J. Math. Kyoto Univ. (JMKYAZ)
43-2 (2003), 305–332

On the principal bundles with parabolic structure

By

Indranil Biswas

1. Introduction

Parabolic vector bundles on a compact Riemann surface were introduced in [MS], and parabolic vector bundles on higher dimensional projective varieties (not necessarily smooth) were introduced in [MY].

Here we consider the principal bundle analog of parabolic vector bundles which was defined in [BBN]. In Section 3 we recall the definition of a parabolic principal bundle, and also describe an equivalent formulation. Let G be a complex algebraic group. According to [BBN], a parabolic G-bundle over X is a functor from the category of finite dimensional left G-modules to the category of parabolic vector bundles over X satisfying certain conditions. This definition is modeled on a description by Nori of the usual principal bundles ([No1]). A functor giving a parabolic G-bundle over X can be concretely represented by a G-space over X which has the property that it is a principal G-bundle outside the parabolic divisor.

Let G be a semisimple algebraic group over $\mathbb{C}$. The Lie algebra $\mathfrak{g}$ of G is a G-module by the adjoint action. Let E_* be a parabolic G-bundle over a compact Riemann surface X and $E_*(\mathfrak{g})$ the corresponding parabolic vector bundle over X for the G-module $\mathfrak{g}$. We prove that E_* admits a flat connection if and only if every direct summand of $E_*(\mathfrak{g})$ is of parabolic degree zero (Theorem 4.2).

Given a vector bundle E and a polynomial $P(x)$ with nonnegative integral coefficients, a vector bundle $P(E)$ is defined by replacing addition and multiplication by the direct sum and tensor product operations respectively. The vector bundle E is called finite if there are two such distinct polynomials P_1 and P_2 with $P_1(E)$ isomorphic to $P_2(E)$ ([We], [No1]). Nori proved that a vector bundle E over a projective manifold is finite if and only if it admits a flat connection with finite monodromy ([No1], [No2]).

We call a parabolic vector bundle F_* to be finite if $P_1(F_*) \cong P_2(F_*)$ for some polynomials P_1 and P_2 with nonnegative integral coefficients and $P_1 \neq P_2$. A parabolic G-bundle E_*, where G is a complex algebraic group, is

Received April 18, 2002
Revised October 21, 2002

called finite if all the associated vector bundles are finite. If G is reductive and V_0 is a fixed faithful G-module, then E_* is finite if the parabolic vector bundle associated to E_* for the G-module V_0 is finite (Proposition 5.1).

Let G be a semisimple algebraic group and E_* a parabolic principal G-bundle over a projective manifold X. We prove that E_* is finite if and only if it admits a flat connection whose monodromy is a finite group (Theorem 5.2).

It should be clear to the reader but nevertheless we should clarify that, like [BBN], the present work was completely inspired and influenced by [No1].

Acknowledgements. The author is very grateful to the referee for going through the paper very carefully. The comments of the referee led to the removal of an error in the proof of Proposition 4.4.

2. Preliminaries

2.1. Parabolic bundles

Let X be a connected smooth projective variety over $\mathbb{C}$. Let D be a normal crossing divisor on X. This means that D is effective, reduced with each irreducible component of D being smooth, and furthermore, the irreducible components of D intersect transversally. Let

$$D = \sum_{i=1}^{c} D_i \tag{2.1}$$

be the decomposition of D into irreducible components. So each D_i is smooth by assumption.

Let E be an algebraic vector bundle over X. A *quasiparabolic* structure on E over D is a filtration

$$E|_{D_i} = F_1^i \supset F_2^i \supset F_3^i \supset \cdots \supset F_{l_i}^i \supset F_{l_i+1}^i = 0 \tag{2.2}$$

by subbundles of the restriction of E to D_i for each $i \in [1\,, c]$. In other words, each F_j^i is a subbundle of $E|_{D_i}$ and $\text{rank}(F_j^i) > \text{rank}(F_{j+1}^i)$ for $j \in [1\,, l_i]$. The intersection of any collection of these subbundles on divisors is a subbundle of E on each component of the intersection of the supports.

For a quasiparabolic structure as above, *parabolic weights* are a collection of rational numbers

$$0 \leq \lambda_1^i < \lambda_2^i < \lambda_3^i < \cdots < \lambda_{l_i}^i < 1 \tag{2.3}$$

where $i \in [1\,, c]$. The parabolic weight λ_j^i corresponds to F_j^i in (2.2). A *parabolic structure* on E is a quasiparabolic structure with parabolic weights. A vector bundle equipped with a parabolic structure on it is also called a *parabolic vector bundle*.

For notational convenience, a parabolic vector bundle defined as above will be denoted by E_*. The divisor D is called the *parabolic divisor* for E_*.

Given any parabolic vector bundle, M. Maruyama and K. Yokogawa associate to it a filtration of sheaves parametrized by $\mathbb{R}$ (see [MY]). The construction of this filtration will be recalled now. The filtration is first constructed for the interval $[0\,,1]$ and then extended to $\mathbb{R}$ using a periodicity condition.

Take a parabolic vector bundle defined as in (2.2) and (2.3). Let

$$\Lambda := \bigcup_{i=1}^{c} \{\lambda^i_{l_1}, \dots, \lambda^i_{l_i}\} \subset \mathbb{Q} \tag{2.4}$$

be the union consisting of all parabolic weights. For any $\lambda \in \Lambda$ and $i \in [1\,, c]$, set $i(\lambda) \in [1\,, l_i]$ that satisfies the two conditions

(1) $\lambda^i_{i(\lambda)} \geq \lambda$, and
(2) if $i(\lambda) \neq 1$, then $\lambda^i_{i(\lambda)-1} < \lambda$.

The two condition clearly fix the integer $i(\lambda)$ uniquely. Let $F^i(\lambda) \subset E$ be the subsheaf defined by the exact sequence

$$0 \longrightarrow F^i(\lambda) \longrightarrow E \stackrel{q}{\longrightarrow} (E|_{D_i})/F^i_{i(\lambda)} \longrightarrow 0 \tag{2.5}$$

with $F^i_{i(\lambda)}$ a term in the filtration (2.2). The projection q in (2.5) is the composition of the restriction homomorphism $E \longrightarrow E|_{D_i}$ to D_i with the obvious projection of $E|_{D_i}$ to its quotient $(E|_{D_i})/F^i_{i(\lambda)}$. Let

$$E(\lambda) := \bigcap_{i=1}^{c} F^i(\lambda) \subset E$$

be the subsheaf of E defined by the intersection.

If $0 \in \Lambda$, then clearly $E(0) = E$. If $0 \notin \Lambda$, set $E(0) := E$. Also, set $E(1) := E \bigotimes \mathcal{O}_X(-D)$. For any $0 \leq t < 1$, define $\lambda(t) \in \Lambda \cup \{0\}$ as

$$\lambda(t) := \text{minimum}\, \{\lambda \in \Lambda \cup \{0\,, 1\} \mid \lambda \geq t\}\,.$$

Now set $E(t) := E(\lambda(t))$.

For any $t \in \mathbb{R}$, define

$$E_t := E(t - [t]) \otimes \mathcal{O}_X(-[t]D) \tag{2.6}$$

where $[t] \in \mathbb{Z}$ is the integral part of t, so $0 \leq t - [t] < 1$, and $E(t - [t])$ is defined above.

From the definition of the filtration $\{E_t\}_{t\in\mathbb{R}}$ it follows immediately that

(1) the filtration is decreasing as t increases;

(2) it is left continuous, which means that for each $t \in \mathbb{R}$, there is $\epsilon(t) > 0$ such that $E_t = E_{t-\epsilon}$ for all $\epsilon \in [0\,, \epsilon(t)]$;

(3) $E_{t+1} = E_t \bigotimes \mathcal{O}_X(-D)$ for all $t \in \mathbb{R}$;

(4) given any finite interval $[a\,,b] \subset \mathbb{R}$, the set

$$\{t \in [a\,,b] \,|\, E_{t_0} \neq E_{t_0+\delta} \text{ for all } \delta > 0\}$$

is finite;

(5) the filtration has a jump at t_0, that is, $E_{t_0} \neq E_{t_0+\delta}$ for all $\delta > 0$, if and only if $t_0 - [t_0]$ is a parabolic weight, i.e., $t_0 - [t_0] \in \Lambda$.

For any $t \in \mathbb{R}$, let E_{t+} denote the right limit of $E_{t+\epsilon}$ as $\epsilon > 0$ converges to 0. It follows from the 4th property stated above that for each $t \in \mathbb{R}$ there is $\epsilon(t) > 0$ such that $E_{t+} = E_{t+\epsilon}$ for all $\epsilon \in (0\,,\epsilon(t))$.

The parabolic structure on E can easily be recovered from the filtration $\{E_t\}_{t\in\mathbb{R}}$. A number $0 \leq \lambda < 1$ is a parabolic weight if and only if $E_\lambda \neq E_{\lambda+}$. If λ is a parabolic weight, then the corresponding term in the quasiparabolic filtration is recovered using the quotient $E_\lambda/E_{\lambda+}$. More precisely, $E_\lambda/E_{\lambda+}$ coincides with the graded piece of the quasiparabolic filtration.

Therefore, a parabolic vector bundle can also be defined in terms of a filtration of sheaves. When a parabolic vector bundle E_* is defined in terms of a filtration $\{E_t\}_{t\in\mathbb{R}}$ of sheaves, then E_0 will be called the *underlying vector bundle* of the parabolic vector bundle E_*. Note that in the filtration defined in (2.6), we have $E_0 = E$.

Let

$$\tau : X \setminus D \longrightarrow X$$

be the inclusion map. Consider the quasicoherent sheaf $\tau_*\tau^*E$ on X given by the direct image of the restriction of E to $X \setminus D$. Note that $\tau_*\tau^*E$ is *not* coherent if the divisor D is nonzero. Each E_t, $t \in \mathbb{R}$, is naturally contained in $\tau_*\tau^*E$. Furthermore, $\tau_*\tau^*E$ is generated by the collection of subsheaves E_t, $t \in \mathbb{R}$.

Now we are in a position to define the direct sum, dual and tensor product operations on parabolic vector bundles.

