J. Math. Kyoto Univ. (JMKYAZ)
43-2 (2003), 235–260

Missing terms in generalized Hardy's inequalities and its applications

By

Toshio HORIUCHI

Abstract

In this article we shall investigate the Hardy inequalities and improve them by finding out missing terms. Although the missing terms for the higher order Hardy inequality can not be determined in a unique way, we shall give a canonical form of the remainder. As a direct application we shall study blow-up solutions of a semilinear elliptic boundary value problem and give some lower estimate of the first eigenvalue of the linearized operator. We also improve the weighted Hardy inequalities, which will be fundamental to study singular solutions of quasilinear elliptic equations.

1. Introduction

Let N be a positive integer and let Ω be a bounded open set of $\mathbb{R}^N$. Let l be an arbitrary nonnegative integer. By $C_0^\infty(\Omega)$ and $C_0^l(\Omega)$ we denote the spaces of all smooth functions and k times continuously differentiable functions having compact supports in Ω respectively. By $H^l(\Omega)$ we denote the space of all functions on Ω, whose generalized derivatives $\partial^\gamma u$ of order $\le l$ satisfy

$$||u||_l = \sum_{|\gamma|\le l} \left(\int_\Omega |\partial^\gamma u(x)|^2 \, dx \right)^{1/2} < +\infty. \tag{1.1}$$

By $H_0^l(\Omega)$ we denote the completion of $C_0^\infty(\Omega)$ with respect to the norm defined by (1.1). Convensionally we set $L^2(\Omega) = H^0(\Omega)$.

In the first place we recall the classical Hardy inequalities.

Theorem 1.1. *If $l < N/2$, then it holds that for any $u \in H_0^l(\Omega)$*

$$\int_\Omega |\nabla^l u|^2 \, dx \ge C_l \int_\Omega \frac{|u(x)|^2}{|x|^{2l}} \, dx. \tag{1.2}$$

1991 *Mathematics Subject Classification(s).* 35J65, 35J35
Received February 28, 2002
Revised July 11, 2002

Here $\nabla^l = \{\partial^\gamma\}$, *where* $|\gamma| = l$ *and* $\nabla = \nabla^1$, *namely*

$$|\nabla^l u|^2 = \sum_{|\gamma|=l} |\partial^\gamma u(x)|^2, \tag{1.3}$$

where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_N)$ *is a multi-index as usual, and then* $\partial^\gamma = (\partial/\partial x_1)^{\gamma_1} \cdot (\partial/\partial x_2)^{\gamma_2} \cdots (\partial/\partial x_N)^{\gamma_N}$. C_l *is a positive number independent of each* u.

In this paper we shall mainly study the Hardy inequalities of the following type: For any $u \in H_0^{2l}(\Omega)$,

$$\int_\Omega |\Delta^l u|^2\, dx \geq H(N, \Delta^l) \int_\Omega \frac{|u(x)|^2}{|x|^{4l}}\, dx \qquad \text{for} \quad l = 1, 2. \tag{1.4}$$

Here the best constants $H(N, \Delta^l)$ $(l = 1, 2)$ are given by the infimum of the next variational problems:

$$\inf \left[\int_\Omega |\Delta^l u|^2\, dx : u \in H_0^{2l}(\Omega), \int_\Omega \frac{|u(x)|^2}{|x|^{4l}}\, dx = 1 \right], \qquad l = 1, 2. \tag{1.5}$$

It is well-known that if $0 \in \Omega$ and $N > 4l$, $H(N, \Delta^l)$ $(l = 1, 2)$ are given by

$$\begin{cases} H(N, \Delta) = \left(\dfrac{N(N-4)}{4} \right)^2, \\ H(N, \Delta^2) = \left(\dfrac{N(N-4)(N+4)(N-8)}{16} \right)^2. \end{cases} \tag{1.6}$$

For the references, see [1] and [4]. Moreover there exists no extremal function in $H_0^{2l}(\Omega)$ which attains the infimum of these problems. Roughly speaking, the candidates of extremals are singular at the origin, hence they can not be admissible in the energy class $H_0^{2l}(\Omega)$. Therefore it is natural to consider that there exist "missing terms" in the right-hand side of the classical Hardy inequalities (1.2) and (1.4). In this spirit we shall investigate the Hardy inequalities (1.4) and improve them by finding out **missing terms**. Although the missing terms for the higher order Hardy inequality can not be determined in a unique way, we shall give a canonical form of the remainder.

As an application we shall consider in the last section the semi-linear boundary value problem defined by

$$\begin{cases} \Delta^2 u = \lambda f(u, r) & \text{in} \quad B, \\ u = \Delta u = 0 & \text{on} \quad \partial B, \end{cases} \tag{1.7}$$

where $r = |x|$, $B = \{x \in \mathbb{R}^N : |x| < 1\}$ and λ is a nonnegative parameter. We shall adopt as the nonlinearity $f(u, r)$ the following f_p and f_e, that is,

$$\begin{cases} f_p(u, r) = (1 + u + Q_p(r))^p, \\ f_e(u, r) = e^{u + Q_e(r)}. \end{cases} \tag{1.8}$$

Here $Q_p(r)$ and $Q_e(r)$ are nonnegative polynomials on B which will be defined in Section 7. Then we shall study fundamental properties of blow-up solution of these problems. We shall also establish the weighted Hardy inequalities, which are not only of interest by itself but also essential to study the blow-up solutions of p-harmonic equations (See [5]).

This paper is organized in the following way. In Section 2 we shall describe our main results on Hardy's inequalities. In Section 3 we shall prepare lemmas which are needed in the proofs of the theorems stated in Section 2. In Sections 4 and 5 we shall establish Theorems 2.1 and 2.2 using lemmas in the previous section. In Section 6 we shall prove Theorems 2.3 and 2.4. In Section 7 we shall apply our theorems to study semilinear boundary value problems which are stated in Section 1.

2. Main results

In this section we state our main results concerned with Hardy's inequalities. To this end we prepare more notations. Let $r > 0$ and let M be an arbitraly positive integer. We set

$$B_r^M = \{x \in \mathbb{R}^M : |x| < r\}. \tag{2.1}$$

By $|\Omega|$ and ω_N we denote the N-dimensional measure of the domain Ω and that of a unit ball B_1^N respectively. Further, by Δ_M and ∇_M, we denote the M-dimensional Laplacian and the M-dimensional gradient in $\mathbb{R}^M$ respectively;

$$\begin{cases} \Delta_M = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2} + \cdots + \dfrac{\partial^2}{\partial x_M^2}, \\ \nabla_M = \left(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}, \ldots, \dfrac{\partial}{\partial x_M} \right). \end{cases} \tag{2.2}$$

Conventionally we set $\Delta = \Delta_N$ and $\nabla = \nabla_N$. In the next we introduce the first eigenvalues for various elliptic problems.

