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Equivalence of graded module braids and interlocking sequences

By

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Abstract

The category of totally ordered graded module braids and that of the exact interlocking sequences are shown to be equivalent. As an application of this equivalence, we show the existence of a connection matrix for a totally ordered graded module braid without assuming the existence of chain complex braid that induces the given graded module braid.

1. Introduction

The connection matrix theory has been an useful tool for topological studies of dynamical systems [2], [3]. When an isolated invariant set of a dynamical system admits a Morse decomposition, a connection matrix for the decomposition describes the relation of the homological Conley index of the isolated invariant set and that of the isolated invariant subsets. This provides information about the structure of connecting orbits, and furthermore, the difference of the connection matrices between distinct parameter values often provides information about the bifurcation of the connecting orbits. The application of the connection matrix theory includes the transition matrix, which is a systematic tool for detecting bifurcations [7], and simplicial model for attractors [4]. A general survey of the Conley index theory is given by Mischaikow and Mrozek [5].

The existence of a connection matrix for a Morse decomposition was proved by Franzosa [3]. To a Morse decomposition, he associated a filtration of the space, which induces a chain complex braid. By taking homology, this chain complex braid then induces a graded module braid consists of the Conley indices. Then he proved the existence of a connection matrix for this graded module braid, which is called a connection matrix for the Morse decomposition. More precisely, he purely algebraically proved that every graded module braid induced from a chain complex braid has a connection matrix, and since the graded module braid of the homological Conley index is induced from a chain complex braid, the existence of the connection matrix for a Morse decomposition with respect to homological Conley index follows. (For another

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proof of the existence of a connection matrix, where the matrix is explicitly written using the Alexsander-Spanier cochain complex, see Capiński [1]. For a generalization of the connection matrix theory for discrete dynamical systems, see Richeson [8].)

It is unknown whether every graded module braid is induced from a chain complex braid. For this reason, we can not apply the connection matrix theory when we are dealing with some variants of the Conley index, for example, generalized cohomological Conley index.

On the other hand, interlocking sequences also has been used to study the modules associated to filtrations of topological spaces. The definition of interlocking sequences seems less intuitive than that of graded module braids, but algebraic properties of the category of interlocking sequences are well known due to the work of Street [10], [11], [12].

In this paper, we show the equivalence of totally ordered graded module braids and interlocking sequences. This gives us a different view of interlocking sequences and enables us to use the result on interlocking sequences for the study of graded module braid and connection matrix theory. We apply this equivalence to the existence problem of connection matrix described above. Namely, we show the existence of a connection matrix for a totally graded module braid that is not assumed to be induced from a chain complex braid.

Remark. For simplicity, we will work on the category of R-modules where R is a ring with projective dimension less or equals to 1. However, the results in the paper can be applied to any homological functors on any category with projective dimension less than or equal to 1.

2. Definitions

For an abelian category $\mathcal{C}$, the category of chain complexes and chain maps over $\mathcal{C}$ is denoted by $\mathsf{C}\mathcal{C}$ and the category of chain complexes and chain homotopy classes is denoted by $\mathsf{K}\mathcal{C}$. The category of graded objects over $\mathcal{C}$ is denoted by $\mathsf{G}\mathcal{C}$.

We fix a commutative ring R with projective dimension less than or equal to 1 as the coefficient of all modules. Denote the category of R-modules by $\mathcal{M}$.

Let $(\mathcal{P}, \prec)$ be a finite partially ordered set. We say that $I \subset \mathcal{P}$ is an *interval* if $\pi, \pi'' \in I$ and $\pi \prec \pi' \prec \pi''$ imply $\pi' \in I$. An k-tuple of mutually disjoint intervals $(I_1, I_2, \ldots, I_k)$ is said to be *adjacent* if $I_1 \cup I_2 \cup \cdots \cup I_k$ is also an interval and if $\pi \in I_i$ and $\pi' \in I_j$ with $i < j$ imply $\pi' \not\prec \pi$. The set of intervals in $\mathcal{P}$ is denoted by $\mathcal{I}$. If $\mathcal{P}$ is totally ordered, we write it as $\mathcal{P}_N = \{1, 2, \ldots, N\}$ where the order of $\mathcal{P}_N$ is the usual order $<$ of the natural numbers. The set of intervals of $\mathcal{P}_N$ is denoted by $\mathcal{I}_N$. Two intervals I and J are said to be *noncomparable* if both (I, J) and (J, I) are adjacent.

