,Y](t^2) = \lim_{n\to\infty} \left( X\left(\frac{t}{n}\right) Y\left(\frac{t}{n}\right) X\left(\frac{-t}{n}\right) Y\left(\frac{-t}{n}\right)\right)^{n^2}.$$

(b) *The exponential map* $\exp\colon \mathrm{Hom}(\mathbb{R}, G) \to G\colon X \mapsto X(1)$ *is smooth.*

*Proof.* The function exp is induced by the compatible family of the smooth maps $\exp_{G_i}\colon \mathrm{L}(G_i) = \mathrm{Hom}(\mathbb{R}, G_i) \to G\colon X \mapsto X(1)$ (via the universal property of $\varinjlim \mathrm{L}(G_i)$ in the category of smooth manifolds). Hence exp is smooth. $\mathrm{Hom}(\mathbb{R}, G)$ being the directed union of the Lie algebras $\mathrm{Hom}(\mathbb{R}, G_i)$, Part (a) easily follows from the finite-dimensional theory (cf. [4, Chapter 3, Section 4.3, Proposition 4]). □

In the situation of the preceding proposition, the exponential map of $G$ need not be locally regular at 0, nor locally injective at 0, nor locally open at 0: then the method of [27]–[30] cannot be used to produce a direct limit Lie group (whenever the method applies, the exponential function will induce a local diffeomorphism at 0). Here is an example of a direct limit group with a bad exponential function:

**Example 5.5.** Let $G := \mathbb{R} \ltimes \mathbb{C}^\infty$, where $\mathbb{R}$ acts on $\mathbb{C}^\infty$ via $t.(z_k)_{k\in\mathbb{N}} = (e^{ikt}z_k)_{k\in\mathbb{N}}$. Then $G$ is an infinite-dimensional Lie group in a natural way; its manifold structure is determined by the global chart $\mathrm{id}\colon G \to \mathbb{R}\times\mathbb{C}^\infty$, where the real vector space $\mathbb{R}\times\mathbb{C}^\infty$ is equipped with the finite topology. Clearly the Lie group $G$ is the direct limit of its subgroups $\mathbb{R}\ltimes V_k$, where $V_k := \{(z_j)_{j\in\mathbb{N}} \in \mathbb{C}^\infty\colon z_j = 0 \text{ for all } j > k\}$. The Lie algebra $\mathfrak{g}$ of $G$ can be identified with $\mathbb{R}\ltimes\mathbb{C}^\infty$, with $\mathbb{R}$ acting on $\mathbb{C}^\infty$ via $t.(z_k)_{k\in\mathbb{N}} = (iktz_k)_{k\in\mathbb{N}}$. Using this identification, the exponential map is given by $\exp\colon \mathfrak{g} \to G$, $(t, (z_k)_{k\in\mathbb{N}}) \mapsto (t, (f(kt)z_k)_{k\in\mathbb{N}})$, where $f(s) = (e^{is}-1)/is$. We set $X := (2\pi, 0) \in \mathfrak{g}$.

Suppose that $U$ is an open 0-neighbourhood in $\mathfrak{g}$. Since $k^{-1}X \to 0$ as $k \to \infty$, there exists $n \in \mathbb{N}$ such that $n^{-1}X \in U$. Since $U$ is open, there is $\varepsilon > 0$ such that $n^{-1}X + re_n \subseteq U$ for all $r \in ]-\varepsilon, \varepsilon[$, where $e_n = \delta_{n,\cdot} \in \mathbb{C}^\infty$. Now $\exp(n^{-1}X + re_n) = (2\pi/n, 0)$ for all $r$ shows that exp is not injective on $U$. Hence exp *is not locally injective at* 0.

If $W$ is an open identity neighbourhood in $G$, the continuity of exp implies that $g := (2\pi/n, 0) = \exp(n^{-1}X) \in W$ for some $n \in \mathbb{N}$. Since $W$ is open, there is $r \neq 0$ with $g' := g + re_n \in W$. We claim that $g' \notin \mathrm{im}\exp$. In fact, suppose to the contrary that we could find some $Z = (t, (z_k)_{k\in\mathbb{N}}) \in \mathfrak{g}$ such that $\exp(Z) = g'$. The above explicit formula for exp shows that $t = 2\pi/n$ and $r = ((e^{int}-1)/int)z_n = 0$. But $r \neq 0$. Hence indeed $g' \notin \mathrm{im}\exp$ and therefore $W \not\subseteq \mathrm{im}\exp$. We conclude: *The exponential image* $\mathrm{im}\exp$ *is not an identity neighbourhood of* $G$.

Note that $\exp_{G_k} = \exp|^{G_k}_{\mathbb{R}\times V_k}$ has a non-invertible derivative at $k^{-1}X$. Hence exp *is not locally regular at* 0: every 0-neighbourhood $U$ in $\mathfrak{g}$ contains an element $Y$ such that $d\exp(Y)$ is not injective, hence not invertible.

In infinite-dimensional Lie theory, it is interesting (and in many cases hard to decide) whether a given Lie algebra is *integrable*, i.e., isomorphic to the Lie algebra of some Lie group. Clearly direct limit Lie groups are natural candidates of Lie groups one would try to associate with *locally finite* Lie algebras, i.e., Lie algebras which are the direct limit of their finite-dimensional subalgebras. From Theorem 5.1 above, we easily deduce the following integrability criterion:

**Corollary 5.6.** *Let $\mathfrak{g}$ be a locally finite real Lie algebra of countable dimension. Suppose that there exists an ascending sequence $\mathfrak{g}_1 \subseteq \mathfrak{g}_2 \subseteq \cdots$ of finite-dimensional subalgebras of $\mathfrak{g}$, a strict directed sequence $G_1 \overset{\phi_{2,1}}{\hookrightarrow} G_2 \overset{\phi_{3,2}}{\hookrightarrow} \cdots$ of finite-dimensional Lie groups, and isomorphisms $\gamma_n\colon \mathrm{L}(G_n) \to \mathfrak{g}_n$ of Lie algebras for $n \in \mathbb{N}$ with the following properties*:

(a) $\mathfrak{g} = \bigcup_{n\in\mathbb{N}} \mathfrak{g}_n$;

(b) $\varepsilon_{n+1,n}\circ\gamma_n = \gamma_{n+1}\circ\mathrm{L}(\phi_{n+1,n})$ *holds for all* $n \in \mathbb{N}$*, where* $\varepsilon_{n+1,n}$ *denotes the inclusion map* $\mathfrak{g}_n \hookrightarrow \mathfrak{g}_{n+1}$.

*Then* $G := \varinjlim G_n$ *exists as a smooth Lie group, and* $\mathrm{L}(G) \cong \mathfrak{g}$.