Definition 2.1. Let us set

$$\begin{cases} \lambda_1 = \inf \left[\int_{B_1^2} |\nabla_2 v|^2 \, dx : v \in H_0^1(B_1^2), \int_{B_1^2} |v|^2 \, dx = 1 \right], \\ \lambda_2 = \inf \left[\int_{B_1^4} |\Delta_4 v|^2 \, dx : v \in H_0^2(B_1^4), \int_{B_1^4} |v|^2 \, dx = 1 \right], \\ \lambda_3 = \inf \left[\int_{B_1^6} |\nabla_6 (\Delta_6 v)|^2 \, dx : v \in H_0^3(B_1^6), \int_{B_1^6} |v|^2 \, dx = 1 \right], \\ \lambda_4 = \inf \left[\int_{B_1^8} |\Delta_8^2 v|^2 \, dx : v \in H_0^4(B_1^8), \int_{B_1^8} |v|^2 \, dx = 1 \right], \\ \lambda_2^* = \inf \left[\int_{B_1^4} |\Delta_4 v|^2 \, dx : v \in H^2(B_1^4) \cap H_0^1(B_1^4), \int_{B_1^4} |v|^2 \, dx = 1 \right]. \end{cases} \tag{2.3}$$

Then the numbers $\lambda_k \, (k = 1, 2, 3, 4)$ and λ_2^* are characterized as follows:

Proposition 2.1. *The numbers $\lambda_k \, (k = 1, 2, 3, 4)$ and λ_2^* are the first eigenvalues of the elliptic boundary value problems below. Namely there exist* *positive smooth functions* v_k *in* B_1^{2k} $(k = 1, 2, 3, 4)$ *and* v_2^* *in* B_1^4 *such that they satisfy*

(2.4)

$$\begin{cases} -\Delta_2 v_1 = \lambda_1 v_1 \quad \text{in } B_1^2, & v_1 = 0 \quad \text{on } \partial B_1^2, \\ \Delta_4^2 v_2 = \lambda_2 v_2 \quad \text{in } B_1^4, & v_2 = \dfrac{d}{dn} v_2 = 0 \quad \text{on } \partial B_1^4, \\ -\Delta_6^3 v_3 = \lambda_3 v_3 \quad \text{in } B_1^6, & v_3 = \dfrac{d}{dn} v_3 = \dfrac{d^2}{dn^2} v_3 = 0 \quad \text{on } \partial B_1^6, \\ \Delta_8^4 v_4 = \lambda_4 v_4 \quad \text{in } B_1^8, & v_4 = \dfrac{d}{dn} v_4 = \dfrac{d^2}{dn^2} v_4 = \dfrac{d^3}{dn^3} v_4 = 0 \quad \text{on } \partial B_1^8, \\ \Delta_4^2 v_2^* = \lambda_2^* v_2^* \quad \text{in } B_1^4, & v_2^* = \Delta_4 v_2^* = 0 \quad \text{on } \partial B_1^4. \end{cases}$$

Here by n *we denote the unit outer normal on* ∂B_1^{2k} $(k = 1, 2, 3, 4)$ *for simplicity.*

Now we are in a position to state our results:

Theorem 2.1. *Suppose* $N > 4$*. Let* Ω *be a bounded domain of* $\mathbb{R}^N$*. Then we have the following two inequalities.*

(1) *For any* $u \in H_0^2(\Omega)$*, it holds that*

$$\int_\Omega |\Delta u|^2\, dx \ge H(N, \Delta) \int_\Omega \frac{|u|^2}{|x|^4}\, dx + \lambda_1 \cdot \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}} \frac{N(N-4)}{2} \int_\Omega \frac{|u|^2}{|x|^2}\, dx + \lambda_2 \cdot \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{4}{N}} \int_\Omega |u|^2\, dx. \tag{2.5}$$

(2) *For any* $u \in H^2(\Omega) \cap H_0^1(\Omega)$*, it holds that*

$$\int_\Omega |\Delta u|^2\, dx \ge H(N, \Delta) \int_\Omega \frac{|u|^2}{|x|^4}\, dx + \lambda_1 \cdot \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}} \frac{N(N-4)}{2} \int_\Omega \frac{|u|^2}{|x|^2}\, dx + \lambda_2^* \cdot \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{4}{N}} \int_\Omega |u|^2\, dx, \tag{2.6}$$

where

$$H(N, \Delta) = \left(\frac{N(N-4)}{4}\right)^2. \tag{2.7}$$

Theorem 2.2. *Suppose* $N > 8$*. Let* Ω *be a bounded domain of* $\mathbb{R}^N$*. Then it holds that for any* $u \in H_0^4(\Omega)$

$$\int_\Omega |\Delta^2 u|^2\, dx \ge H(N, \Delta^2) \int_\Omega \frac{|u|^2}{|x|^8}\, dx + a_1 \cdot \lambda_1 \cdot \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}} \int_\Omega \frac{|u|^2}{|x|^6}\, dx + a_2 \cdot \lambda_2 \cdot \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{4}{N}} \int_\Omega \frac{|u|^2}{|x|^4}\, dx + a_3 \cdot \lambda_3 \cdot \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{6}{N}} \int_\Omega \frac{|u|^2}{|x|^2}\, dx + \lambda_4 \cdot \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{8}{N}} \int_\Omega |u|^2\, dx. \tag{2.8}$$

Here

$$H(N, \Delta^2) = \left(\frac{N(N-4)(N+4)(N-8)}{16} \right)^2. \tag{2.9}$$

By a_1, a_2 and a_3 we denote positive constants defined by

$$\begin{cases} a_1 = \dfrac{1}{16} N^2 (N-4)^2 (N+4)(N-8), \\ a_2 = \dfrac{3}{8} N(N-4)(N+4)(N-8), \\ a_3 = (N+4)(N-8). \end{cases} \tag{2.10}$$

Remark 2.1. The missing terms for the higher order Hardy inequality can not be determined in a unique way, therefore these are considered as canonical forms of the remainder.

In the next we state the results concerned with the weighted Hardy inequalities.

Theorem 2.3. *Suppose that a positive integer N and a real number α satisfy $N + \alpha > 2$. Then it holds that for any $u \in H_0^1(\Omega)$*

$$\begin{aligned} \int_\Omega |\nabla u|^2 |x|^\alpha \, dx \ &\geq H(N, \nabla, \alpha) \int_\Omega |u|^2 |x|^{\alpha-2} \, dx \\ &\quad + \lambda_1 \left(\frac{\omega_N}{|\Omega|} \right)^{\frac{2}{N}} \int_\Omega |u|^2 |x|^\alpha \, dx, \end{aligned} \tag{2.11}$$

where

$$H(N, \nabla, \alpha) = \left(\frac{N-2+\alpha}{2} \right)^2. \tag{2.12}$$

Remark 2.2. When $\alpha = 0$, this result was initially established in [3] by H. Brezis and J. L. Vázquez. They also investigated in [3] fundamental properties of blow-up solutions of some nonlinear elliptic problems.