Now we define four categories of our interest.

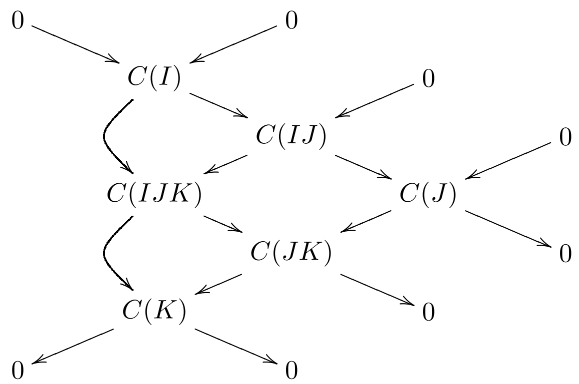
Definition 1 (Filtered Complexes)**.** The category $\mathcal{A}_N$ is defined to be $\mathsf{K}\mathcal{B}_N$, and $\mathcal{B}_N$ is defined as follows: An object of $\mathcal{B}_N$ is an $(N-1)$-tuple $A = (A^1, A^2, \ldots, A^{N-1})$ of objects of $\mathcal{M}$, such that A^i is a submodule of A^{i+1}

and A^{i+1}/A^i is projective in $\mathcal{M}$. For an object A of $\mathcal{A}_N$, we always put $A^0 = 0$. A morphism $f : A \to B$ of $\mathcal{B}_N$ is an $(N-1)$-tuple of morphism $f^i : A^i \to B^i$ which commute with inclusions.

Definition 2 (Chain Complex Braids)**.** The category $\mathcal{C}_N$ is defined to be $\mathsf{K}\mathcal{D}_N$, and $\mathcal{D}_N$ is defined as follows: An object C of $\mathcal{D}_N$ consists of a family $\{C(I) \mid I \in \mathcal{I}_N\}$ of objects of $\mathcal{M}$ where $C(\emptyset) = 0$, and arrows

$$i(I, IJ) : C(I) \to C(IJ), \qquad p(IJ, J) : C(IJ) \to C(J)$$

of $\mathcal{M}$ for each adjacent intervals (I, J). Further, the diagram



is required to be exact and commutative for each adjacent triple (I, J, K), and $p(JI, I)i(I, IJ) = \mathrm{id}|_{C(I)}$ for noncomparable intervals I and J. A morphism of $\mathcal{D}_N$ is a collection of commutative arrows between diagrams.

Definition 3 (Interlocking Sequences)**.** The category $\mathcal{X}_N$ is defined as follows: an object of $\mathcal{X}_N$ is a diagram D consists of a collection D_{uv} of objects of $\mathcal{M}$ for each ordered pair of integers u, v such that $u - N < v < u$, and a collection of arrows $d^{st}_{uv} : D_{st} \to D_{uv}$ in $\mathcal{M}$ for $s - N < t < s$ and $u - N < v < u$ such that $d^{st}_{uv} = 0$ unless $u - N < t \leq v < s \leq u$. These collections are required to satisfy the commuting condition $d^{uv}_{wx} \cdot d^{st}_{uv} = d^{st}_{wx}$ when $w - N < v \leq x < u \leq w$, $u - N < t \leq v < s \leq u$ and $w - N < t \leq x < s \leq w$ hold. A morphism of $\mathcal{X}_N$ is a collection of commutative arrow between diagrams.

An object D of $\mathcal{X}_N$ is said to be *exact* if

$$D_{uv} \xrightarrow{d^{uv}_{tv}} D_{tv} \xrightarrow{d^{tv}_{tu}} D_{tu} \xrightarrow{d^{tu}_{N+v,u}} D_{N+v,u}$$

is an exact sequence in $\mathcal{C}$ for each $t - N < v < u < t$. The full subcategory of $\mathcal{X}_N$ consists of the exact objects will be denoted by $\mathcal{X}^e_N$.

