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On $[X, U(n)]$ when $\dim X$ is $2n$

By

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1. Introduction

Take a topological group G. Then, for a CW-complex X, the homotopy set $[X, G]$ forms a group. This association is a functor from the category of CW-complexes and continuous maps up to homotopy to the category of groups and homomorphisms.

In this paper, we consider the case $G = U(n)$ and denote $[X, U(n)]$ by $U_n(X)$. In this case, remark that, even if X is base pointed, $[X, U(n)]$ and $[X, U(n)]_0$ are isomorphic, since $1 \to \mathrm{Map}_0(X, U(n)) \to \mathrm{Map}(X, U(n)) \to U(n) \to 1$ is a splitting extension of group and $U(n)$ is connected.

Also, if n is sufficiently large, $U_n(X)$ merely equals to $\widetilde{K}^1(X)$. In fact, this is true, when X is a CW-complex whose dimension is lower than $2n$, since $(U(\infty), U(n))$ is $2n$-connected. Thus we may say that $U_n(X)$ is "the unstable $\widetilde{K}^1$-theory" and $U_n(X)$ may provide additional informations to the ordinary K-theory.

Of course, an uncomputable object is useless, and we should offer some methods, tools to compute them and show examples. In the following, we shall investigate the case of $[X, U(n)]$ when $\dim X$ is $2n$.

Our results are the followings:

Theorem 1.1. *If* $\dim X \leq 2n$ *then the next exact sequence holds*:

$$\widetilde{K}^0(X) \xrightarrow{\Theta} \mathrm{H}^{2n}(X; \mathbf{Z}) \to U_n(X) \to \widetilde{K}^1(X) \to 0.$$

(*The explicit form of* Θ *is given in Proposition* 3.1.) *Denoting Coker*Θ *by* $N_n(X)$, *the following is a central extension*:

$$0 \to N_n(X) \xrightarrow{\iota} U_n(X) \to \widetilde{K}^1(X) \to 0. \tag{1.1}$$

In addition, the above exact sequence has the naturality; *if* X, Y *are CW-complexes with their dimensions no more than* $2n$ *and a continuous map* f :

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$X \to Y$ is given, the following commutes.

$$\begin{array}{ccccccccc}
\widetilde{K}^0(Y) & \xrightarrow{\Theta} & \mathrm{H}^{2n}(Y;\mathbf{Z}) & \longrightarrow & U_n(Y) & \longrightarrow & \widetilde{K}^1(Y) & \longrightarrow & 0 \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \\
\widetilde{K}^0(X) & \xrightarrow{\Theta} & \mathrm{H}^{2n}(X;\mathbf{Z}) & \longrightarrow & U_n(X) & \longrightarrow & \widetilde{K}^1(X) & \longrightarrow & 0.
\end{array}$$

Theorem 1.2. *Let X be a finite CW-complex and $\dim X \leq 2n$. Then $N_n(X)$ is a finite Abelian group and the order of any element in $N_n(X)$ divides $n!$.*

Also we give the following theorem concerning $N_n(\quad)$.

Theorem 1.3. *Let X_1, X_2 be finite CW-complexes whose dimensions are $2n_1$, $2n_2$ respectively. Assume $\widetilde{K}^0(X_1)$ or $\widetilde{K}^0(X_1)$ is free and $\mathrm{H}^{2n_1}(X_1;\mathbf{Z}) = \mathrm{H}^{2n_2}(X_2;\mathbf{Z}) = \mathbf{Z}$. If $N_{n_1}(X_1) \cong \mathbf{Z}/l_1\mathbf{Z}$ and $N_{n_2}(X_2) \cong \mathbf{Z}/l_2\mathbf{Z}$, then $N_{n_1+n_2}(X_1 \wedge X_2) \cong \mathbf{Z}/\binom{n_1+n_2}{n_1} l_1 l_2 \mathbf{Z}$.*

When $\widetilde{K}^1(X) = 0$, $U_n(X)$ and $N_n(X)$ coincide. As an example of such a case, we compute $U_{n+m-1}(\Sigma CP^{n-1} \wedge \Sigma CP^{m-1})$. (See Corollary 4.3.) Since we can regard ΣCP^{n-1} as a subspace of $U(n)$, there is a map $\gamma' : \Sigma CP^{n-1} \wedge \Sigma CP^{m-1} \to U(n+m-1)$ which is a restriction of the commutator map from $U(n) \wedge U(m)$ to $U(n+m-1)$. Our calculation shows that $U_{n+m-1}(\Sigma CP^{n-1} \wedge \Sigma CP^{m-1})$ is a cyclic group and γ' is its generator.

R. Bott has showed $U(n)$ and $U(m)$ does not homotopy-commute in $U(n+m-1)$ by means of the Samelson product. The order of γ' above mentioned indicates "how much far from homotopy-commutativity" ΣCP^{n-1} and ΣCP^{m-1} are.

Next, we shall look into the case $\widetilde{K}^1(X) \neq 0$. In this case, even if $\dim X = 2n$, $U_n(X)$ may be non-abelian and, in fact, we show such cases. Our results are the followings.

We set $\mathrm{H}^*(U(n);\mathbf{Z}) = \bigwedge(x_1, x_3, x_5, \ldots, x_{2n-1})$ where $x_{2k-1} = \sigma c_k$, σ is the cohomology suspension and c_k is the k-th universal Chern class. We loosely denote the cohomology map induced by a map f which lies in a homotopy class α by α^*.

Theorem 1.4. *In the same condition as Theorem 1.1, for any $\widetilde{\alpha}$, $\widetilde{\beta} \in U_n(X)$, their commutator $[\widetilde{\alpha}, \widetilde{\beta}]$ lies in $\iota(N_n(X))$ and we have*

$$[\widetilde{\alpha}, \widetilde{\beta}] = \iota\langle u \rangle,$$

where $u = \sum_{k+l+1=n} (\widetilde{\alpha}^*(x_{2k+1}) \cup \widetilde{\beta}^*(x_{2l+1}))$ in $\mathrm{H}^{2n}(X;\mathbf{Z})$ and $\langle u \rangle \in N_n(X)$ means the class represented by u.

Corollary 1.1. *In addition to the assumption of Theorems 1.4, we assume that $\mathrm{H}^{2n}(X;\mathbf{Z})$ is free. Then, if $\alpha \in \widetilde{K}^1(X)$ has a finite order, its inverse image $\widetilde{\alpha} \in U_n(X)$ belongs to the center of $U_n(X)$.*

As an application, we give $U_n(X)$ where X is a sphere bundle over a sphere.

