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Finsler geometry of projectivized vector bundles

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Introduction

The purpose of this article is to reformulate the algebraic geometric concept of ampleness (and the numerical effectiveness) of a holomorphic vector bundle E in terms of Finsler geometry (cf. also Aikou [2], [3]; for general theory on Finsler geometry we refer to [5], [6] and [1]). We also provide some implications of this reformulation. For applications of the formulation using projectivized bundles to complex analysis see [7], [8] and [17]. As expected, the condition involves the concept of a Finsler metric along the fibers of E. By a (complex) Finsler metric (see Section 4 for more details) on E we mean a non-negative function h on E with the following properties:

(FM1) *h is an upper semi-continuous function on E;*
(FM2) *$h(z, \lambda v) = |\lambda| h(z, v)$ for all $\lambda \in \mathbf{C}$ and $(z, v) \in E_z$;*
(FM3) *$h(z, v) > 0$ on $E \backslash \{$zero-section$\}$;*
(FM4) *for z and v fixed the function $h^2(z, \lambda v)$ is smooth even at $\lambda = 0$.*

For example the Kobayashi metric on a hyperbolic manifold is a Finsler metric on the tangent bundle. More generally, any intrinsic (i.e., depending only on the complex structure) (pseudo)-metric of a complex manifold is a Finsler (pseudo)-metric (i.e., (FM3) is replaced by the weaker condition $h(z, v) \geq 0$ on E). Obviously the norm of a *Hermitian* metric on E is Finsler and satisfies, among others, the following additional conditions:

(FM5) *h is of class $\mathcal{C}^0$ on E and of class $\mathcal{C}^\infty$ on $E \backslash \{$zero-section$\}$;*
(FM6) *h is strictly pseudoconvex on $E_z \setminus \{0\}$ for all $z \in M$.*

This last two properties are, in general, not shared by the intrinsic metrics, e.g., the Kobayashi metric is not even continuous unless it is complete (so M is complete hyperbolic); and, even in the complete case, it is in general not smooth outside the zero section. On the other hand, as we shall see, there are many Finsler metrics with these additional properties which are not Hermitian. Without these last 2 conditions differential geometric concepts

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become very complicated, if not impossible, to deal with. For this reason, in this article, we shall only work with Finsler metrics with these additional properties. With these conditions the mixed holomorphic bisectional curvature of E can be defined (see Sections 2 and 4). The term "mixed" refer to the fact that we shall be considering curvature in two directions: one in the space direction and the other in the fiber direction. If E is the tangent bundle and h a Hermitian metric this coincides with the usual notion of holomorphic bisectional curvature.

Obviously a Finsler metric on a holomorphic line bundle E is Hermitian and the ampleness of E is equivalent to the existence of a Hermitian metric along the fibers whose Chern form is a positive definite $(1,1)$-form on M. Thus we shall consider only vector bundles of rank at least 2. For a Finsler metric on E the curvature we use is the curvature of the Chern connection. This curvature has many components: horizontal (or base) components, vertical components and mixed components. The main purpose is to clarify which piece of the curvature carries the information for the bundle to be ample. The main result is the following Theorem (see Theorem 5.5).

Theorem. *Let E be a rank $r \geq 2$ holomorphic vector bundle over a compact complex manifold M. For any positive integer k, denote by $\odot^k E$ the k-fold symmetric product and by $\mathcal{L}_{\mathbf{P}(\odot^k E)}$ the dual of the tautological line bundle over the projectivized bundle $\mathbf{P}(\odot^k E)$. Then the following statements are equivalent*:

(1) E^* *is ample*;

(2) $\mathcal{L}_{\mathbf{P}(E)}$ *is ample*;

(3) $\odot^k E^*$ *is ample for some positive integer k*;

(4) $\mathcal{L}_{\mathbf{P}(\odot^k E)}$ *is ample for some positive integer k*;

(5) $\odot^k E^*$ *is ample for all positive integer k*;

(6) $\mathcal{L}_{\mathbf{P}(\odot^k E)}$ *is ample for all positive integer k*;

(7) *there exists a Finsler metric along the fibers of E with negative mixed holomorphic bisectional curvature*;

(8) *for some positive integer k there exists a Finsler metric along the fibers of $\odot^k E$ with negative mixed holomorphic bisectional curvature*;

(9) *for all positive integer k there exists a Finsler metric along the fibers of $\odot^k E$ with negative mixed holomorphic bisectional curvature*;

(10) *there exists a positive integer m and a Hermitian metric along the fibers of $\odot^m E$ with negative mixed holomorphic bisectional curvature.*

Note that the metric in part (10) of the preceding Theorem is Hermitian not merely Finsler and the positive integer m can be taken to be any integer so that $\mathcal{L}^m_{\mathbf{P}(E)}$ is *very* ample (for nef or spanned bundles see Remark 5.6). The preceding Theorem can also be formulated in terms of the dual bundle (see Theorem 5.7):

Theorem. *Let E be a rank $r \geq 2$ holomorphic vector bundle over a compact complex manifold M. Then the following statements are equivalent*:

(1) *E is ample*;
(2) *$\mathcal{L}_{\mathbf{P}(E^*)}$ is ample*;
(3) *$\odot^k E$ is ample for some positive integer k*;
(4) *$\mathcal{L}_{\mathbf{P}(\odot^k E^*)}$ is ample for some positive integer k*;
(5) *$\odot^k E$ is ample for all positive integer k*;
(6) *$\mathcal{L}_{\mathbf{P}(\odot^k E^*)}$ is ample for all positive integer k*;
(7) *there exists a Finsler metric along the fibers of E with positive mixed holomorphic bisectional curvature*;
(8) *for some positive integer m there exists a Finsler metric along the fibers of $\odot^k E$ with positive mixed holomorphic bisectional curvature*;
(9) *for all positive integer m there exists a Finsler metric along the fibers of $\odot^k E$ with positive mixed holomorphic bisectional curvature*;
(10) *there exists a positive integer m and a Hermitian metric along the fibers of $\odot^m E$ with positive mixed holomorphic bisectional curvature.*

In the special case of the tangent bundle we can say a little more (see Corollary 5.8):

Corollary. *Let M be a compact complex manifold of dimension $n \geq 2$. Then TM is ample (resp. nef) if and only if the anti-canonical bundle $\mathcal{K}^{-1}_{\mathbf{P}(T^*M)}$ is ample (resp. nef) if and only if there exists a Finsler metric on M with negative holomorphic bisectional curvature if and only if there exists a positive integer m and a Hermitian metric along the fibers of $\odot^m TM$ with positive mixed holomorphic bisectional curvature.*

By a result of Mori the ampleness of the tangent bundle implies that M is the complex projective space. The preceding Corollary provides alternative characterizations. We also obtain the following vanishing Theorem (see Corollary 5.9) for nef bundles (cf. [9]):

Corollary. *If E is a nef holomorphic vector bundle of rank $r \geq 2$ over a compact complex manifold M of dimension n, then*

$$\begin{cases} H^i(M, \odot^m E \otimes \det E \otimes \mathcal{K}_M) = 0, \\ H^i(M, \odot^m(\otimes^k E) \otimes (\det \odot^m(\otimes^k E)) \otimes \mathcal{K}_M) = 0 \end{cases}$$

for all $i, m, k \geq 1$. Consequently, if $E = TM$ then $H^i(M, \odot^m TM) = 0$ for all $i, m \geq 1$.

The local calculations towards these results are valid even in the non-compact case. We use these to address the following question.

Is the bundle space of a hermitian holomorphic vector bundle (E, h), over a Kähler manifold (M, g), Kähler?

The answer is clearly yes if M is compact. As the fibers and the base are both Kähler the most natural way is to consider the natural metric induced by these metrics. We show that this natural metric is Kähler if and only if

(M, g) is flat (see Theorem 2.1 and Corollary 2.2). A necessary and sufficient condition is also obtained if we take only a Finsler metric on E (see Theorem 6.8 and Corollary 5.9). Thus, one must look for other means of constructing a Kähler metric on the bundle space E. We are only able to get a partial result (see Corollary 3.5). The general case remains open. The computations are carried out first for Hermitian metrics and then for Finsler metrics. We could have worked out directly the Finsler case and the Hermitian case follows as a special case. We decide on the former presentation as the Hermitian case is less technical and may be useful for those who prefer only to work with Hermitian metrics.

We would also like to express our gratitude to the referee for pointing out an error in our original proof of Proposition 4.4.

1. Riemannian metric on TTM

Before dealing with Kähler manifolds we review briefly the case of Riemannian manifolds (see Besse [4]). Let (M, g) be a Riemannian manifold of dimension n and $\pi : TM \to M$ the tangent bundle. It is quite obvious that the most natural way to put a Riemannian metric on TTM is to decompose the bundle in some natural way as a direct sum of a "vertical" and a "horizontal" sub-bundle each with a natural metric and the direct sum of these is a metric on TTM. The *vertical sub-bundle* is, by definition, the kernel of π_*:

$$\mathcal{V} = \ker \pi_* \subset TTM,$$

where $\pi_* : TTM \to TM$ is the differential of π. In other words, it is the sub-bundle consisting of all vectors tangent to the fibers of $\pi : TM \to M$. There is a distinguished section, the position vector field P, of the vertical bundle. The most convenient way to describe this is via local coordinates. In terms of a local coordinate system $(U; x^1, \ldots, x^n)$, an element $v \in T_pM$ is of the form

$$v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}|_p.$$

Obviously we may identify T_pM with $\mathbf{R}^n$ via the identification:

$$\sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}|_p \mapsto (v^1, \ldots, v^n).$$

hence, on $\pi^{-1}(U) \cong U \times \mathbf{R}^n$ we have the natural coordinate system $(x^1, \ldots, x^n; v^1, \ldots, v^n)$. In terms of these, a local section $V \in \Gamma(\pi^{-1}(U),\ TTM)$ is of the form:

$$V = \sum_{i=1}^{n} a^i(x; v) \frac{\partial}{\partial x^i} + \sum_{i=1}^{n} b^\alpha(x; v) \frac{\partial}{\partial v^\alpha}.$$

We have used the same notation $\partial/\partial x^i$ which maybe considered as a vector field on U or on $\pi^{-1}(U)$. We shall, at times, write $\partial_i = \partial/\partial x^i$ and $\dot{\partial}_\alpha = \partial/\partial v^\alpha$.

It is clear that the vertical sections are spanned locally by $\partial/\partial v^\alpha, \alpha = 1, \ldots, n$:

$$\Gamma(\pi^{-1}(U), \mathcal{V}) = \left\{ V \mid V = \sum_\alpha b^\alpha(x; v) \frac{\partial}{\partial v^\alpha} \right\}$$

is a sub-bundle of rank n. Alternatively,

$$\mathcal{V}|_{\pi^{-1}(U)} = \{V \mid \pi^* dx_i(V) = 0, i = 1, \ldots, n\} \subset TTM|_{\pi^{-1}(U)}.$$

The position vector field

$$P(x; v) = \sum_{\alpha=1}^n v^\alpha \frac{\partial}{\partial v^\alpha}, \qquad v = (v_1, \ldots, v_n),$$

is, by definition, vertical. The Riemannian metric g on M induces naturally an inner product on $\mathcal{V}$:

$$\langle V, W \rangle_{\mathcal{V}} \stackrel{\text{def}}{=} \sum_{i,j=1}^n g_{ij}(\pi(v)) V^i W^j,$$

$V = \sum_{i=1}^n V^i(\partial/\partial v^i), W = \sum_{i=1}^n W^i(\partial/\partial v^i) \in \mathcal{V}_v$.

The horizontal sub-bundle is defined via the Riemannian connection ∇ associate to the metric g. The naturally induced connection on TTM is denoted by $\tilde{\nabla}$. The restriction $\tilde{\nabla}|_{\mathcal{V}}$ is a connection on $\mathcal{V}$. Let P be the position vector field defined above then the bundle map

$$\gamma : TTM \to \mathcal{V}, \qquad \gamma(X) := \tilde{\nabla}_X P,$$

is a surjection. The kernel, denoted by $\mathcal{H}$, of γ is referred to as the *horizontal sub-bundle.* Thus we have an exact sequence of vector bundles:

$$0 \to \ker \gamma = \mathcal{H} \to TTM \xrightarrow{\gamma} \mathcal{V} \to 0,$$

which implies that $\mathcal{V} \cong TTM/\mathcal{H}$. Moreover, the differential restricted to the horizontal sub-bundle $\pi_*|_{\mathcal{H}} : \mathcal{H} \to TM$ is an isomorphism. Using this isomorphism we define a metric on $\mathcal{H}$ by pulling back the Riemannian inner product on TM, i.e.,

$$\langle Z, W \rangle_{\mathcal{H}} = \langle \pi_* Z, \pi_* W \rangle_g, \qquad Z, W \in \mathcal{H}.$$

This together with the inner product on $\mathcal{V}$ defines an inner product $\langle , \rangle_G$ on TTM; more precisely, for $Z_{\mathcal{H}} \in \mathcal{H}$ and $W_{\mathcal{V}} \in \mathcal{V}$ the inner product

$$\langle Z, W \rangle_G = \langle W, Z \rangle_G = 0$$

and for any $Z, W \in TTM$ there is a unique decomposition $Z = Z_{\mathcal{H}} + Z_{\mathcal{V}}$ and $W = Z_{\mathcal{H}} + W_{\mathcal{V}}$ into horizontal and vertical components then

$$\langle Z, W \rangle_G = \langle Z_{\mathcal{H}}, W_{\mathcal{H}} \rangle_{\mathcal{H}} + \langle W_{\mathcal{V}}, W_{\mathcal{V}} \rangle_{\mathcal{V}}.$$

Beginning in the next section we shall extend the preceding construction to Kähler manifolds, in fact we shall deal with a more general situation, namely, the holomorphic tangent bundle of a hermitian holomorphic vector bundle over a Kähler manifold. As we shall see, the main difficulty there is that the canonical Riemannian metric constructed above is in general *not Kähler*.

2. The tangent bundle of a holomorphic vector bundle

In this section the construction of the preceding section shall be extended. Vertical and horizontal sub-bundles of the tangent bundle of a general holomorphic vector bundle E will be defined. *The main point is that the vertical bundle is a holomorphic sub-bundle of TE but the horizontal bundle is in general only a smooth but not a holomorphic sub-bundle.* In this section we deal with the case of a Hermitian metric leaving the (more complicated) case of a Finsler metric in a later section. Let (M, g) be a complex hermitian manifold of complex dimension n. Let $\pi : E \to M$ be a holomorphic vector bundle of rank r and the induced map $\pi_* : TE \to TM$. Let $e_1, \ldots, e_r$ be a local holomorphic frame for E over a local holomorphic coordinate system $(U; z = (z^1, \ldots, z^n))$. Elements of $E|_U$ are of the form $v = \sum_i v^i e_i$ and $\partial/\partial z^1, \ldots, \partial/\partial z^n; \partial/\partial v^1, \ldots, \partial/\partial v^r$ is a local basis for $TE|_{\pi^{-1}(U)}$. The *vertical sub-bundle* is, by definition, the kernel of π_*:

$$\mathcal{V} = \ker \pi_* \subset TE, \tag{2.1}$$

and is a holomorphic sub-bundle of rank r. It is clear that $\pi_* \partial/\partial v^i = 0$ hence

$$\begin{aligned}\mathcal{V}|_{\pi^{-1}(U)} &= \left\{ V \in TE|_{\pi^{-1}(U)} \mid V = \sum_{i=1}^{r} b^i(z; v) \frac{\partial}{\partial v^i} \right\} \\ &= \{ V \in TE|_{\pi^{-1}(U)} \mid dz_i(V) = 0, 1 \le i \le n \}.\end{aligned}$$

The position vector field

$$P(z; v) = \sum_{i=1}^{r} v^i \frac{\partial}{\partial v^i}, \qquad v = (v^1, \ldots, v^r), \tag{2.2}$$

is a *holomorphic* section of $\mathcal{V}$. Let h be a hermitian metric along the fibers of E:

$$\langle v, w \rangle_h = \sum_{i,j=1}^{r} h_{i\bar{j}}(z) v^i \overline{w}^j, \qquad v, w \in E_z. \tag{2.3}$$

This defines, tautologically, a hermitian metric along the fibers of $\mathcal{V}$:

$$\langle V, W \rangle_{\mathcal{V}} = \sum_{i,j=1}^{r} h_{i\bar{j}}(z) V^i \overline{W}^j, \tag{2.4}$$

$V = \sum_{i=1}^r V^i(\partial/\partial v^i), W = \sum_{i=1}^r W^i(\partial/\partial v^i) \in \mathcal{V}$. Denote by $\nabla^{\mathcal{V}}$ the associate hermitian connection on $\mathcal{V}$ with connection forms:

$$\theta_i^j(z,v) = \sum_{k=1}^n \gamma_{ik}^j(z)dz^k, \qquad \gamma_{ik}^j(z) = \sum_{l=1}^r \frac{\partial h_{i\bar{l}}}{\partial z^k}(z)h^{\bar{l}j}(z) \tag{2.5}$$

$(1 \leq i, j \leq r)$ depending only on z, where $(h^{\bar{l}j})$ is the inverse of the matrix $(h_{i\bar{l}})$. The curvature forms of the hermitian connection:

$$\Theta_i^j \overset{\text{def}}{=} d\theta_i^j - \sum_{k=1}^r \theta_i^k \wedge \theta_k^j = d\theta_i^j + \sum_{k=1}^n \theta_k^j \wedge \theta_i^k \tag{2.6}$$

are of bidegree $(1,1)$ so $\Theta_i^j = \overline{\partial}\theta_i^j$ which is equivalent to the condition that:

$$\partial\theta_i^j - \sum_{k=1}^r \theta_i^k \wedge \theta_k^j = \partial\theta_i^j + \sum_{k=1}^r \theta_k^j \wedge \theta_i^k = 0. \tag{2.7}$$