## 6. Direct limit convenient Lie groups

We have already seen in Example 3.5 that the naïve direct limit of an uncountable strict directed system of finite-dimensional Lie groups need not be a topological group, in which case it cannot be made a Lie group in the ordinary sense (as described in Section 5). In this situation, it is unclear whether the directed system has a direct limit in the category $\mathbb{LIE}_\infty$ of Lie groups modelled on s.c.l.c. topological vector spaces, and the naïve direct limit group does not seem to be helpful for its construction. However, the system still has a direct limit in another category of Lie groups (under suitable hypotheses), the category of Lie groups in the sense of 'convenient differential calculus' ([7], [22]), as defined in [21] and [22]. These Lie groups are the group objects in the category of smooth manifolds in the sense of convenient differential calculus; we call them *convenient Lie groups* for brevity. Let us assemble the required preliminaries concerning convenient differential calculus.

A sequence $(x_n)_{n\in\mathbb{N}}$ in a locally convex topological vector space $V$ is called a *Mackey-Cauchy sequence* if there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ converging to 0, and a bounded absolutely convex subset $B \subseteq V$ such that $x_n \in \mu_n B$ for all $n \in \mathbb{N}$ (cf. [22, Lemma 1.6]). A topological vector space $V$ is said to be *convenient* if it is locally convex, Hausdorff, and every Mackey-Cauchy sequence converges ([22, Theorem 2.14 (5)]). If $V$ is a convenient topological vector space, we let $\mathcal{C}^\infty(\mathbb{R}, V)$ denote the set of smooth curves $\mathbb{R} \to V$. The $c^\infty$*-topology on* $V$ is the final topology on $V$ with respect to the mappings in $\mathcal{C}^\infty(\mathbb{R}, V)$; we write $c^\infty(V)$ for $V$, equipped with the $c^\infty$-topology. Note that the $c^\infty$-topology is finer than the original topology. If $V$ is a Fréchet-space,

$c^\infty(V) = V$ holds ([22, Theorem 4.11]); in general, $c^\infty(V)$ is *not* a topological vector space, and if $V$, $W$ are convenient vector spaces, although the map $c^\infty(V \times W) \to c^\infty(V) \times c^\infty(W)$, $(v,w) \mapsto (v,w)$ is easily seen to be continuous, it need not be a homeomorphism. If $V, W$ are convenient topological vector spaces, $U$ is a $c^\infty$-open subset of $V$, and $f\colon V \to W$ is a map, we say that $f$ is *smooth* if $f \circ c\colon \mathbb{R} \to W$ is smooth for all smooth maps $c\colon \mathbb{R} \to V$ with image in $U$. Then composites of smooth maps are smooth. A *smooth manifold* (in the sense of convenient differential calculus) is a pair $(M, \mathcal{A})$, where $M$ is a topological space and $\mathcal{A}$ is a set of homeomorphisms (called *charts*) $\phi\colon U \to W$ from an open subset $U$ of $M$ onto a $c^\infty$-open subset $W$ of a convenient topological vector space $V_\phi$ (equipped with the $c^\infty$-topology), such that $M$ is the union of the domains of the charts $\phi \in \mathcal{A}$ and, for all charts $\phi\colon U_1 \to W_1$ and $\psi\colon U_2 \to W_2$, the coordinate change $\tau := \psi|_{U_1 \cap U_2} \circ \phi^{-1}|^{U_1 \cap U_2}_{\phi(U_1 \cap U_2)}$ is a smooth map. If there is a convenient vector space $V$ such that $V_\phi$ is linearly diffeomorphic to $V$ for all charts $\phi$, we say that $M$ *is modelled on* $V$.

Given smooth manifolds $M$ and $N$, a map $f\colon M \to N$ is said to be *smooth* if it is continuous and if, for every $x \in M$ and charts $\phi\colon U_1 \to W_1$ and $\psi\colon U_2 \to W_2$ around $x$ and $f(x)$, respectively, the mapping $\psi \circ f|^{U_2}_Q \circ \phi^{-1}|^Q_{\phi(Q)}$ is smooth, where $Q := f^{-1}(U_2) \cap U_1$.

If $(M_1, \mathcal{A}_1)$ and $(M_2, \mathcal{A}_2)$ are smooth manifolds, we equip $M_1 \times M_2$ with the final topology with respect to the maps $\phi_1^{-1} \times \phi_2^{-1}\colon W_1 \times W_2 \to U_1 \times U_2 \subseteq M_1 \times M_2$, where $\phi_i\colon U_i \to W_i$ is a chart of $M_i$ for $i = 1, 2$ and $U_1 \times U_2$ is equipped with its topology as a subspace of $c^\infty(V_1 \times V_2)$, where $V_i$ is the convenient vector space such that $W_i \subseteq V_i$. Note that we do not use the topology induced by $c^\infty(V_1) \times c^\infty(V_2)$: this is essential. Let $\mathcal{C}$ denote the collection of all the maps $\phi_1 \times \phi_2$; we call $(M \times N, \mathcal{C})$ the *direct product* of the manifolds $M$ and $N$.

A *convenient Lie group* is a group $G$, together with a smooth manifold structure on $G$ (in the preceding sense), such that the group operations are smooth (see [22], Definition 36.1, where convenient Lie groups are simply called "Lie groups"). Unlike [22], we shall not presume that $G$ be smoothly Hausdorff (which means that the smooth functions $f\colon G \to \mathbb{R}$ separate points on $G$). Note that the topology underlying the product manifold $G \times G$ can be properly finer than the product topology; hence although the group multiplication $\mu\colon G \times G \to G$ is smooth, $G$ need not be a topological group.

**Lemma 6.1.** *Let $V$ be a real vector space, equipped with the finest locally convex topology. Then $V$ is a convenient topological vector space. The $c^\infty$-topology on $V$ coincides with the topology of finitely open sets.*

*Proof.* Any real vector space is complete in its finest locally convex topology ([18, Theorem 8]); therefore it is a convenient topological vector space. Let $F$ be a finite-dimensional subspace of $V$. Then $F$ is a convenient vector space in its Hausdorff vector topology. By [22, Theorem 2.14 (3)], $F$ is $c^\infty$-closed in $V$, whence the $c^\infty$-topology on $V$ induces the $c^\infty$-topology on $F$, by *loc. cit.* Lemma 4.28, which is the Hausdorff vector topology on $F$ since $F$ is Fréchet. Thus $F \cap U$ is open in $F$ for every finite-dimensional subspace $F$ if $U$ is $c^\infty$-open in $V$: hence $U$ is finitely open and we have proved that the $c^\infty$-topology on $V$ is coarser than the finite topology. On the other hand, if $c\colon \mathbb{R} \to V$ is a smooth curve, for every $k \in \mathbb{Z}$ the compact set $c([k-1, k+1])$ has finite-dimensional span $F_k$ in $V$, equipped with the finest locally convex topology ([18, Lemma 2]). Since the finite topology on $V$ induces the Hausdorff vector topology on each $F_k$, we conclude that $c$ is continuous as a mapping into $V$, equipped with the finite topology. By definition of the $c^\infty$-topology as a final topology, we deduce that it is finer than the finite topology. This completes the proof. □