Definition 4 (Graded Module Braids)**.** The category $\mathcal{G}_N$ is defined as follows: an object G of $\mathcal{G}_N$ consists of a family $\{G(I) \mid I \in \mathcal{I}_N\}$ of objects of

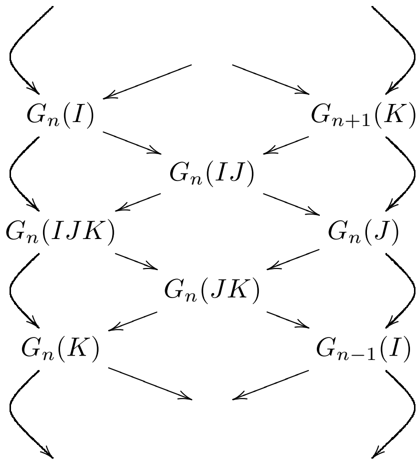
$\mathsf{G}\mathcal{M}$ where $G(\emptyset) = 0$, and arrows

$$\begin{aligned} i(I, IJ) &: G(I) \to G(IJ) \quad \text{of degree } 0 \\ p(IJ, J) &: G(IJ) \to G(J) \quad \text{of degree } 0 \\ \partial(J, I) &: G(J) \to G(I) \quad \text{of degree } -1 \end{aligned}$$

of $\mathsf{G}\mathcal{M}$ such that

$$\cdots \longrightarrow G_n(I) \xrightarrow{i(I,IJ)} G_n(IJ) \xrightarrow{p(IJ,J)} G_n(J) \xrightarrow{\partial(J,I)} G_{n-1}(I) \longrightarrow \cdots$$

is an exact sequence in $\mathcal{M}$ for each adjacent intervals (I, J). Furthermore, the diagram



is required to be commutative for each adjacent (I, J, K), and $p(JI, I)i(I, IJ) = \mathrm{id}|_{G(I)}$ for noncomparable intervals I and J. A morphism of $\mathcal{G}_N$ is a collection of commutative arrows between diagrams.

Now we see some relations among these categories that immediately follows from the definitions.

First, a filtered complex induces a chain complex braid. Precisely, if A is an object of $\mathcal{A}_N$, define an object C of $\mathcal{C}_{N-1}$ by $C(I) = A^l/A^{k-1}$ for $I = \{k, k+1, \ldots, l\}$ and $C(\emptyset) = 0$. It is easy to see that this procedure give rise to a functor $Q_N : \mathcal{A}_N \to \mathcal{C}_{N-1}$.

Next we construct $F_N : \mathcal{A}_N \to \mathcal{X}_N^e$. For an object A of $\mathcal{A}_N$ define $D = F_N A$ by $(E_{pq}D)_n = h_n A^{pq}$ where h denotes the homology functor and $A^{pq} := A^p/A^q$. Arrows d_{uv}^{st} are defined by the arrows induced from short exact sequences

$$0 \longrightarrow A^{qr} \longrightarrow A^{pr} \longrightarrow A^{pq} \longrightarrow 0$$

for $r < q < p$ and the commuting condition $d_{wx}^{uv} \cdot d_{uv}^{st} = d_{wx}^{st}$. For a morphism $f : A \to B$ in $\mathsf{C}\mathcal{B}_N$, define $F_N f : F_N A \to F_N B$ to be the collection $h_n f^{pq} : h_n A^{pq} \to h_n B^{pq}$. Then F_N defines a functor $\mathsf{C}\mathcal{B}_N \to \mathcal{X}_N^e$. Since $F_N f = 0$ if f

is homotopic to zero, F_N induces a functor $\mathcal{A}_N \to \mathcal{X}_N^e$, which is again denoted by F_N.