The components of the curvature forms are given by

$$\Theta_i^j = \overline{\partial}\theta_i^j = \sum_{k,l=1}^n K_{ik\bar{l}}^j dz^k \wedge d\bar{z}^l \tag{2.8}$$

with components given by

$$\begin{aligned} K_{ik\bar{l}}^j = -\frac{\partial \gamma_{ik}^j}{\partial \bar{z}^l} &= -\sum_q \left(\frac{\partial^2 h_{i\bar{q}}}{\partial z^k \partial \bar{z}^l} h^{\bar{q}j} - \frac{\partial h_{i\bar{q}}}{\partial z^k}\frac{\partial h^{\bar{q}j}}{\partial \bar{z}^l} \right) \\ &= -\sum_q \frac{\partial^2 h_{i\bar{q}}}{\partial z^k \partial \bar{z}^l} h^{\bar{q}j} + \sum_{p,q,s} \frac{\partial h_{i\bar{q}}}{\partial z^k}\frac{\partial h_{p\bar{s}}}{\partial \bar{z}^l} h^{\bar{q}p}h^{\bar{s}j}. \end{aligned}$$

The curvature depends only on the base variable $z = (z_1, \ldots, z_n)$ but not on the fiber variable variables $v = (v_1, \ldots, v_r)$. The connection $\nabla^{\mathcal{V}}$ defines a *surjective* bundle map:

$$\gamma : TE \to \mathcal{V}, \qquad \gamma(X) = \nabla_X^{\mathcal{V}} P, \tag{2.9}$$

where P (see (2.2)) is the position vector field. In terms of local coordinates the map γ takes the following form:

$$\begin{aligned} \nabla_X^{\mathcal{V}} P &= \sum_{j=1}^r \left\{ dv^j(X) + \sum_{i=1}^r v^i \theta_i^j(X) \right\} \frac{\partial}{\partial v^j} \\ &= \sum_{j=1}^r \left(b^j + \sum_{i=1}^r \sum_{k=1}^n \gamma_{ik}^j v^i a^k \right) \frac{\partial}{\partial v^j} \end{aligned}$$

for any vector field

$$X = \sum_{i=1}^n a^i(z;v)\frac{\partial}{\partial z^i} + \sum_{i=1}^r b^i(z;v)\frac{\partial}{\partial v^i}.$$

It is clear (for if $X \in \mathcal{V}$ then $a^i = 0$ for all i) that $\gamma|_{\mathcal{V}}$ is the *identity map* on $\mathcal{V}$. Notice that γ is smooth but, in general, *not* holomorphic (this is again clear because the definition of γ involves the connection). The kernel of γ, denoted $\mathcal{H}$, shall be referred to as the *horizontal sub-bundle* which is a smooth (but not holomorphic in general) sub-bundle of TE. However $\mathcal{H}$ is smoothly isomorphic to the quotient bundle $TE/\mathcal{V}$ which is holomorphic as $\mathcal{V}$ is a holomorphic sub-bundle of TE and

$$0 \to \mathcal{V} \to TE \to TE/\mathcal{V} = \mathcal{Q} \to 0$$

is an exact sequence of holomorphic vector bundles. On the other hand, we have an exact sequence

$$0 \to \ker \gamma = \mathcal{H} \to TE \xrightarrow{\gamma} \mathcal{V} \to 0$$

of *smooth* vector bundles and a *smooth* decomposition $TE = \mathcal{H} \oplus \mathcal{V}$. Thus the restriction of the map $\pi_* : TE \to TM$ to $\mathcal{H}$:

$$\pi_*|_{\mathcal{H}} : \mathcal{H} \xrightarrow{\cong} TM$$

is a smooth isomorphism. Using this isomorphism an inner product can be defined on $\mathcal{H}$ by pulling back the Kähler inner product on TM, i.e.,

$$\langle Z, W\rangle_{\mathcal{H}} = \langle \pi_* Z, \pi_* W\rangle_g, \qquad Z, W \in \mathcal{H}. \tag{2.10}$$

This together with the inner product, induced by the hermitian metric h of E, on $\mathcal{V}$ defines an inner product $\langle , \rangle_G$ on TE. More precisely, if $Z \in \mathcal{H}$ and $W \in \mathcal{V}$ then $\langle Z, W\rangle_G = \langle W, Z\rangle_G = 0$ and for any $Z, W \in TTM$ we have unique decompositions $Z = Z_{\mathcal{H}} + Z_{\mathcal{V}}$ and $W = Z_{\mathcal{H}} + W_{\mathcal{V}}$ into horizontal and vertical components then

$$\langle Z, W\rangle_G \overset{\text{def}}{=} \langle Z_{\mathcal{H}}, W_{\mathcal{H}}\rangle_{\mathcal{H}} + \langle W_{\mathcal{V}}, W_{\mathcal{V}}\rangle_{\mathcal{V}} \tag{2.11}$$

is a well-defined inner product on TE.

Given a vector field $V(z) = \sum_i a^i(z)\partial/\partial z^i$ on M the vector field

$$V^{\mathcal{H}}(z, v) = \sum_{i=1}^{n} \left\{ a^i(z)\frac{\partial}{\partial z^i} - \sum_{j=1}^{r}\sum_{k=1}^{n} \gamma^i_{jk} v^j a^k \frac{\partial}{\partial v^i} \right\}$$

is horizontal and shall be referred to as the *horizontal lifting* of V. The horizontal lifts of the local basis $\{\partial_i = \partial/\partial z^i, i = 1, \ldots, n\}$ of TE:

$$\left\{ \partial_i^{\mathcal{H}} = \frac{\partial}{\partial z^i} - \sum_{j,k=1}^{r} \gamma^k_{ji} v^j \frac{\partial}{\partial v^k} \,|\, i = 1, \ldots, n \right\}$$

is a basis of $\mathcal{H}$. By definition, we have:

$$\langle \partial_i^{\mathcal{H}}, \partial_j^{\mathcal{H}}\rangle_G(z, v) = \left\langle \frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j} \right\rangle_g (z) = g_{i\bar{j}}(z),$$

$$\left\langle \frac{\partial}{\partial v^i}, \frac{\partial}{\partial v^i} \right\rangle_G (z, v) = \left\langle \frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j} \right\rangle_h (z) = h_{i\bar{j}}(z)$$

and $\langle \partial_i^{\mathcal{H}}, \partial/\partial v^j \rangle_G = 0$ for all i and j. The dual basis is given by:

$$\begin{cases} \eta^i = dz^i, & 1 \le i \le n, \\ \zeta^i = \sum_{j=1}^r \sum_{k=1}^n \gamma_{jk}^i v^j dz^k + dv^i = dv^i + \sum_{j=1}^r \theta_k^i v^k, & 1 \le i \le r \end{cases} \tag{2.12}$$

and, with respect to this basis, the fundamental form of the metric G defined by the Kähler metric g on M and the fiber metric h on E takes the form:

$$\omega_G = \sqrt{-1} \left(\sum_{i,j=1}^n g_{i\bar{j}}(z)\, dz^i \wedge d\bar{z}^j + \sum_{i,j=1}^r h_{i\bar{j}}(z)\, \zeta^i \wedge \overline{\zeta}^j \right). \tag{2.13}$$

The next Theorem identifies the obstruction of η from being Kähler *assuming that* (M,g) *is Kähler*:

Theorem 2.1. *Let* (M,g) *be a complex Kähler manifold and* (E,h) *be an hermitian holomorphic vector bundle of rank* r *over* M*. Let* ω_G *be the fundamental form of the hermitian metric* G *as defined above then*

$$\begin{aligned} d\omega_G &= \sqrt{-1} \left(\sum_{1 \le i,j,k \le r} h_{i\bar{j}} v^k \Theta_k^i \wedge \overline{\zeta}^j - \sum_{1 \le i,j,k \le r} h_{i\bar{j}} \zeta^i \wedge \bar{v}^k \overline{\Theta}_k^j \right) \\ &= \sqrt{-1} \left(\sum_{1 \le i,j,k \le r} h_{i\bar{j}} \overline{\partial} \zeta^i \wedge \overline{\zeta}^j - \sum_{1 \le i,j,k \le r} h_{i\bar{j}} \zeta^i \wedge \partial \overline{\zeta}^j \right), \end{aligned}$$

where the vertical forms $\zeta^i, i = 1, \dots, r$ *are defined in* (2.12), $(v^1, \dots, v^r)$ *are the fiber coordinates and* θ_k^i, Θ_k^i *are resp. the hermitian connection and curvature forms of the metric* h.

Proof. The first term on the right hand side of (2.13) is closed as it is the Kähler form of the metric g. Thus exterior differentiation yields,

$$d\omega_G = \sqrt{-1} \sum_{i,j=1}^r (dh_{i\bar{j}} \wedge \zeta^i \wedge \overline{\zeta}^j + h_{i\bar{j}} d\zeta^i \wedge \overline{\zeta}^j - h_{i\bar{j}} \zeta^i \wedge d\overline{\zeta}^j).$$

The Theorem shall be verified at each point x^* and we may (the closedness of ω_G is independent of the choice of holomorphic frames) choose local holomorphic frames which is normal at the point x^*; namely a local holomorphic frame $e_1, \dots, e_r$ over an open neighborhood of x^* which is unitary and parallel at x^*: $h_{i\bar{j}}(x^*) = \delta_i^j$ and $dh_{i\bar{j}}(x^*) = 0$ for all $1 \le i, j \le r$. In particular, all connection forms relative to this local frame vanishes at x^* and the curvature at x^* is given by $(\overline{\partial}\partial h)(x^*)$. We have (by the hermitian condition $\overline{h_{k\bar{i}}} = h_{i\bar{k}}$, the identities (2.5) and (2.12)),

$$dh_{i\bar{j}} = 0, \quad d\zeta^i = \sum_k d\theta_k^i v^k, \quad d\overline{\theta}^j = \sum_k d\overline{\theta_k^i} \bar{v}^k m\, \zeta^i \wedge \overline{\zeta}^j = dv^i \wedge d\bar{v}^j,$$

hence,

$$h_{i\bar{j}} d\zeta^i \wedge \overline{\zeta}^j = \delta^i_j \sum_k v^k d\theta^i_k \wedge \overline{\zeta}^j = \delta^i_j \sum_k v^k d\theta^i_k \wedge d\bar{v}^j,$$
$$-h_{i\bar{j}} \zeta^i \wedge d\overline{\zeta}^j = -\delta^i_j \sum_k \bar{v}^k \zeta^i \wedge d\overline{\theta}^j_k = -\delta^i_j \sum_k \bar{v}^k dv^i \wedge d\overline{\theta}^j_k.$$

Summing the above over i and j yields

$$\begin{aligned} d\omega_G &= \sqrt{-1}\left(\sum_{i,j,k,l} h_{i\bar{j}} v^k d\theta^i_k \wedge d\bar{v}^j - \sum_{i,j,k,l} h_{i\bar{j}} \bar{v}^k dv^i \wedge d\overline{\theta}^j_k\right) \\ &= \sqrt{-1}\left(\sum_{i,j,k,l} h_{i\bar{j}} v^k \Theta^i_k \wedge \bar{\zeta}^j - \sum_{i,j,k,l} h_{i\bar{j}} \zeta^i \wedge \bar{v}^k \overline{\Theta}^j_k\right). \end{aligned}$$

Next we observe that

$$\partial\zeta^i + \overline{\partial}\zeta^i = d\zeta^i = \sum_k \partial\theta^i_k v^k + \sum_k \overline{\partial}\theta^i_k v^k$$

and comparing bi-degrees yields

$$\overline{\partial}\theta^i = \sum_k v^k \overline{\partial}\theta^i_k = \sum_k \Theta^i_k v^k \tag{2.14}$$

hence

$$d\omega_G = \sqrt{-1}\left(\sum_{i,j,k} h_{i\bar{j}} \overline{\partial}\zeta^i \wedge \overline{\zeta}^j - \sum_{i,j,k} h_{i\bar{j}} \zeta^i \wedge \partial\overline{\zeta}^j\right)$$

as claimed. □

Corollary 2.2. *Let G be the metric in Theorem 2.1. Then the following conditions are equivalent:*

(i) *G is Kähler;*

(ii) *the one forms $\{\zeta^i, i = 1, \ldots, n\}$ are holomorphic;*

(iii) *the curvature of the vertical bundle $\mathcal{V}$ satisfies the conditions:*

$$\sum_{1 \le i,k \le r} h_{i\bar{m}} v^k \Theta^i_k = 0, \qquad 1 \le m \le r;$$

(iv) *the curvature of the vertical bundle $\mathcal{V}$ is zero.*

Proof. It is clear that the holomorphicity of $\{\zeta^i, i = 1, \ldots, n\}$ implies that $d\eta = 0$. For the converse, we see from the expression (see (2.8))

$$\Theta^i_k \wedge \overline{\zeta}^j = \sum_{p,q=1}^n K^i_{kp\bar{q}}\, dz^p \wedge d\bar{z}^q \wedge \left(d\bar{v}^j + \sum_{s=1}^r \sum_{s=1}^n \overline{\gamma}^j_{r\bar{s}} \bar{v}^r d\bar{z}^s\right)$$

that, for any i, j, k, m

$$\iota_{\partial_{\bar m}}\Theta^i_k \wedge \overline{\zeta}^j = \delta^j_m \sum_{p,q=1}^{n} K^i_{kp\bar q} dz^p \wedge d\bar z^q,$$

where $\iota_{\partial_{\bar m}}$ denotes interior product with the vector field $\partial/\partial \bar v^m$. This shows that

$$\iota_{\partial_{\bar m}} d\eta = \sqrt{-1}\sum_{i,k,l} h_{i\bar m} v^k \Theta^i_k = \sqrt{-1}\sum_i h_{i\bar m}\overline{\partial}\zeta^i$$

($\sum_k v^k \Theta^i_k = \overline{\partial}\zeta^i, i = 1, \dots, r$ by (2.14)) for all m and that

$$\sum_m h^{\bar m j}\iota_{\partial_{\bar m}} d\eta = \sqrt{-1}\sum_{i,m} h^{\bar m j} h_{i\bar m}\overline{\partial}\zeta^i = \sqrt{-1}\sum_i \delta^j_i \overline{\partial}\zeta^i = \sqrt{-1}\overline{\partial}\zeta^j$$

for all j. From these it is clear that assertions (i), (ii) and (iii) are equivalent. From the identities $\sum_k v^k \Theta^i_k = \overline{\partial}\zeta^i, i = 1, \dots, r$ and the fact that the curvature forms $\{\Theta^i_k\}$ are independent of v, as $v = (v^1, \dots, v^k)$ ranges over all $v \in E_z, z \in M$ we infer that $\Theta^i_k = 0$ for all i and k if and only if $\overline{\partial}\zeta^i = 0$ for all i. □

If $E = TM$ is the tangent bundle and $(E, h) = (TM, g)$ then

Corollary 2.3. *Let G be the metric in Theorem* 2.1 *with $(E, h) = (TM, g)$. Then the following conditions are equivalent*:

(i) *G is Kähler*;

(ii) *the one forms $\{\zeta^i, i = 1, \dots, n\}$ are holomorphic*;

(iii) *the curvature of the vertical bundle $\mathcal{V}$ satisfies the conditions*:

$$\sum_{1\le i,k\le r} h_{i\bar m} v^k \Theta^i_k = 0, \qquad 1 \le m \le r;$$

(iv) *the curvature of the vertical bundle $\mathcal{V}$ is zero*;

(v) *the curvature of (M, g) is zero.*

Proof. The first 4 statements are the same as Corollary 2.2. Under the present assumption $h = g$ hence the curvature Θ^i_k of $\mathcal{V}$, defined by h, is the same as that of the curvature Ω^i_k of the metric g on M. □

Corollary 2.2 shows that the natural metric G is generally not Kähler so we look for other means of producing Kähler metrics. Let (M, g) be a hermitian manifold and (E, h) a hermitian holomorphic vector bundle over M. Consider the global $(1,1)$-form $\sqrt{-1}\partial\overline{\partial}||P||^2_G = \sqrt{-1}\partial\overline{\partial}||P||^2_{\mathcal{V}}$ on E (where P is the position vector field as defined in (2.2)). We claim that

$$\sqrt{-1}\partial\overline{\partial}||P||^2_G = \sqrt{-1}\sum_{i,j=1}^{r} \zeta^i \wedge \bar\zeta^i - \sqrt{-1}\langle K(\cdot,\cdot)v, v\rangle_h, \tag{2.15}$$

where ζ^i is given by (2.12) and $\langle K(\cdot,\cdot)v, v\rangle_h$ is the curvature operator defined as follows:

$$Kv = \sum_{i=1}^{r} \Theta_i^j v^i \otimes e_j = \sum_{i=1}^{r} K_{ik\bar{l}}^j v^i dz^k \wedge d\bar{z}^l \otimes e_j$$

and that

(2.16)

$$\langle K(\cdot,\cdot)v, v\rangle_h = \sum_{i,j,q=1}^{r} h_{j\bar{q}} \Theta_i^j v^i \overline{v}^q = \sum_{k,l=1}^{n} \left(\sum_{i,j,q=1}^{r} h_{j\bar{q}} K_{ik\bar{l}}^j v^i \overline{v}^q \right) dz^k \wedge d\bar{z}^l.$$

This is verified by a direct calculation:

$$\begin{aligned}\sqrt{-1}\partial\overline{\partial}||P||_G^2 = \sqrt{-1} \sum_{i,j=1}^{r} &\left\{ h_{i\bar{j}}(z) dv^i \wedge d\bar{v}^j + v^i \bar{v}^j \sum_{k,l=1}^{n} \frac{\partial^2 h_{i\bar{j}}}{\partial z^k \partial \bar{z}^l}(z) dz^k \wedge d\bar{z}^l \right\} \\ &+ \sqrt{-1} \left\{ \sum_{i,j=1}^{r} \sum_{k=1}^{n} v^i \frac{\partial h_{i\bar{j}}}{\partial z^k}(z) dz^k \wedge d\bar{v}^j \right. \\ &\left. + \sum_{l=1}^{n} \bar{v}^j \frac{\partial h_{i\bar{j}}}{\partial \bar{z}^l}(z) dv^i \wedge d\bar{z}^l \right\}.\end{aligned}$$