The heart of the following definition is a variant of the "spectral growth condition" defined in [27]:

**Definition 6.2.** Let $\mathcal{S} := ((G_i)_{i\in I}, (\phi_{ji})_{j\geq i})$ be a strict directed system of finite-dimensional Lie groups. We say that $\mathcal{S}$ is *admissible* if there exists a strict directed system $\mathcal{T} := ((V_i)_{i\in I}, (\eta_{ji})_{j\geq i})$ of finite-dimensional complex vector spaces and complex linear maps and a family $(\pi_i)_{i\in I}$ of continuous complex linear actions $\pi_i\colon G_i \times V_i \to V_i$ which is compatible with the directed systems $((G_i \times V_i)_{i\in I}, (\phi_{ji} \times \eta_{ji})_{j\geq i})$ and $\mathcal{T}$, with the following property: Let $d\pi := \varinjlim d\pi_i\colon \mathfrak{g} \times V \to V$ be the limit map of the family of Lie algebra actions $d\pi_i\colon \mathrm{L}(G_i) \times V_i \to V$ which is compatible with the directed systems $((\mathrm{L}(G_i) \times V_i)_{i\in I}, (\mathrm{L}(\phi_{ji}) \times \eta_{ji})_{j\geq i})$ and $\mathcal{T}$, where $\mathfrak{g} := \varinjlim \mathrm{L}(G_i)$ and $V := \varinjlim V_i$.[*3] It is required that the Lie algebra representation $\mathfrak{g} \to \mathrm{gl}(V)$, $X \mapsto d\pi(X, \cdot)$ is faithful, and that there exists a finitely open 0-neighbourhood $Q$ in $\mathfrak{g}$ such that

$$\sup\{|\operatorname{Im}\lambda|\colon X \in Q, \lambda \in \operatorname{spec} d\pi(X, \cdot)\} < \infty. \tag{1}$$

**Remark 6.3.** In the situation of Definition 6.2, there is a useful criterion for the existence of $Q$, the "bounded growth condition" or "operator norm growth condition" ([28], p. 62, [30] (3.4b)): If there exists a family $(\|\cdot\|_i)_{i\in I}$ of norms on the spaces $V_i$ such that, for every $i \in I$ and $X \in \mathfrak{g}_i$,

$$\limsup_{j\geq i} \|d\pi_j(\mathrm{L}(\phi_{ji})(X),\ \cdot\ )\|_j^{\mathrm{op}} < \infty$$

(where $\|\cdot\|_j^{\mathrm{op}}$ denotes the operator norm with respect to $\|\cdot\|_j$), then there is a neighbourhood $Q$ in $\mathfrak{g}$ with the required property.

We can now state an existence theorem for direct limit convenient Lie groups:

**Theorem 6.4.** *Let $\mathcal{S} = ((G_i)_{i\in I}, (\phi_{ji})_{j\geq i})$ be an admissible strict directed system of finite-dimensional Lie groups. Then the naïve direct limit $(G, (\phi_i)_{i\in I})$ of $\mathcal{S}$ can be given a smooth manifold structure in the sense of convenient differential calculus which makes it the direct limit of $\mathcal{S}$ in the category of convenient Lie groups. If $I$ is countable or if the compatible family $(\pi_i)_{i\in I}$ in the definition of admissibility can be chosen such that $\varinjlim \pi_i$ is a faithful action of $G$, then $G$ is smoothly Hausdorff.*

[*3] Here $d\pi_i(X, v) := d_1(\pi)(1, v).X$ for $X \in \mathfrak{g}$, $v \in V$, where $d_1$ denotes the partial derivative with respect to the variables in $\mathfrak{g}$.

*Proof.* Let $(G,(\phi_i)_{i\in I})$ denote the naïve direct limit of $\mathcal{S}$; we may assume without loss of generality that $G_i \subseteq G$ for all $i$, all maps $\phi_{ji}$ and $\phi_i$ being the respective inclusion maps. Also, we consider all Lie algebras $\mathfrak{g}_i$ as subalgebras of their direct limit $\mathfrak{g}$. Let $Q$ be as in Definition 6.2; after shrinking $Q$ by multiplication with a suitable positive real, we may assume that the supremum in Definition 6.2, Inequality (1) is smaller than $\pi$. Let $\exp := \varinjlim \exp_{G_i} : \mathfrak{g} \to G$. By [27, Proposition 7.1], $U := \exp(Q)$ is an open subset of $G$, and $\alpha := \exp|_Q^U$ is a homeomorphism if $Q$ is equipped with the topology induced by the finite topology. Given $x \in G$, define $\beta_x : xU \to Q$ via $y \mapsto \alpha^{-1}(x^{-1}y)$. By the considerations in [27], for every $i \in I$ such that $x \in G_i$, the map $\beta_x|_{xU\cap G_i}^{Q\cap\mathfrak{g}_i}$ is a chart of $G_i$.

We claim that the family $(\beta_x)_{x\in G}$ can be used as a family of charts which makes $G$ a convenient Lie group modelled on $\mathfrak{g}$, equipped with the finest locally convex topology. Note first that the sets $xU$ cover $G$ (for $x \in G$). Given $x, y \in G$, consider the coordinate change $\tau : \beta_y|_{xU\cap yU} \circ \beta_x^{-1}|_W^{xU\cap yU}$, where $W := \beta_x(xU \cap yU)$. Then $W$ is finitely open by the above, i.e., $W$ is $c^\infty$-open by Lemma 6.1. If $c : \mathbb{R} \to W$ is a smooth curve, consider $c_k := c|_{]k-1,k+1[}$ for $k \in \mathbb{N}$. Then $c_k$ has relatively compact image, whence there exists $i \in I$ such that $\mathrm{im} c_k \subseteq \mathfrak{g}_i$. Increasing $i$ if necessary, we may assume that $x, y \in G_i$. Now $\tau \circ c_k = \tau|_{W\cap g_i}^{Q\cap\mathfrak{g}_i} \circ c_k$, where $\tau|_{W\cap g_i}^{Q\cap\mathfrak{g}_i}$ is analytic by the above (being a coordinate change on $G_i$). Thus $\tau \circ c_k$ is smooth for all $k$, whence also $\tau \circ c$ is smooth. We conclude that $\tau$ is smooth in the sense of convenient differential calculus.