Corollary 1.2. *If* $S^{2n+1}\to X\to S^{2m+1}$ *is a fibration where* $0<n<m$*, then* $U_{2(n+m+1)}(X)$ *has three generators* α*,* β *and* ϵ*, and its relations are*

$$[\alpha,\epsilon]=[\beta,\epsilon]=0$$
$$(n+m+1)!\epsilon=0$$
$$[\alpha,\beta]=n!m!\epsilon.$$

2. Exact sequence

We denote $U(\infty)/U(n)$ by W_n. Then, from the fibration $U(n)\xrightarrow{j}U(\infty)\xrightarrow{p}W_n$, we can deduce the following fibration sequence:

$$\cdots\to\Omega U(\infty)\xrightarrow{\Omega p}\Omega W_n\xrightarrow{\delta}U(n)\xrightarrow{j}U(\infty)\xrightarrow{p}W_n.$$

Since j is a group homomorphism, Ωp is a loop map and also δ is the loop map of $B\delta:W_n\to BU(n)$, for a CW-complex X, there is an exact sequence of groups:

$$[X,\Omega U(\infty)]\xrightarrow{\Omega p_*}[X,\Omega W_n]\xrightarrow{\delta_*}U_n(X)\xrightarrow{j_*}[X,U(\infty)].$$

Recall the natural isomorphisms $[X,BU]\cong\widetilde{K}^0(X)$, $[X,U(\infty)]\cong\widetilde{K}^1(X)$ and, also, the Bott map $\beta:BU\xrightarrow{\simeq}\Omega U(\infty)$. Moreover, since W_n is $2n$-connected, $[X,W_n]$ is trivial, when $\dim X\le 2n$, and this implies j_* is a surjection. These argument implies the next exact sequence, which has the naturality:

$$\widetilde{K}^0(X)\xrightarrow{\Omega p_*\beta_*}[X,\Omega W_n]\xrightarrow{\delta_*}U_n(X)\xrightarrow{j_*}\widetilde{K}^1(X)\to 0.$$

Here, we use the isomorphism $[X,\Omega W_n]\cong \mathrm{H}^{2n}(X;\mathbf{Z})$ as groups introduced as following. In the rest, we assume $\dim X\le 2n$.

Let $x\in \mathrm{H}^{2n+1}(W_n;\mathbf{Z})\cong\mathbf{Z}$ be the generator such that $p^*(x)=x_{2n+1}\in \mathrm{H}^*(U(\infty);\mathbf{Z})$. Consider $a_{2n}=\sigma(x)\in \mathrm{H}^{2n}(\Omega W_n;\mathbf{Z})$ as a map $a_{2n}:\Omega W_n\to K(\mathbf{Z},2n)$. Then $a_{2n*}:\pi_*(\Omega W_n)\to\pi_*(K(\mathbf{Z},2n))$ $(*\le 2n)$ is isomorphic and also $\pi_{2n+1}(K(\mathbf{Z},2n))=0$. Therefore, from Whitehead's theorem, $a_{2n*}:[X,\Omega W_n]\to[X,K(\mathbf{Z},2n)]\cong \mathrm{H}^{2n}(X;\mathbf{Z})$ is a bijection. Note that $a_{2n}:\Omega W_n\to K(\mathbf{Z},2n)$ is a loop map and a_{2n*} above is a group isomorphism. Here we remark that the naturality holds for this isomorphism, i.e., if X, Y are CW-complexes whose dimensions are no more than $2n$ and given a map $f:X\to Y$, the following is commutative;

$$\begin{array}{ccc}[Y,\Omega W_n] & \xrightarrow{\cong} & \mathrm{H}^{2n}(Y;\mathbf{Z})\\ \big\downarrow f^* & & \big\downarrow f^*\\ [X,\Omega W_n] & \xrightarrow{\cong} & \mathrm{H}^{2n}(X;\mathbf{Z})\end{array}$$

Now we set $\Theta = a_{2n*}\Omega p_* \beta_*$, $N_n(X) = \mathrm{Coker}\Theta$ and have the exact sequence and the extension in Theorem 1.1. The map $\mathrm{H}^{2n}(X;\mathbf{Z}) \to U_n(X)$ is the composition $\delta_*(a_{2n*})^{-1}$. The naturality can be easily checked.

Next, we shall prove that

$$0 \to N_n(X) \xrightarrow{\iota} U_n(X) \to \widetilde{K}^1(X) \to 0$$

is a central extension. Let $e_b (b = 1, 2, \dots, N)$ be the $2n$-cells of X, f_b be the attaching map of $2n$-cell e_b and X' be the $(2n-1)$-skeleton of X. We consider the cofibration sequence:

$$\bigvee_b S^{2n-1} \xrightarrow{\vee f_b} X' \longrightarrow X \xrightarrow{\rho} X/X'.$$

Remark $X/X' \cong \bigvee_b S^{2n}$.

Using this, we have a commutative diagram, in which every rows and columns are exact, as follows:

$$\begin{array}{ccccc}
 & & \widetilde{K}^0(X) & & \oplus_b \pi_{2n}(BU) \\
 & & \big\downarrow{\scriptstyle \Theta} & & \big\downarrow \\
0 & \longleftarrow & \mathrm{H}^{2n}(X;\mathbf{Z}) & \longleftarrow & \oplus_b \mathrm{H}^{2n}\left(S^{2n};\mathbf{Z}\right) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
U_n(X') & \longleftarrow & U_n(X) & \longleftarrow & U_n(X/X') \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\widetilde{K}^1(X') & \longleftarrow & \widetilde{K}^1(X) & \longleftarrow & 0 \\
\big\downarrow & & \big\downarrow & & \\
0 & & 0 & &
\end{array}$$

Hence

$$\mathrm{Im}(\mathrm{H}^{2n}(X;\mathbf{Z}) \to U_n(X)) = \mathrm{Im}(U_n(X/X') \to U_n(X)). \tag{2.1}$$

Therefore any element $\alpha \in \mathrm{Im}(\mathrm{H}^{2n}(X;\mathbf{Z}) \to U_n(X))$ can be represented by a map whose value on neighborhood V of X' is constantly the unit, while any element in $U_n(X)$ can be represented by a map whose value on the complement of V is the unit. (The complement of V can be covered by a disjoint union of $2n$-dim open cells.) Hence α and β are commutative and we can say that $\mathrm{Im}(\mathrm{H}^{2n}(X;\mathbf{Z}) \to U_n(X))$ lies in the center of $U_n(X)$.

Now we have just finish the proof of Theorem 1.1 and we shall show the proof of Theorem 1.2.

Proof of Theorem 1.2. It immediately follows that when X is a finite CW-complex, $N_n(X)$ is a finitely generated abelian group, since $\mathrm{H}^{2n}(X;\mathbf{Z})$ is finitely generated. Thus we show that $n!\theta = 0$ for any $\theta \in N_n(X)$.

From (2.1), $\mathrm{Im}(\rho^*: U_n(X/X') \to U_n(X)) \cong \mathrm{Coker}\Theta = N_n(X)$ and $N_n(X)$ is isomorphic to a quotient of $U_n(X/X')$.