Similarly, we define $H_N : \mathcal{C}_N \to \mathcal{G}_N$ by taking homology of a complex. An object G of $\mathcal{G}_N$ is said to be *chain complex generated* if there exists an object C of $\mathcal{C}_N$ such that $G = H_N C$.

3. Equivalence

In this section, we show that $\mathcal{X}_N^e$ is equivalent to $\mathcal{G}_{N-1}$.

Theorem 1. *There exist functors $R_N : \mathcal{X}_N^e \to \mathcal{G}_{N-1}$ and $S_N : \mathcal{G}_N \to \mathcal{X}_{N+1}^e$ such that $R_N S_{N-1}$ and $S_{N-1} R_N$ are identity functors.*

Proof. We first define R_N. Let D be an object of $\mathcal{X}_N^e$. We define $G = R_N D$ as follows. For an interval $I = \{i, i+1, \dots, j\}$, let $G(I) = E_{j,i-1}D$, and for $\emptyset$, let $G(\emptyset) = 0$. Arrows $i(I, IJ)$, $p(IJ, J)$ and $\partial(J, I)$ are defined to be the graded arrows consists of corresponding arrows d_{uv}^{st} in D. The commutativity of the braid G follows from the commuting condition $d_{wx}^{uv} = d_{uv}^{st} \cdot d_{wx}^{st}$. Now let (I, J) be an adjacent pair of intervals. Exactness of the long exact sequence associated to (I, J) follows from the exactness of the sequence

$$\cdots \longrightarrow D_{u,t-N} \longrightarrow D_{uv} \longrightarrow D_{tv} \longrightarrow D_{tu} \longrightarrow D_{N+v,u} \longrightarrow D_{N+v,t} \longrightarrow \cdots$$

in D where $I = \{v+1, \dots, u\}$ and $J = \{u+1, \dots, t\}$. Thus, G is an object of $\mathcal{G}_{N-1}$. Similarly we define R_N on morphisms.

Before constructing S_{N-1}, we define a functor $E_{pq} : \mathcal{X}_N \to \mathsf{G}\mathcal{M}$ for $0 \le q < p < N$ by

$$(E_{pq}D)_{2n} = D_{p-nN,q-nN},$$
$$(E_{pq}D)_{2n-1} = D_{q-(n-1)N,p-nN}$$

on objects, and similarly for morphisms. Note that for any u, v such that $u - N < v < u$ there is unique integer n such that either

$$\text{Type A:} \quad -nN \le v < u < -nN + N \quad \text{or}$$
$$\text{Type B:} \quad -nN < v < -nN + N \le u$$

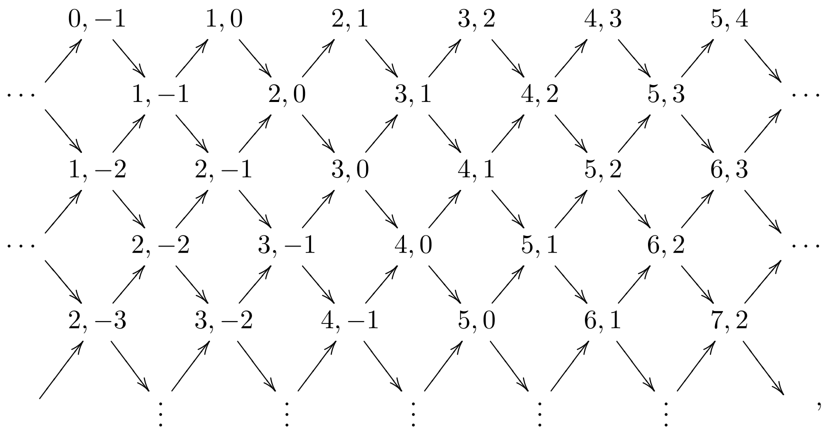
occurs. In both cases, we can express u, v using integers $0 \le q < p < N$ and n by letting $p = u + nN$ and $q = v + nN$ (hence $u = p - nN$, $v = q - nN$) if u, v are of Type A, or $p = v + nN$ and $q = u + (n-1)N$ (hence $u = q - (n-1)N$, $v = p - nN$) if u, v are of Type B. This implies the modules contained in an object D of $\mathcal{X}_N$ are completely determined by $(E_{pq}D)_n$ for all $0 \le q < p < N$ and n.