To see that $G$, equipped with the smooth manifold structure defined by the above coordinate cover, is a convenient Lie group, it remains to show that the group multiplication and inversion are smooth. Let us show smoothness of the multiplication $\mu$ (smoothness of inversion is even easier to prove). Regard $G\times G$ as a smooth manifold modelled on $\mathfrak{g} \times \mathfrak{g}$ (equipped with the product topology, which is again the finest locally convex topology), using the family of charts $(\beta_x \times \beta_y)_{x,y\in G}$ as a coordinate cover. Let $c : \mathbb{R} \to G \times G$ be a smooth curve. Given $t \in \mathbb{R}$, there exist $(x, y) \in G\times G$ and a neighbourhood $V = ]t-r, t+r[$ of $t$ in $\mathbb{R}$ such $c(V) \subseteq xU\times yU$ and such that $(\beta_x\times\beta_y)\circ c|_V^{xU\times yU} : V \to Q\times Q \subseteq \mathfrak{g}\times\mathfrak{g}$ is smooth. Let $0 < s < r$ and set $W := ]t - s, t + s[$; then $(\beta_x \times \beta_y)(c(W))$ is relatively compact, hence contained in $\mathfrak{g}_i\times\mathfrak{g}_i$ for some $i \in I$. We may assume that $x, y \in G_i$. Then $c(W) \subseteq G_i \times G_i$, and $c' := c|_W^{G_i\times G_i}$ is a smooth curve, using that $(\beta_x \times \beta_y)|_{(xU\times yU)\cap(G_i\times G_i)}^{(Q\times Q)\cap(\mathfrak{g}_i\times\mathfrak{g}_i)}$ is a chart of $G_i \times G_i$. We now write $\mu \circ c|_W = \lambda_i \circ \mu_i \circ c'$, where $\mu_i : G_i \times G_i \to G_i$ is the smooth multiplication on $G_i$ and $\lambda_i : G_i \hookrightarrow G$ denotes inclusion. It is easy to check that $\lambda_i$ is smooth. Hence $\mu \circ c|_W$ is smooth as well. Since $t \in \mathbb{R}$ was arbitrary, we conclude that $\mu \circ c$ is smooth. Hence $\mu$ is smooth.

Let us prove now that $G$, equipped with the above convenient Lie group structure, is the direct limit of $\mathcal{S}$ in the category of convenient Lie groups. To this end, let $(H,(f_i)_{i\in I})$ be a cone over $\mathcal{S}$ in the category of convenient Lie groups. Since $(G,(\phi_i)_{i\in I})$ is the direct limit of $\mathcal{S}$ in the category of groups, there is a unique homomorphism $f : G \to H$ such that $f|_{G_i} = f_i$ for all $i \in I$. If $c : \mathbb{R} \to G$ is a smooth curve, for every $t \in G$ there exists an open neighbourhood

$W$ of $x$ in $\mathbb{R}$ such that $c(W) \subseteq G_i$ for some $i \in I$, as above. Hence $f \circ c|_W = f_i \circ c|_W^{G_i}$ shows that $f \circ c|_W$ is smooth, and hence that so is $f \circ c$. Therefore $f$ is smooth.

Suppose now that $(\pi_i)_{i \in I}$ is a compatible family of continuous linear actions $\pi_i \colon G_i \times V_i \to V_i$ on finite-dimensional complex vector spaces which is compatible with $\mathcal{S}$ in the sense described in Definition 6.2; assume that the representation $g \mapsto \pi(g, \cdot)$, where $\pi := \varinjlim \pi_i$, separates points on $G$. Let $V := \varinjlim V_i$, equipped with the finite topology; we consider $V$ as a smooth manifold, modelled on the real vector space $V$, equipped with the finest locally convex topology. Given distinct elements $g, h \in G$, by hypothesis there exists $v \in V$ such that $\pi(g, v) \neq \pi(h, v)$. Let $\lambda \in V'$ such that $\lambda(\pi(g, v)) \neq \lambda(\pi(h, v))$. Then $f := \lambda \circ \pi(\cdot, v) \colon G \to \mathbb{R}$ is smooth since $\lambda$ and $\pi$ are so, and $f(g) \neq f(h)$.

If $I$ is countable, then $G$ is a regular topological space in view of Corollary 3.4; furthermore, $\mathfrak{g}$ (which is finite-dimensional or $\cong \mathbb{R}^\infty$) admits smooth bump functions, i.e., for every $X \in \mathfrak{g}$ and every neighbourhood $U$ of $X$, there exists a smooth function $b \colon \mathfrak{g} \to \mathbb{R}$, vanishing on the complement of $U$, such that $b(X) = 1$. These properties together will entail that $G$ is smoothly Hausdorff. Here, the existence of smooth bump functions is trivial if $\mathfrak{g}$ is finite-dimensional. To settle the infinite-dimensional case, it suffices to construct smooth bump functions around $X = 0 \in \mathbb{R}^\infty$. To this end, let $U$ be any open zero-neighbourhood in $\mathbb{R}^\infty$. Inductively, we find a sequence of real numbers $r_n > 0$ such that $\mathbb{R}^\infty \cap \prod_{n \in \mathbb{N}} [-r_n, r_n] \subseteq U$. In fact, if $C := \prod_{n=1}^N [-r_n, r_n] \subseteq U$ for some $N \in \mathbb{N}$, then $U \cap \mathbb{R}^{N+1}$ is an open neighbourhood of the compact subset $C$ of $\mathbb{R}^{N+1}$. Since $C$ is compact, the neighbourhood $U \cap \mathbb{R}^{N+1}$ of $C$ is in fact a *uniform* neighbourhood of $C$ in $\mathbb{R}^{N+1}$, whence we find some $r_{N+1} > 0$ with $\prod_{n=1}^{N+1} [-r_n, r_n] = C + [-r_{N+1}, r_{N+1}] e_{N+1} \subseteq U \cap \mathbb{R}^{N+1}$. Let $h$ be a smooth function on $\mathbb{R}$ supported in $[-1, 1]$, such that $h(0) = 1$. We let $b \colon \mathbb{R}^\infty \to \mathbb{R}$ be the function given by $b(t_1, \ldots, t_n) := h(t_1/r_1) \cdot h(t_2/r_2) \cdot \ldots \cdot h(t_n/r_n)$ for $(t_1, \ldots, t_n) \in \mathbb{R}^n \subseteq \mathbb{R}^\infty$. Then $b$ is smooth, being smooth on each $\mathbb{R}^n$; furthermore, $b(0) = 1$ and $b|_{\mathbb{R}^\infty \setminus U} = 0$.

To deduce that $G$ is smoothly Hausdorff, assume that $g, h \in G$ are distinct elements. Let $W$ be a neighbourhood of $g$ which is diffeomorphic to an open subset of $\mathfrak{g}$; since $G$ is Hausdorff, we may assume that $h \notin W$. Now $G$ being regular, there exists a closed neighbourhood $U$ of $g$ in $G$, such that $U \subseteq W$. Since $\mathfrak{g}$ admits smooth bump functions, there is a smooth function $H \colon W \to \mathbb{R}$ such that $H|_{W \setminus U} = 0$. We extend $H$ to a function $F$ defined on all of $G$ by setting $F(x) := 0$ for $x \in G \setminus W$. Then $F$ is smooth on the open sets $W$ and $G \setminus U$, whose union is $G$: therefore $F$ is smooth. Furthermore, $F(g) = 1$ and $F(h) = 0$. Thus the smooth functions separate points on $G$, as required. □

**Remark 6.5.** We remark that the atlas constructed in the proof of Theorem 6.4 is real-analytic in the sense of [22, (27.1)], whence $G$ is an analytic convenient Lie group; it is the direct limit of $\mathcal{S}$ in the category of *analytic* convenient Lie groups. The proof of these assertions is completely analogous to the preceding proof in view of the definition of analytic maps (*loc. cit.* (10.3)) in convenient differential calculus. Similarly, if we are given an admissible directed system of finite-dimensional complex Lie groups and complex analytic homomorphisms, we obtain a complex analytic structure on the direct limit convenient Lie group.