We also note that when one linearizes the p-laplacian at the singular function such as $\log |x|$, the weighted Hardy inequalities appear in a natural way.

A similar result can be expected for Δ. In fact, the following weighted inequality holds.

Theorem 2.4. *Suppose that a positive integer N and a real number α*

satisfy $N+\alpha>4$. Then it holds that for any $u\in H_0^2(\Omega)$

(2.13)
$$\int_\Omega |\Delta u|^2|x|^\alpha\,dx+\frac{\alpha(\alpha-4)}{2}\int_\Omega\left(|\nabla u|^2-2\left(\frac{x}{|x|}\cdot\nabla u\right)^2\right)|x|^{\alpha-2}\,dx$$
$$\geq I(N,\Delta,\alpha)\int_\Omega |u|^2|x|^{\alpha-4}\,dx+\lambda_1\frac{N(N-4)}{2}\left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}}\int_\Omega |u|^2|x|^{\alpha-2}\,dx$$
$$+\lambda_2\left(\frac{\omega_N}{|\Omega|}\right)^{\frac{4}{N}}\int_\Omega |u|^2|x|^\alpha\,dx,$$

where

$$I(N,\Delta,\alpha)=\left(\frac{N(N-4)}{4}\right)^2-\frac{\alpha(\alpha-4)(\alpha+2N-4)(\alpha+2N-8)}{16}. \tag{2.14}$$

If we further assume either $\alpha\leq 0$ or $\alpha\geq 4$, we have the following.

Corollary 2.1. *Suppose that the same assumptions as in the previous Theorem* 2.4. *Moreover we assume either $\alpha\leq 0$ or $\alpha\geq 4$. Then it holds that for any $u\in H_0^2(\Omega)$*

$$\int_\Omega |\Delta u|^2|x|^\alpha\,dx+\alpha(\alpha-4)\int_\Omega\left(|\nabla u|^2-\left(\frac{x}{|x|}\cdot\nabla u\right)^2\right)|x|^{\alpha-2}\,dx \tag{2.15}$$
$$\geq H(N,\Delta,\alpha)\int_\Omega |u|^2|x|^{\alpha-4}\,dx+b_1\lambda_1\left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}}\int_\Omega |u|^2|x|^{\alpha-2}\,dx$$
$$+\lambda_2\left(\frac{\omega_N}{|\Omega|}\right)^{\frac{4}{N}}\int_\Omega |u|^2|x|^\alpha\,dx,$$

where

$$\begin{cases} H(N,\Delta,\alpha)=\left(\dfrac{N(N-4)}{4}-\dfrac{\alpha(\alpha-4)}{4}\right)^2, \\ b_1=\dfrac{N(N-4)}{2}+\dfrac{\alpha(\alpha-4)}{2}. \end{cases} \tag{2.16}$$

Proof of Corollary 2.1. From Theorem 2.3 we have

$$\int_\Omega |\nabla u|^2|x|^{\alpha-2}\,dx \tag{2.17}$$
$$\geq H(N,\nabla,\alpha-2)\int_\Omega |u|^2|x|^{\alpha-4}\,dx+\lambda_1\left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}}\int_\Omega |u|^2|x|^{\alpha-2}\,dx.$$

We note that $\alpha(\alpha-4)\geq 0$ and

$$I(N,\Delta,\alpha)+\frac{\alpha(\alpha-4)}{2}H(N,\Delta,\alpha-2)=H(N,\Delta,\alpha). \tag{2.18}$$

Then the desired inequality ealily follows from Theorem 2.4. □

In a similar way we have the following.

Corollary 2.2. *Suppose that the same assumptions as in the previous Theorem 2.4. Moreover we assume that $0 \le \alpha \le 4$. Then it holds that for any $u \in H_0^2(\Omega)$*

(2.19)

$$\begin{aligned}&\int_\Omega |\Delta u|^2 |x|^\alpha \, dx + \frac{\alpha(4-\alpha)}{2} \int_\Omega |\nabla u|^2 |x|^{\alpha-2} \, dx \\ &\quad \ge I(N, \Delta, \alpha) \int_\Omega |u|^2 |x|^{\alpha-4} \, dx + \lambda_1 \left(\frac{\omega_N}{|\Omega|} \right)^{\frac{2}{N}} \frac{N(N-4)}{2} \int_\Omega |u|^2 |x|^{\alpha-2} \, dx \\ &\quad + \lambda_2 \left(\frac{\omega_N}{|\Omega|} \right)^{\frac{4}{N}} \int_\Omega |u|^2 |x|^\alpha \, dx.\end{aligned}$$

Proof. It suffices to note that $\alpha(\alpha - 4) < 0$ and $|\nabla u|^2 - (x/|x| \cdot \nabla u)^2 \ge 0$. □

Remark 2.3. In Theorem 2.4 and its corollaries, we can replace the admissible space $H_0^2(\Omega)$ by $H^2(\Omega) \cap H_0^1(\Omega)$. Then the same results hold if we replace λ_2 by λ_2^* as before.

3. Lemmas

In this section we shall prepare fundamental lemmas which are not only needed to prove our results but also very interesting by itself. First we recall the rearrangement of domains and functions. For a domain Ω we define the ball having the same measure as Ω by

$$\Omega^* = \{x \in \mathbb{R}^N : \omega_N |x|^N < |\Omega|\}, \tag{3.1}$$

where by ω_N we denote the measure of a unit ball. If $|\Omega| = +\infty$, we put $\Omega^* = \mathbb{R}^N$. For a measurable function u, we denote by $u^*(x)$ the spherically symmetric decreasing rearrangement of u (the Schwarz symmetrization of u). Namely,

$$\begin{cases} u^*(x) = \inf\{t \ge 0 : \mu(t) < \omega_N |x|^N\} \qquad \text{in} \quad \Omega^*, \\ \mu(t) = |\{x \in \Omega : |u(x)| > t\}|. \end{cases} \tag{3.2}$$

Then it is well-known that

Lemma 3.1. *Under these notations we have for every $p > 0$*

$$\begin{cases} \int_\Omega |u(x)|^p \, dx = \int_{\Omega^*} u^*(x)^p \, dx, \\ \int_\Omega |\nabla u(x)|^p \, dx \ge \int_{\Omega^*} |\nabla u^*(x)|^p \, dx. \end{cases} \tag{3.3}$$

Let $g \in C^0((0,\infty))$ be a nonnegative decreasing function. Then we have

$$\int_\Omega |u(x)|^p g(|x|)\,dx \le \int_{\Omega^*} u^*(x)^p g(|x|)\,dx. \tag{3.4}$$

From this we see in particular that the symmetric rearrangement does not change the L^2-norm and increases the integral $\int_\Omega (|u^2|/|x|^l)\,dx$. The following is due to G. Talenti (See [9]). For the sake of completeness, we give a short proof.