On the other hand, we can see that

$$U_n(X/X') \cong \bigoplus_b U_n(S^{2n}) \cong \bigoplus_b \mathbf{Z}/n!\mathbf{Z}.$$

Hence the statement follows. □

3. Calculation on exact sequence

Let X be a finite CW-complex of dimension $2n$. In this section, we give the explicit form of the Θ in Theorem 1.1.

See the next diagram:

$$\begin{array}{ccccccc} & & [X,\Omega U(\infty)] & \xrightarrow{\Omega p_*} & [X,\Omega W_n] & & \\ & & \beta_* \uparrow & & \downarrow a_{2n\,*} & & \\ \widetilde{K}^0(X) & = & [X,BU] & \xrightarrow{\Theta} & [X,K(2n,\mathbf{Z})] & = & \mathrm{H}^{2n}(X;\mathbf{Z}) \end{array}$$

The above commutative diagram illustrates the definition of Θ. We set u, the fundamental element of $\mathrm{H}^{2n}(K(2n,\mathbf{Z});\mathbf{Z})$. Then, for any $\theta \in \widetilde{K}^0(X) \cong [X,BU]$,

$$\begin{aligned} \Theta(\theta) &= (a_{2n} \circ \Omega p \circ \beta \circ \theta)^*(u) \\ &= (\Omega p \circ \beta \circ \theta)^*(a_{2n}) \\ &= \theta^* \beta^* \Omega p^*(a_{2n}). \end{aligned}$$

Since, from the definition of a_{2n}, $a_{2n} = \sigma(x)$ and $p^*(x) = \sigma(c_{n+1})$, we can see that $\Theta(\theta) = \theta^*\beta^*(\sigma^2(c_{n+1}))$.

For CW-complexes X and Y, we denote the adjoint isomorphism between the homotopy sets by

$$\tau : [\Sigma X, Y] \to [X, \Omega Y].$$

(We loosely denote the adjoint isomorphism between the mapping spaces by the same symbol τ.)

Let ξ_N be the universal complex vector bundle over $BU(N)$ and η be the canonical complex line bundle over $CP^1 \cong S^2$. Also we set that ζ_N is the classifying map of $(\eta - 1) \wedge (\xi_N - N)$ over $\Sigma^2 BU(N)$ and $\zeta : \Sigma^2 BU \to BU$ is the limit of ζ_N. Then the Bott map satisfies

(3.1) $$\beta \simeq \tau^2 \zeta.$$

Since, regarding the homotopy class $\langle \zeta_N \rangle$ as an element of $\widetilde{K}^0(\Sigma^2 BU(N)) \subset \widetilde{K}^0(S^2 \times BU(N))$,

$$\begin{aligned} \langle \zeta_N \rangle &= (\eta - 1) \wedge (\xi_N - N) \\ &= \eta \hat{\otimes} \xi_N - 1 \hat{\otimes} \xi_N - \eta \hat{\otimes} N + 1 \hat{\otimes} N, \end{aligned}$$

we can proceed the calculation of the total Chern class of $\langle\zeta_N\rangle$ in $\mathrm{H}^*\left(\Sigma^2 BU(N);\mathbf{Z}\right)$ as follows. We regard $\mathrm{H}^*\left(BT^N;\mathbf{Z}\right)\supset\mathrm{H}^*\left(BU(N);\mathbf{Z}\right)$ where T^N is the maximal torus of $U(N)$. Let $c_i\in\mathrm{H}^*\left(BU;\mathbf{Z}\right)$ be the universal Chern class, c be the generator of $\mathrm{H}^2\left(S^2;\mathbf{Z}\right)$ and $t_i(i=1,\dots,N,\ |t_i|=2)$ be the generator of $\mathrm{H}^*\left(BT^N;\mathbf{Z}\right)$. Then we have

$$\begin{aligned}
{\zeta_N}^*\left(1+\sum_{i=1}^{\infty}c_i\right) &= \frac{\prod_{i=1}^{N}(1+c+t_i)}{(1+Nc)\prod_{i=1}^{N}(1+t_i)} \\
&= (1-Nc)\prod_{i=1}^{N}\left(1+\frac{c}{1+t_i}\right) \\
&= 1+\sum_{i=1}^{N}\frac{c}{1+t_i}-Nc \\
&= 1+c\sum_{i=1}^{N}\left(\sum_{j=0}^{\infty}(-t_i)^j\right)-Nc \\
&= 1+c\left(\sum_{j=1}^{\infty}\left((-1)^j\sum_{i=1}^{N}{t_i}^j\right)\right).
\end{aligned}$$

Let $s_j=\sum_{i=1}^{N}{t_i}^j\in\mathrm{H}^*\left(BU(N);\mathbf{Z}\right)$ and we also denote the corresponding primitive element in $\mathrm{H}^{2j}\left(BU;\mathbf{Z}\right)$ by s_j. The above equation implies ${\zeta_N}^*(c_i)=c\,\hat{\otimes}\,(-1)^{i-1}s_{i-1}$ and hence we obtain

$$\zeta^*(c_i)=(-1)^{i-1}\Sigma^2 s_{i-1}. \tag{3.2}$$

Now we can see, from (3.1) and (3.2),

$$\beta^*(\sigma^2(c_{n+1}))=(-1)^n s_n$$

and if we set $s_j:\widetilde{K}^0(X)\cong[X,BU]\to\mathrm{H}^{2j}\left(X;\mathbf{Z}\right)$ as $s_j(\theta)=\theta^*(s_j)$, immediately the next proposition follows.

Proposition 3.1. *For $\theta\in\widetilde{K}^0(X)$,*

$$\Theta(\theta)=(-1)^n s_n(\theta).$$

Now we can deduce some corollaries.

Corollary 3.1. *For $n\geq 1$, $U_n(CP^n)$ vanishes.*

Proof. Let t be the generator of $\mathrm{H}^2\left(CP^n;\mathbf{Z}\right)$. Then, since the first Chern class of the canonical line bundle γ_n over CP^n is t and other Chern classes are zero,

$$s_n(\gamma_n)=t^n.$$

Thus $\Theta(\gamma_n)$ is the generator of $\mathrm{H}^{2n}(CP^n; \mathbf{Z}) \cong \mathbf{Z}$ and $N_n(CP^n)$ vanishes.

Remark that, since $\mathrm{H}^{\mathrm{odd}}(CP^n; \mathbf{Z})$ vanishes, we can see $\widetilde{K}^1(CP^n) = 0$ using the Atiyah-Hirzeburch spectral sequence. (See [2].) Thus, from Theorem 1.1, $U_n(CP^n) = 0$. □

Consider CW-complexes X_1 and X_2 whose dimensions are $2n_1$ and $2n_2$ respectively. We'd like to compute $N_{n_1+n_2}(X_1 \wedge X_2)$ from $N_{n_1}(X_1)$ and $N_{n_2}(X_2)$ under some assumptions.