**Remark 6.6.** The direct limit Lie groups constructed in Theorem 5.1 are also the direct limits in the category of convenient Lie groups, by arguments similar to those used in the proof of Theorem 6.4.

**Remark 6.7.** It is not known to the author whether all of the direct limit convenient Lie groups constructed above are smoothly Hausdorff (without extra hypotheses).

## 7. An instructive example

Let $I$ be an infinite set and $J$ be the set of finite subsets of $I$, directed by inclusion. We consider the group $G = \mathrm{GL}(I,\mathbb{R}) \subseteq \mathbb{R}^{I\times I}$ of $I\times I$-matrices $A$ such that $A-\mathbf{1} \in \mathbb{R}^{(I\times I)}$ and $A$ is invertible. Then $(G, (\phi_F)_{F\in J})$ is the direct limit group of the directed system $\mathcal{S} := ((G_F), (\phi_{EF}))$, where $G_F := \mathrm{GL}(\mathbb{R}^F)$ for $F \in J$ and $\phi_{EF}\colon A \mapsto A \oplus \mathrm{id}_{\mathbb{R}^{E\setminus F}}$ for $F \leq E$ (the homomorphisms $\phi_F\colon G_F \to G$ being defined analogously). Equip $G$ with the naïve direct limit topology. We let $\mathrm{gl}(I,\mathbb{R}) := \mathbb{R}^{(I\times I)}$ denote the real (non-unital) algebra of $I\times I$-matrices with only finitely many non-zero entries; as a Lie algebra, $\mathrm{gl}(I,\mathbb{R}) \cong \varinjlim \mathrm{gl}(\mathbb{R}^F) \cong \varinjlim \mathrm{L}(\mathrm{GL}(\mathbb{R}^F))$. If $I$ is countable, we make $G$ a Lie group modelled on the s.c.l.c. space $\mathrm{gl}(I,\mathbb{R}) \cong \mathbb{R}^\infty$; the group operation will be continuous, and the Lie bracket on $\mathrm{gl}(I,\mathbb{R})^2$ is continuous, as any bilinear map on this space. Of course, we can also consider $\mathrm{GL}(I,\mathbb{R})$ as the direct limit convenient Lie group. Now assume that $I$ is uncountable.

**Theorem 7.1.** *The above directed system $\mathcal{S}$ is admissible, whence* $\mathrm{GL}(I,\mathbb{R})$ *can be made the direct limit convenient Lie group of* $\mathcal{S}$. *Then* $\mathrm{GL}(I,\mathbb{R})$ *is smoothly Hausdorff, and the following holds*:

(a) $\mathrm{GL}(I,\mathbb{R})$ *is not a topological group, because the group multiplication* $\mu\colon \mathrm{GL}(I,\mathbb{R})^2 \to \mathrm{GL}(I,\mathbb{R})$ *is discontinuous with respect to the product topology on* $\mathrm{GL}(I,\mathbb{R})^2$.

(b) *Equip* $\mathrm{gl}(I,\mathbb{R}) := \mathbb{R}^{(I\times I)}$ *with the finest locally convex topology, or with the topology of finitely open sets. Then the matrix multiplication*

$$m\colon \mathrm{gl}(I,\mathbb{R}) \times \mathrm{gl}(I,\mathbb{R}) \to \mathrm{gl}(I,\mathbb{R})$$

*is discontinuous, and so is the Lie bracket*

$$[\cdot,\cdot]\colon \mathrm{gl}(I,\mathbb{R}) \times \mathrm{gl}(I,\mathbb{R}) \to \mathrm{gl}(I,\mathbb{R}).$$

*Here, the product is equipped with the respective product topology.*

*Proof.* The family of inclusions $\gamma_F\colon \mathrm{GL}(\mathbb{R}^F) \hookrightarrow \mathrm{GL}(\mathbb{C}^F)$ gives rise to a compatible family $(\pi_F)_{F\in J}$ of linear actions $\mathrm{GL}(\mathbb{R}^F) \times \mathbb{C}^F \to \mathbb{C}^F$. It is easy to verify the bounded growth condition (Remark 6.3), using the 2-norms

$\|\cdot\|_F\colon (r_i)_{i\in F}\mapsto \sqrt{\sum_{i\in F}|r_i|^2}$ on $\mathbb{C}^F$: hence $\mathcal{S}$ is admissible. All representations $\gamma_F$ being faithful, so is the the direct limit representation $\varinjlim \gamma_F$ corresponding to the action $\varinjlim \pi_F$. We deduce from Theorem 6.4 that $\mathrm{GL}(I,\mathbb{R})$ is smoothly Hausdorff.

(a) This part of the theorem is known, but we give the short proof. Consider for $F\in J$ the closed subgroup $H_F$ of $G_F$ consisting of all diagonal matrices with positive diagonal entries; we let $H$ denote the closed subgroup of $G$ which is the naïve direct limit of the groups $H_F$ (note that the considerations preceding Lemma 3.1 have analogues for closed subspaces). The compatible family of isomorphisms $(\eta_F)_{F\in J}$, where $\eta_F\colon \mathbb{R}^F\to H_F$ maps $(t_j)_{j\in F}$ to the diagonal matrix with entries $e^{t_j}$, induces an isomorphism of semitopological groups $\mathbb{R}^{(I)}\to H$, where $\mathbb{R}^{(I)}$ is equipped with the finite topology. By Example 3.5, $H$ is not a topological group, and hence neither is $G$.

(b) The proof is achieved via a series of lemmas. First, we discuss the case where $\mathrm{gl}(I,\mathbb{R})$ is equipped with the finest locally convex topology.

**Definition 7.2.** Let $V$ be a real vector space, and $(e_i)_{i\in A}$ be a basis for $V$. Given $r=(r_i)_{i\in A}\in(\mathbb{R}^+)^A$, we set $U(r):=\mathrm{conv}\{\pm r_ie_i\colon i\in A\}$ (here $\mathbb{R}^+ := ]0,\infty[$).

It is plain that the sets $U(r)$ form a basis of the filter $\mathcal{U}_0(V)$ of 0-neighbourhoods of $V$, equipped with the finest locally convex topology.