Lemma 3.2 (Talenti). *Let Ω be a domain of $\mathbb{R}^N$. Assume that $N \ge 3$ and $f \in L^p(\Omega)$, where $p = 2N/(N+2)$.*

If a measurable function u is the weak solution to the Dirichlet problem $-\Delta u = f$ in $\Omega, u\big|_{\partial\Omega} = 0$; v is the weak solution to the Dirichlet problem $-\Delta v = |f|^*$ in $\Omega^*, v\big|_{\partial\Omega^*} = 0$; then

$$v \ge |u|^* \text{ pointwise.}$$

Proof. From the hypothesis and Kato's inequality, we see that u satisfies the inequality $-\Delta|u| \le |f|$. Hence $|u|$ is a subsolution of the Dirichlet problem $-\Delta U = |f|$ in Ω, $u\big|_{\partial\Omega} = 0$, and so $|u| \le U$. Therefore we assume $u \ge 0$ and $f \ge 0$ without a loss of generality. Let us set

$$\varphi(s) = \begin{cases} 0 & \text{if} \quad s \le t, \\ \dfrac{s-t}{h} & \text{if} \quad t < s \le t+h, \\ 1 & \text{if} \quad s > t+h. \end{cases} \tag{3.5}$$

Then we see using $\varphi(u)$ as a test function,

$$\frac{1}{h}\int_{\{t<u<t+h\}} |\nabla u|^2\,dx \le \int_{\{t<u\}} f(x)\,dx.$$

By Hölder inequality,

$$\left(\frac{1}{h}\int_{\{t<u<t+h\}} |\nabla u|\,dx\right)^2 \left(\frac{1}{h}|\{t<u<t+h\}|\right)^{-1} \le \int_{\{t<u\}} f(x)\,dx.$$

Then

$$\left(-\frac{d}{dt}\int_{\{t<u\}} |\nabla u|\,dx\right)^2 (\mu'(t))^{-1} \le \int_{\{t<u\}} f(x)\,dx.$$

By the isoperimetric inequality ((2.26); p. 172 in [9] by G. Talenti) we have

$$-\frac{d}{dt}\int_{\{t<u\}} |\nabla u|\,dx \ge N\omega_N^{1/N}\mu(t)^{1-\frac{1}{N}}.$$

So that

$$N^2\omega_N^{2/N}\mu(t)^{2-\frac{2}{N}}(-\mu'(t))^{-1} \le \int_{\{t<u\}} f(x)\,dx.$$

Let us set $f^*(x) = \overline{f}(\omega_N |x|^N)$ and $u^*(x) = \overline{u}(\omega_N |x|^N)$. Note that

$$\int_{\{t<u\}} f(x)\,dx \le \int_0^{\mu(t)} \overline{f}(\sigma)\,d\sigma.$$

For the proof of this, see (2.6b) in [9] for example. Therefore we get

$$\begin{aligned} t &\le \int_0^t N^{-2}\omega_N^{-2/N}\mu(t)^{-2+2/N}(-\mu'(t)) \int_0^{\mu(t)} \overline{f}(\sigma)\,d\sigma\,dt \\ &= \int_{\mu(t)}^{|\Omega|} N^{-2}\omega_N^{-2/N} t^{-2+2/N} \int_0^t \overline{f}(\sigma)\,d\sigma\,dt. \end{aligned}$$

Hence

$$\overline{u}(s) \le \int_s^{|\Omega|} N^{-2}\omega_N^{-2/N} t^{-2+2/N} \int_0^t \overline{f}(\sigma)\,d\sigma\,dt.$$

On the otherhand

$$v(x) = \int_{\omega_N |x|^N}^{|\Omega|} N^{-2}\omega_N^{-2/N} s^{-2+2/N} \int_0^s \overline{f}(\sigma)\,d\sigma\,ds.$$

After all we see

$$u^*(x) = \overline{u}(\omega_N |x|^N) \le v(x).$$

□

Let us set

$$\text{(3.6)} \qquad \begin{cases} I^l(u;\Omega) = \displaystyle\int_\Omega |\Delta^l u|^2\,dx,\, u \in C_0^\infty(\Omega), \\ I^l = \inf\left[I^l(u;\Omega) : u \in C_0^\infty(\Omega), \displaystyle\int_\Omega \frac{|u|^2}{|x|^{2l}}\,dx = 1 \right], \\ I_r^l = \inf\left[I^l(u;\Omega^*) : u \in C_{0,rad}^\infty(\Omega^*), \displaystyle\int_{\Omega^*} \frac{|u|^2}{|x|^{2l}}\,dx = 1 \right]. \end{cases}$$

By $C_{0,rad}^\infty(\Omega^*)$ we denote the set of all spherically symmetric functions $u \in C_0^\infty(\Omega^*)$. Under these preparations, we can show the following:

Lemma 3.3 (Reduction). *Under these notations, it holds that $I^l \ge I_r^l$ for every positive integer l. If Ω is a ball with its center being the origin, then it holds that $I^l = I_r^l$.*

Proof. Let $u \in C_0^\infty(\Omega)$ be nonnegative without a loss of generality. It suffices to show that there is a function $v \in C_{0,rad}^\infty(\Omega^*)$ such that

$$\text{(3.7)} \qquad \frac{I^l(u;\Omega)}{\int_\Omega |u|^2/|x|^{2l}\,dx} \ge \frac{I_r^l(v;\Omega^*)}{\int_{\Omega^*} |v|^2/|x|^{2l}\,dx}.$$

Assume $l = 1$. We put $-\Delta u = f \in C_0^\infty(\Omega)$. From the definition of the decreasing rearrangement, we see that $|f|^*$ is spherically symmetric in Ω^* and

Lipschitz continuous. Let $v \in C^2(\overline{\Omega^*})$ be the unique solution of the Dirichilet problem defined by

$$-\Delta v = |f|^* \quad \text{in } \Omega^*, \qquad v = 0 \quad \text{on } \partial\Omega^*. \tag{3.8}$$

Here we note that v is radial. Then we see from Lemma 3.2 that $u^* \le v$ in Ω^* and

$$\int_\Omega |\Delta u|^2\,dx = \int_\Omega |f|^2\,dx = \int_{\Omega^*} |f|^{*2}\,dx = \int_{\Omega^*} |\Delta v|^2\,dx. \tag{3.9}$$

Further we see that

$$\int_\Omega \frac{|u|^2}{|x|^4}\,dx \le \int_{\Omega^*} \frac{|u^*|^2}{|x|^4}\,dx \le \int_{\Omega^*} \frac{|v|^2}{|x|^4}\,dx. \tag{3.10}$$

Since v can be approximated by elements in $C_0^\infty(\Omega^*)$, we see $I^1 \ge I_r^1$. This proves the assertion when $l = 1$.