We may choose a local frame of E which is normal at any given point z^* and, with respect to such a frame, we have, at the point z^*:

$$\sqrt{-1}\partial\overline{\partial}||P||_G^2 = \sqrt{-1} \sum_{i=1}^{n} \left\{ dv^i \wedge d\bar{v}^i + v^i \bar{v}^i \sum_{k,l=1}^{n} \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \bar{z}^l} dz^k \wedge d\bar{z}^l \right\}.$$

Note that the first sum is a vertical form (i.e., annihilates the horizontal vector fields) because at the point z^* the connection forms vanish and we see from (2.12) that $dv^i = \theta^i$ at z^*. On the other hand the second term is a horizontal form. Indeed, the curvature at z^* is given by (see (2.8))

$$\Theta_i^j = -\sum_{k,l=1}^{r} \frac{\partial^2 h_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} dz^k \wedge d\bar{z}^l,$$

thus the second sum in the expression above is a curvature term:

$$\sqrt{-1}\partial\overline{\partial}||P||_G^2 = \sqrt{-1} \sum_{i=1}^{r} \left\{ dv^i \wedge d\bar{v}^i - v^i \bar{v}^i \sum_{k,l=1}^{n} K_{ik\bar{l}}^i dz^k \wedge d\bar{z}^l \right\},$$

where $v = (v^1, \ldots, v^r)$ and (2.16) is verified. It is clear that $\sqrt{-1}\partial\overline{\partial}||P||_G^2$ is positive definite in the vertical direction. For any tangent vectors X, Y of type

$(1, 0)$ on M,

$$K(X,Y)v = \sum_{i=1}^{r} \Theta_i^j(X,\overline{Y})v^i,$$
$$\langle K(X,Y)v,u\rangle_h = \sum_{k,l=1}^{n} \sum_{i,j,q=1}^{r} h_{j\bar{q}} K_{ik\bar{l}}^j X^k \overline{Y}^l v^i \overline{u}^q.$$

If $||X||_g \neq 0$ and $||v||_h \neq 0$ the *mixed holomorphic bisectional curvature of* (E, h) is defined to be:

$$k(X,v) = \frac{\langle K(X,X)v,v\rangle_h}{||X||_g^2||v||_h^2}, \qquad X \in T_xM,\ v \in E_x. \tag{2.17}$$

Identifying the position vector field $P = \sum v^i\partial/\partial v^i$ with the position vector $(v^1, \dots, v^r)$ we sometimes write $k(X, P)$ instead of $k(X, v)$. The preceding calculation shows that:

Theorem 2.4. *Let P be the position vector on E where (E, h) is a hermitian holomorphic vector bundle of rank r over a complex hermitian manifold (M, g). Let G be the metric along the fibers of TE defined by g and h then the $(1, 1)$-form $\sqrt{-1}\partial\overline{\partial}||P||_G^2$ is positive definite on $E\backslash\{$zero-section$\}$ if and only if the mixed holomorphic bisectional curvature $k(X, P)$ is strictly negative for all non-zero $X \in T_xM$.*

Proof. This is quite clear from the identity

$$\sqrt{-1}\partial\overline{\partial}||P||_G^2 = \sqrt{-1}\left(\sum_{i=1}^{r}\theta^i\wedge\bar{\theta}^i - \sum_{i=1}^{r}\sum_{k,l=1}^{n} v^i\bar{v}^i K_{ik\bar{l}}^i dz^k\wedge d\bar{z}^l\right)$$

as the first term on the right guaranteed that the $(1, 1)$-form is positive definite in the fiber directions while the second term is positive definite in the base directions if and only if the mixed holomorphic bisectional curvature $k(X, P)$ is strictly negative. □

If $E = TM$ and $h = g$ then $v \in T_xM$ is also a tangent vector of M and writing $v = Y$, we have:

$$k(X,Y) = \frac{\langle R(X,X)Y,Y\rangle_g}{||X||_g^2||Y||_g^2}$$

if $X, Y \in T_xM$ are non-zero tangent vectors at x. Thus $k(X, Y)$ is just the usual holomorphic bisectional curvature and we get from Theorem 2.4 that

Theorem 2.5. *Let P be the position vector field on TM where (M, g) is a complex hermitian manifold. Then the $(1, 1)$-form $\sqrt{-1}\partial\overline{\partial}||P||_g^2$ is positive definite on $TM\backslash\{$zero-section$\}$ if and only if the holomorphic bisectional curvature of g is strictly negative.*

Write $\sqrt{-1}\partial\overline{\partial}||P||_G^2 = \omega + \rho$ with

$$\omega = \sqrt{-1}\sum_{i,j=1}^{r}\zeta^i \wedge \bar{\zeta}^i, \qquad \rho = -\sqrt{-1}\langle K(\cdot,\cdot)v, v\rangle_h$$

and, with respect to a frame normal at a point $x^* \in M$:

$$\sqrt{-1}\partial\overline{\partial}||P||_G^2 = \sqrt{-1}\sum_{i=1}^{r} dv^i \wedge d\bar{v}^i - \sqrt{-1}\sum_{k,l=1}^{n} K(v)_{k\bar{l}}dz^k \wedge d\bar{z}^l$$

at the point $(x^*, v) \in TE$ where $K(v)_{k\bar{l}} = \sum_{i=1}^{r} v^i\bar{v}^i K^i_{ik\bar{l}}$. The first sum is horizontal while the second sum is horizontal. Thus

$$(\sqrt{-1}\partial\overline{\partial}||P||_G^2)^{n+r} = \omega^r \wedge \rho^n = (-1)^n \det(K(v)_{k\bar{l}})\, dV \wedge dZ, \tag{2.18}$$

where $dV_v = (\sqrt{-1}\sum_{i=1}^{r} dv^i \wedge d\bar{v}^i)^r$ and $dV_z = (\sqrt{-1}\sum_{k,l=1}^{n} dz^k \wedge d\bar{z}^l)^n$ are the vertical and horizontal volume elements. This shows that $(\sqrt{-1}\partial\overline{\partial}||P||_G^2)^{n+r} > 0$ if and only if $(-1)^n \det(K(v)_{k\bar{l}}) > 0$.

In the next section the preceding Theorem shall be formulated on the projectivized bundle rather than on E. The reason for working on $\mathbf{P}(E)$ rather than E is that $\mathbf{P}(E)$ is compact if M is compact.

3. The tangent bundle of a projectivized vector bundle

Let (M, g) be a Kähler manifold with holomorphic tangent bundle $p_M : TM \to M$ and let (E, h) be a hermitian holomorphic vector bundle of rank $r \geq 2$ over M with projection

$$p_E : E \to M. \tag{3.1}$$

Denote by $E_* = E\backslash\{\text{zero-section}\}$ then there is a natural $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ action on E_* and the quotient

$$[\,] : E_* \to \mathbf{P}(E) = E_*/\mathbf{C}^* \tag{3.2}$$

shall be referred to as the projectivized vector bundle. The natural projection map shall be denoted by

$$[p_E] : \mathbf{P}(E) \to M. \tag{3.3}$$

As the notations suggested, the following diagram commutes:

$$\begin{array}{ccc} E_* & = & E_* \\ [\,]\ \downarrow & & \downarrow p_E \\ \mathbf{P}(E) & \xrightarrow{[p_E]} & M \end{array}.$$

The quotient map (3.2) induces a bundle map between the tangent bundles

$$[\,]_* : TE_* \to T\mathbf{P}(E).$$

The kernel of $[\,]_*$ is the trivial line bundle $\langle P\rangle$ spanned by the position vector field P (defined in (2.2)) and we have a short exact sequence of holomorphic bundles

$$0 \to \langle P\rangle \to TE_* \xrightarrow{[\,]_*} T\mathbf{P}(E) \to 0. \tag{3.4}$$

The pull-back $[p_E]^*E$ is a sub-bundle of $\mathbf{P}(E)\times E$ over $\mathbf{P}(E)$ and inherits projection maps $p_1 : [p_E]^*E \to \mathbf{P}(E)$ and $p_2 : [p_E]^*E \to E$ such that the following diagram is commutative:

$$\begin{array}{ccc} [p_E]^*E & \xrightarrow{p_2} & E \\ \downarrow p_1 & & \downarrow p_E \\ \mathbf{P}(E) & \xrightarrow{[p_E]} & M\ . \end{array}$$

The tautological line bundle, denoted $\mathcal{L}^{-1}_{\mathbf{P}(E)}$, is a sub-bundle of $[p_E]^*E$ defined by:

$$\mathcal{L}^{-1}_{\mathbf{P}(E)} = \{((z,[v]),\lambda v) \in [p_E]^*E \,|\, (z,[v]) \in \mathbf{P}(E),\, \lambda\in\mathbf{C}\}. \tag{3.5}$$

The dual, denoted $\mathcal{L}_{\mathbf{P}(E)}$, shall be referred to as the "hyperplane bundle" over $\mathbf{P}(E)$. We shall often write, for simplicity, $\mathcal{L}$ instead of $\mathcal{L}_{\mathbf{P}(E)}$. The following definition is standard:

Definition 3.1. A holomorphic line bundle L over M is said to be ample (resp. nef) if there exists a hermitian metric h along the fibers such that the first Chern form $c_1(L,h)$ is positive definite (resp. positive semi-definite). A holomorphic vector bundle E of rank $r\geq 2$ is ample (resp. nef) if the line bundle $\mathcal{L}_{\mathbf{P}(E^*)}$ over $\mathbf{P}(E^*)$ is ample (resp. nef). The *dual* bundle E^* is said to be ample (resp. nef) if the line bundle $\mathcal{L}_{\mathbf{P}(E)}$ over $\mathbf{P}(E)=E_*/\mathbf{C}^*$ is ample (resp. nef).

Let P be the position vector field on TE then the function $||P(z,v)||^2_h$ is globally well-defined on E and is non-vanishing outside the zero section hence $\log||P(z,v)||^2_h$ is well-defined on E_*. Moreover, since $\log||P(z,\lambda v)||^2_h = \log||P(z,v)||^2_h + \log|\lambda|^2$ for all $\lambda\in\mathbf{C}^*$ the $(1,1)$-form $(\sqrt{-1}/2\pi)\partial\overline{\partial}\log||P||^2_h$ descends to a well-defined $(1,1)$-form ϕ on $\mathbf{P}(E)$. Indeed, we may consider $||P||_h$ as a metric along the fibers of the tautological line bundle $\mathcal{L}^{-1}$ and $(\sqrt{-1}/2\pi)\overline{\partial}\partial\log||P||^2_h$ descends to $-\phi = c_1(\mathcal{L}^{-1})$, the first Chern form of the line bundle $\mathcal{L}^{-1}$; equivalently, $(\sqrt{-1}/2\pi)\partial\overline{\partial}\log||P||^2_h$ to $\phi=c_1(\mathcal{L})$ which is the first Chern form of the dual line bundle $\mathcal{L}$. Being a form on $\mathbf{P}(E)$ we have $\phi^{N-1}\equiv 0$ where $\dim\mathbf{P}(E)=\dim E-1=N-1$ thus $(\partial\overline{\partial}\log||P||^2_h)^{N-1}\equiv 0$. Indeed the position vector field is a zero eigen-vector of $\partial\overline{\partial}\log||P||^2_h$, i.e., $\iota_P\partial\overline{\partial}\log||P||^2_h = 0$.

Theorem 3.2. *Let P be the position vector field on a holomorphic vector bundle (E,h) of rank $r \geq 2$ over a hermitian manifold (M,g) of dimension n. Then the $(1,1)$-form $\sqrt{-1}\partial\overline{\partial}\log ||P||_h^2$ descends to a well-defined form $\phi(= 2\pi c_1(\mathcal{L}_{\mathbf{P}(E)}))$ on $\mathbf{P}(E)$ moreover the following conditions are equivalent:*

(i) *ϕ is positive definite (resp. positive semi-definite);*

(ii) *the mixed holomorphic bisectional curvature $k(X,P)$ of (E,h) is strictly negative (non-positive) for all non-zero $X \in TM$ and where P is the position vector field along the fibers of E.*

Proof. Differentiating twice we get

$$\partial\overline{\partial}\log ||P||_h^2 = \frac{\partial\overline{\partial}||P||_h^2}{||P||_h^2} - \frac{\partial||P||_h^2 \wedge \overline{\partial}||P||_h^2}{||P||_h^4}. \tag{3.6}$$

In terms of a normal holomorphic frame at a point z^* (see the proof of Theorem 2.1),

$$\begin{aligned}
\partial\left(\sum_{i,j=1}^{r} h_{i\bar{j}} v^i \bar{v}^j\right) &= \sum_{i,j=1}^{r} \frac{\partial h_{i\bar{j}}}{\partial z_k} v^i \bar{v}^j dz_k + \sum_{i,j=1}^{r} h_{i\bar{j}} \bar{v}^j dv^i, \\
\overline{\partial}\left(\sum_{i,j=1}^{r} h_{i\bar{j}} v^i \bar{v}^j\right) &= \sum_{i,j=1}^{r} \frac{\partial h_{i\bar{j}}}{\partial \bar{z}_k} v^i \bar{v}^j d\bar{z}_k + \sum_{i,j=1}^{r} h_{i\bar{j}} v^i d\bar{v}^j
\end{aligned}$$

and from the computation of $\partial\overline{\partial}||P||_h^2$ in the last section (in the proof of Theorem 2.4),

$$\begin{aligned}
\sqrt{-1}\partial\overline{\partial}\log ||P||_h^2 = \sqrt{-1}&\frac{||P||_h^2 \sum_{i=1}^{r} dv^i \wedge d\bar{v}^i - \sum_{i,j=1}^{r} \bar{v}^j v^i dv^i \wedge d\bar{v}^j}{||P||_h^4} \\
&- \sqrt{-1}\frac{\langle K(\cdot,\cdot)P, P\rangle_h}{||P||_h^2},
\end{aligned}$$

where K is the hermitian curvature of h. The first term on the right is $[\,]^*\omega_{FS}$ where ω_{FS} is the Fubini-Study metric of the fiber $\mathbf{P}(E_{z^*})$ and $[\,] : E_* \to \mathbf{P}(E)$ is the quotient map. Thus, along each fiber, we have

$$\sqrt{-1}\partial\overline{\partial}\log ||P||_h^2 = [\,]^*\omega_{FS} - \sqrt{-1}\frac{\langle K(\cdot,\cdot)P, P\rangle_h}{||P||_h^2} \tag{3.7}$$

from which we infer easily that ϕ is positive definite in the fiber directions and is trivial in the horizontal directions; on the other hand the second term is trivial in the fiber directions and is positive definite in the base (horizontal) directions (by Theorem 2.4) if and only if the mixed bisectional curvature:

$$k(X,P) = \frac{\langle K(X,\overline{X})P, P\rangle_h}{||X||_g^2 ||P||_h^2}$$

is strictly negative for non-zero $X \in TM$. This shows that (i) and (ii) are equivalent. □

Note that the hermitian curvature matrix is skew hermitian, consequently the mixed holomorphic bisectional curvature of a holomorphic hermitian bundle (E, h) is positive (resp. negative) if and only if the mixed holomorphic bisectional curvature of its dual (E^*, h^*) is negative (resp. positive). Theorem 3.2 applied to the dual (E^*, h^*) of (E, h) yields:

Theorem 3.3. *Let P^* be the position vector field on (E^*, h^*) the dual of a holomorphic vector bundle (E, h) of rank $r \geq 2$ over a complex hermitian manifold (M, g) of dimension n. Then the $(1,1)$-form $\sqrt{-1}\partial\overline{\partial}\log ||P^*||^2_{h^*}$ descends to a well-defined form $\psi(= 2\pi c_1(\mathcal{L}_{\mathbf{P}(E^*)}))$ on $\mathbf{P}(E^*)$ moreover the following conditions are equivalent:*

(i) *ψ is positive definite (resp. positive semi-definite);*

(ii) *the mixed holomorphic bisectional curvature of (E^*, h^*) is strictly negative (resp. non-positive);*

(iii) *the mixed holomorphic bisectional curvature of (E, h) is strictly positive (resp. non-negative).*

Each of these conditions implies that E is ample (resp. nef).

Just as in Section 3 the condition on the mixed bisectional curvature in Theorems 3.2 and 3.3 is reduced to the usual bisectional curvature if $(E, h) = (TM, g)$.

Fix hermitian metrics (M, g) and (E, h) and let $\mu_{g,h}$ be the supremum of the mixed holomorphic bisectional curvature $k(X, P)$; more precisely:

$$m_{g,h}(x) = \sup_{X \in T_xM, \sigma \in E_x} k(X, P), \qquad \mu_{g,h} = \sup_{x \in M} m_{g,h}. \tag{3.8}$$

Obviously the function $m_{g,h}(x)$ is continuous and, if M is compact, $\mu_{g,h}$ is a finite constant. However $\mu_{g,h}$ may be infinite if M is non-compact, in which case it is necessary to work with $m_{g,h}(x)$. In any case, we have, by definition:

$$\sqrt{-1}\frac{\langle K(\cdot,\cdot)\sigma, \sigma\rangle_h}{||\sigma||^2_h} \leq \mu_{g,h}\,\omega_g,$$

where ω_g is the fundamental form associate to the metric g. This together with (3.7) implies that, for $\mu_{g,h}$ finite

$$c_1(\mathcal{L}_{\mathbf{P}(E)}) + \lambda[p_E]^*\omega \geq \omega_{FS} + c[p_E]^*\omega - \mu_{g,h}[p_E]^*\omega_g.$$

($[p_E] : \mathbf{P}(E) \to M$ is the projection map) for any constant λ and any $(1,1)$-form ω on M. If M is compact Kähler we may take ω to be a Kähler form on M then there exists a constant $\lambda >> 0$ such that $\lambda\omega - \mu_{g,h}\omega_g$ is positive definite (in particular, if ω_g is Kähler we mat take $\omega = \omega_g$ and λ to be any number strictly larger than $\mu_{g,h}$). With this choice we see from the preceding inequality that $c_1(\mathcal{L}_{\mathbf{P}(E)}) + \lambda[p_E]^*\omega$ is positive definite as ω_{FS} is positive semi-definite and is positive definite in the fiber directions while $\lambda[p_E]^*\omega - \mu_{g,h}[p_E]^*\omega_g$ is positive semi-definite and is positive definite in the horizontal directions. If in addition ω is Kähler then $c_1(\mathcal{L}_{\mathbf{P}(E)}) + \lambda[p_E]^*\omega$ is a Kähler metric on $\mathbf{P}(E)$. This is equivalent to the condition that $\sqrt{-1}\partial\overline{\partial}\log||P||_h^2 + \lambda p_E^*\omega$ ($p_E : E \to M$ is the projection map) is positive semi-definite on $E\backslash$ {zero-section} and is positive definite in all directions transversal to the radial direction (i.e., the direction spanned by the position vector field P. This is equivalent to the condition that $\sqrt{-1}\partial\overline{\partial}||P||_h^2 + \lambda p_E^*\omega$ is positive definite on E.