Given two parabolic vector bundle E_* and V_*, with D as the common parabolic divisor, and E and V respectively as the underlying vector bundles, consider

$$W := \tau_*\tau^*E \oplus \tau_*\tau^*V = \tau_*\tau^*(E \oplus V).$$

The *direct sum* $E_* \bigoplus V_*$ is defined to be the parabolic vector bundle that corresponds to the filtration $\{W_t\}_{t\in\mathbb{R}}$ in W defined as

$$W_t := E_t \oplus V_t$$

where $\{E_t\}_{t\in\mathbb{R}}$ and $\{V_t\}_{t\in\mathbb{R}}$ are the filtrations for E_* and V_* respectively. Note that the underlying vector bundle for the parabolic vector bundle $E_* \bigoplus V_*$ coincides with $E \bigoplus V$. The set of all parabolic weights of $E_* \bigoplus V_*$ is the union of the parabolic weights of E_* and V_*.

Now define

$$U := (\tau_*\tau^*E) \otimes_{\tau_*\mathcal{O}_{X\setminus D}} (\tau_*\tau^*V) = \tau_*\tau^*(E \otimes V)\,.$$

So each $E_t \bigotimes V_{t'}$ is a subsheaf of U. For any $t \in \mathbb{R}$, let U_t denote the subsheaf of U generated by all the subsheaves $E_{t_1} \bigotimes V_{t_2}$ with $t_1 + t_2 \geq t$. It is easy to see that each U_t is a coherent subsheaf of U. In fact, U_t is a subsheaf of $E_{t-1} \bigotimes V_{-1}$. This follows immediately from the third property that says $E_{t+1} = E_t \bigotimes \mathcal{O}_X(-D)$.

The parabolic vector bundle defined by the filtration $\{U_t\}_{t\in\mathbb{R}}$ of U is the *parabolic tensor product* $E_* \bigotimes V_*$ (see [Yo], [Bi2], [BBN] for the details).

Note that the underlying vector bundle for the parabolic vector bundle $E_* \bigotimes V_*$ *need not* coincide with $E \bigotimes V$. In fact, if E_* has a parabolic weight α and V_* has a parabolic weight β such that $\alpha + \beta \geq 1$, then the underlying vector bundle for the parabolic vector bundle $E_* \bigotimes V_*$ *does not* coincide with $E \bigotimes V$. In that case $E \bigotimes V$ is a proper subsheaf of the vector bundle underlying $E_* \bigotimes V_*$. Let Λ_1 (respectively, Λ_2) denote the set of all parabolic weights of E_* (respectively, V_*). The set of all parabolic weights of $E_* \bigotimes V_*$ coincides with the following set

$$\{\lambda + \mu \,|\, \lambda \in \Lambda_1,\, \mu \in \Lambda_2,\, \lambda + \mu < 1\} \bigcup \{\lambda + \mu - 1 \,|\, \lambda \in \Lambda_1,\, \mu \in \Lambda_2,\, \lambda + \mu \geq 1\}\,.$$

It is straight forward to deduce this from the definition of tensor product.

For any $t \in \mathbb{R}$, the coherent sheaf E_{-t-1+} (this right limit was defined earlier) coincides with E over $X \setminus D$. Therefore, we have a natural isomorphism

$$(E_{-t-1+})^* \;\cong\; E^*$$

over $X \setminus D$. This implies that over X,

$$(E_{-t-1+})^* \;\subset\; \tau_*\tau^* E^*\,.$$

Indeed, if A and B are two torsionfree coherent sheaves over X with an inclusion $A \hookrightarrow B$ over $X \setminus D$, then this inclusion homomorphism extends to an injective homomorphism

$$A \longrightarrow \tau_*\tau^* B$$

over X. This is an immediate consequence of the definition of $\tau_*\tau^* B$.

Therefore, we have

$$\mathcal{E}_t \;:=\; (E_{-t-1+})^* \;\subset\; \mathcal{E}' \;:=\; \tau_*\tau^* E^*$$

with the inclusion obtained above. Note that $\{\mathcal{E}_t\}_{t\in\mathbb{R}}$ is a decreasing filtration in the sense that there is a natural inclusion of the coherent sheaf $\mathcal{E}_{t_1}$ in $\mathcal{E}_{t_2}$ provided $t_2 \leq t_1$. Indeed, the dual of the homomorphism $E_{-t_2-1+} \hookrightarrow E_{-t_1-1+}$ is the natural inclusion. In fact, the filtration satisfies all the conditions required to define a parabolic structure. The parabolic vector bundle corresponding to the filtration $\{\mathcal{E}_t\}_{t\in\mathbb{R}}$ of $\mathcal{E}'$ is defined to be the *parabolic dual* of E_*. This parabolic dual will be denoted by E_*^*.

If E_* has at least one nonzero parabolic weight, then the underlying vector bundle for the parabolic vector bundle E_*^* *does not* coincide with E^*. The underlying vector bundle for E^*_* is a subsheaf of E^*. The set of all parabolic weights of E^*_* is

$$\{1-\lambda \,|\, \lambda \in \Lambda_1 \setminus (\Lambda_1 \cap \{0\})\} \cup (\Lambda_1 \cap \{0\})\,,$$

where Λ_1 as before is the set of all parabolic weights of E_*.

For any two parabolic vector bundles E_* and V_* the *parabolic bundle homomorphism* from E_* to V_* is the parabolic vector bundle

$$\mathrm{Hom}_P(E_*\,, V_*) \,:=\, E^*_* \otimes V_*\,.$$

The set of all parabolic weights of $\mathrm{Hom}_P(E_*\,, V_*)$ can be calculated from the above description of parabolic weights of a dual and a tensor product. The set of all parabolic weights of $\mathrm{Hom}_P(E_*\,, V_*)$ coincides with

$$\{\mu-\lambda \,|\, \lambda\in\Lambda_1,\, \mu\in\Lambda_2,\, \mu\geq\lambda\}\bigcup\{\mu-\lambda+1 \,|\, \lambda\in\Lambda_1,\, \mu\in\Lambda_2,\, \mu<\lambda\}$$

where Λ_1 and Λ_2 are the set of all parabolic weights of E_* and V_* respectively.

The parabolic tensor product is self-dual, that is, $(E^*_*)^* \,=\, E_*$. The tensor product is associative, that is,

$$E_*\otimes(V_*\otimes W_*) \,=\, (E_*\otimes V_*)\otimes W_*$$

where E_*, V_* and W_* are any parabolic vector bundles. Furthermore, the tensor product is distributive, that is,

$$E_*\otimes(V_*\oplus W_*) \,=\, (E_*\otimes V_*)\oplus(E_*\otimes W_*)\,.$$

See [Bi2], [BBN] for the details.

2.2. Parabolic bundle and bundles with finite group action

Let Y be a connected smooth projective variety and

$$\Gamma \,\subset\, \mathrm{Aut}(Y)$$

a finite subgroup of the automorphism group of the variety Y.

A Γ-*linearized vector bundle* over Y is an algebraic vector bundle W over Y equipped with an action of Γ which is compatible with the obvious action of Γ on Y. In other words, Γ acts on the total space of the vector bundle W and for every $g \,\in\, \Gamma$ the action of g on W is a vector bundle isomorphism of W with $(g^{-1})^*W$.

Assume that the quotient Y/Γ is smooth. Let

$$q\,:\,Y\,\longrightarrow\,X\,:=\,Y/\Gamma$$

be the quotient map. Let $D_q\,\subset\,Y$ be the reduced ramification divisor for q.

Take a Γ-linearized vector bundle W over Y. Let

$$\widetilde{D}\,\subset\,D_q$$

be the union of all those irreducible components D' of D_q that satisfy the condition that for a general point z of D', the action of its isotropy subgroup (for the action of Γ on Y) on the fiber W_z is nontrivial. So $\widetilde{D}$ depends on W.

Assume that the image $D := q(\widetilde{D})$ is a normal crossing divisor on X. In [Bi1], using W we constructed a parabolic vector bundle over X with D as the parabolic divisor. This construction will be briefly recalled.

Let

$$D = \sum_{j=1}^{h} D_j$$

be the decomposition of D into irreducible components. Set $\widetilde{D}_j := q^{-1}(D_j)$. So

$$\widetilde{D} := q^{-1}(D) = \sum_{j=1}^{h} \widetilde{D}_j = \sum_{j=1}^{h} n_j (\widetilde{D}_j)_{\mathrm{red}}$$

where $(\widetilde{D}_j)_{\mathrm{red}}$ is the reduced divisor defined by $\widetilde{D}_j$ and $n_j \geq 1$.

Since W is Γ-linearized, the divisor $\widetilde{D}$ is left invariant by the action of Γ on Y. Consequently, we have an action of Γ on the direct image

$$W(t) := q_* \left(W \otimes \mathcal{O}_Y \left(\sum_{j=1}^{h} [-t n_j] (\widetilde{D}_j)_{\mathrm{red}} \right) \right)$$

on X, where $t \in \mathbb{R}$. Finally define

$$E_t := W(t)^{\Gamma}, \tag{2.7}$$

to be the invariant part for the action of Γ on $W(t)$. The filtration $\{E_t\}_{t \in \mathbb{R}}$ gives a parabolic vector bundle over X. See Section 2c of [Bi1] for the details.

The converse is also true. Fix X and D as in Section 2.1. Also fix an integer N. We will consider all parabolic vector bundles over X with D as the parabolic divisor and satisfying the condition that all the parabolic weights are integral multiples of $1/N$ (that is, any number in Λ (defined in (2.4)) is an integral multiple of $1/N$). There is a finite Galois covering

$$q : Y \longrightarrow X, \tag{2.8}$$

where Y is a smooth projective variety, such that all parabolic vector bundles of the above type arise from Γ-linearized vector bundles over Y, where Γ is the Galois group for the covering map q in (2.8). More precisely, given a parabolic vector bundle E_* of the above type, with parabolic weights multiples of $1/N$, there is a unique Γ-linearized vector bundle W over Y such that the parabolic vector bundle constructed from W coincides with E_*. See Section 3 of [Bi1] for the details. The covering q was first constructed in [Ka] to prove vanishing theorems (see also [KMM, Chapter 1.1]).

This correspondence between Γ-linearized vector bundles and parabolic vector bundles is compatible with the direct sum, tensor product and dualization operations. To describe this, let V and W be two Γ-linearized vector bundles over Y. So $V \bigoplus W$ and $V \bigotimes W$ have natural Γ-linearizations. Also, V^* is a Γ-linearized vector bundle. Let E_* and F_* be the parabolic vector bundles corresponding to V and W respectively. Then, the parabolic vector bundles corresponding to $V \bigoplus W$ and $V \bigotimes W$ are $E_* \bigoplus F_*$ and $E_* \bigotimes F_*$ respectively. Similarly, the parabolic vector bundle corresponding to V^* is the parabolic dual E_*^*. The parabolic vector bundle corresponding to the Γ-linearized vector bundle $\mathrm{Hom}(V, W)$ is the parabolic homomorphism bundle $\mathrm{Hom}_P(E_*, F_*)$.