Now we assume that $l \ge 2$. Again we choose and fix a smooth nonnegative function $u \in C_0^\infty(\Omega)$ and put $(-\Delta)^l u = f \in C_0^\infty(\Omega)$. Let us set $u_0 = (-\Delta)^{l-1}u$ and $v_0 = (-\Delta)^{l-1}V$. By $V \in C^{2l}(\overline{\Omega^*})$ we denote the unique radial solution of the boundary value problem defined by

$$\begin{cases} (-\Delta)^l V = |f|^* & \text{in} \quad \Omega^*, \\ (-\Delta)^m V = 0 & \text{on} \quad \partial\Omega^* \quad \text{for } m = 0, 1, \dots, l-1. \end{cases} \tag{3.11}$$

In fact it is not difficult to see the solvability of this boundary value problem (See [8] for example). Then $u_0 \in C^\infty(\Omega)$ and $v_0 \in C^2(\overline{\Omega^*})$ satisfy the following equations with homogeneous Dirichlet conditions:

$$\begin{cases} -\Delta u_0 = f & \text{in} \quad \Omega, \\ -\Delta v_0 = |f|^* & \text{in} \quad \Omega^*. \end{cases} \tag{3.12}$$

From Lemma 3.2 we have

$$|u_0|^* \le v_0. \tag{3.13}$$

By $w \in C^{2l-2}(\overline{\Omega^*})$ we denote the unique radial solution of the following:

$$\begin{cases} (-\Delta)^{l-1} w = |u_0|^* & \text{in} \quad \Omega^*, \\ (-\Delta)^m w = 0 & \text{on} \quad \partial\Omega^* \quad \text{for } m = 0, 1, \dots, l-2. \end{cases} \tag{3.14}$$

Now we claim that

$$u^* \le w \tag{3.15}$$

We prove this inductively. If $l = 2$, this follows from Lemma 3.2. Assume that (3.15) holds for $l \le k$, where $k \ge 2$. We consider the case that $l = k+1$. We set $(-\Delta)^{k-1}u = \tilde{u}$ and $(-\Delta)^{k-1}w = \tilde{w}$. Then we see

$$\begin{cases} -\Delta \tilde{u} = u_0 \quad \text{in } \Omega & \tilde{u} = 0 \quad \text{on } \partial\Omega, \\ -\Delta \tilde{w} = |u_0|^* \quad \text{in } \Omega^*, & \tilde{w} = 0 \quad \text{on } \partial\Omega^*. \end{cases} \tag{3.16}$$

Therefore we have $|\tilde{u}|^* \le \tilde{w}^* = \tilde{w}$. By the assumption of induction and maximum principle we see $u^* \le w$ with replacing u_0 by $\tilde{u}$.

Here we recall that V satisfies

$$\begin{cases} (-\Delta)^{l-1} V = v_0 & \text{in} \quad \Omega^*, \\ (-\Delta)^m V = 0 & \text{on} \quad \partial\Omega^* \quad \text{for } m = 0, 1, \ldots, l-2. \end{cases} \tag{3.17}$$

Since $|u_0|^* \le v_0$, by making use of the maximum principle $l-1$ times we also see $w \le V$ so that we have $u^* \le V$. As before we see

$$\begin{cases} \displaystyle\int_\Omega |\Delta^l u|^2\, dx = \int_{\Omega^*} |\Delta^l V|^2\, dx, \\ \displaystyle\int_\Omega \frac{|u|^2}{|x|^{2l}}\, dx \le \int_{\Omega^*} \frac{|u^*|^2}{|x|^{2l}}\, dx \le \int_{\Omega^*} \frac{|V|^2}{|x|^{2l}}\, dx, \end{cases} \tag{3.18}$$

and this proves Lemma 3.3. □

4. Proof of Theorems 2.1 and 2.2

We begin with the definition:

Definition 4.1 (m Laplacian). For $m \in \mathbb{R}$ and $v \in C^2((0,\infty))$, we set

$$\delta_m v(r) = r^{1-m} \frac{\partial}{\partial r}\left(r^{m-1} \frac{\partial}{\partial r} v(r) \right) = \frac{\partial^2 v(r)}{\partial r^2} + \frac{m-1}{r} \frac{\partial v(r)}{\partial r}. \tag{4.1}$$

Then we can show

Lemma 4.1. *Let M and m be positive integers. Let us set $r = |x|$ for $x \in \mathbb{R}^M$. For $\alpha \in \mathbb{R}$ and $v \in C^\infty((0,\infty))$ it holds that*

$$\Delta_M v(r) = \delta_M v(r),$$
$$\Delta_M^m (r^\alpha v(r)) = r^\alpha \left(\delta_{M+2\alpha} + \frac{\alpha(M+\alpha-2)}{r^2} \right)^m v(r).$$

Proof of Theorem 2.1. Since the assertion (2) follows in a quite similar way, we prove the assertion (1) only. From Lemma 3.3, it is enough to prove the result in the symmetric case. To this end we set

$$\omega_N R^N = |\Omega| \tag{4.2}$$

and replace Ω by Ω^*. In addition to this fact, since $C_0^\infty(\Omega)$ is densely contained in $H_0^2(\Omega)$, we also replace the function space $H_0^2(\Omega)$ by $C_{0,rad}^\infty(\Omega^*)$. Moreover, a simple scaling allows to consider the case $R = 1$.