First, let $\mu_N : BU(N) \wedge BU(N) \to BU$ be the classifying map of $(\xi_N - N) \wedge (\xi_N - N)$ and $\mu : BU \wedge BU \to BU$ be the limit of μ_N.

Lemma 3.1. *In the above situation,*

$$\mu^*(s_j) = \sum_{k=1}^{j-1} \binom{j}{k} s_k \mathbin{\hat{\otimes}} s_{j-k}.$$

Proof. Since $\mathrm{H}^*(BU; \mathbf{Z})$ is free and the Chern character $\mathrm{ch} = \sum_{i=0}^{\infty}(s_i/i!)$ satisfies

$$\mathrm{ch}(\xi_N \mathbin{\hat{\otimes}} \xi_N) = \mathrm{ch}(\xi_N) \mathbin{\hat{\otimes}} \mathrm{ch}(\xi_N),$$

we can see that

$$\frac{{\mu_N}^*(s_j)}{j!} = \sum_{k=1}^{j-1} \frac{s_k}{k!} \mathbin{\hat{\otimes}} \frac{s_{j-k}}{(j-k)!}$$

in $\mathrm{H}^{2j}(BU(N) \wedge BU(N); \mathbf{Q})$ and

$${\mu_N}^*(s_j) = \sum_{k=1}^{j-1} \binom{j}{k} s_k \mathbin{\hat{\otimes}} s_{j-k}$$

in $\mathrm{H}^{2j}(BU(N) \wedge BU(N); \mathbf{Z})$. This implies the statement of the theorem. □

This leads us to the next lemma.

Lemma 3.2. *Let* X_1, X_2 *be CW-complexes. For* $\theta_1 \in \widetilde{K}^0(X_1)$ *and* $\theta_2 \in \widetilde{K}^0(X_2)$, $\theta_1 \wedge \theta_2 \in \widetilde{K}^0(X_1 \wedge X_2)$ *satisfies*

$$s_j(\theta_1 \wedge \theta_2) = \sum_{k=1}^{j-1} \binom{j}{k} s_k(\theta_1) \mathbin{\hat{\otimes}} s_{j-k}(\theta_2).$$

Proof. We regard θ_1 and θ_2 as their classifying maps respectively. Then $\mu \circ (\theta_1 \wedge \theta_2)$ is the classifying map of $\theta_1 \wedge \theta_2 \in \widetilde{K}^0(X_1 \wedge X_2)$:

$$X_1 \wedge X_2 \xrightarrow{\theta_1 \wedge \theta_2} BU \wedge BU \xrightarrow{\mu} BU.$$

Thus

$$\begin{aligned} s_j(\theta_1 \wedge \theta_2) &= (\theta_1 \wedge \theta_2)^* \mu^* s_j \\ &= (\theta_1 \wedge \theta_2)^* \sum_{k=1}^{j-1} \binom{j}{k} s_k \hat{\otimes} s_{j-k} \\ &= \sum_{k=1}^{j-1} \binom{j}{k} s_k(\theta_1) \hat{\otimes} s_{j-k}(\theta_2). \end{aligned}$$

□

Now we give the proof of Theorem 1.3.

Proof of Theorem 1.3. Since $\mathrm{H}^{2n_1+2n_2}(X_1 \wedge X_2; \mathbf{Z}) = \mathbf{Z}$, what we have to do is to investigate $\mathrm{Im}\Theta$ in $\mathrm{H}^{2n_1+2n_2}(X_1 \wedge X_2; \mathbf{Z})$. Let u_1 and u_2 be the generators of $\mathrm{H}^{2n_1}(X_1; \mathbf{Z})$ and $\mathrm{H}^{2n_2}(X_2; \mathbf{Z})$ respectively.

First, we see $\mathrm{Im}\Theta \supset \langle \binom{n_1+n_2}{n_1} l_1 l_2 u_1 \otimes u_2 \rangle$. Since $N_{n_i}(X_i) \cong \mathbf{Z}/l_i\mathbf{Z}$, there exists $\theta_i \in \widetilde{K}^0(X_i)$ which satisfies $s_{n_i}(\theta_i) = l_i u_i$. ($i = 1$, 2.) Thus $\theta_1 \hat{\otimes} \theta_2 \in \widetilde{K}^0(X_1 \wedge X_2)$ satisfies

$$\begin{aligned} \Theta(\theta_1 \hat{\otimes} \theta_2) &= \pm s_{n_1+n_2}(\theta_1 \hat{\otimes} \theta_2) \\ &= \pm \binom{n_1+n_2}{n_1} l_1 l_2 u_1 \otimes u_2. \end{aligned}$$

On the other hand, $\mathrm{Im}\Theta \subset \langle \binom{n_1+n_2}{n_1} l_1 l_2 u_1 \otimes u_2 \rangle$ is also true. Since $\widetilde{K}^0(X_1)$ or $\widetilde{K}^0(X_2)$ is free, any $\theta \in \widetilde{K}^0(X_1 \wedge X_2)$ has the form of $\sum \theta_a \hat{\otimes} \theta_b$ where $\theta_a \in \widetilde{K}^0(X_1)$ and $\theta_b \in \widetilde{K}^0(X_2)$. From the assumption, it holds that $s_{n_1}(\theta_1) \in \langle l_1 u_1 \rangle$ and $s_{n_2}(\theta_2) \in \langle l_2 u_2 \rangle$. Therefore $s_{n_1+n_2}(\theta_a \otimes \theta_b) \in \langle \binom{n_1+n_2}{n_1} l_1 l_2 u_1 \otimes u_2 \rangle$ and, since $s_{n_1+n_2}$ is primitive, $s_{n_1+n_2}(\theta) \in \langle \binom{n_1+n_2}{n_1} l_1 l_2 u_1 \otimes u_2 \rangle$.

Hence $\mathrm{Im}\Theta = \langle \binom{n_1+n_2}{n_1} l_1 l_2 u_1 \otimes u_2 \rangle$ and the statement follows. □

4. Applications

From Theorem 1.3, some corollaries follow directly.