**Lemma 7.3.** *Let $V$ be a real vector space, $(e_i)_{i\in A}$ be a basis for $V$, and $\beta\colon V\times V\to X$ be a bilinear map into a real locally convex space $X$. Equip $V$ with the finest locally convex topology. Then the following holds*:

(i) *$\beta$ is continuous if and only if $\beta$ is continuous at $(0,0)$, i.e., if and only if for every convex symmetric 0-neighbourhood $W$ in $X$, there is $r\in(\mathbb{R}^+)^A$ such that $\beta(U(r)\times U(r))\subseteq W$.*

(ii) *If $W$ is a convex symmetric 0-neighbourhood in $X$ and $r\in(\mathbb{R}^+)^A$, we have $\beta(U(r)\times U(r))\subseteq W$ if and only if $\beta(r_ie_i,r_je_j)\in W$ for all $i,j\in A$.*

*Proof.* (i) It is well-known that multilinear maps between topological vector spaces are continuous if and only if they are continuous at the origin ([3, Chapter I, Section 1, No. 6, Proposition 5]).

(ii) The implication '$\Rightarrow$' is trivial. Conversely, suppose that $\beta(r_ie_i,r_je_j)\in W$ for all $i,j\in A$; then also $\beta(r_ie_i,-r_je_j)\in W$ for all $i,j$, by symmetry of $W$. Fix $i\in A$. Since $\beta(r_ie_i,\cdot)$ is linear and $W$ is convex, we deduce from $\beta(r_ie_i,\pm r_re_j)\in W$ for all $j$ that $\beta(r_ie_i,U(r))\subseteq W$. Fix $u\in U(r)$. Since $\beta(\pm r_ie_i,u)\in W$ for all $i\in A$ by the preceding, we conclude as above that $\beta(U(r),u)\subseteq W$. Since $u$ was arbitrary, $\beta(U(r)\times U(r))\subseteq W$ follows. □

**Lemma 7.4.** *Consider $\mathrm{gl}(I,\mathbb{R})$, equipped with the finest locally convex topology, where $I\geq\aleph_0$. Then the following statements are equivalent*:

(i) *The Lie bracket $[\cdot,\cdot]\colon \mathrm{gl}(I,\mathbb{R})\times\mathrm{gl}(I,\mathbb{R})\to\mathrm{gl}(I,\mathbb{R})$ is continuous*;

(ii) *Matrix multiplication $m:\mathrm{gl}(I,\mathbb{R})\times\mathrm{gl}(I,\mathbb{R})\to\mathrm{gl}(I,\mathbb{R})$ is continuous.*

*Proof.* Since matrix addition is continuous and so is taking negatives, the implication '(ii)⇒(i)' is obvious.

(i)⇒(ii): Suppose that the Lie bracket is continuous. We partition $I$ into three disjoint sets $I_1$, $I_2$, $I_3$ of equal cardinality and define

$$V_1 := \operatorname{span}\{E_{ij} : i \in I_1, j \in I_2\},$$
$$V_2 := \operatorname{span}\{E_{ij} : i \in I_2, j \in I_3\},$$
$$V_3 := \operatorname{span}\{E_{ij} : i \in I_1, j \in I_3\},$$

where the $E_{ij}$'s are the matrix units. Then $[V_1, V_2] \subseteq V_3$, and $[\cdot,\cdot]|_{V_1\times V_2}^{V_3}$ is continuous. For $k \in \{1,2,3\}$, there is a bijection $f_k : I \to I_k$ and a linear isomorphism $\phi_k : \mathrm{gl}(I,\mathbb{R}) \to V_k$ determined by

$$\begin{aligned} E_{ij} &\mapsto E_{f_1(i)f_2(j)} && \text{if} \quad k=1,\\ E_{ij} &\mapsto E_{f_2(i)f_3(j)} && \text{if} \quad k=2,\\ E_{ij} &\mapsto E_{f_1(i)f_3(j)} && \text{if} \quad k=3. \end{aligned}$$

Then $m = \phi_3^{-1} \circ [\cdot,\cdot]|_{V_1\times V_2}^{V_3} \circ (\phi_1 \times \phi_2)$; hence $m$ is continuous. □

We now recall the following fact from [2]:

**Lemma 7.5.** *A set $I$ is uncountable if and only if there is a function $g : I^2 \to \mathbb{R}^+$ such that for every function $f : I \to \mathbb{R}^+$, there is $(i,j) \in I^2$ such that $g(i,j) < f(i)f(j)$.*

**Lemma 7.6.** *The matrix multiplication $m : \mathrm{gl}(I,\mathbb{R})^2 \to \mathrm{gl}(I,\mathbb{R})$ is discontinuous if $\mathrm{gl}(I,\mathbb{R})$ is equipped with the finest locally convex topology, for every uncountable set $I$.*

*Proof.* The matrix units $E_{ij}$ (where $(i,j) \in I^2$) form a basis of $\mathrm{gl}(I,\mathbb{R})$; therefore the sets $U(r) := \operatorname{conv}\{\pm r_{ij}E_{ij} : (i,j) \in I^2\}$ (where $r = (r_{ij}) \in (\mathbb{R}^+)^{I\times I}$) constitute a filter basis for the filter of 0-neighbourhoods in $\mathrm{gl}(I,\mathbb{R})$. Let $g : I^2 \to \mathbb{R}^+$ be a function with the properties described in Lemma 7.5. I claim that $m(U(r) \times U(r)) \not\subseteq U(g)$, for every $r = (r_{ij}) \in (\mathbb{R}^+)^{I\times I}$. Replacing each $r_{ij}$ by $\min\{r_{ij}, r_{ji}\}$, we may assume that $r$ is symmetric. Fix any $i_0 \in I$ and define $f : I \to \mathbb{R}^+$ via $f(i) := r_{ii_0}$. By definition of $g$, there is a pair $(i,j) \in I^2$ such that $r_{ii_0}r_{i_0j} = r_{ii_0}r_{ji_0} = f(i)f(j) > g(i,j)$. Now $(r_{ii_0}E_{ii_0}, r_{i_0j}E_{i_0j}) \in U(r)\times U(r)$ and $m(r_{ii_0}E_{ii_0}, r_{i_0j}E_{i_0j}) = r_{ii_0}r_{i_0j}E_{ij} \notin U(g)$. We have proved that $m$ is not continuous at $(0,0)$; hence $m$ is discontinuous. □

Note that in the situation of the preceding lemma, the commutator bracket is discontinuous as well, by Lemma 7.4. Thus all assertions of Theorem 7.1 (b) concerning $\mathrm{gl}(I,\mathbb{R})$, equipped with the finest locally convex topology, are proved. The remainder of (b) can be deduced easily from the following lemma:

**Lemma 7.7.** *Let $V$ be a real vector space, $X$ be a locally convex vector space, and $\beta : V \times V \to X$ be a bilinear map. Let $\mathcal{O}_{\mathrm{fop}}$ be the topology of finitely open sets on $V$, and $\mathcal{O}_{\mathrm{lcx}}$ the finest locally convex topology. If $\beta : (V, \mathcal{O}_{\mathrm{fop}})^2 \to X$ is continuous at $(0,0)$, then so is $\beta : (V, \mathcal{O}_{\mathrm{lcx}})^2 \to X$.*