Let us set for $u \in C_{0,rad}^\infty(B)$

$$u = r^{2-\frac{N}{2}} v, \qquad v \in C_{0,rad}^\infty(B). \tag{4.3}$$

Here we note that v and its derivatives vanish at the origin, if $N > 4$. We see from Lemma 4.1 with $\alpha = 2 - N/2$ that

$$(4.4) \qquad \Delta(r^{2-\frac{N}{2}} v(r)) = r^{2-\frac{N}{2}} \left(\delta_4 v(r) + Q \frac{v(r)}{r^2} \right), \qquad Q = -\frac{N(N-4)}{4}.$$

Then

(4.5)

$$\begin{aligned}
&\int_B |\Delta u|^2 \, dx = \int_B |\Delta(r^{2-\frac{n}{2}} v)|^2 \, dx \\
&= |S^{N-1}| \int_0^1 \left(\delta_4 v + \frac{Q}{r^2} v \right)^2 r^3 \, dr \quad \text{(Polar coordinate)} \\
&= |S^{N-1}| \int_0^1 \left(|\delta_4 v|^2 - \frac{2Q}{r^2} |\partial_r v|^2 + \frac{Q^2}{r^4} v^2 \right) r^3 \, dr \\
&= \frac{|S^{N-1}|}{|S^3|} \int_{B_1^4} |\Delta v(|y|)|^2 \, dy - \frac{2Q|S^{N-1}|}{|S^2|} \int_{B_1^2} |\nabla_2 v(|y|)|^2 \, dy + Q^2 \int_B \frac{v(|y|)^2}{r^N} \, dy.
\end{aligned}$$

Here by $|S^{M-1}|$ we denote the measure of the M-dimensional unit sphere. Then it holds that

(4.6)

$$\begin{aligned}
&\int_B |\Delta u|^2 \, dx = \int_B |\Delta(r^{2-\frac{N}{2}} v)|^2 \, dx \\
&\geq \lambda_2 \frac{|S^{N-1}|}{|S^3|} \int_{B_1^4} |v(|y|)|^2 \, dy - 2Q\lambda_1 \frac{|S^{N-1}|}{|S^2|} \int_{B_1^2} |v(|y|)|^2 \, dy + Q^2 \int_B \frac{v(|y|)^2}{r^N} \, dy \\
&\geq H(N, \Delta) \int_B \frac{|u|^2}{|x|^4} \, dx + \lambda_1 \cdot \frac{N(N-4)}{2} \int_B \frac{|u|^2}{|x|^2} \, dx + \lambda_2 \cdot \int_B |u|^2 \, dx,
\end{aligned}$$

where λ_1 and λ_2 are defined in (2.3). This proves the assertion. □

Remark 4.1. To prove the assertion (1), it suffices to replace $C_0^\infty(\Omega)$ by $H^2(\Omega) \cap C_0^1(\Omega)$.

5. Proof of Theorem 2.2

Again from Lemmas 3.2 and 3.3, it is enough to prove the result in the symmetric case. Let us set for $B = B_1^N(0)$ and $u \in C_{0,rad}^\infty(B)$

$$(5.1) \qquad u = r^{4-\frac{N}{2}} v, \quad v \in C_{0,rad}^\infty(B).$$

Here we note that v and its derivatives vanish at the origin, if $N > 8$. We see from Lemma 4.1 with $\alpha = 4 - N/2$ that

$$(5.2) \quad \Delta(r^{4-\frac{N}{2}} v(r)) = r^{4-\frac{N}{2}} \left(\delta_8 v(r) + P \frac{v(r)}{r^2} \right), \qquad P = -\frac{(N+4)(N-8)}{4}.$$

As before we see

$$\int_B |\Delta^2 u|^2\,dx = \int_B |\Delta^2(r^{4-\frac{N}{2}}v)|^2\,dx$$
$$= |S^{N-1}| \int_0^1 \left| \left(\delta_8 + \frac{P}{r^2}\right)^2 v(r)\right|^2 r^7\,dr \quad \text{(Polar coordinate)}$$
$$= |S^{N-1}| \int_0^1 \left(\delta_8^2 v(r) + \frac{2P}{r^2}\delta_6 v(r) + \frac{S}{r^4}v(r)\right)^2 r^7\,dr,$$

where

$$S = \frac{N(N-4)(N+4)(N-8)}{16} = H(N,\Delta^2)^{\frac{1}{2}}. \tag{5.3}$$

Integration by parts gives

Lemma 5.1. *For any $v \in C_0^\infty((0,1))$, we have*

$$\begin{aligned}&\int_0^1 \left(\delta_8^2 v + \frac{2P}{r^2}\delta_6 v + \frac{S}{r^4}v\right)^2 r^7\,dr\\ &= \int_0^1 |\delta_8^2 v|^2 r^7\,dr + S^2\int_0^1 \frac{v^2}{r}\,dr\\ &\quad + a_1\int_0^1 |\partial_r v|^2 r\,dr + a_2\int_0^1 |\delta_4 v|^2 r^4\,dr + a_3\int_0^1 |\partial_r \delta_6 v|^2 r^5\,dr.\end{aligned} \tag{5.4}$$

Here a_1, a_2 and a_3 are defined by (2.10).

Proof. First we have

$$\left(\delta_8^2 v + \frac{2P}{r^2}\delta_6 v + \frac{S}{r^4}v\right)^2 r^7$$
$$= r^7(\delta_8^2 v)^2 + S^2\frac{v^2}{r} + \frac{a_3^2}{4}r^3(\delta_6 v)^2 - a_3 r^5\delta_8^2 v\cdot\delta_6 v + 2Sr^3\delta_8^2 v\cdot v - Sa_3 r\delta_6 v\cdot v.$$

When we integrate the both sides on the interval $(0,1)$, the each term is calculated as follows.

$$J_1 = \int_0^1 r^3(\delta_6 v)^2\,dr = \int_0^1 [r^3(\partial_r^2 v)^2 + 15r(\partial_r v)^2]\,dr \tag{5.5}$$

$$\begin{aligned}J_2 &= \int_0^1 r^5\delta_8^2 v\cdot\delta_6 v\,dr\\ &= -\int_0^1 [r^5(\partial_r^3 v)^2 + 23r^3(\partial_r^2 v)^2 + 165r(\partial_r v)^2]\,dr\end{aligned} \tag{5.6}$$

$$J_3 = \int_0^1 r^3\delta_8^2 v\cdot v\,dr = \int_0^1 [r^3(\partial_r^2 v)^2 - 5r(\partial_r v)^2]\,dr \tag{5.7}$$

$$J_4 = -\int_0^1 r\delta_6 v\cdot v\,dr = \int_0^1 r(\partial_r v)^2\,dr \tag{5.8}$$

Then we have

$$
\begin{aligned}
(5.9)\qquad &\int_0^1 \left(\delta_8^2 v + \frac{2P}{r^2}\delta_6 v + \frac{S}{r^4}v\right)^2 r^7\,dr \\
&= \int_0^1 \left((\delta_8^2 v)^2 + S^2\frac{v^2}{r^8}\right) r^7\,dr + \frac{a_3^2}{4}J_1 - a_3 J_2 + 2SJ_3 + Sa_3J_4 \\
&= \int_0^1 \left((\delta_8^2 v)^2 + S^2\frac{v^2}{r^8}\right) r^7\,dr \\
&\quad + b_1\int_0^1 r(\partial_r v)^2\,dr + b_2\int_0^1 r^3(\partial_r^2 v)^2\,dr + a_3\int_0^1 r^5(\partial_r^3 v)^2\,dr.
\end{aligned}
$$