2.3. Principal bundles

Let G be a linear algebraic group over $\mathbb{C}$. Let M be a connected smooth projective variety over $\mathbb{C}$.

A *principal G-bundle* over M is a smooth complex variety E equipped with an action of G on the right together with a surjective morphism

$$p : E \longrightarrow M$$

satisfying the following conditions:

(1) the map p is affine and smooth;

(2) the map p is a morphism of G-spaces, with the action of G on M being the trivial one;

(3) the map from $E \times G$ to the fiber product $E \times_M E$ defined by $(z, g) \longmapsto (z, zg)$ is an isomorphism.

Note that we do *not* assume E to be locally trivial in Zariski topology.

In [No1], Nori gave a Tannakian description of G-bundles which will be very useful for us. This description will be recalled below.

Let $\mathrm{Rep}(G)$ denote the category of all finite dimensional complex left representations of the group G, or equivalently, left G-modules. Note that $\mathrm{Rep}(G)$ is closed under the operations of direct sum and tensor product. It is also closed under taking the dual. By a G-module we will always mean a left G-module.

Let $\mathrm{Vect}(M)$ denote the category of all algebraic vector bundles over M.

Given a principal G-bundle E over M and a left G-module V, the group G acts on $E \times V$. The action of any $g \in G$ sends a point $(\zeta, v) \in E \times V$ to the point $(\zeta g, g^{-1}v) \in E \times V$. The corresponding quotient space

$$E(V) := E \overset{G}{\wedge} V = \frac{E \times V}{G} \tag{2.9}$$

defines a vector bundle over M (see [Gi, p. 114, Définition 1.3.1]). The vector bundle $E(V)$ is said to be *associated* to E for the G-module V.

Note that if

$$f : V \longrightarrow W \tag{2.10}$$

is a homomorphism of G-modules, then we have a homomorphism of vector bundles

$$\tilde{f} : E(V) \longrightarrow E(W)$$

that sends any $(z, v) \in E \overset{G}{\wedge} V$ (see (2.9)) to $(z, f(v)) \in E \overset{G}{\wedge} W$. Let

$$\mathcal{F}(E) : \mathrm{Rep}(G) \longrightarrow \mathrm{Vect}(M) \tag{2.11}$$

be the functor that sends any G-module V to the vector bundle $E(V)$ (Defined in (2.9)) and sends any homomorphism f of G-modules to the homomorphism $\tilde{f}$ between the corresponding vector bundles.

The functor $\mathcal{F}(E)$ defined above enjoys several natural abstract properties some of which we list here. The functor $\mathcal{F}(E)$ is compatible with the algebra structures of $\mathrm{Rep}(G)$ and $\mathrm{Vect}(M)$ defined using direct sum and tensor product operations. It takes a dual representation to the dual vector bundle. Furthermore, $\mathcal{F}(E)$ takes an exact sequence of G-modules to an exact sequence of vector bundles. It takes the trivial G-module $\mathbb{C}$ to the trivial line bundle on M. The dimension of a G-module V coincides with the rank of the vector bundle $\mathcal{F}(E)(V)$.

Nori proves that the collection of principal G-bundles over M are in bijective correspondence with the collection of $\mathbb{C}$-additive functors

$$\mathcal{F} : \mathrm{Rep}(G) \longrightarrow \mathrm{Vect}(M)$$

satisfying the following properties (see [No1, p. 31] and [No2, p. 77] for the details):

(1) The rank of the vector bundle $\mathcal{F}(V)$ coincides with the dimension of the G-module V.

(2) A morphism of vector bundles is said to be *strict* if the cokernel is also locally free. Let f be a homomorphism of G-modules as in (2.10). Then the corresponding homomorphism of vector bundles

$$\mathcal{F}(f) : \mathcal{F}(V) \longrightarrow \mathcal{F}(W)$$

is strict. In other words, the cokernel of $\mathcal{F}(f)$ is locally free. Note that this implies that both the image and the kernel of $\mathcal{F}(f)$ are both locally free.

(3) The kernel of the homomorphism $\mathcal{F}(f)$ (which is a vector bundle by the previous condition) coincides with $\mathcal{F}(\mathrm{kernel}(f))$ and the cokernel of $\mathcal{F}(f)$ coincides with $\mathcal{F}(\mathrm{cokernel}(f))$. The rank of the vector bundle $\mathcal{F}(V)$ coincides with the dimension of the G-module V.

(4) For any two G-modules V and W,

$$\mathcal{F}(V \otimes W) = \mathcal{F}(V) \otimes \mathcal{F}(W)$$

and $\mathcal{F}(V^*) = \mathcal{F}(V)^*$. Furthermore, $\mathcal{F}(\mathbb{C})$, where $\mathbb{C}$ is the trivial G-module, is the trivial line bundle $\mathcal{O}_M$.

(5) For any two G-modules V and W, the map

$$\mathcal{F}(\mathrm{Hom}(V, W)) = \mathcal{F}(V^* \otimes W) \longrightarrow \mathcal{F}(V^*) \otimes \mathcal{F}(W) = \mathrm{Hom}(\mathcal{F}(V), \mathcal{F}(W))$$

is injective.

Given such a functor $\mathcal{F}$, there is a G-bundle E, unique up to a unique isomorphism, such that $\mathcal{F} \cong \mathcal{F}(E)$ ([No1, p. 34, Proposition 2.9], [No2]). For any G-bundle E, the functor $\mathcal{F}(E)$ clearly has all the above properties.

3. Parabolic G-bundle

The above alternative description of a principal G-bundles due to Nori clearly gives a way to define the parabolic analog of G-bundles.

Let $\mathrm{PVect}(X)$ denote the category of all parabolic vector bundles over X with a fixed normal crossing divisor D as the parabolic divisor. Fix a positive integer N. Let

$$\mathrm{PVect}_N(X) \subset \mathrm{PVect}(X) \tag{3.1}$$

denote the subcategory of all parabolic vector bundles E_* with the property that all the parabolic weights of E_* are integral multiples of $1/N$. From the description of parabolic weights of a tensor product, direct sum, dual, and a homomorphism (given in Section 2.1) it follows immediately that $\mathrm{PVect}_N(X)$ is closed under the operations of taking direct sum, tensor product, dual and homomorphism.

A *parabolic G-bundle* is a $\mathbb{C}$-additive functor

$$\mathcal{F}_P : \mathrm{Rep}(G) \longrightarrow \mathrm{PVect}_N(X) \tag{3.2}$$

for some $N \geq 1$ satisfying the following conditions:

(1) the functor $\mathcal{F}$ takes the operations of direct sum, tensor product, dual and homomorphism in $\mathrm{Rep}(G)$ to the corresponding operation on $\mathrm{PVect}_N(X)$ (we already noted that $\mathrm{PVect}_N(X)$ is closed under all these operations);

(2) the functor $\mathcal{F}$ satisfies all the five conditions that characterize a G-bundle (described in Section 2.3) with the direct sum, tensor product, dual and homomorphism operations being those for parabolic bundles.

(See Section 2 of [BBN] for the details.)

Let $\mathcal{F}_P$ be a functor as in (3.2). Fix a covering q as in (2.8) such that for any $E_* \in \mathrm{PVect}_N(X)$ we have a Γ-linearized vector bundle on Y. We recall that there is bijective correspondence between $\mathrm{PVect}_N(X)$ and the collection of all Γ-linearized vector bundle on Y.

Let $\mathrm{Vect}_\Gamma(Y) \subset \mathrm{Vect}(Y)$ denote the subcategory of Γ-linearized vector bundle on Y. Consider the composition of $\mathcal{F}_P$ with the functor

$$\mathrm{PVect}_N(X) \longrightarrow \mathrm{Vect}_\Gamma(Y)$$

that sends any $E_* \in \mathrm{PVect}_N(X)$ to the Γ-linearized vector bundle over Y corresponding to E_*. This composition will be denoted by $\mathcal{F}'_P$. By the result of Nori described in Section 2.3 the functor $\mathcal{F}'_P$ defines a principal G-bundle E_G over Y.

A Γ-*linearized principal* G-*bundle* is a principal G-bundle E'_G over Y together with a lift of the Galois action of Γ on Y to the total space of E'_G satisfying the condition that the action of Γ on E'_G commutes with the action of G on E'_G. So a Γ-linearized $\mathrm{GL}(N,\mathbb{C})$-bundle is a Γ-linearized vector bundle of rank n by the standard representation.

Since the image of the functor $\mathcal{F}'_P$ defined above is contained in $\mathrm{Vect}_\Gamma(Y)$, it follows that E_G is Γ-linearized. Indeed, for any $\gamma \in \Gamma$, the G-bundle $\gamma^* E_G$ over Y corresponds to the composition of $\mathcal{F}'_P$ with the automorphism of $\mathrm{Vect}(Y)$ defined by $E \longmapsto \gamma^* E$. If $E \in \mathrm{Vect}_\Gamma(Y)$, then E is identified with $\gamma^* E$. Since the image of $\mathcal{F}'_P$ is contained in $\mathrm{Vect}_\Gamma(Y)$, from the result of Nori we get an identification of E_G with $\gamma^* E_G$. As γ runs over Γ, these identifications define a Γ-linearization of E_G.

Consider the quotient space E_G/Γ. Since the action of Γ on E_G is a lift of the action of Γ on Y, we have a projection

$$f \,:\, E_G/\Gamma \longrightarrow Y/\Gamma \,=\, X\,. \tag{3.3}$$

Since the actions of Γ and G on E_G commute, the quotient space E_G/Γ is equipped with an action of G and the map f in (3.3) is a morphism of G-spaces with the action of G on X being the trivial one. The action of G over $f^{-1}(X \setminus D)$ is free. Hence f makes E_G/Γ a principal G-bundle over $X \setminus D$. In general, the action is *not* free over D. However, the isotropy subgroup of any $y \in f^{-1}(D)$ is a finite group, as Γ itself is a finite group. Also, since Y/Γ is smooth, the quotient E_G/Γ must also be smooth.

The isotropy subgroup of any $z \in f^{-1}(D)$ is in fact abelian. This follows immediately from the fact that for any point $y \in q^{-1}(x) \subset Y$, where q is defined in (2.8), the isotropy group of y for the action of Γ on Y is abelian. It is evident that the isotropy of z is a subgroup of the isotropy of y. That the isotropy of y is abelian follows immediately from the construction of the covering q given in [KMM, Chapter 1.1, pp. 303–305].