Corollary 4.1. *Let X be a finite CW-complex with its dimension $2n$ and $\mathrm{H}^{2n}(X; \mathbf{Z}) \cong \mathbf{Z}$. If $N_n(X) \cong \mathbf{Z}/l\mathbf{Z}$,*

$$N_{n+1}(\Sigma^2 X) \cong \mathbf{Z}/(n+1)l\mathbf{Z}.$$

Proof. Set $X_1 = S^2$ and $X_2 = X$ in Theorem 1.3 and the proof is straightforward. □

Corollary 4.2. *The next equality holds*:

$$U_{n_1+n_2}(CP^{n_1} \wedge CP^{n_2}) \cong \mathbf{Z}/\binom{n_1+n_2}{n_1}\mathbf{Z}.$$

Proof. As seen in Corollary 3.1, $N_n(CP^n)$ vanishes. Thus, applying Theorem 1.3, $N_{n_1+n_2}(CP^{n_1} \wedge CP^{n_2}) \cong \mathbf{Z}/\binom{n_1+n_2}{n_1}\mathbf{Z}$. And this coincides with $U_{n_1+n_2}(CP^{n_1} \wedge CP^{n_2})$, since $\widetilde{K}^1(CP^{n_1} \wedge CP^{n_2})$ vanishes. $\square$

Let $\epsilon_{n-1} : \Sigma CP^{n-1} \to U(n)$ be the usual embedding described in [6, pp. 22–23]. This embedding satisfies in cohomology

$$\epsilon_{n-1}^*(x_{2k+1}) = \Sigma t^k$$

where t is the generator of $\mathrm{H}^2\left(CP^{n-1}; \mathbf{Z}\right)$ and $1 \leq k \leq n-1$. Also we set the commutator map $\gamma : U(n) \wedge U(m) \to U(n+m-1)$ and $\gamma' = \gamma \circ (\epsilon_{n-1} \wedge \epsilon_{m-1})$.

Corollary 4.3. *We can see*

$$U_{n+m-1}(\Sigma CP^{n-1} \wedge \Sigma CP^{m-1}) \cong \mathbf{Z}/\frac{(n+m-1)!}{(n-1)!(m-1)!}\mathbf{Z}$$

and its generator is the class $\langle \gamma' \rangle$.

Proof. We set $X = \Sigma CP^{n-1} \wedge \Sigma CP^{m-1}$. From Corollaries 4.1 and 4.2, the first half of this corollary can be easily obtained and what we have to do is to prove that $\langle \gamma' \rangle$ is a generator of $U_{n+m-1}(X)$. From Theorem 1.1, to prove this, it is sufficient to show that, in the exact sequence below, $\langle \gamma' \rangle \in U_{n+m-1}(X)$ comes from $\Sigma(t^{n-1}) \otimes \Sigma(t^{m-1}) \in \mathrm{H}^{2n+2m-2}(X; \mathbf{Z})$.

$$\widetilde{K}^0(X) \to \mathrm{H}^{2n+2m-2}(X; \mathbf{Z}) \to U_{n+m-1}(X) \to \widetilde{K}^1(X).$$

In the similar manner to that in [4], we consider the next diagram:

$$\begin{array}{ccccc}
 & \overset{\lambda_0}{\nearrow} & \Omega S^{2(n+m)-1} & \xrightarrow{\Omega j} & \Omega W_{n+m-1} \\
 & & \downarrow \delta & & \downarrow \delta \\
U(n) \wedge U(m) & \xrightarrow{\gamma} & U(n+m-1) & \xrightarrow{\cong} & U(n+m-1) \\
 & & \downarrow & & \downarrow \\
 & & U(n+m) & \xrightarrow{i} & U(\infty) \\
 & & \downarrow & & \downarrow \\
 & & S^{2(n+m)-1} & \xrightarrow{j} & W_{n+m-1}
\end{array}$$

where two columns are fibration sequences and i and j are usual embeddings. In [3], it is showed that there exists a map λ_0 which makes the above diagram homotopy commutative and also satisfies

$${\lambda_0}^*(v) = x_{2n-1} \otimes x_{2m-1},$$

where v is the generator of $\mathrm{H}^{2n+2m-2}\left(\Omega S^{2(n+m)-1}; \mathbf{Z}\right)$. (Actually λ_0 is the adjoint of the join of the projections $U(n) \to U(n)/U(n-1)$ and $U(m) \to$

$U(m)/U(m-1)$.) If we set $\lambda = \Omega j \circ \lambda_0$, since $\Omega j^*(a_{2n}) = v$, we have that $\lambda^*(a_{2n}) = x_{2n-1} \otimes x_{2m-1}$.

Hence $(\lambda \circ (\epsilon_{n-1} \wedge \epsilon_{m-1}))^*(a_{2n}) = \Sigma^2(t^{n-1} \otimes t^{m-1})$, i.e., by the isomorphism $[X, \Omega W_{n+m-1}] \xrightarrow{a_{2n*}} \mathrm{H}^{2n+2m-2}(X; \mathbf{Z})$, $\langle \lambda \circ (\epsilon_{n-1} \wedge \epsilon_{m-1}) \rangle$ corresponds to the generator $\Sigma^2(t^{n-1} \otimes t^{m-1})$.

Moreover, since $\delta \circ \lambda = \gamma$, $\delta^*(a_{2n*})^{-1}(\Sigma^2(t^{n-1} \otimes t^{m-1}) = \langle \delta^*(\lambda \circ (\epsilon_{n-1} \wedge \epsilon_{m-1})) \rangle = \langle \gamma \circ (\epsilon_{n-1} \wedge \epsilon_{m-1}) \rangle = \langle \gamma' \rangle$ and the proof is finished. □

5. Commutator in $U_n(X)$

In the rest of this paper, we treat the case $\dim X = 2n$ and $\widetilde{K}^1(X) \neq 0$. In such cases, $U_n(X)$ may not be commutative. We prove Theorem 1.4 which describes the commutator in $U_n(X)$ in such cases.

In the rest, let γ be the commutator map $U(n) \wedge U(n) \to U(n)$ and consider the next diagram.

$$\begin{array}{ccc} & & \Omega W_n \\ & \overset{\widetilde{\gamma}}{\nearrow} & \big\downarrow \Omega\delta \\ U(n) \wedge U(n) & \xrightarrow{\gamma} & U(n) \\ & & \big\downarrow i \\ & & U(\infty) \end{array}$$

Since $i \circ \gamma$ is null-homotopic, there exists a lift $\widetilde{\gamma} : U(n) \wedge U(n) \to \Omega W_n$, such that $\Omega\delta \circ \widetilde{\gamma} \simeq \gamma$.

To find an adequate lift $\widetilde{\gamma}$, we prepare some maps and propositions. We set $j : \Sigma U(n) \vee \Sigma U(n) \to BU(n)$, $k : \Sigma U(n) \times \Sigma U(n) \to BU$ as the following compositions respectively:

$$\Sigma U(n) \vee \Sigma U(n) \xrightarrow{\tau^{-1}1 \vee \tau^{-1}1} BU(n) \vee BU(n) \xrightarrow{\quad \nabla \quad} BU(n),$$

$$\Sigma U(n) \times \Sigma U(n) \xrightarrow{\tau^{-1}1 \times \tau^{-1}1} BU(n) \times BU(n) \xrightarrow{\quad \overline{\mu} \quad} BU,$$

where ∇ is the folding map and $\overline{\mu}$ is the classifying map of the cross product of the universal vector bundles over $BU(n)$.