The construction above, however, does not work if M is non-compact because $\mu_{g,h}$ may be infinite. In the non-compact case we have to deal with the function $m_{g,h}$. This can be done if M is Stein (i.e., there exists a strictly plurisubharmonic exhaustion function f on M) or, more generally, by dropping the exhaustion condition (i.e., each of the level set of f is compact) on the strictly plurisubharmonic function f. On such M we may take the Kähler form to be the Levi form of f: $\omega_g = \sqrt{-1}\partial\overline{\partial}f$. By definition, at any point x

$$-\sqrt{-1}\frac{\langle K(\cdot,\cdot)P,P\rangle_h}{||P||_h^2} \leq -m_{g,h}\sqrt{-1}\partial\overline{\partial}f = -m_{g,h}\omega_g.$$

Let $\chi : \mathbf{R} \to \mathbf{R}$ be a positive convex increasing function then

$$\sqrt{-1}\partial\overline{\partial}(\chi\circ f) = \chi^{'}(f)\sqrt{-1}\partial\overline{\partial}f + \chi^{''}(f)\sqrt{-1}\partial f\wedge\overline{\partial}f \geq \chi^{'}(f)\sqrt{-1}\partial\overline{\partial}f$$

thus, by choosing χ such that

$$\chi^{'}(f(x)) > |m_{g,h}(x)| \tag{3.9}$$

(it is clear that such a function χ exists) then

$$-\sqrt{-1}\frac{\langle K(\cdot,\cdot)P,P\rangle_h}{||P||_h^2} \leq |m_{g,h}|\omega_g \leq \sqrt{-1}\partial\overline{\partial}(\chi\circ f)$$

and consequently,

$$\begin{aligned} &c_1(\mathcal{L}_{\mathbf{P}(E)}) + [p_E]^*\sqrt{-1}\partial\overline{\partial}(\chi\circ f) \\ &\quad = \omega_{FS} + [p_E]^*\sqrt{-1}\partial\overline{\partial}(\chi\circ f) - \sqrt{-1}\frac{\langle K(\cdot,\cdot)P,P\rangle_h}{||P||_h^2} \end{aligned}$$

is positive definite on $\mathbf{P}(E)$ and $\sqrt{-1}\partial\overline{\partial}||P||^2 + p_E^*\sqrt{-1}\partial\overline{\partial}(\chi\circ f)$ is positive definitive on E. We summarized the above in the following Theorem:

Corollary 3.4. *Let (M,g) be a Kähler manifold and $[p_E] : (E,h) \to M$ be a hermitian holomorphic vector bundle, of rank ≥ 2 over M. If M is compact then there exists a constant $\lambda > 0$ such that $\sqrt{-1}\partial\overline{\partial}||P||_h^2 + \lambda p_E^*\omega_g$ is a Kähler metric on the bundle space E. Here P is the position vector field on E and ω_g is the Kähler form associate to the metric g. If M is non-compact and admits a strictly plurisubharmonic function f then there exists an increasing convex function $\chi : \mathbf{R} \to \mathbf{R}$ such that $\sqrt{-1}\partial\overline{\partial}||P||_h^2$ $+p_E^*\sqrt{-1}\partial\overline{\partial}(\chi\circ f)$ is a Kähler metric on E.*

The case of a general non-compact Kähler manifold remains open.

Let $[p_E]_* : T\mathbf{P}(E) \to TM$ be the differential of the projection $[p_E] : \mathbf{P}(E) \to M$. The projectivized vertical sub-bundle $[\mathcal{V}_E] = \ker[p_E]_*$ consisting of tangent vectors of $\mathbf{P}(E)$ tangential to the fibers of $[p_E]$. Note that $\ker p_E = \mathcal{V}_E$ where $p_E : TE \to TM$ and $[\mathcal{V}_E] = [\ ]_* \mathcal{V}_E$ where $[\ ] : E \backslash \{\text{zero-section}\} \to \mathbf{P}(E)$ is the quotient map (see (3.2)). The kernel of the quotient map is spanned by the position vector field P hence $[\mathcal{V}_E] = \mathcal{V}_E / \langle P \rangle_{\mathbf{C}}$. There is an exact sequence (which shall be referred to as the Euler sequence over $\mathbf{P}(E)$ see [10]):

$$0 \to \mathbf{C} \to [p_E]^* E \otimes \mathcal{L}_{\mathbf{P}(E)} \xrightarrow{\rho} [\mathcal{V}_E] \to 0, \tag{3.10}$$

where $\mathcal{L}_{\mathbf{P}(E)}$ is the "hyperplane" bundle as defined in (3.5) and $\mathbf{C}$ is the trivial line bundle spanned by the tautological section

$$\tau(z, [v]) = \sum_{i=1}^{r} v^i \otimes e_i$$

(if M is a single point then $E = \mathbf{C}^r$ and the preceding reduces to the classical Euler sequence for projective space is the exact sequence (see [10])

$$0 \to \mathbf{C} \to \mathbf{C}^r \otimes \mathcal{L}_{\mathbf{P}^{r-1}} = \oplus^r \mathcal{L}_{\mathbf{P}^{r-1}} \to T\mathbf{P}^{r-1} \to 0,$$

where $\mathcal{L}_{\mathbf{P}^{r-1}}$ is the hyperplane bundle on $\mathbf{P}^{r-1}$). The homomorphism ρ in (3.10) is given as follows. A local section σ of $[p_E]^* E \otimes \mathcal{L}_{\mathbf{P}(E)}$ is of the form

$$\sigma(z, [v]) = \sum_{i=1}^{r} \sigma_i(z, [v]) \otimes e_i(z, [v]),$$

where each σ_i is a local section of $\mathcal{L}_{\mathbf{P}(E)}$. The section σ determines a vector field on $[p_E]^* E$:

$$V_\sigma = \sum_{i=1}^{r} \sigma_i(z, [v^1, \ldots, v^r]) \otimes \frac{\partial}{\partial v^i}$$

which, by definition, is vertical ($v_1, \ldots, v_n$ are fiber coordinates) and $\rho(\sigma) = [\ ]_* V_\sigma$. It is clear that the kernel of ρ is spanned by the tautological section (corresponding to the position vector field), i.e.,

$$\tau = \sum_{i=1}^{r} v^i \otimes e_i \Leftrightarrow V_\tau = P = \sum_{i=1}^{r} v^i \otimes \frac{\partial}{\partial v^i}$$

hence $[\ ]_* V_\tau = [\ ]_* P = 0$. The Euler sequence implies that

$$c_1([\mathcal{V}_E^*]) = -c_1([p_E]^* E \otimes \mathcal{L}_{\mathbf{P}(E)}) = -c_1([p_E]^* E) - r c_1(\mathcal{L}_{\mathbf{P}(E)}), \tag{3.11}$$

where we have also used the case $k = 1$ of the following identity for Chern classes of tensor product of a line bundle $\mathcal{F}$ and a rank r vector bundle $\mathcal{E}$:

$$c_k(\mathcal{E} \otimes \mathcal{F}) = \sum_{i=0}^{k} \binom{r-i}{k-i} c_i(\mathcal{E}) c_1^{k-i}(\mathcal{F}).$$

Note that in the classical Euler sequence this reduces to (as E is the trivial bundle) the well-known fact that $c_1(\mathcal{K}_{\mathbf{P}^{r-1}}) = -rc_1(\mathcal{L}_{\mathbf{P}^{r-1}})$. On the other hand, we have (by definition) an exact sequence:

$$0 \to [\mathcal{V}_E] \to T\mathbf{P}(E) \to TE/[\mathcal{V}_E] \to 0,$$

where $T\mathbf{P}(E)/[\mathcal{V}_E]$ is $\mathcal{C}^\infty$-isomorphic to the horizontal sub-bundle $\mathcal{H}_E$ (which is holomorphically isomorphic to TM under the map $[p_E]_* : T\mathbf{P}(E) \to TM$). By duality we get a $\mathcal{C}^\infty$ exact sequence:

$$0 \to [p_E]^* T^*M \to T^*\mathbf{P}(E) \to [\mathcal{V}_E^*] \to 0,$$

which implies that $c_1(T^*\mathbf{P}(E)) = c_1([p_E]^*T^*M) + c_1(\mathcal{V}_E^*)$, i.e.,

$$c_1(\mathcal{K}_{\mathbf{P}(E)}) = c_1([p_E]^*\mathcal{K}_M) + c_1([\mathcal{V}_E^*]), \tag{3.12}$$

where $\mathcal{K}_{\mathbf{P}(E)}$ and $\mathcal{K}_M$ are the canonical bundles of $\mathbf{P}(E)$ and M respectively. The preceding identities imply (cf. Griffiths [11], Kobayashi-Ochiai [15]):

Theorem 3.5. *For any holomorphic vector bundle E of rank $r \geq 2$ over a complex manifold M, we have*

$$\mathcal{K}_{\mathbf{P}(E)} \cong [p_E]^*(\mathcal{K}_M \otimes \det E^*) \otimes \mathcal{L}^{-r}_{\mathbf{P}(E)},$$

where $\mathcal{L}^{-r}_{\mathbf{P}(E)}$ is the dual of the r-fold tensor product of the "hyperplane bundle" $\mathcal{L}_{\mathbf{P}(E)}$.

Proof. Recall that an exact sequence of vector bundles $0 \to E_1 \to E_2 \to E_3 \to 0$ induces an isomorphism: $\det E_1 \otimes \det E_3 \cong \det E_2$. We get from the dual Euler sequence (see (3.10)) $\det \mathcal{V}_E^* \cong [p_E]^* \det E^* \otimes \mathcal{L}^{-r}_{\mathbf{P}(E)}$. On the other hand, by (3.11), it is clear that

$$\mathcal{K}_{\mathbf{P}(E)} \cong [p_E]^*(\mathcal{K}_M) \otimes \det \mathcal{V}_E^*$$

hence $\mathcal{K}_{\mathbf{P}(E)} \cong [p_E]^*(\mathcal{K}_M) \otimes [p_E]^* \det E^* \otimes \mathcal{L}^{-r}_{\mathbf{P}(E)}$ as claimed. $\square$

Let $E = TM$ in the preceding Theorem then $\det E^* = \mathcal{K}_M$ and we get

$$\mathcal{K}_{\mathbf{P}(TM)} \cong [p_{TM}]^*\mathcal{K}_M^2 \otimes \mathcal{L}^{-n}_{\mathbf{P}(TM)},$$

where $n = \dim M$; equivalently, $\mathcal{K}_{\mathbf{P}(TM)} \otimes \mathcal{L}^n_{\mathbf{P}(TM)} \cong [p_{TM}]^*\mathcal{K}_M^2$. Let $E = T^*M$ in the preceding Theorem then $\det E^* = \mathcal{K}_M^{-1}$ and we get $\mathcal{K}_{\mathbf{P}(T^*M)} \cong \mathcal{L}^{-n}_{\mathbf{P}(T^*M)}$ where $n = \dim M$; equivalently, $\mathcal{K}^{-1}_{\mathbf{P}(T^*M)} \cong \mathcal{L}^n_{\mathbf{P}(T^*M)}$.

Corollary 3.6. *Let $n = \dim M$ then* (i) *TM is ample (resp. nef) if and only if $\mathcal{K}^{-1}_{\mathbf{P}(T^*M)}$ is ample (resp. nef),* (ii) *if $\mathcal{K}_M$ is nef then $\mathcal{K}_{\mathbf{P}(TM)} \otimes \mathcal{L}^n_{\mathbf{P}(TM)}$ is nef,* (iii) *if T^*M is ample then $\mathcal{K}_{\mathbf{P}(TM)} \otimes \mathcal{L}^n_{\mathbf{P}(TM)}$ is nef hence $\mathcal{K}_{\mathbf{P}(TM)} \otimes \mathcal{L}^{n+1}_{\mathbf{P}(TM)}$ is ample.*

Proof. Assertion (i) is clear from the remark above. For (ii) if $\mathcal{K}_M$ is nef then $c_1(\mathcal{K}_{\mathbf{P}(TM)} \otimes \mathcal{L}^n_{\mathbf{P}(TM)}) = c_1(\mathcal{K}_M)^2 \geq 0$ hence $\mathcal{K}_{\mathbf{P}(TM)} \otimes \mathcal{L}^n_{\mathbf{P}(TM)}$ is also nef. If T^*M is ample then $\mathcal{K}_M$ is ample hence $c_1(\mathcal{K}_{\mathbf{P}(TM)} \otimes \mathcal{L}^n_{\mathbf{P}(TM)}) = c_1(\mathcal{K}_M)^2 \geq 0$. In fact it is positive definite in the horizontal direction. By definition T^*M is ample if and only if $\mathcal{L}_{\mathbf{P}(TM)}$ is ample hence $c_1(\mathcal{K}_{\mathbf{P}(TM)} \otimes \mathcal{L}^{n+1}_{\mathbf{P}(TM)}) = 2c_1(\mathcal{K}_M) + c_1(\mathcal{L}_{\mathbf{P}(TM)})$ is positive definite. □

Corollary 3.7. *If TM is ample,* $\dim M = n$, *then $H^i(M, \odot^m TM) = 0$ for all $m \geq 1$ and $H^i(M, \odot^{mn+k}(\otimes TM)) = 0$ for all $m, k \geq 1$.*

Proof. If TM is ample then $\mathcal{L}^m_{\mathbf{P}(T^*M)}$ and $\mathcal{K}^{-1}_{\mathbf{P}(T^*M)}$ are both ample for all $m \geq 1$. This implies that $\mathcal{L}^m_{\mathbf{P}(T^*M)} \otimes \mathcal{K}^{-1}_{\mathbf{P}(T^*M)}$ is also ample. By Kodaira's vanishing Theorem we have, $H^i(\mathbf{P}(T^*M), \mathcal{L}^m_{\mathbf{P}(T^*M)}) = 0$ for all $i \geq 1$ and, by Grothendieck's Theorem:

$$H^i(M, \odot^m TM) \cong H^i(\mathbf{P}(T^*M), \mathcal{L}^m_{\mathbf{P}(T^*M)}) = 0$$

for all $i, m \geq 1$ as well. If we take $E = \otimes^m T^*M$ in Theorem 3.8 then rank $E = nm$ where $n = \dim M$ and

$$\begin{aligned} c_1(\det E^*) = m(1 &+ (n-1)C_1^{m-1} + (n-1)^2 C_2^{m-1} \\ &+ \cdots + (n-1)^{m-1} C_{m-1}^{m-1}) c_1(\mathcal{K}_M^{-1}). \end{aligned}$$

Here we use the following (see [S-W] for details) standard formula with $F = T^*M$:

Proposition 3.8. *Let F be a vector bundle of rank r then*

$$\begin{aligned} c_1(\otimes^m F) = m(1 &+ (r-1)C_1^{m-1} + (r-1)^2 C_2^{m-1} \\ &+ \cdots + (r-1)^{m-1} C_{m-1}^{m-1}) c_1(F), \end{aligned}$$

where C_j^k is the usual binomial coefficient of choosing j elements from a set of k objects.

By Theorem 3.5,

$$\begin{aligned} c_1(\mathcal{K}_{\mathbf{P}(E)}) + mn\, c_1(\mathcal{L}_{\mathbf{P}(E)}) &= [p_E]^* \{c_1(\mathcal{K}_M) + c_1(\det E^*)\} \\ &= [p_E]^* \left(\left(m \sum_{i=0}^{m-1} (n-1)^i C_i^{m-1} \right) - 1 \right) c_1(\mathcal{K}_M^{-1}), \end{aligned}$$

which implies that $\mathcal{K}_{\mathbf{P}(E)} \otimes \mathcal{L}^{mn}_{\mathbf{P}(E)}$ is nef if $\mathcal{K}_M^{-1}$ is nef. If TM is ample then so is $\mathcal{L}_{\mathbf{P}(E)}$ we infer from the preceding that $\mathcal{K}_{\mathbf{P}(E)} \otimes \mathcal{L}^{mn+k}_{\mathbf{P}(E)}$ is ample for $l \geq 1$ and since $\mathcal{L}^{mn+k}_{\mathbf{P}(E)}$ is also ample we have as before: $H^i(M, \odot^{mn+k}(\otimes TM)) \cong H^i(\mathbf{P}(E), \mathcal{L}^{mn+k}_{\mathbf{P}(E)}) = 0$ for all $i, k, m \geq 1$. □

4. Curvature of tensor products of vector bundles

It is well-known that the ampleness of a vector bundle E implies that the tensor products $\otimes^k E$, symmetric product $\odot^k E$ are also ample for any positive integer k; the same is also true for the exterior product $\wedge^k E$ for $1 \le k \le r =$ rank E. The corresponding differential geometric statement, which we now show, asserts that the negativity (resp. positivity) of the mixed bisectional curvature of E implies the negativity (resp. positivity) of the mixed bisectional curvature of $\otimes^k E, \odot^k E$ for any positive k and $\wedge^k E$ for $1 \le k \le r$. In this section we work out the precise relation between the respective curvatures. As a result of these calculations we shall show that, in fact, the negativity (resp. positivity) of the mixed bisectional curvature of E and the negativity (resp. positivity) of the mixed bisectional curvature of $\odot^k E, k > 0$ are equivalent.