The abelianness of the isotropy of y can also be deduced using the given condition that D is a normal crossing divisor. Indeed, the fundamental group of the complement

$$(\mathbb{C}^*)^k \times \mathbb{C}^{d-k} \,=\, \mathbb{C}^d \setminus \{(x_1, x_2, \ldots, x_d) \in \mathbb{C}^d \,|\, x_1 x_2 \cdots x_k \,=\, 0\}$$

is abelian. Hence the Galois group for any étale Galois cover of $(\mathbb{C}^*)^k \times \mathbb{C}^{d-k}$ is abelian. Since for a sufficiently small analytic open neighborhood $U_x \subset X$ of $x \in D$, the complement $U_x \setminus (U_x \cap D)$ is homotopic to some $(\mathbb{C}^*)^k \times \mathbb{C}^{d-k}$ where $d = \dim X$ and $k \in [1\,, d]$, it follows that the isotropy subgroup of any $y \in q^{-1}(x)$ for the action of Γ on Y is abelian.

The above observations clearly suggests the following alternative description of a parabolic G-bundle.

A *parabolic* G-*bundle* over X with D as the parabolic divisor is a smooth variety Q over X equipped with an action of G such that the surjective projection f of Q to X is G-equivariant with the action of G on X being the trivial one, and satisfying the following conditions:

(1) the action of G on Q is proper, and $X = Q/G$;
(2) $f : f^{-1}(X \setminus D) \longrightarrow X \setminus D$ is a principal G-bundle over $X \setminus D$ (so the action of G is free over $f^{-1}(X \setminus D)$);
(3) for any point $x \in D$ and $z \in f^{-1}(x)$, the isotropy of z, for the action of G on Q, is a finite abelian subgroup of G.

Note that the quotient map f in (3.3) satisfies all the above conditions.

The above definition of a parabolic G-bundle is equivalent to the earlier definition modeled on Nori's definition of a G-bundle.

There is a close analogy of parabolic G-bundles with the Seifert fibered spaces. More precisely, if we replace G in the above definition of a parabolic G-bundle by the circle group S^1, and take X to be a compact Riemann surface, then the total space Q is a Seifert fibered three manifold (see [He, Chapter 12]).

4. Flat connection on a parabolic bundle

We will recall the definition of a logarithmic connection introduced in [De1]. As before, let X be a connected smooth projective manifold and D a normal crossing divisor on X. Let $\Omega^i_X(\log D)$ denote the sheaf of logarithmic i-forms on X singular along D ([De1, Ch. II, §3]). Take an algebraic vector bundle E over X. A *logarithmic connection* on E singular along D is an algebraic differential operator

$$\mathcal{D} : E \longrightarrow \Omega^1_X(\log D) \otimes E$$

satisfying the Leibniz identity which says that $\mathcal{D}(fs) = f\mathcal{D}(s) + df \otimes s$, where f is a locally defined holomorphic function on X and s is a locally defined holomorphic section of E. The Leibniz identity implies that the differential operator $\mathcal{D}$ is of order 1.

The curvature of $\mathcal{D}$ is a holomorphic section

$$\mathcal{D} \circ \mathcal{D} \in H^0(X, \Omega^2_X(\log D) \otimes \mathrm{End}(E)) .$$

The logarithmic connection is called *flat* if the curvature of $\mathcal{D}$ vanishes identically.

For any irreducible component D_i of D, we have a residue map

$$\mathrm{Res}(D_i) : \Omega^1_X(\log D) \longrightarrow \mathcal{O}_{D_i}$$

which is defined using the Poincaré adjunction formula ([De1, p. 77, (3.7.2)]).

Let $\mathcal{D}$ be a logarithmic connection. For any irreducible component D_i of D consider the composition

$$E \xrightarrow{\mathcal{D}} \Omega^1_X(\log D) \otimes E \xrightarrow{\mathrm{Res}(D_i)\otimes \mathrm{Id}_E} E|_{D_i} .$$

This composition gives a section

$$\mathrm{Res}(\mathcal{D}, D_i) \in H^0(D_i, \mathrm{End}(E|_{D_i})) \tag{4.1}$$

which is called the *residue* of $\mathcal{D}$ along D_i ([De1, p. 78, (3.8.3)]).

Let E_* be a parabolic vector bundle as defined in Section 2.1 with E as the underlying vector bundle and D as the parabolic divisor.

A *holomorphic connection* on E_* is a logarithmic connection $\mathcal{D}$ on E such that

(1) for each irreducible component D_i of D, the residue $\mathrm{Res}(\mathcal{D}, D_i)$ (defined in (4.1)) is semisimple (that is, completely reducible);

(2) the residue $\mathrm{Res}(\mathcal{D}, D_i)$ preserves the quasiparabolic filtration in (2.2);

(3) on each graded piece F^i_j/F^i_{j+1} in (2.2), $j \in [1\,, l_i]$, the action of $\mathrm{Res}(\mathcal{D}, D_i)$ is multiplication by the scalar λ^i_j, where λ^i_j are the parabolic weights as in (2.3).

Note that since $\mathrm{Res}(\mathcal{D}, D_i)$ preserves the filtration in (2.2), it acts on each graded piece F^i_j/F^i_{j+1}.

A *flat* connection on E_* is a logarithmic connection $\mathcal{D}$ on E as above satisfying the extra condition that $\mathcal{D}$ is flat.

A connection on a Γ-linearized vector bundle is called Γ-*equivariant* if the action of Γ on the vector bundle preserves the connection.

The above definition of a holomorphic connection on a parabolic vector bundle is simply the translation of the definition of a Γ-equivariant holomorphic connection using the bijective correspondence between parabolic vector bundles and Γ-linearized vector bundles. To explain this, let W be the Γ-linearized vector bundle on Y corresponding to E_* after choosing a suitable cover q as in (2.8). On $q^{-1}(X \setminus D)$, the two vector bundles W and q^*E are canonically identified and the action of Γ on $W|_{q^{-1}(X\setminus D)}$ corresponds to the natural action of Γ on $q^*E|_{q^{-1}(X\setminus D)}$ obtained from the fact that the vector bundle is a pullback from Y/Γ. Here E denotes the underlying vector bundle for E_*. This assertion follows immediately from the identity (2.7). Therefore, a holomorphic connection on $E|_{X\setminus D}$ induces a Γ-equivariant holomorphic connection on $W|_{q^{-1}(X\setminus D)}$. Now, the conditions on a holomorphic connection on E_* are exactly the ones that are required to extend the connection on $W|_{q^{-1}(X\setminus D)}$ to a connection on W over Y. Note that any extension of a Γ-equivariant flat connection on $q^{-1}(X \setminus D)$ to Y must be Γ-equivariant. Indeed, if ∇ is a connection on W over Y extending the Γ-equivariant connection on $q^{-1}(X \setminus D)$, then for any $\gamma \in \Gamma$, the difference $\gamma^*\nabla - \nabla$ is a $\mathrm{End}(W)$-valued one-form on Y vanishing on $q^{-1}(X \setminus D)$. So, we have $\gamma^*\nabla = \nabla$.

Conversely, if we have a Γ-equivariant holomorphic connection on W over Y, then it induces a holomorphic connection on E over $X\setminus D$ using the identity (2.7). It is straightforward to check that this connection extends to X as a logarithmic connection. See Lemma 4.11 of [Bi2]. This logarithmic connection satisfies the conditions in the definition of a holomorphic connection on E_*. Clearly, a holomorphic connection on the parabolic vector bundle E_* is flat if and only if the corresponding holomorphic connection on the Γ-linearized vector bundle W is flat.

Lemma 4.1. *Let E_* and V_* be parabolic vector bundles equipped with* *holomorphic connections $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively. Then the direct sum $E_* \bigoplus V_*$ and the tensor product $E_* \bigotimes V_*$ have induced holomorphic connections. Similarly, the parabolic dual E_*^* also has an induced holomorphic connection.*

Proof. If $E_* \in \mathrm{PVect}_{N_1}(X)$ and $V_* \in \mathrm{PVect}_{N_2}(X)$, then $E_*, V_* \in \mathrm{PVect}_N(X)$, where $N = N_1 N_2$. Fix a covering q as in (2.8). So E_* and V_* correspond to Γ-linearized vector bundles F and W respectively over Y. Let ∇_1 (respectively, ∇_2) be the holomorphic connection on F (respectively, W) corresponding to the holomorphic connection $\mathcal{D}_1$ (respectively, $\mathcal{D}_2$) on E_* (respectively, V_*). Now, ∇_1 and ∇_2 together induce holomorphic connections on $F \bigoplus W$ and $F \bigotimes W$. Since the direct sum and tensor product operations of Γ-linearized vector bundles correspond to direct sum and tensor product operations of parabolic vector bundles, we have holomorphic connections on $E_* \bigoplus V_*$ and $E_* \bigotimes V_*$. Similarly, E_*^* also gets a holomorphic connection from the connection on F^* induced by ∇_1. □

Let G be a connected semisimple algebraic group over $\mathbb{C}$. Let $\mathfrak{g}$ be the Lie algebra of G. So $\mathfrak{g}$ is a left G-module by the adjoint action. The Lie algebra multiplication operation

$$\mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$$

is a homomorphism of G-modules.

Let E_* be a parabolic principal G-bundle over X. So we have the parabolic vector bundle $E_*(\mathfrak{g})$ which is the image of the G-module $\mathfrak{g}$ by the functor as in (3.2) defining E_*. The Lie algebra multiplication operation gives a homomorphism

$$\mu \, : \, E_*(\mathfrak{g}) \otimes E_*(\mathfrak{g}) \longrightarrow E_*(\mathfrak{g}) \tag{4.2}$$

of parabolic vector bundles. From Lemma 4.1 we know that a holomorphic connection on $E_*(\mathfrak{g})$ induces a holomorphic connection on $E_*(\mathfrak{g}) \bigotimes E_*(\mathfrak{g})$.

A *holomorphic connection* on E_* is defined to be a holomorphic connection $\mathcal{D}$ on the parabolic vector bundle $E_*(\mathfrak{g})$ satisfying the condition that the homomorphism μ in (4.2) commutes with the connections (the connection on $E_*(\mathfrak{g}) \bigotimes E_*(\mathfrak{g})$ being the induced one).

A *flat* connection on E_* is a holomorphic connection $\mathcal{D}$ as above satisfying the extra condition that $\mathcal{D}$ is flat.