*Proof.* Let $W$ be a convex symmetric 0-neighbourhood in $X$. If the map $\beta\colon (V, \mathcal{O}_{\mathrm{fop}})^2 \to X$ is continuous at 0, there is a symmetric 0-neighbourhood $U$ in $(V, \mathcal{O}_{\mathrm{fop}})$ such that $\beta(U \times U) \subseteq W$. Set $U' := \mathrm{conv}(U)$; then $U'$ is convex, symmetric, and absorbing, and hence is a 0-neighbourhood in $(V, \mathcal{O}_{\mathrm{lcx}})$. Furthermore, as in the proof of Lemma 7.3 (b), we find that $\beta(U' \times U') \subseteq W$. Thus $\beta\colon (V, \mathcal{O}_{\mathrm{lcx}})^2 \to X$ is continuous at $(0, 0)$. □

To complete the proof of Theorem 7.1 (b), let $I$ be any uncountable set. The matrix multiplication and Lie bracket $(\mathrm{gl}(I, \mathbb{R}), \mathcal{O}_{\mathrm{lcx}})^2 \to (\mathrm{gl}(I, \mathbb{R}), \mathcal{O}_{\mathrm{lcx}})$ are discontinuous; by Lemma 7.3, these mappings are discontinuous at $(0, 0)$. We deduce from Lemma 7.7 that matrix multiplication and Lie bracket are also discontinuous at $(0, 0)$ when considered as mappings

$$(\mathrm{gl}(I, \mathbb{R}), \mathcal{O}_{\mathrm{fop}})^2 \to (\mathrm{gl}(I, \mathbb{R}), \mathcal{O}_{\mathrm{lcx}}).$$

Since $\mathcal{O}_{\mathrm{lcx}} \subseteq \mathcal{O}_{\mathrm{fop}}$, we deduce that matrix multiplication and Lie bracket are discontinuous *a fortiori* as mappings $(\mathrm{gl}(I, \mathbb{R}), \mathcal{O}_{\mathrm{fop}})^2 \to (\mathrm{gl}(I, \mathbb{R}), \mathcal{O}_{\mathrm{fop}})$. This completes the proof. □

## 8. Non-archimedian analogues

Most of the results obtained by now are not specific for real Lie groups and hold equally well for Lie groups over totally disconnected local fields, as we shortly sketch in the following.

Let $K$ be a totally disconnected commutative local field [35], with valuation ring $R$ and valuation ideal $P = \pi R$. For information concerning topological vector spaces over $K$, the reader is referred to [26]; the necessary background concerning $K$-Lie groups can be found in [33] and [4, Chapter 3].

We set $K^\infty := K^{(\mathbb{N})}$, equipped with the finite topology (which is defined as in the real case); it coincides with the finest vector space topology on $K^{(\mathbb{N})}$. Suppose that $X_1$ and $X_2$ are $K$-vector spaces of countable dimension (finite or infinite), equipped with their finite topologies, and $U$ an open subset of $X_1$. Let $f\colon U \to X_2$ be a continuous map, and $F$ a finite-dimensional subspace of $X_1$. For every $x \in F \cap U$, there is an open neighbourhood $C$ of $x$ in $F \cap U$ which is relatively compact in $F \cap U$. Then $f(C)$ is a relatively compact subset of a $K$-vector space equipped with the finite topology; by Proposition 3.6 (b), $f(C)$ has finite-dimensional span $S$. We say that $f\colon U \to X_2$ is *analytic* if it is continuous and if for every $F$, $x$, $C$, $S$ as above, the map $f|_C^S$ is analytic in the usual sense. If $V_1$, $V_2$, and $V_3$ are vector spaces of countable dimension, equipped with their finite topologies, and if $f\colon U_1 \to V_2$ and $g\colon U_2 \to V_3$ are analytic maps such that $f(U_1) \subseteq U_2$, where $U_1$ and $U_2$ are open subsets of $V_1$ and $V_2$, respectively, then the composition $g \circ f|^{U_2}$ is analytic. Hence analytic $K$-manifolds modelled on topological vector spaces of the above type, and analytic maps between these, can be defined in the usual way. All manifolds discussed below will be assumed to be of this form. A group $G$ equipped with an analytic $K$-manifold structure modelled on $K^\infty$ (or some $K^n$) with respect to which the group operations are analytic will be called a *Lie group of countable dimension* in the following.

**Lemma 8.1.** *Let $M$ be a finite-dimensional analytic $K$-manifold and $N$ be an analytic submanifold of $M$. Let $m := \dim M$ and $n := \dim N$. Suppose that $\psi\colon W \to V$ is a chart of $N$, where $W$ is an open compact subset of $N$ and $V$ an open compact subset of $K^n$, and suppose that $\Omega$ is an open neighbourhood of $W$ in $M$. Then there exists an open compact subset $U \subseteq \Omega$ of $M$ and a chart $\phi\colon U \to V{\times}R^{m-n}$ such that $U \cap N = W$ and $\phi|_W = \lambda \circ \psi$, where $\lambda\colon V \to V{\times}R^{m-n} : v \mapsto (v, 0)$.*

*Proof.* Let $W'$ be an open subset of $M$ such that $W' \cap N = W$. Since $N$ is a submanifold of $M$, every point $x \in W$ has an open compact neighbourhood $C \subseteq \Omega \cap W'$ in $M$ on which a chart $\gamma\colon C \to Q$ is defined such that $\gamma|_{C\cap N}^{Q\cap K^n}$ is a chart of $N$ (where we identify $K^n$ with the subspace $K^n{\times}\{0\}$ of $K^m$, and $Q$ is an open compact subset of $K^m$). By compactness, $W$ is covered by the domains $C_1, \ldots, C_k \subseteq \Omega$ of finitely many of these charts $\gamma_i\colon C_i \to Q_i$. Set $C_1' := C_1$ and $C_i' := C_i \backslash (C_1 \cup \cdots \cup C_{i-1})$ for $i = 2, \ldots, k$. Then the maps $\gamma_i|_{C_i'}^{\mathrm{im} C_i'}$ are also charts of the above type, whence we may assume w.l.o.g. that the sets $C_1, C_2, \ldots, C_k$ are disjoint.

Fix $i$. For every $z \in Q_i$, there exists a minimal number $s_z \in \mathbb{Z}$ such that the ball $z + \pi^{s_z} R^m$ is contained in $Q_i$, and clearly these balls partition $Q_i$. Note that there are finitely many maximal balls by compactness. Hence we find finitely many disjoint balls $B_1, \ldots, B_s \subseteq Q_i$ which cover $\gamma_i(W \cap C_i)$, such that $B_j \cap \gamma_i(W \cap C_i) \neq \emptyset$ for $j = 1, \ldots, s$. Now $\gamma_i$ can be replaced by the maps $\gamma_i|_{\gamma_i^{-1}(B_j)}^{B_j}$ (where $j = 1, \ldots, s$).