For Hermitian holomorphic vector bundles (E, h) and (F, k) the tensor product $E \otimes F$ is equipped with the Hermitian metric $H = h \otimes k$, i.e.,

$$\langle a \otimes \alpha, b \otimes \beta \rangle_{h \otimes k} = \langle a, b \rangle_h \langle \alpha, \beta \rangle_k, \tag{4.1}$$

more generally, for $s_{i,j}, t_{\mu,\nu} \in \mathbf{C}$,

$$\left\langle \sum_{i,j} s_{i,j} a_i \otimes \alpha_j, \sum_{\mu,\nu} t_{\mu,\nu} b_\mu \otimes \beta_\nu \right\rangle_{h \otimes k} = \sum_{i,j,\mu,\nu} s_{i,j} \bar{t}_{\mu,\nu} \langle a_i, b_\mu \rangle_h \langle \alpha_j, \beta_\nu \rangle_k.$$

Let $\{e_1, \ldots, e_r\}, \{f_1, \ldots, f_s\}$ be local frames for E and F then

$$\{e_{ij} = e_i \otimes f_j, 1 \le i \le r, 1 \le j \le s\} \tag{4.2}$$

is a local frame for $E \otimes F$. For $1 \le i, j, p, q \le n$,

$$H_{\{ij\}\{\bar{p}\bar{q}\}} = \langle e_{ij}, e_{pq} \rangle_H = \langle e_i, e_p \rangle_h \langle f_j, f_q \rangle_h = h_{i\bar{p}} k_{j\bar{q}}.$$

If (E, h) and (F, k) are Finsler bundles (h and k satisfy conditions (FM1) through (FM6) as stated in the introduction) we cannot, in general, define the tensor product of h and k on $E \otimes F$ even though for simple elements $a \otimes b, a \in E, b \in F, H(z, a \otimes b) = h(z, a)k(z, b)$ is well-defined but there is no natural way of extending this definition to the general elements as in the case of Hermitian metrics. However, as h and k are strictly pseudoconvex along the fibers we may proceed as follows. Let $\eta = h^2$ and $\kappa = k^2$ then, as was seen in the preceding section,

$$h_{i\bar{j}}(z; v) = \frac{\partial^2 \eta}{\partial v^i \partial \bar{v}^j}(z; v), \qquad k_{i\bar{j}}(z; u) = \frac{\partial^2 \kappa}{\partial v^i \partial \bar{v}^j}(z; u) \tag{4.3}$$

are hermitian metrics on the vertical bundles $\mathcal{V}_E$ and $\mathcal{V}_F$ respectively. Now the tensor product of these two hermitian metrics is a hermitian metric of the bundle $\mathcal{V}_E \otimes \mathcal{V}_F$. It is easily seen that $\mathcal{V}_E \otimes \mathcal{V}_F$ is the vertical sub-bundle of $T(E \otimes F)$, i.e., $\mathcal{V}_E \otimes \mathcal{V}_F = \mathcal{V}_{E \otimes F}$. In what follows we shall be working with the Hermitian and Finsler cases at the same time with the understanding that, in the later case we are working on $\mathcal{V}_E \otimes \mathcal{V}_F$ instead of $E \otimes F$. The connection of the tensor product is the tensor product of the connections:

$$\nabla(a \otimes \alpha) = (\nabla a) \otimes \alpha + a \otimes \nabla\alpha. \tag{4.4}$$

For simplicity of notations the same symbol is used for the 3 different connections (on $E \otimes F, E, F$ respectively). The connection is extended to general elements by enforcing linearity (over $\mathbf{C}$) and Leibnitz rule. In terms of the frames (4.2) the connection forms are related by: $\theta_{E\otimes F} = \theta_E \otimes I_s + I_r \otimes \theta_F$ where $\theta_i^p(E)$ and $\theta_j^q(F)$ are the connection forms of E and F respectively (in the Finsler case, $\theta_i^p(\mathcal{V}_E)$ and $\theta_j^q(\mathcal{V}_F)$ are, respectively, the connection forms of the metrics defined by (4.3) on $\mathcal{V}_E$ and $\mathcal{V}_F$). A simple calculation shows that the curvature forms are similarly related: $\Theta_{E\otimes F} = \Theta_E \otimes I_s + I_r \otimes \Theta_F$. It is clear from the preceding formula that the mixed holomorphic bisectional curvature of $(E \otimes F, h \otimes k)$ is ≤ 0 (resp. < 0) if the mixed holomorphic bisectional curvatures of (E, h) and (F, k) are both ≤ 0 (resp. < 0). By induction, we have:

Proposition 4.1. *Let* $(E_i, h_i), i = 1, \ldots, m$ *be Hermitian* (*resp. Finsler satisfying* (FM1)–(FM6)) *holomorphic vector bundles over a complex manifold* M. *If the holomorphic bisectional curvature of* (E_i, h_i) *are* ≤ 0 (*resp.* < 0 *resp.* ≥ 0, *resp.* > 0) *for* $i = 1, \ldots, m$ *then the holomorphic bisectional curvature of* $(\otimes_{i=1}^m E_i, \otimes_{i=1}^m h_i)$ (*resp.* $(\otimes_{i=1}^m \mathcal{V}_{E_i}, \otimes_{i=1}^m h_i)$ *in the case of Finsler metrics*) *is* ≤ 0 (*resp.* < 0 *resp.* ≥ 0, *resp.* > 0).

The case of exterior product is similar. For a hermitian holomorphic vector bundles (E, h) the wedge product $\wedge^m E (m \leq r = \text{rank } E)$ is equipped with the metric $H = \wedge^m h$:

$$\langle a_1 \wedge \cdots \wedge a_m, b_1 \wedge \cdots \wedge b_m\rangle_{\wedge^m h} = \det(\langle a_i, b_j\rangle_h)_{1\leq i,j\leq m}. \tag{4.5}$$

If h is only a Finsler metric satisfying conditions (FM1)–(FM7) as stated in the introduction then it defines a hermitian metric on $\mathcal{V}_E$ (see (4.3)) and hence also a hermitian metric along the fibers of $\wedge^m \mathcal{V}_E$. In the following we shall work with the case of hermitian metric with the understanding that the Finsler case is analogous and is obtained by replacing E by $\mathcal{V}_E$ in the discussions below. Denote the set of increasing indices by:

$$\mathcal{I}_{m,r} = \{I = (i_1, \ldots, i_m) \,|\, 1 \leq i_1 < \cdots < i_m \leq r\}.$$

Let $\{e_1, \ldots, e_r\}$ be a local frame for E then

$$\{e_I \,|\, I \in \mathcal{I}_{m,r}\} \tag{4.6}$$

is a local frame for $\wedge^m E$. In terms of these frames the components of the metric H are given by: $H_{I\overline{J}} = \langle e_I, e_J\rangle_{\wedge^m h} = \det(\langle e_{i_k}, e_{j_l}\rangle_h)_{1\leq k,l\leq m}$. The associated connection ∇ on $\wedge^m E$ (which will be denoted by the same notation) is given as follows:

$$\nabla(a_1 \wedge \cdots \wedge a_m) = \sum_{i=1}^m a_1 \wedge \cdots \wedge \nabla a_i \wedge \cdots \wedge a_m.$$

The preceding defining identity can be verified by skew symmetrizing the connection for tensor products. For example if $m = 2$ then $a \wedge b = (1/2)(a \otimes b - b \otimes a)$ hence

$$\begin{aligned}\nabla(a \wedge b) &= \frac{1}{2}(\nabla(a \otimes b) - \nabla(b \otimes a)) \\ &= \frac{1}{2}\{(\nabla a) \otimes b + a \otimes \nabla b - (\nabla b) \otimes a - b \otimes \nabla a\} \\ &= (\nabla a) \wedge b + a \wedge \nabla b.\end{aligned}$$

The general case is verified in an analogously way using the identity

$$a_1 \wedge \cdots \wedge a_m = \frac{1}{m!} \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)\, a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(m)},$$

where σ ranges over the symmetric group on m elements. In terms of the given frames (4.6) the connection forms are related by the formula:

$$\theta_{\wedge^m E} = \sum_{i=1}^{m} I_r \wedge \cdots \wedge I_r \wedge \theta_E \wedge I_r \wedge \cdots \wedge I_r$$

and, analogously, the curvature forms are related by

$$\Theta_{\wedge^m E} = \sum_{i=1}^{m} I_r \wedge \cdots \wedge I_r \wedge \Theta_E \wedge I_r \wedge \cdots \wedge I_r.$$

This implies the following result:

Proposition 4.2. *Let (E, h) be an Hermitian (resp. Finsler satisfying conditions* (FM1)–(FM6)) *holomorphic vector bundle over a complex manifold M. If the holomorphic bisectional curvature of (E, h) is ≤ 0 (resp. < 0 resp. ≥ 0, resp. > 0) then, for $1 \leq m \leq r$, the mixed holomorphic bisectional curvature of $(\wedge^m E, \wedge^m h)$ (resp. $(\wedge^m \mathcal{V}_E, \wedge^m h)$ is ≤ 0 (resp. < 0 resp. ≥ 0, resp. > 0).*

For a hermitian holomorphic vector bundle (E, h) the symmetric product $\odot^2 E$ is equipped with the metric $H = h \odot h = \odot^2 h$:

$$\langle a \odot \alpha, b \odot \beta \rangle_{h \odot h} = \frac{1}{2}(\langle a, b \rangle_h \langle \alpha, \beta \rangle_h + \langle a, \beta \rangle_h \langle \alpha, b \rangle_h)$$

and, more generally, for any positive integer m,

$$\langle \odot_{i=1}^m a_i, \odot_{j=1}^m b_j \rangle_{\odot^m h} = \frac{1}{m!} \sum_{\sigma} \prod_{i=1}^{n} \langle a_i, b_{\sigma(i)} \rangle_h, \tag{4.7}$$

where σ ranges over all elements of the symmetric group on m elements. If h is Finsler metric satisfying (FM1)–(FM5) the same argument applies to the bundle $\odot^m \mathcal{V}_E$. Let

$$\mathcal{J}_{m,r} = \{I = (i_1, \ldots, i_m), 1 \leq i_1 \leq \cdots \leq i_m \leq r\}.$$

Let $\{e_1, \ldots, e_r\}$ be a local frame for E then

$$\{e_I = e_{i_1} \odot \cdots \odot e_{i_m}, I = (i_1, \ldots, i_m) \in \mathcal{J}_{m,r}\} \tag{4.8}$$

is a local frame for $\odot^m E$. For example, if $m = 2$ then for $1 \leq i \leq j \leq r, 1 \leq p, \leq q \leq r$, the components $H_{\{ij\}\{\bar{p}\bar{q}\}}$ of the metric $H = h \odot h$ are given by

$$H_{\{ij\}\{\bar{p}\bar{q}\}} = \langle e_{ij}, e_{pq} \rangle_H = \frac{1}{2}(h_{i\bar{p}}h_{j\bar{q}} + h_{i\bar{q}}h_{j\bar{p}}).$$

The connection of the symmetric product is the symmetric product of the connections:

$$\nabla(\odot_{i=1}^m a_i) = \sum_{i=1}^m a_1 \odot \cdots \odot (\nabla a_i) \odot \cdots \odot \alpha_m. \tag{4.9}$$

Thus, relative to the frames (4.8) the connection forms and the curvature forms are related by

$$\theta_{\odot^m E} = \sum_{i=1}^m I_r \odot \cdots \odot I_r \odot \theta_E \odot I_r \cdots \odot I_r,$$
$$\Theta_{\odot^m E} = \sum_{i=1}^m I_r \odot \cdots \odot I_r \odot \Theta_E \odot I_r \cdots \odot I_r.$$

Thus for any positive integer m we have:

Proposition 4.3. *Let* (E, h) *be an hermitian* (*resp. Finsler satisfying* (FM1)–(FM6)) *holomorphic vector bundle over a complex manifold* M. *If the holomorphic bisectional curvature of* (E, h) *is* ≤ 0 (*resp.* < 0 *resp.* ≥ 0, *resp.* $>$ 0) *then, for* $1 \leq m \leq r$, *the holomorphic bisectional curvature of* $(\odot^m E, \odot^m h)$ (*resp.* $(\odot^m \mathcal{V}_E, \odot^m h)$) *is* ≤ 0 (*resp.* < 0 *resp.* ≥ 0, *resp.* > 0).

For the symmetric bundle the converse is also true in the following sense. Suppose that $\otimes^m E$ admits a hermitian metric such that the mixed holomorphic bisectional curvature is ≤ 0 (resp. < 0). Then the bundle $\oplus^k \odot^m E$ with the product metric $\oplus^k H$ also satisfies the condition that the mixed holomorphic bisectional curvature is ≤ 0 (resp. < 0). Assume that $\odot^{m-1} E$ is *spanned*, i.e., there exists a basis $\sigma_0, \ldots, \sigma_N$ of $H^0(X, \odot^{m-1} E)$ such that

$$\sigma_0(x), \ldots, \sigma_N(x) \text{ span the fiber of } \odot^{m-1} E_x \qquad \text{for all} \quad x \in M. \tag{4.10}$$

This condition implies that, at each point $x \in M$, there exists $i_1, \ldots, i_q, q =$ rank $\odot^{m-1} E$ such that $\sigma_{i_1}(x), \ldots, \sigma_{i_q}(x)$ is a basis of $\odot^{m-1} E_x$. Note that this condition is satisfied if $\odot^{m-1} E$ is very ample. Consider the map $\iota : E \to \odot^m E$ defined by

$$\iota : E \to \oplus^{N+1} \odot^m E, \ \iota(s) = (s \odot \sigma_0, \ldots, s \odot \sigma_N).$$

It is clear that $\iota(as+bt)=a\iota(s)+b\iota(t), a,b\in\mathbf{C}, s,t\in E$. We claim that it is also injective. By the spanned condition we may assume without loss of generality that $\sigma_0(x),\ldots,\sigma_{q-1}(x)$ is a basis of $\otimes^{m-1}E_x$; if $\iota(s)=0, s\in E_x$ then $s\odot\sigma_i=0$ for all i which implies that $s=0$. This shows that ι is a linear isomorphism hence $\iota(E)=\Delta$ is a holomorphic sub-bundle of $\oplus^{N+1}\odot^m E$:

$$\iota: E\cong\Delta\subset\oplus^{N+1}\odot^m E.$$

Let $h=\oplus^{N+1}H|_\Delta$ be the induced metric on Δ then ι^*h is a hermitian metric on E. Let $e_1,\ldots,e_r$ be a local unitary frame (relative to ι^*h) for E then

$$f_i=\iota(e_i),\qquad i=1,\ldots,r$$

is a local unitary frame for Δ. Extend this to an unitary frame $\{f_i, i=1,\ldots,M=\operatorname{rank}\oplus^{N+1}\odot^m E\}$ for $\oplus^{N+1}\odot^m E$. Denote by $\oplus^{N+1}H=(H_{ij}(z))$ where $1\le i,j\le M$ the hermitian metric on $\oplus^{N+1}\odot^m E$ relative to this frame then the induced metric $h=\oplus^{N+1}H|_\Delta=(H_{ij}(z))_{1\le i,j\le r}$. Let $\hat{D}$ (resp. D) be the hermitian connection of $\oplus^{N+1}H$ (resp. h) then $\hat{D}f_i=\sum_{i=1}^M\hat\theta_i^j f_j$ for $1\le i\le M$ and $Df_i=\sum_{i=1}^r\theta_i^j f_j$ for $1\le i\le r$. Let $Q=\oplus^{N+1}\odot^m E/\Delta$ be the quotient bundle with the quotient metric and the quotient connection and define an operator $A=\hat{D}|_\Delta-D:\Gamma(\Delta)\to\Gamma(Q)$ then the connection matrices are related by

$$\hat\theta=\begin{pmatrix}\theta_\Delta & {}^t\overline{A}\\ A & \theta_Q\end{pmatrix},$$

where A is represented, with respect to the chosen frame, by a matrix (a_i^j) where $r+1\le i\le M, 1\le j\le r$ of 1-forms. The curvature forms are given by: $\hat{D}^2 f_i=\sum_{i,k=1}^M\hat\Theta_i^k\otimes f_k$, $\hat\Theta_i^k=d\hat\theta_i^k-\sum_{j=1}^M\hat\theta_i^j\wedge\hat\theta_j^k$ for $1\le i,k\le M$ and $D^2f_i=\sum_{i,k=1}^r\Theta_i^k\otimes f_k$, $\Theta_i^k=d\theta_i^k-\sum_{j=1}^r\theta_i^j\wedge\theta_j^k$ for $1\le i,k\le r$. The curvature matrices are related by (see [G-H])

$$\hat\Theta=\begin{pmatrix}\Theta_\Delta-{}^t\,\overline{A}\wedge A & *\\ * & \Theta_Q+A\wedge{}^t\,\overline{A}\end{pmatrix},$$

where ${}^t\overline{A}\wedge A=(A_i^k=\sum_{r+1\le j\le M}\bar a_i^j\wedge a_j^k)_{1\le i,k\le r}$. Thus, for $\sigma=\sum_{j=1}^r a^j f_j\in\Delta$,

$$\begin{aligned}\langle\hat K\sigma,\sigma\rangle_{\oplus^{N+1}H}&=\left\langle\sum_{i,j=1}^r a^i\hat\Theta_i^j f_j,\sum_{k=1}^r a^k f_k\right\rangle_{\oplus^{N+1}H}\\&=\left\langle\sum_{j=1}^r a^i(\Theta_i^j-A_i^j)f_j,\sum_{k=1}^r a^k f_k\right\rangle_h\\&=\langle K\sigma,\sigma\rangle_h-\left\langle\sum_{j=1}^r a^iA_i^kf_j,\sum_{k=1}^r a^kf_k\right\rangle_h.\end{aligned}$$

The second term on the right above is positive (see [G-H]) hence we have the following *partial* converse of Proposition 4.3:

Proposition 4.4. *Let E be a holomorphic vector bundle and suppose that, for some positive integer m $\otimes^{m-1}E$ is spanned and that there exists a hermitian metric H on $\odot^m E$ such that the mixed holomorphic bisectional curvature of $(\odot^m E, H)$ is ≤ 0 (resp. < 0) then there exists a hermitian metric on E such that the mixed holomorphic bisectional curvature of (E, h) is ≤ 0 (resp. < 0).*

Remark 4.5. If E is ample then the condition that $\otimes^{m-1}E$ is spanned is automatically satisfied if m is sufficiently large. We shall see in the next section that the condition that the mixed holomorphic bisectional curvature of $(\otimes^m E, H)$ is < 0 implies that E is ample hence, the condition that "$\otimes^{m-1}E$ is spanned" is redundant in this case for sufficiently large m.