A holomorphic vector bundle V over a compact connected Riemann surface admits a holomorphic connection if and only if for every decomposition $V \cong V_1 \bigoplus V_2$, the degree of V_1 is zero ([We], [At]). We will prove an analog of this criterion for parabolic G-bundles.

Any holomorphic connection on a principal bundle over a Riemann surface M is automatically flat, as there are no nonzero forms of type $(2\,,0)$ on M. By a connection we will always mean a holomorphic connection. So we will often say just "connection" instead of "holomorphic connection".

Given a parabolic vector bundle E_*, a parabolic vector bundle F_* is called a *direct summand* of E_* if there is another parabolic vector bundle V_* such that E_* is isomorphic to $F_* \bigoplus V_*$. A clarification of this definition is needed. Given a subbundle F of the underlying vector bundle E of the parabolic vector bundle E_*, there is an induced parabolic structure on F ([MS], [MY]). Let F_* denote this parabolic vector bundle with F as the underlying vector bundle. If V is another subbundle of E with $E = F \bigoplus V$, then it may happen that $F_* \bigoplus V_*$ is *not* isomorphic to E_*. In other words, the condition that F is a direct summand of E does not imply that F_* is a direct summand of E_*.

Theorem 4.2. *Let X be a compact connected Riemann surface. As before, the algebraic group G is assumed to be semisimple. A parabolic G-bundle E_* over X admits a flat connection if and only if every direct summand of the parabolic vector bundle $E_*(\mathfrak{g})$ is of parabolic degree zero.*

Proof. Fix a Galois covering $q : Y \longrightarrow X$ as in (2.8) such that the parabolic G-bundle E_* corresponds to a Γ-linearized G-bundle F_G over Y. The Galois group for q will be denoted by Γ. Let

$$\mathrm{ad}(F_G) := \frac{F_G \times \mathfrak{g}}{G}$$

be the adjoint vector bundle. So $\mathrm{ad}(F_G)$ is the vector bundle over Y associated to F_G for the adjoint action of G on its Lie algebra $\mathfrak{g}$ (see (2.9)). Therefore, the parabolic vector bundle $E_*(\mathfrak{g})$ corresponds to the Γ-linearized vector bundle $\mathrm{ad}(F_G)$.

We already noted that a flat connection on the vector bundle $E_*(\mathfrak{g})$ corresponds to a Γ-equivariant flat connection on the corresponding Γ-linearized vector bundle $\mathrm{ad}(F_G)$. A flat connection $\mathcal{D}$ on $E_*(\mathfrak{g})$ is compatible with the homomorphism μ in (4.2) if and only if the corresponding flat connection ∇ on $\mathrm{ad}(F_G)$ preserves the Lie algebra structure of the fibers of $\mathrm{ad}(F_G)$. Indeed, this is an immediate consequence of the fact that the connection on a parabolic tensor power of $E_*(\mathfrak{g})$ induced by the connection $\mathcal{D}$ on $E_*(\mathfrak{g})$ corresponds to the connection induced by ∇ on the corresponding tensor power of $\mathrm{ad}(F_G)$.

Let ∇ be a connection on $\mathrm{ad}(F_G)$. Consider the connection on

$$\mathrm{Hom}(\mathrm{ad}(F_G)^{\otimes 2}, \mathrm{ad}(F_G))$$

induced by ∇. Let $\mathbf{m}$ denote the section of this homomorphism bundle defined by the Lie algebra structure of the fibers of $\mathrm{ad}(F_G)$. The connection ∇ is said to *preserve* the Lie algebra structure of the fibers of $\mathrm{ad}(F_G)$ if $\mathbf{m}$ is a flat section for the induced connection. Note that ∇ preserves the Lie algebra structure of the fibers of $\mathrm{ad}(F_G)$ if and only if the homomorphism

$$\mathrm{ad}(F_G) \otimes \mathrm{ad}(F_G) \longrightarrow \mathrm{ad}(F_G)$$

defining the Lie algebra structure commutes with the connections (the connection on $\mathrm{ad}(F_G)^{\otimes 2}$ is the one induced by ∇).

The next step would be to prove the following proposition which says that $\mathrm{ad}(F_G)$ admits a Γ-equivariant flat connection compatible with the Lie algebra structure of its fibers if and only if it admits a flat connection (not necessarily Γ-equivariant or Lie algebra structure preserving).

Proposition 4.3. *The adjoint vector bundle* $\mathrm{ad}(F_G)$ *admits a* Γ*-equivariant flat connection preserving the Lie algebra structure of the fibers if and only if it admits a flat connection.*

Proof. Let $\mathrm{GL}(\mathfrak{g})$ denote the group of all linear automorphisms of the vector space $\mathfrak{g}$. Its Lie algebra will be denoted by $\mathrm{gl}(\mathfrak{g})$.

Let $F_{\mathrm{GL}(\mathfrak{g})}$ be the principal $\mathrm{GL}(\mathfrak{g})$-bundle over Y obtained by extending the structure group of the G-bundle F_G using the homomorphism $G \longrightarrow \mathrm{GL}(\mathfrak{g})$ which is defined by the adjoint action of G on $\mathfrak{g}$. Let

$$\tau : F_G \longrightarrow F_{\mathrm{GL}(\mathfrak{g})} := \frac{F_G \times \mathrm{GL}(\mathfrak{g})}{G} \tag{4.3}$$

be the map for this extension of structure group. So $\tau(z) = \{(z, e)\}$, where $z \in F_G$ and $e \in \mathrm{GL}(\mathfrak{g})$ is the identity element.

A flat connection on $F_{\mathrm{GL}(\mathfrak{g})}$ is a holomorphic $\mathrm{gl}(\mathfrak{g})$-valued one-form ω on the total space of $F_{\mathrm{GL}(\mathfrak{g})}$ satisfying the following two conditions:

(1) the form ω is equivariant for the natural action of $\mathrm{GL}(\mathfrak{g})$ on $F_{\mathrm{GL}(\mathfrak{g})}$ and the adjoint action of $\mathrm{GL}(\mathfrak{g})$ on its Lie algebra $\mathrm{gl}(\mathfrak{g})$;

(2) the restriction of ω to any fiber of the projection of $F_{\mathrm{GL}(\mathfrak{g})}$ to Y is the Maurer-Cartan form.

(See [KN, p. 64, Proposition 1.1] for connection on principal bundles.)

Consider the homomorphism $\iota : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ defined by the adjoint action of G. So, $\iota(v)(w) = [v, w]$. Note that ι is a homomorphism of G-modules. Since G is semisimple, the homomorphism ι is injective.

Since G is semisimple, there is a retraction

$$\rho : \mathrm{End}(\mathfrak{g}) \longrightarrow \mathfrak{g}$$

of G-modules. So $\rho \circ \iota$ is the identity automorphism of $\mathfrak{g}$.

Giving a connection on the vector bundle $\mathrm{ad}(F_G)$ is equivalent to giving a connection on the principal bundle $F_{\mathrm{GL}(\mathfrak{g})}$. Note that the map τ in (4.3) has the property that its differential is injective everywhere. More precisely, τ is an unramified covering map over its image. Using the property of τ it follows that if ω is a connection form on $F_{\mathrm{GL}(\mathfrak{g})}$, then $\tau^*(\rho \circ \omega)$ is a connection form on F_G, where τ is defined in (4.3) and ρ is the splitting considered above. Indeed, since the projection ρ is a homomorphism of G-modules, the form $\tau^*(\rho \circ \omega)$ is G-equivariant, and since the differential of τ is injective everywhere, the form ω coincides with the Maurer-Cartan form on a fiber of the projection of F_G to Y.

The connection on $\mathrm{ad}(F_G)$ induced by a connection on the principal bundle F_G is clearly compatible with the Lie algebra structure of the fibers of $\mathrm{ad}(F_G)$. Therefore, if $\mathrm{ad}(F_G)$ admits a flat connection, then it admits one that is compatible with the Lie algebra structure of the fibers of $\mathrm{ad}(F_G)$.

Note that if the connection ω on $F_{\mathrm{GL}(\mathfrak{g})}$ is Γ-equivariant, then the connection $\tau^*(\rho \circ \omega)$ on F_G is also Γ-equivariant. Indeed, this follows immediately from the fact that the map τ in (4.3) is Γ-equivariant. Therefore, to complete the proof of the proposition it suffices to show that if $\mathrm{ad}(F_G)$ admits a flat connection, then it admits one that is Γ-equivariant.

We recall that the space of all connections on $\mathrm{ad}(F_G)$ is an affine space for the vector space $H^0(Y, K_Y \bigotimes \mathrm{End}(\mathrm{ad}(F_G)))$, where K_Y is the holomorphic cotangent bundle of Y. If ∇ is a connection on the vector bundle $\mathrm{ad}(F_G)$, then the connection

$$\nabla' := \frac{1}{\#\Gamma} \sum_{g\in\Gamma} g^*\nabla$$

on $\mathrm{ad}(F_G)$, where $\#\Gamma$ is the order of the group Γ and the average is defined using the affine space structure on the space of all connections, is clearly Γ-equivariant. This completes the proof of the proposition. □

Continuing with the proof of Theorem 4.2, we call a Γ-linearized vector bundle V over Y decomposable if it is isomorphic, as a Γ-linearized vector bundle, to $V_1 \bigoplus V_2$, where V_1 and V_2 are Γ-linearized vector bundles of positive rank. We will call V to be *indecomposable* if it is not decomposable.

When Γ is the trivial group, the following proposition is Proposition 19 of [At].

Proposition 4.4. *Any indecomposable Γ-linearized vector bundle over Y of degree zero admits a connection.*

Proof. Let V be a holomorphic vector bundle over Y. Let $\mathrm{Diff}^1_Y(V, V)$ denote the vector bundle of differential operators of order one on V. Consider the symbol homomorphism

$$\sigma : \mathrm{Diff}^1_Y(V, V) \longrightarrow TY \otimes \mathrm{End}(V).$$

The *Atiyah bundle*

$$\mathrm{At}(V) := \sigma^{-1}(TY \otimes \mathrm{Id}_V) \subset \mathrm{Diff}^1_Y(V, V)$$

is the inverse image of $TY \bigotimes \mathrm{Id}_V \subset TY \otimes \mathrm{End}(V)$ by the symbol map. Consider the *Atiyah exact sequence*

$$0 \longrightarrow \mathrm{End}(V) \longrightarrow \mathrm{At}(V) \stackrel{\sigma}{\longrightarrow} TY \longrightarrow 0. \tag{4.4}$$

A holomorphic connection on V is a holomorphic splitting of the exact sequence (4.4) [At, p. 188, Definition].