Also we set $f : \Sigma(U(n) \wedge U(n)) \to \Sigma U(n) \vee \Sigma U(n)$ as follows: Setting $(0, *)$ be the base point of $\Sigma U(n)$, we regard $\Sigma U(n) \vee \Sigma U(n) \subset \Sigma U(n) \times \Sigma U(n)$. For $x, y \in U(n)$ and $t \in [0, 1]$, we set $f_0 : U(n) * U(n) \to \Sigma U(n) \vee \Sigma U(n)$ as

$$f_0(t, x, y) = \begin{cases} ((1-2t, x), *) & \left(0 \leq t \leq \dfrac{1}{2}\right) \\ (*, (2t-1, y)) & \left(\dfrac{1}{2} \leq t \leq 1\right). \end{cases}$$

Then set $f : \Sigma(U(n) \wedge U(n)) \simeq U(n) * U(n) \xrightarrow{f_0} \Sigma U(n) \vee \Sigma U(n)$.

Proposition 5.1. *A map* $\widetilde{\gamma} : U(n) \wedge U(n) \to \Omega W_n$ *satisfies* $\Omega\delta \circ \widetilde{\gamma} \simeq \gamma$, *if and only if* $\tau^{-1}\widetilde{\gamma}$ *makes the following diagram homotopy commutative*:

$$\begin{array}{ccc} \Sigma(U(n) \wedge U(n)) & \xrightarrow{f} & \Sigma U(n) \vee \Sigma U(n) \\ \downarrow{\scriptstyle \tau^{-1}\widetilde{\gamma}} & & \downarrow{\scriptstyle j} \\ W_n & \xrightarrow{\delta} & BU(n) \end{array}$$

Proof. We recall that f induces the generalized Whitehead product

$$[\ ,\ ] : [\Sigma U(n), BU(n)] \times [\Sigma U(n), BU(n)] \to [\Sigma(U(n) \wedge U(n)), BU(n)]$$

by associating, for $\eta, \eta' \in [\Sigma U(n), BU(n)]$ represented by g and h respectively, the class $[\eta, \eta']$ represented by $\nabla \circ (g \vee h) \circ f$. This implies that $j \circ f$ represents $[\tau^{-1}1, \tau^{-1}1]$, while it is known that $\tau[\tau^{-1}\eta, \tau^{-1}\eta'] = \langle \eta, \eta' \rangle$ where $\langle\ ,\ \rangle$ is the generalized Samelson product. (See [1].) Thus, $\tau(j \circ f)$ lies in $\tau[\tau^{-1}1, \tau^{-1}1] = \langle 1, 1 \rangle$ and

$$\tau(j \circ f) \simeq \gamma.$$

Hence, the commutativity of the above diagram is equivalent to

$$\tau(\delta \circ \tau^{-1}\widetilde{\gamma}) \simeq \gamma,$$

while $\tau(\delta \circ \tau^{-1}\widetilde{\gamma}) = \Omega\delta \circ \widetilde{\gamma}$. □

Let EU be a space that $U(\infty)$ acts freely. We denote the quotient map $EU \to EU/U(n) = BU(n)$ by q' and consider the next commutative diagram, in which each row is a fibration.

(5.1)
$$\begin{array}{ccccc} W_n & \xrightarrow{\delta} & BU(n) & \xrightarrow{Bi} & BU \\ {\scriptstyle p}\uparrow & & {\scriptstyle p'}\uparrow & & \| \\ U(\infty) & \longrightarrow & EU & \longrightarrow & BU \end{array}$$

Lemma 5.1. *In the Leray-Serre spectral sequence of the fibration* $W_n \xrightarrow{\delta} BU(n) \xrightarrow{Bi} BU$, *the cohomology element* $x \in \mathrm{H}^{2n+1}(W_n; \mathbf{Z})$ *transgresses to the* $(n+1)$*-th Chern class* $c_{n+1} \in \mathrm{H}^{2n+2}(BU; \mathbf{Z})$, *i.e.,* $\partial(x) = Bi^*(c_{n+1})$ *in the diagram*

$$\mathrm{H}^{2n+1}(W_n; \mathbf{Z}) \xrightarrow{\partial} \mathrm{H}^{2n+2}(BU(n), W_n; \mathbf{Z}) \xleftarrow{Bi^*} \mathrm{H}^{2n+2}(BU; \mathbf{Z}).$$

Proof. In the Leray-Serre spectral sequence of the fibration $W_n \to BU(n) \to BU$, the transgression image in $\mathrm{H}^{2n+2}(BU; \mathbf{Z})$ is equals to $\mathrm{Ker}(Bi^* : \mathrm{H}^{2n+2}(BU; \mathbf{Z}) \to \mathrm{H}^{2n+2}(BU(n); \mathbf{Z}))$ which is generated by c_{n+1}.

On the other hand, in the Leray-Serre spectral sequence of the fibration $U(\infty) \to EU \to BU$, $x_{2n+1} \in \mathrm{H}^*(U(\infty); \mathbf{Z})$ transgresses to c_{n+1} + (decomposable elements) $\in \mathrm{H}^{2n+2}(BU; \mathbf{Z})$.

Therefore, since $p^*(x) = x_{2n+1}$ and (5.1) is commutative, it follows that x transgresses to c_{n+1}. $\square$

Proposition 5.2. *We can take* $\widetilde{\gamma}$ *so that*

$$\widetilde{\gamma}^*(a_{2n}) = \sum_{k+l+1=n} x_{2k+1} \otimes x_{2l+1}.$$

Proof. In this proof we set $A = \Sigma(U(n) \wedge U(n))$. Let I_f, C_f be the mapping cylinder and the mapping cone of f respectively and q be the quotient map $I_f \to I_f/A = C_f$. Then we have a cofibration

$$A \to I_f \to C_f$$

where it is known that $C_f \simeq \Sigma U(n) \times \Sigma U(n)$. (See Theorem 4.2 of [1] for detail.) Also, the homotopy commutativity of the next diagram, in which ϕ is the map induced by the natural projection $[0,1] \times A \to A$, can be easily checked.