Remark 4.6. We remark that the preceding calculation yields no particular useful information in the case of (mixed) *positive* bisectional curvature.

5. Finsler metrics

Let E be a holomorphic vector bundle of rank $r \geq 2$ over a complex manifold M and let $\mathcal{L}^{-1} = \mathcal{L}^{-1}_{\mathbf{P}(E)}$ over $\mathbf{P}(E)$ be the "tautological" line bundle. By definition the bundle space of $\mathcal{L}^{-1}$ is the blowing up of the bundle space of E along the zero section. Thus there is a canonical isomorphism $\mathcal{L}^{-1}\backslash\{$zero-section$\} \cong E\backslash$ {zero-section} compatible with the respective $\mathbf{C}^*$ structure associated to the respective bundle structures. Let H be a hermitian metric along the fibers of $\mathcal{L}^{-1}$ which, via the preceding isomorphism, determines uniquely a function

$$h : E \to \mathbf{R}_{\geq 0}, \qquad h(z, v) = ||\beta^{-1}(z, v)||_H \tag{5.1}$$

(where $\beta : \mathcal{L}^{-1} \to E$ is the blowing up map along the zero section) with the following properties:

(FM1) *h is of class $\mathcal{C}^0$ on E and is of class $\mathcal{C}^\infty$ on $E\backslash\{$zero-section$\}$;*
(FM2) *$h(z, \lambda v) = |\lambda| h(z, v)$ for all $\lambda \in \mathbf{C}$;*
(FM3) *$h(z, v) > 0$ on $E\backslash$ {zero-section};*
(FM4) *for z and v fixed the function $\eta_{z,v}(\lambda) = h^2(z, \lambda v)$ is smooth even at $\lambda = 0$.*

A function on E satisfying properties (FM1), (FM2), (FM3) and (FM4) above shall be referred to as a *Finsler metric* of class $\mathcal{C}^\infty$. In the literature some authors do not include property (FM4) in the definition. With condition (FM4) a Finsler metric on E determines uniquely a hermitian metric along the fibers of $\mathcal{L}^{-1}$ via (5.1). *Note that we do not require that $h^2(z, v)$ be of class $\mathcal{C}^\infty$ along the zero-section*; *indeed, with this extra condition the Finsler metric is the norm of a hermitian metric along the fibers of E and we are in the situation of Section* 2. A Finsler metric is said to be *strictly pseudoconvex along the fibers* if the following condition is satisfied:

(FM5) $h|_{E_z}$ *is a strictly pseudoconvex function on* $E_z \setminus \{0\}$ *for all* $z \in M$. $h|_{E_z}$ *is strictly convex on* $E_z \setminus \{0\}$ *for all* $z \in M$.

Note that we require that F be strictly pseudoconvex only in the fiber directions. Let $G = h^2$ then

$$G(z, \lambda v) = |\lambda|^2 G(z, v) \tag{5.2}$$

for all $\lambda \in \mathbf{C}$. Taking $\lambda = e^{\sqrt{-1}\theta}$ yields $G(z, e^{\sqrt{-1}\theta}v) = G(z, v)$ and we see that a level set $\{G = c\}$(c a constant) of G is invariant by the circle action. A set invariant under the circle action is said to be *circular*. We derive some basic formulas of the derivatives of G which are needed in later calculations of the curvature (these formulas are shared by functions satisfying the homogeneous Monge-Ampere equation see for example [1], [16] and [18]). *It is understood that all differentiations are carried out off the zero section.* It is clear that the homogeneity property (5.2) remains valid for all partial derivatives of G in the base variables, i.e.,

$$\frac{\partial^{a+b} G}{\partial z_{\alpha_1} \cdots \partial z_{\alpha_a} \partial \bar{z}_{\beta_1} \cdots \bar{z}_{\beta_b}}(z, \lambda v) = |\lambda|^2 \frac{\partial^{a+b} G}{\partial z_{\alpha_1} \cdots \partial z_{\alpha_a} \partial \bar{z}_{\beta_1} \cdots \bar{z}_{\beta_b}}(z, v). \tag{5.3}$$

On the other hand, differentiating (5.2) with respect to the fiber variables $v = (v^1, \ldots, v^r)$ yields (for $\lambda \in \mathbf{C}^*, v \in E_z \setminus \{0\}$)

$$\frac{\partial G}{\partial v^i}(z, \lambda v) = \overline{\lambda} \frac{\partial G}{\partial v^i}(z, v), \qquad \frac{\partial G}{\partial \overline{v}^i}(z, \lambda v) = \lambda \frac{\partial G}{\partial \overline{v}^i}(z, v). \tag{5.4}$$

The identities (5.2), (5.3) and (5.4) imply that

$$\sum_{i=1}^{r} \frac{1}{G(z, \lambda v)} \frac{\partial G}{\partial z^i}(z, \lambda v) dz^i = \sum_{i=1}^{r} \frac{1}{G(z, v)} \frac{\partial G}{\partial z^i}(z, v) dz^i,$$
$$\sum_{i=1}^{r} \frac{1}{G(z, \lambda v)} \frac{\partial G}{\partial v^i}(z, \lambda v) d(\lambda v^i) = \sum_{i=1}^{r} \frac{1}{G(z, v)} \frac{\partial G}{\partial v^i}(z, v) dv^i.$$

In other words, $\partial_z \log G, \partial_v \log G$ (resp. $\overline{\partial}_z \log G, \overline{\partial}_v \log G$) are invariant by the $\mathbf{C}^*$-action on E where ∂_z, ∂_v (resp. $\overline{\partial}_z, \overline{\partial}_v$) are the base component and fiber component of the operator ∂ (resp. $\overline{\partial}$) on E. More precisely, we have

$$\lambda^*(\partial_z \log G(z, v)) = \sum_{i=1}^{r} \frac{1}{G(z, \lambda v)} \frac{\partial G}{\partial z^i}(z, \lambda v) dz^i = \partial_z \log G(z, v),$$
$$\lambda^*(\partial_z \log G(z, v)) = \sum_{i=1}^{r} \frac{1}{G(z, \lambda v)} \frac{\partial G}{\partial z^i}(z, \lambda v) d(\lambda v^i) = \partial_v \log G(z, v)$$

and, as G is real-valued, a similar set of formulas for $\partial_{\bar{z}} \log G(z, v)$. This also implies that the level set $\{G(z, v) = c\}$ is non-singular for any *positive* constant c. The identity (5.2) and (5.3) imply in particular that

$$\frac{1}{G(z, \lambda v)} \frac{\partial^2 G}{\partial z^i \partial \bar{z}^j}(z, \lambda v) = \frac{1}{G(z, v)} \frac{\partial^2 G}{\partial z^i \partial \bar{z}^j}(z, v) \tag{5.5}$$

and differentiating the first identity of (5.4) with respect to $\bar{v}^j$ yields:

$$\frac{\partial^2 G}{\partial v^i \partial \overline{v}^j}(z, \lambda v) = \frac{\partial^2 G}{\partial v^i \partial \overline{v}^j}(z, v), \tag{5.6}$$

in other words, both $G^{-1}\partial_z\overline{\partial}_z G$ and $G^{-1}\partial_v\overline{\partial}_v G$ are invariant by the $\mathbf{C}^*$-action. Since

$$\partial_z\overline{\partial}_z \log G = \frac{1}{G}\partial_z\overline{\partial}_z G - \partial_z \log G \wedge \overline{\partial}_z \log G,$$
$$\partial_v\overline{\partial}_v \log G = \frac{1}{G}\partial_v\overline{\partial}_v G - \partial_v \log G \wedge \overline{\partial}_v \log G,$$

we conclude, from the calculations above that $\partial_z\overline{\partial}_z \log G$ and $\partial_v\overline{\partial}_v \log G$ are invariant by the the $\mathbf{C}^*$-action. To deal with mixed derivatives we differentiate (5.4) with respect to z then

$$\frac{\partial^2 G}{\partial v^i \partial z^{\overline{\beta}}}(z, \lambda v) = \overline{\lambda}\frac{\partial^2 G}{\partial v^i \partial z^{\overline{\beta}}}(z, v), \qquad \frac{\partial^2 G}{\partial z^\alpha \partial \overline{v}^j}(z, \lambda v) = \lambda\frac{\partial^2 G}{\partial z^\alpha \partial \overline{v}^j}(z, v), \tag{5.7}$$

and these imply that $G^{-1}\partial_v\overline{\partial}_z G$ and $G^{-1}\partial_z\overline{\partial}_v G$ are both invariant by the $\mathbf{C}^*$-action:

$$\lambda^*(\partial_v\overline{\partial}_z G(z, v)) = \sum \frac{1}{G(z, \lambda v)}\frac{\partial^2 G}{\partial v^i \partial \bar{z}^j}(z, \lambda v) d(\lambda v^i) \wedge d\bar{z}^j = \partial_v\overline{\partial}_z G(z, v),$$
$$\lambda^*(\partial_v\overline{\partial}_z G(z, v)) = \sum \frac{1}{G(z, \lambda v)}\frac{\partial^2 G}{\partial z^i \partial \overline{v}^j}(z, \lambda v) d\bar{z}^j \wedge d(\lambda \bar{v}^j) = \partial_z\overline{\partial}_v G(z, v).$$

Since $\partial\overline{\partial}\log G = (1/G)\partial\overline{\partial}G - \partial\log G \wedge \overline{\partial}\log G = (1/G)(\partial_z\overline{\partial}_z G + \partial_v\overline{\partial}_v G + \partial_v\overline{\partial}_z G + \partial_z\overline{\partial}_v G) - (\partial_z \log G + \partial_v \log G) \wedge (\overline{\partial}_z \log G + \overline{\partial}_v \log G)$ we conclude that $\partial\overline{\partial}\log G$ is also invariant by the $\mathbf{C}^*$-action. These show that both $\partial\overline{\partial}\log G$, $\partial_z\overline{\partial}_z \log G$ and $\partial_v\overline{\partial}_v \log G$ descend to well-defined $(1,1)$-forms on $\mathbf{P}(E)$. Moreover, if the Finsler metric is strictly pseudoconvex along the fibers then (5.3) implies that the restriction of the Levi-form $\partial_v\overline{\partial}_v \log G$, to the maximal complex tangent bundle of $\{G = c\} \cap E_z$, is strictly pseudoconvex for all $c > 0$. This is so because the maximal complex tangent bundle of $\{G = c\} \cap E_z$ is annihilated by $\partial_v \log G$. In other words we have shown that

Lemma 5.1. *Let F be a Finsler metric along the fibers of a holomorphic vector bundle, of rank ≥ 2, over a complex manifold. Then $\sqrt{-1}\partial\overline{\partial}\log G$ ($G = F^2$) descends to a well-defined $(1,1)$-form ϕ on $\mathbf{P}(E)$. If G is strictly pseudoconvex then ϕ is positive definite when restrict to the tangent bundle $\mathbf{P}(E_z)$ of a fiber $\mathbf{P}(E_z)$.*

The final set of formulas are obtained by differentiating the identity (5.2) with respect to the variable λ and $\overline{\lambda}$ resulting in the identities

$$\sum_{i=1}^n v^i \frac{\partial G}{\partial v^i}(z, \lambda v) = \overline{\lambda}G(z, v); \sum_{i=1}^n \overline{v}^i \frac{\partial G}{\partial \overline{v}^i}(z, \lambda v = \lambda G(z, v); \tag{5.8}$$

in particular, we have

$$PG(z,\lambda v)=\sum_{i=1}^{n} v^i\frac{\partial G}{\partial v^i}(z,v)=G(z,v)=\sum_{i=1}^{n}\bar v^i\frac{\partial G}{\partial \bar v^i}(z,v)=\overline{P}G(z,\lambda v),$$

where $P=\sum_i v^i\partial/\partial v^i$ is the position vector field. Differentiating (5.8) with respect to $\overline{\lambda}$ and using (5.7) yields

$$\sum_{i,j=1}^{r} v^i\bar v^j\frac{\partial^2 G}{\partial v^i\partial \bar v^j}(z,\lambda v)=\sum_{i,j=1}^{r} v^i\bar v^j\frac{\partial^2 G}{\partial v^i\partial \bar v^j}(z,v)=G(z,v); \tag{5.9}$$

in other words $P\overline{P}G(z,\lambda v)=P\overline{P}G(z,v)=G(z,v)$. On the other hand, differentiating (5.7) with respect to λ yields

$$\sum_{i,j=1}^{r} v^i v^j\frac{\partial^2 G}{\partial v^i\partial v^j}(z,\lambda v)=\sum_{i=1}^{r} v^i v^j\frac{\partial^2 G}{\partial v^i\partial v^j}(z,v)=0. \tag{5.10}$$

Note that

$$\sum_{i=1} v^i\frac{\partial}{\partial v^i}\sum_{j=1}^{r} v^j\frac{\partial G}{\partial v^j}=\sum_{j=1}^{r} v^j\frac{\partial G}{\partial v^j}+\sum_{i,j=1}^{r} v^i v^j\frac{\partial^2 G}{\partial v^i\partial v^j}$$

hence, we see from (5.8) that (5.10) is equivalent to the condition that

$$P^2G(z,\lambda v)=\overline{\lambda}G(z,v). \tag{5.11}$$

Inductively we get

$$\sum_{i_1,\ldots,i_k=1}^{r} v^{i_1}\cdots v^{i_k}\frac{\partial^k G}{\partial v^{i_1}\cdots\partial v^{i_k}}=0 \tag{5.12}$$

for $k\geq 2$ and

$$\sum_{i_1,\ldots,i_k,j_1,\ldots,j_l=1}^{r} v^{i_1}\cdots v^{i_k}\bar v^{j_1}\cdots\bar v^{j_l}\frac{\partial^{k+l} G}{\partial v^{i_1}\cdots\partial v^{i_k}\partial\bar v^{j_1}\cdots\partial\bar v^{j_l}}=0 \tag{5.13}$$

for $k\geq 2, l\geq 0$. In fact we see, by differentiating (5.6) with respect to λ (resp. $\overline{\lambda}$) that:

$$\sum_{k=1}^{r} v^k\frac{\partial^3 G}{\partial v^i\partial\bar v^j\partial v^k}=0=\sum_{k=1}^{r}\bar v^l\frac{\partial^3 G}{\partial v^i\partial\bar v^j\bar v^l},$$

and hence

$$P^k\sum_{i,j=1}^{r}\frac{\partial^2 G}{\partial v^i\partial\bar v^j}=0 \tag{5.14}$$

for all $k \geq 1$, where $P = \sum v^i \partial/\partial v^i$ is the position vector field. Note that, by property (FM4) of a Finsler metric, formulas (5.7)–(5.14) are valid even at the zero section.

If h is a hermitian metric along the fibers of E: $\langle z, w\rangle_h = \sum_{i,j} h_{i\bar{j}}(z)v^i\bar{w}^j$ then its norm $F(z,v) = (\sum_{i,j} h_{i\bar{j}}(z)v^i\bar{v}^j)^{1/2}$ is a Finsler metric strictly pseudoconvex along the fibers with the following additional properties:

(FM6) $G = F^2$ *is smooth even at the zero-section,*
(FM7) $G_{i\bar{j}} = (\partial^2 G)/(\partial v^i \partial\bar{v}^j)$ *is independent of* v *for all* i, j.