The space of all extensions of TY by $\mathrm{End}(V)$ is parametrized by

$$H^1(Y, K_Y \otimes \mathrm{End}(V)) \cong H^0(Y, \mathrm{End}(V))^* \tag{4.5}$$

with the isomorphism being the Serre duality. Note that $\mathrm{End}(V) \cong \mathrm{End}(V)^*$ with the isomorphism defined by the symmetric bilinear form

$$A \otimes B \longmapsto \mathrm{trace}(AB)$$

on the fibers of $\mathrm{End}(V)$.

We will recall a few properties of the extension class for (4.4). Let

$$\beta_V \in H^1(Y, K_Y \otimes \mathrm{End}(V))$$

be the *Atiyah class* representing the extension in (4.4), and let

$$\overline{\beta}_V \in H^0(Y, \mathrm{End}(V))^* \tag{4.6}$$

correspond to β_V by the isomorphism in (4.5).

Let I denote the identity automorphism of V. We have

$$\overline{\beta}_V(I) = 2\pi\sqrt{-1}\mathrm{degree}(V) \tag{4.7}$$

which is a consequence of the construction of Chern classes from the Atiyah class [At, p. 197, Theorem 6].

Let

$$F_{\cdot} : 0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{k-1} \subset F_k = V \tag{4.8}$$

be a filtration of V by holomorphic subbundles, that is, each F_i is a holomorphic subbundle of V. Let

$$\mathrm{End}_{F_{\cdot}}(V) \subset \mathrm{End}(V)$$

be the subbundle that preserves the filtration. So for each $y \in Y$ and $w \in \mathrm{End}(V)_y$ in the fiber over y, we have $w \in \mathrm{End}_{F_{\cdot}}(V)_y$ if and only if $w((F_i)_y) \subset (F_i)_y$ for each $i \in [1, k]$. Let

$$\mathrm{End}^0_{F_{\cdot}}(V) \subset \mathrm{End}_{F_{\cdot}}(V) \tag{4.9}$$

be the subbundle of nilpotent endomorphisms with respect to the flag. So, $w \in \mathrm{End}^0_{F_{\cdot}}(V)_y$ if and only if $w((F_i)_y) \subset (F_{i-1})_y$ for each $i \in [1, k]$.

With the above notation, the Atiyah bundle $\mathrm{At}(V)$ contains a subbundle $\overline{F}$ defined by the sheaf of differential operators on V that preserves the filtration $F_{\cdot}$ in (4.8). In other words, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{End}_{F_{\cdot}}(V) & \longrightarrow & \overline{F} & \longrightarrow & TY & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathrm{End}(V) & \longrightarrow & \mathrm{At}(V) & \longrightarrow & TY & \longrightarrow & 0 \end{array}$$

where $\mathrm{End}_{F_{\cdot}}(V) \longrightarrow \mathrm{End}(V)$ is the natural inclusion map, and $\overline{F}$ is defined by the condition that a holomorphic section w of $\mathrm{At}(V)$, defined over an open subset $U \subset Y$, is a section of $\overline{F}$ if and only if for each $i \in [1, k]$ and any holomorphic section s_i of F_i over U, the evaluation $w(s_i)$ is again a section of F_i.

From the above commutative diagram it follows immediately that the extension class β_V is in the image of $H^1(Y, K_Y \bigotimes \mathrm{End}_{F_{\cdot}}(V))$, for the homomorphism

$$H^1(Y, K_Y \otimes \mathrm{End}_{F_{\cdot}}(V)) \longrightarrow H^1(Y, K_Y \otimes \mathrm{End}(V)),$$

induced by the inclusion of $\text{End}_{F_\cdot}(V)$ in $\text{End}(V)$. Using this it can be shown (see the next paragraph) that

$$\overline{\beta}_V \in \text{kernel}(\psi)\,, \tag{4.10}$$

where $\overline{\beta}_V$ is defined in (4.6) and

$$\psi : H^0(Y,\, \text{End}(V))^* \longrightarrow H^0(Y,\, \text{End}^0_{F_\cdot}(V))^*$$

($\text{End}^0_{F_\cdot}(V)$ is defined in (4.9)) is the dual of the homomorphism $H^0(Y,\, \text{End}^0_{F_\cdot}(V)) \hookrightarrow H^0(Y,\, \text{End}(V))$ induced by the inclusion of $\text{End}^0_{F_\cdot}(V)$ in $\text{End}(V)$.

To prove the inclusion in (4.10) first recall that the isomorphism in (4.5) was constructed using the trace form. Note that $\text{End}^0_{F_\cdot}(V)$ is precisely the orthogonal part $\text{End}_{F_\cdot}(V)^\perp \subset \text{End}(V)$ with respect to the trace form. (This is a special case of the general fact that for any parabolic subalgebra $\mathfrak{p}$ in a complex semisimple Lie algebra $\mathfrak{g}$ the orthogonal part $\mathfrak{p}^\perp \subset \mathfrak{g}$ for the Killing form on $\mathfrak{g}$ coincides with the nilpotent radical of $\mathfrak{p}$.) Therefore, the composition

$$\text{End}_{F_\cdot}(V) \hookrightarrow \text{End}(V) \cong \text{End}(V)^* \longrightarrow (\text{End}^0_{F_\cdot}(V))^*$$

is the zero homomorphism (in fact, the above is an exact sequence of vector bundles). This immediately implies the inclusion in (4.10).

Take any $\tau \in \text{Aut}(Y)$, and let

$$\overline{\tau} : H^1(Y,\, K_Y \otimes \text{End}(V)) \longrightarrow H^1(Y,\, K_Y \otimes \text{End}(\tau^*V))$$

be the isomorphism induced by τ. Let

$$\beta_{\tau^*V} \in H^1(Y,\, K_Y \otimes \text{End}(\tau^*V))$$

be the Atiyah class for τ^*V. Since $\tau^*\text{At}(V) \cong \text{At}(\tau^*V)$, the identity

$$\beta_{\tau^*V} = \overline{\tau}(\beta_V) \tag{4.11}$$

is obviously valid.

Let W be a Γ-linearized vector bundle over Y. The group Γ has a natural action on $H^1(Y,\, K_Y \bigotimes \text{End}(W))$. Let

$$\beta \in H^1(Y,\, K_Y \otimes \text{End}(W)) \tag{4.12}$$

represent the Atiyah exact sequence of W. From (4.11) it follows immediately that

$$\beta \in H^1(Y,\, K_Y \otimes \text{End}(W))^\Gamma\,.$$

In other word, β is fixed by the action of Γ on $H^1(Y,\, K_Y \bigotimes \text{End}(W))$. The isomorphism in (4.5) commutes with the action of the automorphism group $\text{Aut}(V)$ of the vector bundle V on $H^1(Y,\, K_Y \bigotimes \text{End}(V))$ and $H^0(Y,\, \text{End}(V))^*$ respectively. Therefore, if

$$\overline{\beta} \in H^0(Y,\, \text{End}(W))^* \tag{4.13}$$

corresponds to the extension class β by the isomorphism in (4.5), then

$$\overline{\beta} \in (H^0(Y, \mathrm{End}(W))^*)^\Gamma .$$

For a linear action of Γ on a finite dimensional complex vector space U we have

$$(U^*)^\Gamma \cong (U_\Gamma)^* ,$$

where U^* is the dual of U and U_Γ is the space of all coinvariants, that is, the quotient

$$U_\Gamma := \frac{U}{\sum_{g\in\Gamma}(g-1)U}$$

with $(g-1)U := \mathrm{Image}((g-1)U)$. From this observation it follows immediately that

$$(H^0(Y, \mathrm{End}(W))^*)^\Gamma \cong (H^0(Y, \mathrm{End}(W))_\Gamma)^* ,$$

and hence we have $\overline{\beta} \in (H^0(Y, \mathrm{End}(W))_\Gamma)^*$. Consequently, we have

$$\overline{\beta}\circ(g-1) = 0 \tag{4.14}$$

on $H^0(Y, \mathrm{End}(W))$ for all $g \in \Gamma$.

Take any section $\phi \in H^0(Y, \mathrm{End}(W))$. So we have

$$\phi = \phi_0 + \sum_{g\in\Gamma}(g-1)\psi_g , \tag{4.15}$$

where $\phi_0 \in H^0(Y, \mathrm{End}(W))^\Gamma$ is a Γ-invariant section and ψ_g, $g \in \Gamma$, are some elements in $H^0(Y, \mathrm{End}(W))$.

Since Y is compact and connected, the characteristic polynomial of $\phi_0(y) \in \mathrm{End}(W_y)$ does not depend on y. Consider the decomposition of W obtained from the generalized eigenspace decomposition for ϕ_0. Since ϕ_0 is left invariant by the action of Γ, this is a decomposition of W into a direct sum of Γ-linearized vector bundles.

Assume that W is indecomposable. This implies that $\phi_0(y)$ has only one eigenvalue, say λ. So, the endomorphism of W

$$\phi' := \phi_0 - \lambda \mathrm{Id}_W$$

is nilpotent with respect to the filtration of subbundles of W defined by ϕ_0. Note that since ϕ_0 has exactly one eigenvalue, using the powers of ϕ' we get a filtration $F_.$ of subbundles of W. More precisely, the subbundles in the filtration $F_.$ are the inverse image of the torsion sheaves $\mathrm{Torsion}(W/(\phi')^i(W))$, $i \geq 0$, for the natural projection

$$W \longrightarrow W/(\phi')^i(W) .$$

If $\phi' \neq 0$, then this filtration $F_.$ of W is nontrivial. Since ϕ' is nilpotent with respect to the filtration $F_.$, setting $V = W$ in (4.10) we conclude that $\overline{\beta}(\phi') = 0$, where $\overline{\beta}$ is defined in (4.13). Now, if $\mathrm{degree}(W) = 0$, then from (4.7) it follows that $\overline{\beta}(\phi_0) = 0$.

Finally, (4.14) and (4.15) together imply that $\overline{\beta}(\phi) = 0$ for all ϕ, that is, $\overline{\beta} = 0$. Consequently, we have $\beta = 0$, where β is the extension class defined in (4.12). This completes the proof of the proposition. □

Continuing with the proof of Theorem 4.2, given a Γ-linearized vector bundle W, a Γ-linearized vector bundle W_1 is called a *direct summand* of W if there is a Γ-linearized vector bundle W_2 such that W and $W_1 \bigoplus W_2$ are isomorphic as Γ-linearized vector bundles.