Here,

$$
(5.10)\qquad \begin{cases} b_1 = 45a_3 + \dfrac{1}{16}a_3 N(N-4)(N^2-4N+18), \\ b_2 = 15a_3 + \dfrac{3}{8}a_3 N(N-4), \\ a_3 = (N+4)(N-8). \end{cases}
$$

Putting

$$
(5.11)\qquad \begin{cases} b_1 = a_1 + 3b_2, \\ b_2 = a_2 + 15a_3, \end{cases}
$$

we have

(5.12)

$$
\begin{aligned}
&\int_0^1 \left(\delta_8^2 v + \frac{2P}{r^2}\delta_6 v + \frac{S}{r^4}v\right)^2 r^7\,dr \\
&= \int_0^1 (\delta_8^2 v)^2 r^7\,dr + a_1\int_0^1 r(\partial_r v)^2\,dr + a_2\int_0^1 [r^3(\partial_r^2 v)^2 + 3r(\partial_r v)^2]\,dr \\
&\quad + a_3\int_0^1 [r^5(\partial_r^3 v)^2 + 15r^3(\partial_r^2 v)^2 + 45r(\partial_r u)^2]\,dr + \int_0^1 S^2\frac{v^2}{r}\,dr.
\end{aligned}
$$

Here

$$
(5.13)\qquad \begin{cases} a_1 = \dfrac{1}{16}N^2(N-4)^2(N+4)(N-8), \\ a_2 = \dfrac{3}{8}N(N-4)(N+4)(N-8). \end{cases}
$$

□

Now we prepare the following:

Lemma 5.2. *For any $\alpha \in \mathbb{R}$ and any $v \in C_0^4((0,1))$ we have*

$$\int_0^1 \left(\partial_r^2 v + \frac{\alpha}{r}\partial_r v\right)^2 r^3\, dr = \int_0^1 [r^3(\partial_r^2 v)^2 + (\alpha(\alpha-2)r(\partial_r v)^2]\, dr, \tag{5.14}$$

$$\begin{aligned}&\int_0^1 \left(\partial_r\left(\partial_r^2 + \frac{\alpha}{r}\partial_r\right)v\right)^2 r^5\, dr \\ &\quad = \int_0^1 [r^5(\partial_r^3 v)^2 + \alpha(\alpha-2)r^3(\partial_r^2 v)^2 + 3\alpha(\alpha-2)r(\partial_r v)^2]\, dr.\end{aligned} \tag{5.15}$$

The end of proof of Theorem 2.2. From the previous lemma, we see

$$\begin{aligned}\int_B |\Delta^2 u|^2\, dx &= S^2 \int_B \frac{v(|y|)^2}{|y|^N}\, dy + a_1 \frac{|S^{N-1}|}{|S^1|}\int_{B_1^2} |\nabla_2 v(|y|)|^2\, dy \\ &\quad + a_2 \frac{|S^{N-1}|}{|S^3|}\int_{B_1^4} |\Delta_4 v(|y|)|^2\, dy + a_3 \frac{|S^{N-1}|}{|S^5|}\int_{B_1^6} |\nabla_6 \Delta_6 v(|y|)|^2\, dy \\ &\quad + \frac{|S^{N-1}|}{|S^7|}\int_{B_1^8} |\Delta_8^2 v(|y|)|^2\, dy \\ &\geq S^2 \int_B \frac{v(|y|)^2}{|y|^N}\, dy + a_1\lambda_1 \frac{|S^{N-1}|}{|S^1|}\int_{B_1^2} |v(|y|)|^2\, dy \\ &\quad + a_2\lambda_2 \frac{|S^{N-1}|}{|S^3|}\int_{B_1^4} |v(|y|)|^2\, dy + a_3\lambda_3 \frac{|S^{N-1}|}{|S^5|}\int_{B_1^6} |v(|y|))|^2\, dy \\ &\quad + \lambda_4 \frac{|S^{N-1}|}{|S^7|}\int_{B_1^8} |v(|y|)|^2\, dy \\ &= H(N,\Delta^2)\int_B \frac{u^2}{|x|^8}\, dx + a_1\lambda_1 \int_B \frac{|u|^2}{|x|^6}\, dx \\ &\quad + a_2\lambda_2 \int_B \frac{|u|^2}{|x|^4}\, dx + a_3\lambda_3 \int_B \frac{|u|^2}{|x|^2}\, dy + \lambda_4 \int_B |u|^2\, dx.\end{aligned}$$

This proves the assertion. □

6. Proofs of Theorems 2.3 and 2.4

First we prepare two elementary lemmas.

Lemma 6.1. *Let Ω be a domain of $\mathbb{R}^N$. Assume that $u \in C_0^\infty(\Omega)$ and $f \in C^2(\Omega)$. Then it holds that*

$$\int_\Omega |\nabla(uf)|^2\, dx = \int_\Omega |\nabla u|^2 f\, dx - \frac{1}{2}\int_\Omega u^2(\Delta(f^2) - 2|\nabla f|^2)\, dx. \tag{6.1}$$

Proof. Integration by parts leads us to obtain (6.1). □

Lemma 6.2. *Let Ω be a domain of $\mathbb{R}^N$. Assume that $u \in C_0^\infty(\Omega)$ and*

$f \in C^4(\Omega)$. *Then it holds that*

$$\int_\Omega |\Delta(uf)|^2\,dx = \int_\Omega (|\Delta u|^2 f^2 + \int_\Omega u^2 f \Delta^2 f)\,dx \\ + 2\int_\Omega \left(|\nabla u|^2 |\nabla f|^2 - 2f \sum_{j,k=1}^N \frac{\partial^2 f}{\partial x_j \partial x_k} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k} \right) dx. \tag{6.2}$$

Proof. First we see

$$|\Delta(uf)|^2 = f^2(\Delta u)^2 + u^2(\Delta f)^2 + 4(\nabla u \cdot \nabla f)^2 \\ + 2uf\Delta u \Delta f + 4f\Delta u(\nabla u \cdot \nabla f) + 4u\Delta f(\nabla u \cdot \nabla f). \tag{6.3}$$