Indeed either of these properties characterizes the norm of a hermitian metric on E (see, for example [1]):

Lemma 5.2. *A Finsler metric F is the norm of a hemitian metric on E iff the function $G = F^2$ is smooth at the zero section iff the functions $\{G_{i\bar{j}}, 1 \leq i, j \leq r\}$ are independent of v.*

Given a Finsler metric F which is stricly pseudoconvex along the fibers we define a hermitian inner product on the vertical bundle $\mathcal{V} \subset TE$ by:

$$\langle V, W\rangle_{\mathcal{V}} = \sum_{i,j=1}^{r} G_{i\bar{j}}(z,v)V^i\overline{W}^j, \qquad G_{i\bar{j}} = \frac{\partial^2 G}{\partial v^i \partial\bar{v}^j} \tag{5.15}$$

for $V = \sum_i V^i\partial/\partial v^i, W = \sum_i W^i\partial/\partial v^i \in \mathcal{V}$. In Section 2 the hermitian inner product along the fibers of the vertical bundle is defined by a hermitian metric on E and formula (2.4) is the same as (5.15) *except that the functions G_{ij} in* (2.4) *depend only on the base coordinates* $(z^1, \dots, z^n)$ *but not on the fiber coordinate* $(v^1, \dots, v^r)$. The hermitian inner product (5.15) defines uniquely a hermitian connection (known as the *Chern connection*) $\nabla^{\mathcal{V}}$ and the connection forms are given by (compare (2.5)):

$$\theta_i^k = \sum_{j=1}^{r}(\partial G_{i\bar{j}})G^{\bar{j}k} = \sum_{\alpha=1}^{n}\Gamma_{i\alpha}^k dz^\alpha + \sum_{l=1}^{r}\gamma_{il}^k dv^l, \tag{5.16}$$

where $(\Gamma_{i\alpha}^k)$ and $(\gamma_{i\alpha}^k)$ are respectively the horizontal and vertical Christoffel symbols:

$$\Gamma_{i\alpha}^k = \sum_{j=1}^{r}\frac{\partial G_{i\bar{j}}}{\partial z^\alpha}G^{\bar{j}k} \qquad \text{and} \qquad \gamma_{il}^k = \sum_{j=1}^{r}\frac{\partial G_{i\bar{j}}}{\partial v^l}G^{\bar{j}k}.$$

If F comes from a hermitian metric then, by Lemma 5.2, the vertical Christoffel symbols γ_{il}^k vanish and (5.16) reduces to (2.5). The curvature forms of a hermitian connection are always of type $(1,1)$ hence

$$\Theta_i^k = d\theta_i^k - \sum_{j=1}^{r}\theta_i^l \wedge \theta_l^k = d\theta_i^k + \sum_{j=1}^{r}\theta_l^k \wedge \theta_i^l = \overline{\partial}\theta_i^k; \tag{5.17}$$

equivalently $\partial\theta_i^k - \sum_{j=1}^r \theta_i^l \wedge \theta_l^k = \partial\theta_i^k + \sum_{j=1}^r \theta_l^k \wedge \theta_i^l = 0$. These formulas are the same as (2.7) and (2.8) except that there are now horizontal, vertical and mixed components:

$$\begin{aligned}\Theta_i^k = &\sum_{\alpha,\beta=1}^n K_{i\alpha\bar\beta}^k dz^\alpha \wedge d\bar z^\beta + \sum_{j,l=1}^r \kappa_{ij\bar l}^k dv^j \wedge d\bar v^l \\ &+ \sum_{\alpha=1}^n \sum_{l=1}^r \mu_{i\alpha\bar l}^k dz^\alpha \wedge d\bar v^l + \sum_{j=1}^r \sum_{\beta=1}^n \nu_{ij\bar\beta}^k dv^j \wedge d\bar z^\beta.\end{aligned}$$

The components are given as follows:

$$K_{i\alpha\bar\beta}^k = -\frac{\partial\Gamma_{i\alpha}^k}{\partial\bar z^\beta}, \quad \kappa_{ij\bar l}^k = -\frac{\partial\gamma_{ij}^k}{\partial\bar v^l}, \quad \mu_{i\alpha\bar l}^k = -\frac{\partial\Gamma_{i\alpha}^k}{\partial\bar v^l}, \quad \nu_{ij\bar\beta}^k = -\frac{\partial\gamma_{ij}^k}{\partial\bar z^\beta}. \tag{5.18}$$

As in Section 2 (compare (2.9)) the connection $\nabla^{\mathcal V}$ defines a surjection

$$\gamma : TE \to \mathcal V, \qquad \gamma(X) = \nabla_X^{\mathcal V} P$$

for any $X \in TE$ and where $P = \sum_{i=1}^r v^i \partial/\partial v^i$ is the position vector field. This map is now more complicated, in terms of coordinates

$$\nabla_X^{\mathcal V} P = \sum_{j=1}^r \left(b^j + \sum_{i=1}^r \sum_{\alpha=1}^n \Gamma_{i\alpha}^j v^i a^\alpha \right) \frac{\partial}{\partial v^j}$$

for any vector field $X = \sum_{\alpha=1}^n a^\alpha(z;v)(\partial/\partial z^\alpha) + \sum_{i=1}^r b^i(z;v)(\partial/\partial v^i)$. The kernel of γ is the horizontal sub-bundle $\mathcal H$. The horizontal lifts of the local basis $\{\partial_\alpha = \partial/\partial z^\alpha\}$:

$$\left\{ \partial_\alpha^{\mathcal H} = \frac{\partial}{\partial z^\alpha} - \sum_{j,k=1}^r \Gamma_{j\alpha}^k v^j \frac{\partial}{\partial v^k} \mid \alpha = 1, \ldots, n \right\}$$

is a local basis of $\mathcal H$ and these together with $\{\partial_i^{\mathcal V} = \partial/\partial v^i, i = 1, \ldots, r\}$ form a local basis for TE. Let g be a Finsler metric on M inducing a hermitian inner product $\langle , \rangle_{\mathcal H}$ along the fibers of $\mathcal H$ and we define an inner product $\langle\ \rangle_{TE}$ along the fibers of TE by taking the direct sum:

$$\langle\ \rangle_{TE} = \langle , \rangle_{\mathcal H} + \langle , \rangle_{\mathcal V}. \tag{5.19}$$

We shall also use the notation $\langle , \rangle_g$ instead of $\langle , \rangle_{\mathcal H}$ and $\langle , \rangle_h$ instead of $\langle , \rangle_{\mathcal V}$ indicating the fact that the inner product depends only on the Finsler metrics g and h respectively. Let P be the position vector field and, as in the previous sections, we consider the $(1,1)$-forms $\sqrt{-1}\partial\bar\partial||P||_h^2$ on E and $\sqrt{-1}\partial\bar\partial \log ||P||_h^2$ on $E\backslash\{\text{zero-section}\}$. The expressions for these forms in terms of the metrics are formally the same as in the previous section but the computation is now more complicated. We have

$$\sqrt{-1}\partial\overline{\partial}||P||_h^2 = \sqrt{-1}\sum_{i,j=1}^{r}\left\{G_{i\bar{j}}dv^i\wedge d\bar{v}^j + v^i\bar{v}^j\sum_{\alpha,\beta=1}^{n}\frac{\partial^2 G_{i\bar{j}}}{\partial z^\alpha\partial\bar{z}^\beta}dz^\alpha\wedge d\bar{z}^\beta\right\}$$
$$+\sqrt{-1}\sum_{i,j=1}^{r}\left\{\sum_{\alpha=1}^{n}v^i\frac{\partial G_{i\bar{j}}}{\partial z^\alpha}dz^\alpha\wedge d\bar{v}^j + \sum_{l=1}^{n}\bar{v}^j\frac{\partial G_{i\bar{j}}}{\partial\bar{z}^\beta}dv^i\wedge d\bar{z}^\beta\right\}$$
$$+\sqrt{-1}\left\{\sum_{i,j,k=1}^{r}v^i\frac{\partial G_{i\bar{j}}}{\partial v^k}dv^k\wedge d\bar{v}^j + \sum_{i,j,l=1}^{r}\bar{v}^j\frac{\partial G_{i\bar{j}}}{\partial\bar{v}^l}dv^i\wedge d\bar{v}^l\right\}$$
$$+\sqrt{-1}\sum_{i,j,k,l=1}^{r}v^i\bar{v}^j\frac{\partial^2 G_{i\bar{j}}}{\partial v^k\partial\bar{v}^l}dv^k\wedge d\bar{v}^l.$$

By (5.9) the last 3 term on the right above vanish thus:

$$\sqrt{-1}\partial\overline{\partial}||P||_h^2 = \sqrt{-1}\sum_{i,j=1}^{r}\left\{G_{i\bar{j}}dv^i\wedge d\bar{v}^j + v^i\bar{v}^j\sum_{\alpha,\beta=1}^{n}\frac{\partial^2 G_{i\bar{j}}}{\partial z^\alpha\partial\bar{z}^\beta}dz^\alpha\wedge d\bar{z}^\beta\right\}$$
$$+\sqrt{-1}\sum_{i,j=1}^{r}\left\{\sum_{\alpha=1}^{n}v^i\frac{\partial G_{i\bar{j}}}{\partial z^\alpha}dz^\alpha\wedge d\bar{v}^j + \sum_{l=1}^{n}\bar{v}^j\frac{\partial G_{i\bar{j}}}{\partial\bar{z}^\beta}dv^i\wedge d\bar{z}^\beta\right\}.$$

By Lemma 2.4 we may (as positivilty or negativity is independent of the choice of holomorphic frames) choose a local frame of E which is normal at any given point z^*, i.e., we may assume that $G_{i\bar{j}} = \delta_i^j, \partial G_{i\bar{j}}/\partial z^\alpha = 0$ and so

$$\sqrt{-1}\partial\overline{\partial}||P||_h^2 = \sqrt{-1}\left\{\sum_{i=1}^{r}dv^i\wedge d\bar{v}^i + \sum_{i,j=1}^{r}v^i\bar{v}^j\sum_{\alpha,\beta=1}^{n}\frac{\partial^2 G_{i\bar{j}}}{\partial z^\alpha\partial\bar{z}^\beta}dz^\alpha\wedge d\bar{z}^\beta\right\}$$

at the point z^*. Moreover, by (5.18), the second term on the right above is the base direction of the curvature, hence

$$\sqrt{-1}\partial\overline{\partial}||P||_h^2 = \sqrt{-1}\sum_{i=1}^{r}\left\{dv^i\wedge d\bar{v}^i - \sum_{i,j=1}^{r}v^i\bar{v}^j\sum_{k,l=1}^{n}K_{i\bar{j}\alpha\bar{\beta}}dz^\alpha\wedge d\bar{z}^\beta\right\}.$$

For $\sigma = \sum_{i=1}^r v^i e_i, \tau = \sum_{i=1}^r \tau^i e_i$ in E_x (which maybe identified with the tangent vectors $\sum\sigma^i\partial/\partial v^i$ and $\sum\tau^j\partial/\partial v^j$) define the $(1,1)$-form $\langle K(\cdot,\cdot)\sigma,\sigma\rangle_h$:

$$\langle K(X,Y)\sigma,\tau\rangle_h = \sum_{i,j,k=1}^{r}\sum_{\alpha,\beta=1}^{n}G_{j\bar{k}}K^j_{i\alpha\bar{\beta}}X^\alpha\overline{Y^\beta}\sigma^i\overline{\tau}^k$$
$$= \sum_{i,j=1}^{r}\sum_{\alpha,\beta=1}^{n}K_{i\bar{j}\alpha\bar{\beta}}X^\alpha\overline{Y^\beta}\sigma^i\overline{\tau}^k$$

and tangent vectors X, Y of type $(1,0)$ on M and where $K^j_{i\alpha\bar{\beta}}$ is the *base or horizontal component* of the curvature as defined in (5.18). The preceding computations show that

$$\sqrt{-1}\partial\overline{\partial}||P||^2_h = \sqrt{-1}\sum_{i,j=1}^{r} G_{i\bar{j}}dv^i \wedge d\bar{v}^j - \sqrt{-1}\langle K(\cdot,\cdot)P, P\rangle_h.$$

The first term on the right above is an $(1,1)$-form in the fiber variables and is positive definite (in the fiber direction) by the assumption that the Finsler metric F is strictly pseudoconvex along the fibers. The second term is an $(1,1)$-form in the base variables, hence $\sqrt{-1}\partial\overline{\partial}||P||^2_h$ is positive definite if and only if the second term is also positive definite (in the base direction) on $E\backslash\{\text{zero-section}\}$. We define the (base component) of the holomorphic bisectional curvature by $k(X,P) = \langle K(X,X)P,P\rangle / ||X||^2_g ||P||^2_h$.

Theorem 5.3. *Let (E,h) be a Finsler holomorphic vector bundle over a complex Finsler manifold (M,g). Assume that h satisfies* (FM1)–(FM5) *and let K be the base component of the curvature of the Chern connection associated to the Finsler metric h. Let $P = \sum_{i=1}^{r} v^i\partial/\partial v^i$ be the position vector field. Then the $(1,1)$-form $\sqrt{-1}\partial\overline{\partial}||P||^2_h$ is positive definite on $E\backslash$ $\{$zero-section$\}$ if and only if the base component of the mixed holomorphic bisectional curvature is strictly negative in the direction of X and P and on $E\backslash\{$zero-section$\}$:*

$$\langle K(X,X)P,P\rangle = \sum_{i,j=1}^{r}\sum_{k,l=1}^{n} K_{i\bar{j}\alpha\bar{\beta}}v^i\bar{v}^j X^\alpha \overline{X}^\beta < 0$$

for all nonzero tangent vector X of type $(1,0)$ on M.

The expression for $\partial\overline{\partial}\log||P||^2_h$ can now be carried out just as in Section 3:

$$\sqrt{-1}\partial\overline{\partial}\log||P||^2_h = [\,]^*\omega_{FS} - \sqrt{-1}\frac{\langle K(\cdot,\cdot)P,P\rangle_h}{||P||^2_h},$$

where K is defined as in (4.18), ω_{FS} is the Fubini-Study metric of the fiber $\mathbf{P}(E_{z^*})$ and $[\,] : E_* = E\backslash\{\text{zero-section}\} \to \mathbf{P}(E)$ is the quotient map. Thus ϕ is positive definite in the fiber direction and (see Theorem 3.2) we have:

Theorem 5.4. *Let P be the position vector field on a holomorphic vector bundle E of rank $r \geq 2$ over a complex manifold M of dimension n with a Finsler metric h satisfying* (FM1)–(FM5)*. Then the $(1,1)$-form $\sqrt{-1}\partial\overline{\partial}\log||P||^2_h$ descends to a well-defined form $\phi(= c_1(\mathcal{L}_{\mathbf{P}(E)}))$ on $\mathbf{P}(E)$; moreover ϕ is positive definite if and only if the base component of the mixed holomorphic bisectional curvature $\langle K(X,X)P,P\rangle_h$ is strictly negative in the direction of P on $E\backslash\{$zero-section$\}$ and non-zero tangent vector $X \in TM$.*

As pointed out at the beginning of this section, a Finsler metric on E is identified with a hermitian metric along the fibers of the "tautological" line bundle $\mathcal{L}^{-1}$ over $\mathbf{P}(E)$. Abusing the notation we shall denote by h these two metrics. The $(1,1)$-form $\sqrt{-1}\partial\overline{\partial}\log ||P||_h^2$ descends to the Chern form of $\mathcal{L}$ with the dual metric h^*. Thus the existence of a Finsler metric such that $c_1(\mathcal{L}_{\mathbf{P}(E)}, h^*)$ is positive definite is equivalent to the condition that the line bundle $\mathcal{L}$ is ample. This is equivalent to the condition that $\mathcal{L}^m$ is very ample for some positive integer m. Let $\{\sigma_0, \dots, \sigma_N\}$ be a basis of global holomorphic sections of $\mathcal{L}^m$ then

$$\Phi = [\sigma_0, \dots, \sigma_N] : \mathbf{P}(E) \to \mathbf{P}^N$$

is a holomorphic embedding and that $\mathcal{L}^m$ is the pull-back of the hyperplane section bundle $\mathcal{O}_{\mathbf{P}^N}(1)$ with the dual canonical metric h_0^*, i.e., $c_1(\mathcal{O}_{\mathbf{P}^N}(1), h_0^*)$ is the Fubini-Study form on $\mathbf{P}^N$. Moreover the Chern form $mc_1(\mathcal{L}, h) = c_1(\mathcal{L}^m, h^m)$ is cohomologous to $c_1(\mathcal{O}_{\mathbf{P}^N}(1), h_0^*)$. The canonical metric h_0 on the tautological line bundle $\mathcal{O}_{\mathbf{P}^N}(-1)$ is by definition:

$$h_0([w_0, \dots, w_N]) = \left(\sum_{i=0}^{N} |w_i|^2\right)^{1/2}, \qquad [w_0, \dots, w_N] \in \mathbf{P}^N(\mathbf{C}),$$

where $[w_0, \dots, w_N]$ are the homogeneous coordinates on $\mathbf{P}^N$. Thus

$$h_\Phi = (\Phi^* h_0)^m = \left(\sum_{i=0}^{N} \sigma_i \otimes \overline{\sigma}_i\right)^{1/2} = \left(\sum_{i=0}^{N} \sigma_i \otimes \overline{\sigma}_i\right)^{\frac{1}{2m}} \tag{5.20}$$

is well-defined on $\mathcal{L}^{-1}$ with $c_1(\mathcal{L}, h_\Phi^*) = (1/m)\Phi^* c_1(\mathcal{O}_{\mathbf{P}^N}(1), h_0^*) > 0$. The corresponding Finsler metric on E can be similarly expressed via Grothendieck's Theorem (see [12]):

Theorem (Grothendieck)**.** *Let E be a holomorphic vector bundle, and E^* its dual, of rank $r \geq 2$ over a complex manifold M. Let $p = [p_E] : \mathbf{P}(E) \to M$ be the projection map then $p_*(\mathcal{L}^m_{\mathbf{P}(E)}) \cong \odot^m E^*$ for all $m \geq 0$ and $p^i_* \mathcal{L}^m_{\mathbf{P}(E)} = 0$ for all $m \geq 0$ and $i > 0$ where p^i_* denotes the i-th direct image and $p_* = p^0_*$. Consequently, the corresponding cohomology groups are also isomorphic, i.e. $H^i(\mathbf{P}(E), \mathcal{L}^m_{\mathbf{P}(E)}) \cong H^i(X, \odot^m E^*)$ for all integers $i \geq 0$ and $m \geq 0$.*

It is a general fact form the spectral sequence of Leray that for a continuous map $p : X \to Y$ the direct image sheaves $p^i_* \mathcal{S}$ are the obstructions for the induced homorphisms $H^i(X, \mathcal{S}) \to H^i(Y, p_*\mathcal{S})$ be isomorphisms. Grothendieck showed that these obstructions vanish for the projection map of a bundle.