If $V \cong V_1 \bigoplus V_2$, then from a holomorphic connection on V we can construct holomorphic connections on V_1 and V_2 as follows. Let p_{V_i} (respectively, q_{V_i}), $i = 1, 2$, be the inclusion (respectively, projection) of V to V_i defined using a fixed isomorphism of V with $V_1 \bigoplus V_2$. If ∇^V is a holomorphic connection on V, then the first order differential operator

$$(\mathrm{Id}_{K_Y} \otimes q_{V_i}) \circ \nabla^V \circ p_{V_i} \,:\, V_i \longrightarrow K_Y \otimes V_i$$

is a holomorphic connection on V_i (see [At, p. 202, Proposition 17]). Conversely, if V_1 and V_2 are equipped with holomorphic connections, then V has an induced holomorphic connection. Any holomorphic vector bundle with a holomorphic connection is of degree zero. Indeed, recall that a holomorphic connection on a Riemann surface is flat.

Therefore, using Proposition 4.4 we conclude that a Γ-linearized vector bundle admits a Γ-equivariant connection if and only if every direct summand of it is of degree zero.

We next note that in the bijective correspondence between $\mathrm{PVect}_N(X)$ and $\mathrm{Vect}_\Gamma(Y)$ we have

$$\text{par-deg}(F_*) = \frac{\mathrm{degree}(W')}{\#\Gamma} \tag{4.16}$$

([Bi1, p. 318, (3.12)]), where $F_* \in \mathrm{PVect}_N(X)$ and $W' \in \mathrm{Vect}_\Gamma(Y)$ correspond to each other. In view of this, Proposition 4.3 together with the above conclusion completes the proof of the theorem. □

Let M be a connected smooth projective manifold of complex dimension at least three. Fix an ample line bundle L over M. Let E_G be a holomorphic principal G-bundle over M, where G is a complex algebraic group.

Atiyah proved that E_G admits a holomorphic connection if and only if for any $n_0 \in \mathbb{N}$ there is an integer $n \geq n_0$ and a smooth divisor D_n in the complete linear system $|L^{\otimes n}|$ such that the restriction of E_G to D_n admits a holomorphic connection ([At, p. 204, Proposition 21]).

Note that for a covering q as in (2.8), if L is an ample line bundle over X, then q^*L is ample over Y, since the morphism q is finite. Also note that if D is a normal crossing divisor on X, then there is a $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, the general member $D_n \in |L^{\otimes n}|$ has the following properties

(1) D_n is smooth and irreducible;
(2) $D \cap D_n$ is a normal crossing divisor on D_n.

If E_* is a parabolic G-bundle over X with D as the parabolic divisor, for such a divisor D_n, we can restrict E_* to D_n to get a parabolic G-bundle over D_n with $D \cap D_n$ as the parabolic divisor.

Therefore, the above quoted Proposition 21 of [At] gives the following Proposition.

Proposition 4.5. *Let D be a normal crossing divisor on a connected smooth projective variety X with $\dim X \geq 3$. Let E_* be a parabolic G-bundle over X with D as the parabolic divisor, where G is a complex semisimple algebraic group. Fix an ample line bundle L over X. The parabolic G-bundle E_* admits a holomorphic connection if and only if for every $n_0 \in \mathbb{N}$ there is an integer $n \geq n_0$ and a divisor $D_n \in |L^{\otimes n}|$ in the complete linear system such that*

(1) *D_n is smooth;*
(2) *$D \cap D_n$ is a normal crossing divisor on D_n;*
(3) *the parabolic G-bundle on D_n, with $D \cap D_n$ as the parabolic divisor, obtained by restricting E_* to D_n admits a holomorphic connection.*

5. Finite principal bundles

Let $P(x)$ be a polynomial in one variable whose coefficients are nonnegative integers. Given a vector bundle E, define $P(E)$ by substituting E for x and replacing the addition and multiplication by direct sum and tensor product operations respectively. In other words, if $P(x) = \sum_{i=0}^{n} a_i x^i$, then

$$P(E) := \bigoplus_{i=0}^{n} (E^{\otimes i} \otimes_{\mathbb{C}} \mathbb{C}^{a_i}) .$$

An algebraic vector bundle E is called *finite* if there are two distinct polynomials with nonnegative integral coefficients, say P_1 and P_2, such that the vector bundle $P_1(E)$ is isomorphic to $P_2(E)$ ([We], [No1], [No2]).

The main result of [No1] says that a vector bundle E over a projective manifold M is finite if and only if there is a finite étale Galois cover $p : \widetilde{M} \longrightarrow M$ such that the pullback p^*E is trivial. Note that the condition that there is a finite étale Galois covering p with p^*E trivial is equivalent to the condition that E admits a flat connection whose monodromy group is finite.

The above definition of finiteness suggests the following definition for principal bundles.

Let G be a complex algebraic group. A principal G-bundle E_G over a smooth projective variety M is defined to be *finite* if for every finite dimensional G-module V, the associated vector bundle $E_G(V) := (E_G \times V)/G$ is finite.

We recall that a G-module V_0 is called faithful if the homomorphism $G \longrightarrow \mathrm{Aut}(V_0)$ is injective.

Proposition 5.1. *Let G be a complex reductive algebraic group and V_0 a finite dimensional faithful G-module. A principal G-bundle E_G over M is finite if and only if the associated vector bundle $E_G(V_0)$ over M is finite.*

Proof. If E_G is finite then obviously $E_G(V_0)$ is finite. To prove the converse, assume that the vector bundle $E_G(V_0)$ is finite.

First note that if W is finite then W^* is also finite, as $P_1(W) \cong P_2(W)$ implies $P_1(W^*) \cong P_2(W^*)$. From the above quoted result of Nori that a vector bundle is finite if and only if it has a flat connection with finite monodromy it follows immediately that if W_1 and W_2 are finite then both $W_1 \bigoplus W_2$ and $W_1 \bigotimes W_2$ are also finite.

Let V be a finite dimensional G-module. Since V_0 is faithful and G is reductive, we know that V is a direct summand of a G-module $\mathcal{V}$ of the form

$$\mathcal{V} = \bigoplus_{j=1}^{l} V_0^{\otimes n_j} \otimes (V_0^*)^{\otimes m_j}$$

([De2, p. 40, Proposition 3.1 (a)]). Since $E_G(V_0)$ is finite, from the above remarks on tensor product, direct sum and dual it follows immediately that the associated vector bundle

$$E_G(\mathcal{V}) := \frac{E_G \times \mathcal{V}}{G}$$

is finite.

Any direct summand of a finite vector bundle is finite ([No1, p. 36, Lemma 3.2 (2)]). Since the G-module V is a direct summand of $\mathcal{V}$, the associated vector bundle $E_G(V)$ is a direct summand of $E_G(\mathcal{V})$. This completes the proof of the proposition. □

Given a parabolic vector bundle E_* and a polynomial $P(x) = \sum_{i=0}^{n} a_i x^i$, where $a_i \in \mathbb{N}$ are nonnegative, define

$$P(E_*) := \bigoplus_{i=0}^{n} (E_*)^{\otimes i} \bigotimes_{\mathbb{C}} \mathbb{C}^{a_i}$$

using the tensor product and direct sum operations of parabolic vector bundles. Imitating the definition of a finite vector (principal) bundle we will define a finite parabolic vector (principal) bundle.

A parabolic vector bundle E_* is defined to be *finite* if there are two distinct polynomials with nonnegative integral coefficients, say P_1 and P_2, such that the parabolic vector bundle $P_1(E_*)$ is isomorphic to $P_2(E_*)$.

A parabolic G-bundle F_* is defined to be *finite* if for every finite dimensional G-module V, the corresponding parabolic vector bundle $F_*(V)$ is finite. Here $F_*(V)$ denotes the image of the G-module V by the functor as in (3.2) defining the parabolic G-bundle.

Let G be a complex semisimple group. Let E_* be a parabolic G-bundle over a connected projective manifold X with a normal crossing divisor D as the parabolic divisor.

Theorem 5.2. *The parabolic G-bundle E_* is finite if and only if it admits a flat connection with finite monodromy.*

Proof. Take $N \in \mathbb{N}$ such that the functor as in (3.2) defining the parabolic G-bundle E_* sends $\mathrm{Rep}(G)$ to $\mathrm{PVect}_N(X)$. Fix a covering as in (2.8) such that we have bijective correspondence between $\mathrm{PVect}_N(X)$ and $\mathrm{Vect}_\Gamma(Y)$, where Γ is the Galois group for the covering map q. Let E_G denote the Γ-linearized principal G-bundle over the covering Y corresponding to the parabolic G-bundle E_*.

Assume that E_* is finite. Let $\mathfrak{g}$ be the Lie algebra of G, which is a G-module by the adjoint action. Let $E_*(\mathfrak{g})$ denote the parabolic vector bundle which is the image of the G-module $\mathfrak{g}$ by the functor as in (3.2) defining the parabolic G-bundle E_*. Since E_* is finite, the parabolic vector bundle $E_*(\mathfrak{g})$ is finite. Let P_1 and P_2 be two distinct polynomials with nonnegative integral coefficients such that

$$P_1(E_*(\mathfrak{g})) \cong P_2(E_*(\mathfrak{g}))\,. \tag{5.1}$$

Such polynomials exist since $E_*(\mathfrak{g})$ is finite.

Consider the adjoint vector bundle $\mathrm{ad}(E_G)$. Note that $\mathrm{ad}(E_G)$ corresponds to $E_*(\mathfrak{g})$ by the bijective correspondence between $\mathrm{PVect}_N(X)$ and $\mathrm{Vect}_\Gamma(Y)$. From (5.1) it follows that

$$P_1(\mathrm{ad}(E_G)) \cong P_2(\mathrm{ad}(E_G))\,.$$

In other words, $\mathrm{ad}(E_G)$ is a finite vector bundle. Therefore, from [No1] we know that $\mathrm{ad}(E_G)$ has a flat connection ∇ whose monodromy group is finite. We need to show that ∇ is Γ-equivariant, as well as it preserves the Lie algebra structure of the fibers of $\mathrm{ad}(E_G)$ in order to be able to conclude that ∇ induces a connection on E_*.

Since the monodromy group of ∇ is finite, there is a Hermitian structure on $\mathrm{ad}(E_G)$ which is preserved by ∇. To explain this fix a point $y \in Y$. Let

$$\Gamma_0 \subset \mathrm{Aut}(\mathrm{ad}(E_G)_y)$$

be the monodromy of ∇, where $\mathrm{Aut}(\mathrm{ad}(E_G)_y)$ denotes the group of all linear isomorphisms of the fiber $\mathrm{ad}(E_G)_y$.