Now we define the functor S_{N-1} as follows.

Let G be an object of $\mathcal{G}_{N-1}$. We determine the modules contained in D by the equality

$$(E_{pq}D)_n = G_n(\{q+1, \dots, p\})$$

for each $0 \leq q < p < N$, and define arrows $d^{st}_{s+1,t}$ and $d^{st}_{s,t+1}$ to be corresponding arrows in G. To define the other arrows in D, we consider the following web diagram



where p, q denotes the module D_{pq} and all arrows are $d^{st}_{s+1,t}$ or $d^{st}_{s,t+1}$ for some s, t. Note that every rectangle in the web diagram takes the form of

$$\begin{array}{ccc} & u, v & \\ \swarrow & & \searrow \\ u+1, v & & u, v+1 \\ \searrow & & \swarrow \\ & u+1, v+1 & \end{array} .$$

Choosing the intervals I, J and K suitably, the commutativity of this rectangle follows form the commutativity of the diagram

$$\begin{array}{ccc} & G_n(IJ) & \\ \swarrow & & \searrow \\ G_n(IJK) & & G_n(J) \\ \searrow & & \swarrow \\ & G_n(JK) & \end{array}$$

of G if the pairs (u, v) and $(u + 1, v + 1)$ are of the same type, and from

$$\begin{array}{ccc} & G_n(JK) & \\ \swarrow & & \searrow \\ G_n(K) & & G_{n-1}(I) \\ \searrow & & \swarrow \\ & G_{n-1}(IJ) & \end{array}$$

if these are of different type. Therefore the web diagram above is commutative, hence we can define d^{st}_{uv} to be the composition of these $d^{st}_{s+1,t}$ and $d^{st}_{s,t+1}$ along an arbitrary path connecting u, v and s, t. It then follows form the construction that the commuting condition $d^{uv}_{wx} = d^{st}_{uv} \cdot d^{st}_{wx}$ is satisfied, and it follows from the exactness of G that $d^{st}_{uv} = 0$ unless $u - N < t \leq v < s \leq u$. Hence D is an object of $\mathcal{X}_N$. Furthermore, we can show that D is exact using the converse of the argument in the definition of R_N. Thus, D is an object of $\mathcal{X}^e_N$. We define S_{N-1} on morphisms in a similar manner, as before.

To finish the proof of Theorem 1, we need to check that $R_N S_{N-1}$ and $S_{N-1} R_N$ are identity functors. It is clear that $R_N S_{N-1}$ and $S_{N-1} R_N$ do not affect on morphisms, so what we need to check is that $R_N S_{N-1} G = G$ and $S_{N-1} R_N D = D$. But each module and arrow in D and G is also not changed by $R_N S_{N-1}$ and $S_{N-1} R_N$, respectively, thus these equalities hold. □

4. Application

In this section, we use the result of the preceding section to prove the existence of the connection matrix for a graded module braid that is not guaranteed to be chain complex generated.

We begin with defining the connection matrix and the connection matrix pair. Let $\mathcal{P}$ be a finite partially ordered set. We say that an R-module C admits a $\mathcal{P}$-*splitting* if there exists a splitting $C = \bigoplus_{\pi \in \mathcal{P}} C(\pi)$. If C and D are R-modules with $\mathcal{P}$-splittings and Δ is a morphism

$$\Delta : \bigoplus_{\pi \in \mathcal{P}} C(\pi) \to \bigoplus_{\pi \in \mathcal{P}} D(\pi),$$

we regard Δ as a matrix $(\Delta_{\pi,\pi'})_{\pi,\pi' \in \mathcal{P}}$ where each entry is a morphism $\Delta_{\pi,\pi'} : C(\pi') \to D(\pi)$. We say that Δ is *upper triangular* if $\Delta_{\pi,\pi'} \neq 0$ implies $\pi \preceq \pi'$. Similarly Δ is said to be *strictly upper triangular* if $\Delta_{\pi,\pi'} \neq 0$ implies $\pi \prec \pi'$. For an interval I, the submatrix $(\Delta_{\pi,\pi'})_{\pi,\pi' \in I} : C(I) \to D(I)$ is denoted by $\Delta(I)$.