(5.2)
$$\begin{array}{ccccc}
A & \hookrightarrow & I_f & \xrightarrow{q} & C_f \\
\| & & \phi \downarrow \simeq & & \downarrow \cong \\
\Sigma(U(n) \wedge U(n)) & \xrightarrow{f} & \Sigma U(n) \vee \Sigma U(n) & \hookrightarrow & \Sigma U(n) \times \Sigma U(n) \\
 & & \downarrow j & & \downarrow k \\
W_n & \xrightarrow{\delta} & BU(n) & \xrightarrow{Bi} & BU
\end{array}$$

We regard that $BU(n) \xrightarrow{Bi} BU$ is a fibration and δ is the inclusion of the fibre $W_n = Bi^{-1}(*)$ where $*$ is the base point of BU. We set $A/A \in I_f/A = C_f$ as the base point of C_f, deform the composition $C_f \cong \Sigma U(n) \times \Sigma U(n) \xrightarrow{k} BU$ so as to be base point preserving and denote the obtained map by k'. Then, by the homotopy lifting property, we can deform $j \circ \phi$ into j' so that $k' \circ q = Bi \circ j'$. Now we have a commutative (not only "homotopy commutative") diagram:

(5.3)
$$\begin{array}{ccccc}
A & \hookrightarrow & I_f & \xrightarrow{q} & C_f \\
\downarrow & & \downarrow j' & & \downarrow k' \\
W_n & \xhookrightarrow{\delta} & BU(n) & \xrightarrow{Bi} & BU
\end{array}$$

The commutativity of the above diagram implies $j'|_A : A \to W_n$. Thus, if we let $j_A = j'|_A$, $\delta \circ j_A \simeq j \circ f$ and, by Proposition 5.1, it follows that τj_A satisfies the claim $\Omega\delta \circ \tau j_A \simeq \gamma$.

On the other hand, since j' is a map between pairs $(I_f, A) \to (BU(n), W_n)$, we obtain the next commutative diagram.

(5.4)
$$\begin{array}{ccccc} \mathrm{H}^{2n+1}(A,*;\mathbf{Z}) & \xrightarrow{\partial} & \mathrm{H}^{2n+2}(I_f,A;\mathbf{Z}) & \xleftarrow{q^*} & \mathrm{H}^{2n+2}(C_f,*;\mathbf{Z}) \\ \uparrow {j_A}^* & & \uparrow j'^* & & \uparrow k'^* \\ \mathrm{H}^{2n+1}(W_n,*;\mathbf{Z}) & \xrightarrow{\partial} & \mathrm{H}^{2n+2}(BU(n),W_n;\mathbf{Z}) & \xleftarrow{Bi^*} & \mathrm{H}^{2n+2}(BU,*;\mathbf{Z}) \end{array}$$

Here we observe the exact sequence of the pair (I_f, A)

$$\mathrm{H}^{2n+1}(I_f/A;\mathbf{Z}) \xrightarrow{q^*} \mathrm{H}^{2n+1}(I_f,*;\mathbf{Z}) \xrightarrow{f^*} \mathrm{H}^{2n+1}(A,*;\mathbf{Z}) \xrightarrow{\partial} \mathrm{H}^{2n+2}(I_f,A;\mathbf{Z}).$$

Since, by the diagram (5.2), q^* is equal to the cohomology map induced by $\Sigma U(n) \vee \Sigma U(n) \hookrightarrow \Sigma U(n) \times \Sigma U(n)$, q^* is epic and f^* is 0-map. This implies $\partial : \mathrm{H}^{2n+1}(A,*;\mathbf{Z}) \to \mathrm{H}^{2n+2}(I_f,A;\mathbf{Z})$ is monic.

Now, using Lemma 5.1, we chase the diagram (5.4) as

$$\partial {j_A}^*(x) = j'^*\partial(x) = j'^* Bi^*(c_{n+1}) = q^* k'^*(c_{n+1}).$$

By the diagram (5.2) and the definition of k, it follows that, under the identification of $I_f/A = C_f \simeq \Sigma U(n) \times \Sigma U(n)$,

$$\partial {j_A}^*(x) = q^* k^*(c_{n+1}) = \sum_{k+l=n+1} (\Sigma x_{2k-1}) \otimes (\Sigma x_{2l-1}). \tag{5.5}$$

Moreover we know that the next diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^{2n+2}(\Sigma A;\mathbf{Z}) & \xrightarrow{\pi^*} & \mathrm{H}^{2n+2}(C_f;\mathbf{Z}) \\ \| & & \| \\ \mathrm{H}^{2n+1}(A;\mathbf{Z}) & \xrightarrow{\partial} & \mathrm{H}^{2n+2}(I_f,A;\mathbf{Z}) \end{array}$$

The map π is the quotient map $C_f \to C_f/(\Sigma U(n) \vee \Sigma U(n)) \cong \Sigma A$, i.e., this is homotopic to the natural projection

$$\pi : C_f \cong \Sigma U(n) \times \Sigma U(n) \to \Sigma U(n) \wedge \Sigma U(n).$$

Therefore,

$$\partial\left(\Sigma\left(\sum_{k+l=n+1} x_{2k-1} \otimes x_{2l-1}\right)\right) = \sum_{k+l=n+1} (\Sigma x_{2k-1}) \otimes (\Sigma x_{2l-1}). \tag{5.6}$$

Finally, since $\partial : \mathrm{H}^{2n+1}(A,*;\mathbf{Z}) \to \mathrm{H}^{2n+2}(I_f,A;\mathbf{Z})$ is monic, (5.5) and (5.6) imply that

$${j_A}^*(x) = \Sigma\left(\sum_{k+l=n+1} x_{2k-1} \otimes x_{2l-1}\right)$$

and, if we set $\widetilde{\gamma} = \tau j_A$, we have

$$\widetilde{\gamma}^*(a_{2n}) = \sum_{k+l=n+1} x_{2k-1} \otimes x_{2l-1}$$

as desired. $\square$

Now, we shall show the proof of Theorem 1.4.

Proof of Theorem 1.4. Let X be a CW-complex with its dimension $2n$, and take any $\widetilde{\alpha}$ and $\widetilde{\beta} \in U_n(X)$. Assume that each class is represented by a and b respectively. Since $\widetilde{K}^1(X)$ is commutative, their commutator $[\widetilde{\alpha}, \widetilde{\beta}]$ comes from $N_n(X)$. Recall that $[X, \Omega W_n]$ is isomorphic to $\mathrm{H}^{2n}(X; \mathbf{Z})$ by the correspondence which associates, for $\phi \in [X, \Omega W_n]$, the cohomology class $\phi^*(a_{2n})$. Hence, what we have to do is to compute $\lambda^*(a_{2n})$ where $\lambda : X \to \Omega W_n$ satisfies $\Omega\delta \circ \lambda \in [\widetilde{\alpha}, \widetilde{\beta}]$.