Choose a Hermitian structure h on $\mathrm{ad}(E_G)_y$. Now define the Hermitian structure

$$\hat{h} := \sum_{g\in\Gamma_0} g^* h$$

on $\mathrm{ad}(E_G)_y$, where $g^*h(v\,,w) := h(g(v)\,,g(w))$; note that Γ_0 is a finite group. This Hermitian structure $\hat{h}$ is evidently preserved by the action of the monodromy group Γ_0. Consequently, by parallel translations of $\hat{h}$ (for the connection ∇) we obtain a Hermitian structure on the vector bundle $\mathrm{ad}(E_G)$ which is preserved by ∇. In other words, ∇ is a unitary connection. This implies that the vector bundle $\mathrm{ad}(E_G)$ is quasistable (with respect to any polarization) with vanishing Chern classes of positive degree, and ∇ is the unique unitary flat connection on $\mathrm{ad}(E_G)$. See [Do2, p. 231, Proposition 1] (and also [Do1, p. 1, Theorem 1] as referred in [Do2] for uniqueness).

From the uniqueness of unitary flat connection on a vector bundle over Y it follows immediately that the connection ∇ is preserved by the action of the Galois group Γ on $\mathrm{ad}(E_G)$. Indeed, for any $g \in \Gamma$, the connection $g^*\nabla$ on $g^*\mathrm{ad}(E_G) = \mathrm{ad}(E_G)$ coincides with ∇, as $g^*\nabla$ is unitary flat with ∇ also being so. In other words, the connection ∇ is Γ-equivariant.

As in the proof of Theorem 4.2, let

$$\mathbf{m} \in H^0(Y, \mathrm{Hom}(\mathrm{ad}(E_G)^{\otimes 2}, \mathrm{ad}(E_G)))$$

be the section defined by the Lie algebra structure of the fibers of $\mathrm{ad}(E_G)$. Consider the connection $\overline{\nabla}$ on $\mathrm{Hom}(\mathrm{ad}(E_G)^{\otimes 2}, \mathrm{ad}(E_G))$ induced by the connection ∇ on $\mathrm{ad}(E_G)$. Since ∇ is unitary flat, the connection $\overline{\nabla}$ is also unitary flat. Since $\mathbf{m}$ is a holomorphic section of $\mathrm{Hom}(\mathrm{ad}(E_G)^{\otimes 2}, \mathrm{ad}(E_G))$, it must be a flat section with respect to the unitary flat connection $\overline{\nabla}$ ([Do1, p. 6, Proposition 3 (ii)]). In other words, the connection ∇ on $\mathrm{ad}(E_G)$ preserves the Lie algebra structure of the fibers of $\mathrm{ad}(E_G)$.

Since ∇ is Γ-equivariant and preserves the Lie algebra structure of the fibers of $\mathrm{ad}(E_G)$, it induces a connection $\mathcal{D}$ on the parabolic G-bundle E_* (see Section 4). Since ∇ is flat with finite monodromy, the connection $\mathcal{D}$ is flat with finite monodromy. So, a finite parabolic G-bundle admits a flat connection with finite monodromy.

To prove the converse, let $\mathcal{D}$ be a flat connection on the parabolic G-bundle E_*. Let V be a finite dimensional G-module. Let $E_*(V)$ denote the parabolic vector bundle which is the image of the G-module V by the functor as in (3.2) defining the parabolic G-bundle E_*. We need to show that $E_*(V)$ is finite.

Let $W = E_G(V) := (E_G \times V)/G$ be the vector bundle associated to E_G for the G-module V. So W and $E_*(V)$ correspond to each other by the bijective correspondence between $\mathrm{PVect}_N(X)$ and $\mathrm{Vect}_\Gamma(Y)$. We will show that W is a finite vector bundle.

The connection $\mathcal{D}$ on E_* induces Γ-equivariant flat connection ∇ on the adjoint vector bundle $\mathrm{ad}(E_G)$ that preserves the Lie algebra structure of the fibers of $\mathrm{ad}(E_G)$. Let $Z(G) \subset G$ be the center of G. So the adjoint action of $Z(G)$ on $\mathfrak{g}$ is trivial, and the quotient

$$G' := \frac{G}{Z(G)}$$

acts faithfully on $\mathfrak{g}$. Since the connected component containing the identity element of the group of all automorphisms of the Lie algebra $\mathfrak{g}$ coincides with G', the connection ∇ on $\mathrm{ad}(E_G)$ gives a connection ∇' on the principal G'-bundle

$$E_G(G') := \frac{E_G \times G'}{G}$$

obtained by extending the structure group of E_G using the quotient map $G \longrightarrow G'$. Indeed, since ∇ preserves the Lie algebra structure of the fibers of $\mathrm{ad}(E_G)$, it induces a flat connection ∇' on $E_G(G')$. Consider the map

$$\tau : E_G \longrightarrow E_G(G') \tag{5.2}$$

for the extension of structure group. So, for any $z \in E_G$ we have $\tau(z) = \{(z\,,e)\}$, where e is the identity element in G'.

Since G is semisimple, its center $Z(G)$ is a finite group. Therefore, the projection τ in (5.2) is a covering map. Consequently, the pullback $\overline{\nabla} := \tau^*\nabla'$ is a flat connection on the principal G-bundle E_G.

Since the monodromy of ∇ is a finite group and $Z(G)$ is finite, the monodromy of the connection $\overline{\nabla}$ on E_G is a finite group.

A connection on a principal bundle induces a connection on any of its associated bundles. Let ∇^V denote the flat connection on the above vector bundle $W = E_G(V)$ (associated to E_G for the G-module V) by the connection $\overline{\nabla}$. The monodromy of ∇^V is a finite group since the monodromy of $\overline{\nabla}$ is so.

Since ∇ is Γ-equivariant, the connection $\overline{\nabla}$ is Γ-equivariant. Hence the connection ∇^V on W is also Γ-equivariant. Therefore, ∇^V induces a flat connection on the parabolic vector bundle $E_*(V)$. Recall that $E_*(V)$ corresponds to W by the bijective correspondence between $\mathrm{PVect}_N(X)$ and $\mathrm{Vect}_\Gamma(Y)$. Let $\mathcal{D}_V$ denote the connection on $E_*(V)$ induced by ∇^V. Note that the monodromy group of $\mathcal{D}_V$ is finite since ∇^V has finite monodromy.

Let Γ_0 be a finite subgroup of $\mathrm{Aut}(V_0)$, where V_0 is a finite dimensional vector space. So V_0 is a Γ_0-module. Given a polynomial $P(x)$ with nonnegative integral coefficients, $P(V_0)$ is a Γ_0-module which is constructed by replacing addition and multiplication by direct sum and tensor product operations respectively. We want to show that there are two such distinct polynomials P_1 and P_2 with the property that the two Γ_0-modules $P_1(V_0)$ and $P_2(V_0)$ are isomorphic.

Since Γ_0 is a finite group, there are only finitely many finite dimensional irreducible Γ_0-modules. Now the above assertion that there are two distinct polynomials P_1 and P_2 with $P_1(V_0) \cong P_2(V_0)$ is a very special case of [No1, p. 35, Lemma 3.1] (set the base X in [No1] to be a single point).

Now, fix a point $x \in X \setminus D$. Set $V_0 = E_x$ and set Γ_0 to be the monodromy representation for the flat connection $\mathcal{D}_V$ over $X \setminus D$. The assertion that $P_1(V_0) \cong P_2(V_0)$ as Γ_0-modules immediately implies that the two parabolic vector bundles $P_1(E_*(V))$ and $P_2(E_*(V))$ are isomorphic (they have flat connections with same monodromy). In other words, the parabolic vector bundle $E_*(V)$ is finite. This completes the proof of the theorem. □

School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Bombay 400005, India
e-mail: indranil@math.tifr.res.in

References

[At] M. F. Atiyah, *Complex analytic connections in fibre bundles*, Trans. Amer. Math. Soc. **85** (1957), 181–207.

[BBN] V. Balaji, I. Biswas and D. S. Nagaraj, *Principal bundles over projective manifolds with parabolic structure over a divisor*, Tôhoku Math. Jour. **53** (2001), 337–367.

[Bi1] I. Biswas, *Parabolic bundles as orbifold bundles*, Duke Math. Jour. **88** (1997), 305–325.

[Bi2] ———, *Parabolic ample bundles*, Math. Ann. **307** (1997), 511–529.

[De1] P. Deligne, *Equations Différentielles à Points Singuliers Réguliers*, Lecture Notes in Math. **163**, Springer-Verlag, Berlin, 1970.

[De2] P. Deligne (notes by J. S. Milne), *Hodge cycles on abelian varieties*, (Hodge cycles, motives, and Shimura varieties, by P. Deligne, J. S. Milne, A. Ogus and K.-Y. Shih), Lecture Notes in Math. **900**, Springer-Verlag, Berlin-New York, 1982.

[Do1] S. K. Donaldson, *Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. Lond. Math. Soc. **50** (1985), 1–26.

[Do2] ———, *Infinite determinants, stable bundles and curvature*, Duke Math. Jour. **54** (1987), 231–247.

[Gi] J. Giraud, *Cohomologie non abélienne*, Die Grundlehren der mathematischen Wissenschaften, Band **179**, Springer-Verlag, Berlin-New York, 1971.

[He] J. Hempel, 3-*Manifolds*, Ann. of Math. Studies **86**, Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1976.

[Ka] Y. Kawamata, *Characterization of the abelian varieties*, Compositio Math. **43** (1981), 253–276.

[KMM] Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem*, Adv. Stud. Pure Math. **10** (1987), 283–360.

[KN] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol. I.*, Interscience Publishers, a division of John Wiley & Sons, New York-London, 1963.

[MY] M. Maruyama and K. Yokogawa, *Moduli of parabolic stable sheaves*, Math. Ann. **293** (1992), 77–99.

[MS] V. B. Mehta and C. S. Seshadri, *Moduli of vector bundles on curves with parabolic structures*, Math. Ann. **248** (1980), 205–239.

[No1] M. V. Nori, *On the representations of the fundamental group*, Compositio Math. **33** (1976), 29–41.

[No2] ———, *The fundamental group-scheme*, Proc. Ind. Acad. Sci. (Math. Sci.) **91** (1982) 73–122.

[We] A. Weil, *Généralisation des fonctions abéliennes*, J. Math. Pures Appl. **17** (1938), 47–87.

[Yo] K. Yokogawa, *Infinitesimal deformations of parabolic Higgs sheaves*, Int. Jour. Math. **6** (1995), 125–148.