By the preceding, we may assume w.l.o.g. that every $Q_i$ is a ball and hence w.l.o.g. that $Q_i = R^m$ (thus $\gamma_i(W \cap C_i) = R^n \times \{0\}$).

Set $U := C_1 \cup \cdots \cup C_k$. Then $\Gamma(v, r) := \gamma_i^{-1}(\gamma_i(\psi^{-1}(v)) + (0, r))$ for $v \in \psi(C_i \cap W)$ defines a $\mathcal{C}^\omega$-diffeomorphism $\Gamma\colon V \times R^{m-n} \to U$, since the open subsets $C_1, \ldots, C_k$ partition $U$. Now $\phi := \Gamma^{-1}$ is the required chart. $\square$

**Proposition 8.2.** *Suppose that $\mathcal{S} = ((M_i)_{i\in I}, (\phi_{ji})_{j\geq i})$ is a countable directed system of finite-dimensional analytic $K$-manifolds such that every $\phi_{ji}$ is an embedding of analytic manifolds. Then the direct limit $(M, (\phi_i)_{i\in I})$ in $\mathbb{TOP}$ can be equipped with an analytic manifold structure which makes $(M, (\phi_i)_{i\in I})$ the direct limit of $\mathcal{S}$ in the category of analytic $K$-manifolds of countable dimension. All maps $\phi_i$ are embeddings of analytic manifolds*; *$M$ is regular and totally disconnected.*

*Proof.* We may assume that $I = (\mathbb{N}, \leq)$ and $M_1 \subseteq M_2 \subseteq \cdots \subseteq M$, the morphisms $\phi_{ji}$ and $\phi_i$ being the respective inclusion maps. Let $d_i := \dim M_i$.

Suppose that $x \in M_n$; let $\Omega$ be any open neighbourhood of $x$ in $M$. There is an open compact neighbourhood $U_n \subseteq \Omega$ of $x$ in $M_n$ and an open neighbourhood $V_n$ of 0 in $K^{d_n}$ such that there is a chart $\phi_n\colon U_n \to V_n$; w.l.o.g. $V_n = R^{d_n}$.

By the preceding lemma and induction, we find open compact subsets $U_k \subseteq \Omega$ of $M_k$ and charts $\phi_k\colon U_k \to R^{d_k}$ for $k > n$ such that $U_k \cap M_{k-1} = U_{k-1}$ and $\phi_k|_{U_{k-1}} = \lambda_{k-1} \circ \phi_{k-1}$, where $\lambda_{k-1}$ denotes inclusion $R^{d_{k-1}} \hookrightarrow R^{d_k} : r \mapsto (r, 0)$.

Set $U := \bigcup_{k\geq n} U_k$. Then $U \subseteq \Omega$, and $U$ is open and closed in the direct limit topology. Since $\Omega$ was arbitrary, we conclude that $M$ is regular and totally disconnected.

By Lemma 3.1, $U$ is the direct limit of its subspaces $U_k$ (with the inclusion maps), and this directed system is equivalent via the family $(\phi_k)_{k\geq n}$ to the directed system of the subspaces $R^{d_k}$ of the subspace $R^\infty$ of $K^\infty$ (or some $R^N$ if the dimensions $d_k$ are bounded), with direct limit $R^\infty$ (or $R^N$). Set $g_x\colon = \varinjlim \phi_k\colon U \to R^\infty$ (or $R^N$). As in the real case, one verifies that the maps $g_x$ form an analytic atlas for $M$ (where $x \in M$), and that $M$ has the asserted properties.

**Corollary 8.3.** *Let $\mathcal{S} = ((G_i)_{i\in I}, (\phi_{ji})_{j\geq i})$ be a countable directed system of finite-dimensional $K$-Lie groups and analytic embeddings $\phi_{ji}$, with direct limit $(G, (\phi_i)_{i\in I})$ in $\mathbb{TG}$. Then there exists a unique analytic manifold structure on $G$ which makes $(G, (\phi_i)_{i\in I})$ the direct limit of $\mathcal{S}$ in the category of $K$-Lie groups of countable dimension; every $\phi_i$ is an analytic embedding.* □

Let $G$ be a topological group. A *local $p$-adic one-parameter subgroup* of $G$ is a continuous homomorphism $\xi\colon U \to G$, where $U$ is an open subgroup of $\mathbb{Q}_p$. Its *germ at* 0 is the set of all local $p$-adic one-parameter subgroups $\zeta$ of $G$ such that $\xi$ and $\zeta$ coincide on some 0-neighbourhood. The set of all germs at 0 of local $p$-adic one-parameter subgroups of $G$ will be denoted by $\mathrm{Hom}_{loc}(\mathbb{Q}_p, G)$. If $G$ is a $p$-adic Lie group, it is well-known that its Lie algebra $\mathrm{L}(G)$ can be identified with $\mathrm{Hom}_{loc}(\mathbb{Q}_p, G)$ in a natural way. The identification can be described as follows: Let $\phi\colon M \to G$ be an exponential function for $G$, defined on some open $\mathbb{Z}_p$-submodule $M$ of $\mathrm{L}(G)$ (see [4, Chapter 3, Sections 4.3 and 4.2, Lemma 3 (iii)]). Then $X \in \mathrm{L}(G)$ corresponds to the germ at 0 of the local $p$-adic one-parameter subgroup $\xi\colon p^k\mathbb{Z}_p \to G$, $t \mapsto \phi(tX)$, where $k \in \mathbb{N}_0$ is chosen so large that $p^kX \in M$.

Along the lines of Proposition 5.2 and paragraph 5.3 above, we deduce:

**Corollary 8.4.** *The direct limit topological group $(G, (\phi_i)_{i\in I})$ of any countable strict directed system $\mathcal{S} = ((G_i)_{i\in I}, (\phi_{ji})_{j\geq i})$ of finite-dimensional $p$-adic Lie groups can be given a $p$-adic Lie group structure which makes it the direct limit of $\mathcal{S}$ in the category of $p$-adic Lie groups of countable dimension. The set $\mathrm{Hom}_{loc}(\mathbb{Q}_p, G)$ of germs at 0 of local $p$-adic one-parameter subgroups can be identified with the direct limit Lie algebra $\varinjlim \mathrm{L}(G_i)$, and every local $p$-adic one-parameter subgroup of $G$ is an analytic mapping.* □

The classes of manifolds and Lie groups "of countable dimension", and the corresponding notion of analytic map, are slightly special. After this research was completed, a general differential calculus of smooth mappings between open subsets of topological vector spaces over non-discrete topological fields has been developed [1]. It can be shown that the smooth Lie groups underlying the direct limit Lie groups constructed in the present section are also the direct limits of the given directed systems in the category of smooth Lie groups modelled on (arbitrary) topological $K$-vector spaces [10]; likewise for manifolds.

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