On the other hand, by the definition, we know $[\widetilde{\alpha}, \widetilde{\beta}]$ is the class represented by the map $\gamma \circ (a \times b) \circ \Delta$, where Δ is the diagonal map of X. Thus we can set $\lambda = \widetilde{\gamma} \circ (a \times b) \circ \Delta$ as shown in the following diagram.

$$\begin{array}{ccccccc} & & & & & & \Omega W_n \\ & & & & & \nearrow^{\widetilde{\gamma}} & \downarrow \Omega\delta \\ X & \xrightarrow{\Delta} & X \times X & \xrightarrow{a \times b} & U(n) \times U(n) & \xrightarrow{\gamma} & U(n) \\ & & & & & & \downarrow i \\ & & & & & & U(\infty) \end{array}$$

Therefore we have that

$$\begin{aligned} \lambda^*(a_{2n}) &= \Delta^*(\widetilde{\alpha} \times \widetilde{\beta})^* \widetilde{\gamma}^*(a_{2n}) \\ &= \Delta^*(\widetilde{\alpha} \times \widetilde{\beta})^* \left(\sum_{k+l+1=n} x_{2k+1} \otimes x_{2l+1} \right) \\ &= \sum_{k+l+1=n} \widetilde{\alpha}^*(x_{2k+1}) \cup \widetilde{\beta}^*(x_{2l+1}). \end{aligned}$$

Here, if we let $u = \sum_{k+l+1=n} \widetilde{\alpha}^*(x_{2k+1}) \cup \widetilde{\beta}^*(x_{2l+1})$, by the correspondence $[X, \Omega W_n] \cong \mathrm{H}^{2n}(X; \mathbf{Z})$, we have

$$[\widetilde{\alpha}, \widetilde{\beta}] = \iota\langle u \rangle.$$

$\square$

Now we give the proof of Corollary 1.1.

Proof of Corollary 1.1. Take $\widetilde{\alpha} \in U_n(X)$ and assume that the order of its image in $\widetilde{K}^1(X)$ is finite. Then, for $x_{2k+1} \in \mathrm{H}^*(U(n); \mathbf{Z})$ is primitive,

$\widetilde{\alpha}^*(x_{2k+1})$ has, also, a finite order. This implies that, for any $\widetilde{\beta} \in U_n(X)$, $\sum_{k+l+1=n} \widetilde{\alpha}^*(x_{2k+1}) \cup \widetilde{\beta}^*(x_{2l+1})$ has a finite order as well, while $\mathrm{H}^{2n}(X; \mathbf{Z})$ is free. Hence $\sum_{k+l+1=n} \widetilde{\alpha}^*(x_{2k+1}) \cup \widetilde{\beta}^*(x_{2l+1}) = 0$ and, as seen in the proof of Theorem 1.4, $[\widetilde{\alpha}, \widetilde{\beta}]$ vanishes. □

6. Examples

In this section, using Theorems 1.1 and 1.4, we give Corollary 1.2 as an example.

Proof of Corollary 1.2. Let $0 < n < m$, $S^{2n+1} \xrightarrow{i} X \xrightarrow{p} S^{2m+1}$ be a fibration and set $N = n + m + 1$, i.e., $\dim X = 2N$. We set the generators of $\mathrm{H}^{2n+1}(S^{2n+1}; \mathbf{Z})$ and $\mathrm{H}^{2m+1}(S^{2m+1}; \mathbf{Z})$ as u_{2n+1} and u_{2m+1} respectively. Also we loosely denote $p^*(u_{2m+1}) \in \mathrm{H}^*(X; \mathbf{Z})$ by u_{2m+1} and the inverse image $(i^*)^{-1}(u_{2n+1})$ by u_{2n+1}, i.e.,

$$\mathrm{H}^*(X; \mathbf{Z}) = \wedge(u_{2n+1}, u_{2m+1}).$$

Since $\mathrm{H}^*(X; \mathbf{Z})$ is free, Atiyah-Hirzeburch spectral sequence of X is trivial. Then, if we set the generators of $\widetilde{K}^1(S^{2n+1})$ and $\widetilde{K}^1(S^{2m+1})$ as ϵ_n and ϵ_m respectively, $\widetilde{K}^1(X) \cong \mathbf{Z} \oplus \mathbf{Z}$ has two generators α and β which satisfy

$$i^*\alpha = \epsilon_n, \qquad \beta = p^*\epsilon_m. \tag{6.1}$$

From Theorem 1.1 we have a central extension

$$0 \to N_N(X) \to U_N(X) \to \widetilde{K}^1(X) \to 0.$$

Thus we can take $\widetilde{\alpha}, \widetilde{\beta} \in U_N(X)$ so that they come to α and β in $\widetilde{K}^1(X)$ respectively.

Lemma 6.1. $N_n(X) \cong \pi_{2N}(U(N)) \cong \mathbf{Z}/N!\mathbf{Z}$.

Proof. We set $X' = X^{(2N-1)}$ the $(2N-1)$-skeleton of X. From the assumption of X, we have a cell decomposition,

$$X = S^{2n+1} \cup e_{2m+1} \cup e_{2N}, \ X' = S^{2n+1} \cup e_{2m+1}.$$

Thus $S^{2n+2} \to \Sigma X' \to S^{2m+2}$ is cofibration and $0 = U_N(S^{2m+2}) \to U_N(\Sigma X') \to U_N(S^{2n+2}) = 0$ is exact. Hence $U_N(\Sigma X') = 0$.

Next, from (2.1), $N_N(X) = \mathrm{Im}(U_N(X/X') \to U_N(X))$ and also

$$0 = U_N(\Sigma X') \to U_N(X/X') \to U_N(X) \to U_N(X')$$

is exact. Therefore $N_N(X) \cong U_N(X/X') = \pi_{2N}(U(N))$ which is known to be $\mathbf{Z}/N!\mathbf{Z}$. □

Now, we set $\epsilon = u_{2n+1}u_{2m+1} \in \mathrm{H}^{2N}(X; \mathbf{Z})$, $\langle\epsilon\rangle \in N_N(X)$ is the class determined by ϵ, and $\widetilde{\epsilon} = \iota\langle\epsilon\rangle$. Then, we have prepared three generators $\widetilde{\alpha}$, $\widetilde{\beta}$ and $\widetilde{\epsilon}$ of $U_N(X)$. All we have to do is to prove $[\widetilde{\alpha}, \widetilde{\beta}] = n!m!\widetilde{\epsilon}$.

Since ϵ_n is the generator of $\widetilde{K}^1(S^{2n+1}) \cong [S^{2n+1}, U(\infty)]$, it is well known that

$$\epsilon_n{}^*(\sigma c_k) = \begin{cases} n! u_{2n+1} & (k = n+1) \\ 0 & (\text{otherwise}). \end{cases}$$

Hence, from (6.1) and the definition of $\widetilde{\alpha}$ and $\widetilde{\beta}$,

$$\widetilde{\alpha}^*(x_{2k+1}) = \begin{cases} n! u_{2n+1} & (k = n) \\ 0 & (\text{otherwise}), \end{cases}$$

$$\widetilde{\beta}^*(x_{2k+1}) = \begin{cases} m! u_{2m+1} & (k = m) \\ 0 & (\text{otherwise}). \end{cases}$$

Therefore $\sum_{k+l+1=n}(\widetilde{\alpha}^*(x_{2k+1}) \cup \widetilde{\beta}^*(x_{2l+1})) = n!m!\epsilon$ and it follows from Theorem 1.4 that $[\widetilde{\alpha}, \widetilde{\beta}] = n!m!\widetilde{\epsilon}$. □

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