Denote by $\gamma : H^0(\mathbf{P}(E), \mathcal{L}^m) \cong H^0(M, \odot^m E^*)$. Under this isomorphism a basis $\sigma_0, \dots, \sigma_N$ of $H^0(\mathbf{P}(E), \mathcal{L}^m)$ is identified with a basis $\omega_0, \dots, \omega_N$ of $H^0(M, \odot^m E^*)$, i.e., $\gamma^{-1}\omega_i = \sigma_i$. The Finsler metric on E corresponding to h_Φ will be denoted, by abuse of notation, also by h_Φ and is given by

$$h_\Phi(a) = (\langle a^{\otimes m}, a^{\otimes m}\rangle_m)^{1/2m} = \left(\sum_{i=0}^{N} |\omega_i(a^{\otimes m})|^2\right)^{1/2m}, \qquad a \in E. \tag{5.21}$$

Observe that the basis of sections $\{\omega_i\}$ of $H^0(M, \odot^m E^*)$ actually defines a *hermitian inner product* on $\odot^m E$:

$$\langle A, B\rangle_m \overset{\text{def}}{=} \sum_{i=0}^{N} \omega_i(A)\,\overline{\omega}_i(\overline{B}), \qquad A, B \in \odot^m E. \tag{5.22}$$

Moreover, the norm $||\ ||_m$ of the inner product is the Finsler metric h_Φ on E:

$$||v||_{h_\Phi} = \left(\sum_{i=0}^{N} \omega_i(\otimes^m v)\,\overline{\omega}_i(\overline{\otimes^m v})\right)^{1/2m} = ||\otimes^m v||_m^{1/2m}, \qquad v \in E. \tag{5.23}$$

This is clear as the homogeneity condition, $||\lambda v||_{h_\Phi} = ||\lambda^m \otimes^m v||_m^{1/2m} = ||\lambda v||_{h_\Phi} = |\lambda|||v||_{h_\Phi}$, is satisfied. Grothendieck's Theorem applied to $\mathbf{P}(\odot^m E)$ yields also the isomorphism: $\tilde{\tau} : H^0(\mathbf{P}(\odot^m E), \mathcal{L}_{\mathbf{P}(\odot^m E)}) \cong H^0(M, \odot^m E^*)$. Thus $\{\rho_i, i = 0, \ldots, N\} = \{\tilde{\tau}^{-1}\omega_i, i = 0, \ldots, N\}$ is a basis of $H^0(\mathbf{P}(\odot^m E)$, $\mathcal{L}_{\mathbf{P}(\odot^m E)})$ and the Hermitian metric h_Φ on $\odot^m E$ induces a Hermitian metric on $\mathcal{L}_{\mathbf{P}(\odot^m E)}$ denoted by $\tilde{h}_\Phi$: $\tilde{h}_\Phi(\rho) = \tilde{h}_\Phi(\tilde{\tau}^{-1}\omega) = h^*_\Phi(\omega)$, $\rho = \tilde{\tau}^{-1}\omega, \omega \in \odot^m E^*$ where h^*_Φ is the dual of h_Φ. We summarize these in the following Theorem:

Theorem 5.5. *Let E be a rank $r \geq 2$ holomorphic vector bundle over a compact complex manifold M and for any positive integer k let $\mathcal{L}_{\mathbf{P}(\odot^k E)}$ be the "hyperplane bundle" over $\mathbf{P}(\odot^k E)$. Then the following statements are equivalent*:

(1) *E^* is ample*;
(2) *$\mathcal{L}_{\mathbf{P}(E)}$ is ample*;
(3) *$\odot^k E^*$ is ample for some positive integer k*;
(4) *$\mathcal{L}_{\mathbf{P}(\odot^k E)}$ is ample for some positive integer k*;
(5) *$\odot^k E^*$ is ample for all positive integer k*;
(6) *$\mathcal{L}_{\mathbf{P}(\odot^k E)}$ is ample for all positive integer k*;
(7) *there exists a Finsler metric along the fibers of E, satisfying* (FM1)–(FM5)*, with negative mixed holomorphic bisectional curvature*;
(8) *there exists a positive integer k and a Finsler metric along the fibers of $\odot^k E$, satisfying* (FM1)–(FM5)*, with negative mixed holomorphic bisectional curvature*;
(9) *for any positive integer k there exists a Finsler metric along the fibers of $\odot^k E$, satisfying* (FM1)–(FM5)*, with negative mixed holomorphic bisectional curvature*;
(10) *there exists a positive integer m and a Hermitian metric along the fibers of $\odot^m E$ with negative mixed holomorphic bisectional curvature.*

Proof. By definitions (1) and (2) (resp. (3) and (4), resp. (5) and (6)) are equivalent. By Propositions 4.3 and 4.4, (7), (8) and (9) are equivalent. If $\mathcal{L}_{\mathbf{P}(E)}$ is ample then there exists a Hermitian metric along the fibers of $\mathcal{L}_{\mathbf{P}(E)}$ such that $c_1(\mathcal{L}_{\mathbf{P}(E)}, H)$ is positive definite this implies in particular that the Finsler metric h on E defined via (5.1) satisfies conditions (FM1)–(FM5). The equivalence of (1) and (7) is then a consequence of Theorem 5.4. Condition (3) implies that there exists m such that $\mathcal{L}^m_{\mathbf{P}(E)}$ is very ample. The discussion preceding the Theorem shows that (7) is equivalent to (10). $\square$

Remark 5.6. Note that part (10) in the preceding Theorem asserts the existence of a Hermitian metric not merely a Finsler metric. This characterization is lost if we replace ampleness by nefness (we no longer have Proposition 4.4); in this case the following statements are equivalent:

(1) E^* is nef;

(2) $\mathcal{L}_{\mathbf{P}(E)}$ is nef;

(3) $\odot^k E^*$ is nef;

(4) $\mathcal{L}_{\mathbf{P}(\odot^k E)}$ is nef for some positive integer k;

(5) $\odot^k E^*$ is nef for some positive integer k;

(6) $\mathcal{L}_{\mathbf{P}(\odot^k E)}$ is nef for all positive integer k;

(7) there exists a Finsler metric along the fibers of E, satisfying (FM1)–(FM5), with non-positive mixed holomorphic bisectional curvature;

(8) there exists a positive integer k and a Finsler metric along the fibers of $\odot^k E$, satisfying (FM1)–(FM5), with non-positive mixed holomorphic bisectional curvature;

(9) for any positive integer k there exists a Finsler metric along the fibers of $\odot^k E$, satisfying (FM1)–(FM5), with non-positive mixed holomorphic bisectional curvature.

If we replace the nef condition by the condition "spanned" then Proposition 4.4 is still applicable hence E^* is nef implies that there exists a positive integer m and a Hermitian metric along the fibers of $\odot^m E$ with non-positive mixed holomorphic bisectional curvature. However we do not know if the converse is true.

The result of Theorem 5.6 applies also to the dual, in which case we have:

Theorem 5.7. *Let E be a rank $r \geq 2$ holomorphic vector bundle over a compact complex manifold M and for any positive integer k let $\mathcal{L}_{\mathbf{P}(\odot^k E)}$ be the "hyperplane bundle" over $\mathbf{P}(\odot^k E)$. Then the following statements are equivalent*:

(1) *E is ample*;

(2) *$\mathcal{L}_{\mathbf{P}(E^*)}$ is ample*;

(3) *$\odot^k E$ is ample for some positive integer k*;

(4) *$\mathcal{L}_{\mathbf{P}(\odot^k E^*)}$ is ample for some positive integer k*;

(5) *$\odot^k E$ is ample for all positive integer k*;

(6) *$\mathcal{L}_{\mathbf{P}(\odot^k E^*)}$ is ample for all positive integer k*;

(7) *there exists a Finsler metric along the fibers of E with positive mixed holomorphic bisectional curvature*;

(8) *for some positive integer k there exists a Finsler metric along the fibers of $\odot^k E$ with positve mixed holomorphic bisectional curvature*;

(9) *for all positive integer k there exists a Finsler metric along the fibers of $\odot^k E$ with positve mixed holomorphic bisectional curvature*;

(10) *there exists a positive integer* m *and a Hermitian metric along the fibers of* $\odot^m E$ *with positive mixed holomorphic bisectional curvature.*

Remarks analogous to Remark 5.6 also applies. The Theorem applies of course to the tangent, as well as the cotangent, bundle and in the later case we get from Theorem 3.3 and Corollary 3.9 that:

Corollary 5.8. *Let* $E = T^*M$ *be the cotangent bundle of a compact complex* n*-dimensional manifold* M *then the following statements are equivalent*:

(1) TM *is ample*;

(2) $\mathcal{L}_{\mathbf{P}(T^*M)}$ *is ample*;

(3) *the anti-canonical bundle* $\mathcal{K}^{-1}_{\mathbf{P}(T^*M)}$ *is ample*;

(4) $\odot^k TM$ *is ample for some positive integer* k;

(5) $\mathcal{L}_{\mathbf{P}(\odot^k T^*M)}$ *is ample for some positive integer* k;

(6) $\odot^k TM$ *is ample for all positive integer* k;

(7) $\mathcal{L}_{\mathbf{P}(\odot^k T^*M)}$ *is ample for all positive integer* k;

(8) *there exists a Finsler metric on* M *with non-negative holomorphic bisectional curvature*;

(9) *there exists a Finsler metric along the fibers of* $\odot^k T^*M$*, for some positive integer* k*, with positive mixed holomorphic bisectional curvature*;

(10) *for any positive integer* k *there exists a Finsler metric along the fibers of* $\odot^k T^*M$ *with positive mixed holomorphic bisectional curvature*;

(11) *there exists a positive integer* m *and a Hermitian metric along the fibers of* $\odot^m TM$ *with positive mixed holomorphic bisectional curvature.*

Remrak 5.6 applies also to Corollary 5.8. We have the following vanishing theorem (compare Corollary 3.7):

Corollary 5.9. *If* E *is a nef holomorphic vector bundle of rank* $r \geq 2$ *over a compact complex manifold* M *of dimension* n*, then*

$$\begin{cases} H^i(M, \odot^m E \otimes \det E \otimes \mathcal{K}_M) = 0, \\ H^i(M, \odot^m(\otimes^k E) \otimes (\det \odot^m(\otimes^k E)) \otimes \mathcal{K}_M) = 0 \end{cases}$$

for all $i, m, k \geq 1$*. Consequently, if* $E = TM$ *then* $H^i(M, \odot^m TM) = 0$ *for all* $i, m \geq 1$.

Proof. If E is nef then $\mathcal{L}_{\mathbf{P}(E^*)}$ is nef. This implies that, for each $0 \leq i \leq n$ there exist constants α_i such that

$$\dim H^i(\mathbf{P}(E^*), \mathcal{L}^m_{\mathbf{P}(E^*)}) \leq \alpha_i m^{n+r-i} + O(m^{n+r-2})$$

for all m. The condition that

$$\int_{\mathbf{P}(E^*)} c_1^{n+r-1}(\mathcal{L}_{\mathbf{P}(E^*)}) > 0$$

implies, via Riemann-Roch:

$$\dim H^0(\mathbf{P}(E^*), \mathcal{L}^m_{\mathbf{P}(E^*)}) + O(m^{n+r-2}) = \chi(\mathcal{L}^m_{\mathbf{P}(E^*)}) = \alpha m^{n+r-i} + O(m^{n+r-2}).$$

Thus the Kodaira dimension $\kappa(\mathcal{L}_{\mathbf{P}(E^*)}) = n + r - 1 = \dim \mathbf{P}(E^*)$. By the vanishing Theorem of Grauert-Riemenschneider we have, $H^i(\mathbf{P}(E^*), \mathcal{L}^{-1}_{\mathbf{P}(E^*)}) = 0$ for all $i \leq n+r-2$ and, by Serre's duality, $H^i(\mathbf{P}(E^*), \mathcal{L}_{\mathbf{P}(E^*)} \otimes \mathcal{K}_{\mathbf{P}(E^*)}) = 0$ for all $i \geq 1$. Since $\mathcal{L}^{r+m}_{\mathbf{P}(E^*)} = \mathcal{L}^m_{\mathbf{P}(E^*)} \otimes p^*(\mathcal{K}_{\mathcal{M}} \otimes \det E) \otimes \mathcal{K}^{-1}_{\mathbf{P}(E^*)}$ is nef for all $m \geq 0$ and satisfies the condition

$$\int_{\mathbf{P}(E^*)} c_1^{n+r-1}(\mathcal{L}^{r+m}_{\mathbf{P}(E^*)}) > 0$$

we infer, as above, that

$$H^i(\mathbf{P}(E^*), \mathcal{L}^{-m}_{\mathbf{P}(E^*)} \otimes p^*(\mathcal{K}_M^{-1} \otimes \det E^*) \otimes \mathcal{K}_{\mathbf{P}(E^*)}) = H^i(\mathbf{P}(E^*), \mathcal{L}^{-m-r}_{\mathbf{P}(E^*)}) = 0$$

for all $i \leq n + r - 2$ and, by Serre's duality, $H^i(\mathbf{P}(E^*), \mathcal{L}^m_{\mathbf{P}(E^*)} \otimes p^*(\mathcal{K}_M \otimes \det E)) = 0$ for all $i, m \geq 1$. By Grothendieck's Theorem we then get:

$$H^i(M, \odot^m E \otimes \det E \otimes \mathcal{K}_M) \cong H^i(\mathbf{P}(E^*), \mathcal{L}^m_{\mathbf{P}(E^*)} \otimes p^*(\mathcal{K}_M \otimes \det E) = 0$$

for all $i, m \geq 1$. If we take $E = TM$ then $\det E = \mathcal{K}_M^{-1}$ and the preceding reduces to $H^i(M, \odot^m TM) = 0$ for all $i, m \geq 1$.

From the expression (3.7) in Section 3 we have $c_1^{n+r-1}(\mathcal{L}_{\mathbf{P}(E^*)}) = \omega_{FS}^{r-1} \wedge \rho^n_{E^*}$ where ρ_{E^*} is the horizontal $(1,1)$-form (cf. (2.18)):

$$\rho_{E^*} = \sqrt{-1} \frac{\langle K_E(\cdot, \cdot)P, P\rangle_h}{||P||_h^2}$$

which is positive definite (resp. positive semi-definite) in the horizontal direction if and only if the mixed bisectional curvature is positive (resp. non-negative). Thus for a nef bundle E we have

$$\int_{\mathbf{P}(E^*)} c_1^{n+r-1}(\mathcal{L}_{\mathbf{P}(E^*)}) \geq 0$$

and is > 0 if and only if the $(1,1)$-form ρ_{E^*} is positive definite at one point (hence on an open set). By Proposition 4.1 it is clear that $\rho_{E^*} \geq 0$ implies that $\rho_{\otimes^k E^*} \geq 0$ and, in addition, if ρ_{E^*} is positive definite at one point on $\mathbf{P}(E^*)$ then $\rho_{\otimes^k E^*}$ is positive definite at one point on $\mathbf{P}(\otimes^k E^*)$. This last condition is equivalent to the condition that

$$\int_{\mathbf{P}(\otimes^k E^*)} c_1^{n+kr-1}(\mathcal{L}_{\mathbf{P}(\otimes^k E^*)}) > 0.$$

Thus we have the vanishing Theorem:

$$H^i\big(\mathbf{P}(\otimes^k E^*), \mathcal{L}^m_{\mathbf{P}(\otimes^k E^*)} \otimes p^*(\mathcal{K}_M \otimes (\det \otimes^k E))\big) = 0$$

for all $i, m, k \geq 1$ and by Grothendieck's Theorem:

$$H^i(M, \odot^m(\otimes^k E) \otimes \mathcal{K}_M \otimes (\det \otimes^k E))) = 0$$

for all $i, m, k \geq 1$. □

If (M, g) is a Kähler metric then Theorem 2.1 extends also to (E, F) where F is only a Finsler metric. The calculation is entirely similar (though more complicated) and shall be omitted. A calculation as in Section 2 shows that

$$(5.24) \qquad \begin{cases} \eta^\alpha = dz^i, & 1 \leq \alpha \leq n, \\ \zeta^i = dv^i + \displaystyle\sum_{j=1}^{r}\sum_{\alpha=1}^{n} \Gamma^i_{j\alpha} v^j dz^\alpha, & 1 \leq i \leq r \end{cases}$$

is a dual basis (compare (2.12)). In terms of the dual frame, the fundamental form of the inner product on TE (see (5.19)) is given by (compare (2.13))
(5.25)

$$\eta = \eta_F = \eta_{TE} = \sqrt{-1}\left(\sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}}(z)\, dz^i \wedge d\bar{z}^j + \sum_{i,j=1}^{r} G_{i\bar{j}}(z, v)\, \zeta^i \wedge \overline{\zeta}^j\right),$$

where the first term on the right is the Kähler form on M and G_{ij} is as given by (5.15) with $G = F^2$. The obstruction of η from being Kähler is given analogously (in fact formally the same; cf. Theorem 2.1) by:

Theorem 5.10. *Let (M, g) be a complex Kähler manifold and E be a holomorphic vector bundle of rank r over M with Finsler metric h which is strictly pseudoconvex along the fibers. Let $\eta = \eta_F$ be the fundamental form of the hermitian inner product $\langle , \rangle_{TE}$ as defined in* (5.26) *then*

$$\begin{aligned} d\eta &= \sqrt{-1}\left(\sum_{1\leq i,j,k\leq r} G_{i\bar{j}} v^k \Theta^i_k \wedge \overline{\zeta}^j - \sum_{1\leq i,j,k\leq r} G_{i\bar{j}} \zeta^i \wedge \bar{v}^k \overline{\Theta}^j_k\right) \\ &= \sqrt{-1}\left(\sum_{1\leq i,j\leq r} G_{i\bar{j}} \overline{\partial}\zeta^i \wedge \overline{\zeta}^j - \sum_{1\leq i,j\leq r} G_{i\bar{j}} \zeta^i \wedge \partial\overline{\zeta}^j\right), \end{aligned}$$

where $\zeta^i = dv^i + \sum_k \theta^i_k v^k$.

The following immediate Corollary is the analogue of Corollary 2.2:

Corollary 5.11. *With the same assumptions as in Theorem* 5.3 *the following conditions are equivalent*:

(i) *the metric η in Theorem* 5.9 *is Kähler*;

(ii) *the one forms $\{\zeta^i, i = 1, \ldots, n\}$, as given by* (5.20) *are holomorphic*;

(iii) *the curvature of the vertical bundle* $\mathcal{V}$ *satisfies the conditions*:

$$\sum_{1\le i,k\le r} G_{i\bar{m}} v^k \Theta^i_k = 0, \qquad G_{i\bar{m}}(z,v) = \frac{\partial^2 G}{\partial v^i \partial \bar{v}^m}(z,v)$$

for all m.

Remark 5.11. Since the curvature froms $\{\Theta^i_k\}$ depend on the base variables $(z^1,\dots,z^n)$ and the fiber variables $(v^1,\dots,v^r)$ we cannot, as in the case of Corollary 2.2, conclude that the curvature forms $\{\Theta^i_k, 1\le i,k\le r\}$ vanish.

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