Note that if $\Delta : C \to C$ is an upper triangular boundary map on a R-module with $\mathcal{P}$-splitting, then $\{(C(I), \Delta(I)) \mid I \in \mathcal{I}\}$ is a chain complex braid. In fact we can choose the inclusion map as $i(I, IJ)$ and the projection map as $p(IJ, J)$. We denote the graded module braid induced from $\{(C(I), \Delta(I)) \mid I \in \mathcal{I}\}$ by $\mathcal{H}\Delta$.

Definition 5 (Connection Matrix)**.** Let $G = \{G(I) \mid I \in \mathcal{I}\}$ be a graded module braid and $C = \{C(\pi) \mid \pi \in \mathcal{P}\}$ a collection of modules, $\Delta : \bigoplus_{\pi \in \mathcal{P}} C(\pi) \to \bigoplus_{\pi \in \mathcal{P}} C(\pi)$ an upper triangular boundary map. If $\mathcal{H}\Delta$ is isomorphic to G, then Δ is called a *C-connection matrix* of G.

The connection matrix pair, an analog of the connection matrix for discrete dynamical system is defined by Richeson [8].

Definition 6 (Connection Matrix Pair)**.** Let $G = \{G(I) \mid I \in \mathcal{I}\}$ be a graded module braid and $\chi : G \to G$ a graded module braid homomorphism, $\Delta, A : \bigoplus_{\pi \in \mathcal{P}} G(\pi) \to \bigoplus_{\pi \in \mathcal{P}} G(\pi)$ maps where Δ is strictly upper triangular boundary map and A is an upper triangular map. If $\mathcal{H}\Delta$ is isomorphic to G, and the map A induces on $\mathcal{H}\Delta$ is conjugate to χ, then (Δ, A) is called a *connection matrix pair* of (G, χ).

The existence of the connection matrix and the connection matrix pair is established only for chain complex generated graded braid modules. Thus it is natural to ask when a given graded module braid is chain complex generated. We use the following result to show that every graded module braid is chain complex generated if it is totally ordered.

Theorem 2 (Street [10], Theorem 16)**.** *The functor $F_N : \mathcal{A}_N \to \mathcal{X}_N^{\mathrm{e}}$ has a right inverse functor $V_N : \mathcal{X}_N^{\mathrm{e}} \to \mathcal{A}_N$.*

Corollary 1. *For any totally ordered graded module braid G and a collection $C = \{(C(\pi), \partial(\pi)) \mid \pi \in \mathcal{P}_N\}$ of free chain complexes such that the homology group of each $(C(\pi), \partial(\pi))$ is isomorphic to $G(\pi)$, there exists a C-connection matrix.*

Proof. By the theorem of Franzosa [3], it suffice to show that G is chain complex generated, that is, there exists a chain complex braid C' such that $H_N(C') = G$. But Theorem 2 assures that there exists a filtered complex A such that $F_{N+1}(A) = S_N(G)$. Since the diagram of functors

$$\begin{array}{ccc} \mathcal{A}_N & \xrightarrow{Q_N} & \mathcal{C}_{N-1} \\ V_N \uparrow\downarrow F_N & & \downarrow H_{N-1} \\ \mathcal{X}_N^{\mathrm{e}} & \underset{S_{N-1}}{\overset{R_N}{\rightleftarrows}} & \mathcal{G}_{N-1} \end{array}$$

commutes, we can choose $C' = P_N(A)$. □

Similarly, we can prove the following from the theorem of Richeson [8] and Theorem 2 above, but in this case we need to assume that G is free since the existence of the connection matrix pair is proved only for free graded module braids.

Corollary 2. *For any totally ordered graded free module braid G with endomorphism $\psi : G \to G$, there exists a connection matrix pair.*

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