Then integration by parts gives us

$$\int_\Omega uf\Delta u\Delta f\,dx = -\int_\Omega \nabla u \cdot \nabla(uf\Delta f)\,dx \\ = -\int_\Omega |\nabla u|^2 f\Delta f + \frac{1}{2}\int_\Omega u^2 \Delta(f\Delta f)\,dx, \tag{6.4}$$

$$\int_\Omega u\Delta f(\nabla u \cdot \nabla f)\,dx = -\frac{1}{2}\int_\Omega u^2 \operatorname{div}(\Delta f \nabla f)\,dx \\ = -\frac{1}{2}\int_\Omega u^2 \left((\Delta f)^2 + \nabla(\Delta f)\cdot\nabla f\right)\,dx, \tag{6.5}$$

$$\int_\Omega f\Delta u(\nabla u \cdot \nabla f)\,dx = -\int_\Omega \nabla u \cdot \nabla\left(f(\nabla u \cdot \nabla f)\right)\,dx \\ = -\int_\Omega (\nabla f \cdot \nabla u)^2\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2(|\nabla f|^2 + f\Delta f)\,dx \\ - \sum_{j,k=1}^N \int_\Omega f\partial^2_{j,k} f \partial_j u \partial_k u\,dx. \tag{6.6}$$

Using these formula we can easily show the assertion. □

Proof of Theorem 2.3. From this the proof of Theorem 2.3 is reduced to the case $\alpha = 0$, which was established by H. Brezis and J. J. Vazquez in [3]. In fact, for $f = |x|^{\alpha/2}$, we have

$$\int_\Omega |\nabla u|^2 |x|^\alpha\,dx = \frac{\alpha(\alpha+2N-4)}{4}\int_\Omega |u|^2|x|^{\alpha-2}\,dx + \int_\Omega |\nabla(u|x|^{\frac{\alpha}{2}})|^2\,dx. \tag{6.7}$$

Here we note that the proof of Lemma 6.1 still works for this weight f, since $N + \alpha > 2$. Then we can apply the inequality (2.11) with a parameter α being 0, and we obtain

$$\int_\Omega |\nabla(u|x|^{\frac{\alpha}{2}})|^2\,dx \\ \geq \frac{(N-2)^2}{4}\int_\Omega |u|^2|x|^{\alpha-2}\,dx + \lambda_1\left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}}\int_\Omega |u|^2|x|^\alpha\,dx. \tag{6.8}$$

The desired inequality follows from this and (6.7). □

Proof of Theorem 2.4. We put $f = |x|^{\alpha/2}$ for $N+\alpha > 4$ and apply Lemma 6.2. Then we have

$$\int_\Omega |\Delta(u|x|^{\frac{\alpha}{2}})|^2\,dx$$
$$(6.9)\qquad = \int_\Omega (\Delta u)^2|x|^\alpha\,dx + \frac{\alpha(\alpha-4)}{2}\int_\Omega \left(|\nabla u|^2 - 2\left|\frac{du}{dr}\right|^2\right)|x|^{\alpha-2}\,dx$$
$$+ \frac{\alpha(\alpha-4)(\alpha+2N-4)(\alpha+2N-8)}{16}\int_\Omega u^2|x|^{\alpha-4}\,dx,$$

where

$$(6.10)\qquad \frac{du}{dr} = \frac{x}{|x|}\cdot\nabla u.$$

Then we apply Theorem 2.1 to $u|x|^{\alpha/2}$ and obtain

(6.11)
$$\int_\Omega |\Delta(u|x|^{\frac{\alpha}{2}})|^2\,dx \geq H(N,\Delta)\int_\Omega |u|^2|x|^{\alpha-4}\,dx$$
$$+ \lambda_1\cdot\left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}}\frac{N(N-4)}{2}\int_\Omega |u|^2|x|^{\alpha-2}\,dx + \lambda_2\cdot\left(\frac{\omega_N}{|\Omega|}\right)^{\frac{4}{N}}\int_\Omega |u|^2|x|^\alpha\,dx.$$

Combining this with (6.9) we have the desired inequality. □

7. Applications

Let Ω be a bounded domain of $\mathbb{R}^N$. In connection with combustion theory and other applications, many authors have been studied positive solutions of the semi-linear elliptic boundary value problem defined by

$$(7.1)\qquad -\Delta u = \lambda f(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Here λ is a nonnegative parameter, and the nonlinearity f is, roughly speaking, continuous, positive, increasing, superlinear and convex function. A typical example is $f(u) = e^u$. It is well-known that there is a finite number λ^* such that (7.1) has a classical positive solution $u \in C^2(\overline{\Omega})$ if $0 < \lambda < \lambda^*$. On the other hand no solution exists, even in the weak sense, for $\lambda > \lambda^*$. This value λ^* is often called the extremal value and solutions for this extremal value are called extremal solutions. It has been a very interesting problem to find and study the properties of these extremal solutions. In this section we shall consider a similar problem for the fourth order equations.

Let B be a unit ball of $\mathbb{R}^N$. Let $f(t,r)$ be a continuous positive function defined for $t \in [0,+\infty)$ and $r \in [0,1]$. Moreover we assume that $f(\cdot,r)$ is increasing and strictly convex with

$$(7.2)\qquad f(0,r) > 0 \quad \text{and} \lim_{t\to\infty}\frac{f(t,r)}{t} = \infty \qquad \text{uniformly in } r \in [0,1].$$

Now we consider the boundary value problem: For $r = |x|$

$$\begin{cases} \Delta^2 u = \lambda f(u,r) & \text{in} \quad B, \\ u = \Delta u = 0, & \text{on} \quad \partial B. \end{cases} \tag{7.3}$$

This problem is a generalization of (7.1). First we define a weak solution of the problem (7.3).

Definition 7.1 (Weak solution of (7.3)). Let us set $\delta(x) = \text{dist}(x, \partial B)$ (the distance to the boundary from x). A function $u \in L^1(B)$ is called a weak solution of (7.3) if $f(u, |x|)$ satisfy

$$\delta(x) f(u, |x|) \in L^1(B) \tag{7.4}$$

and u satisfies (7.3) in the following weak sense:

$$\int_B (u\Delta^2\varphi - \lambda f(u,r)\varphi)\, dx = 0 \tag{7.5}$$

for all $\varphi \in C^4(\overline{B})$ with $\varphi = \Delta\varphi = 0$ on ∂B.

From the standard elliptic regularity theory it follows that bounded weak solutions for this problem are classical solutions. Moreover u satisfies the boundary conditions $u = \Delta u = 0$ in this case. Now we consider unbounded solutions. To this end we introduce an energy solution and a singular energy solution.

Definition 7.2 (Energy solution, singular energy solution). A weak solution u of (7.3) is said to be an energy solution if $u \in H^2(B) \cap H_0^1(B)$. If an energy solution u is not bounded, u is said to be singular.

Remark 7.1. Later we shall specify the nonlinearity $f(u,r)$ in order to study singular extremal solutions precisely. From the definition, an energy solution u satisfies

$$\int_B (\Delta u \Delta\varphi - \lambda f(u, |x|)\varphi)\, dx = 0 \tag{7.6}$$

for all $\varphi \in C^2(\overline{B})$ with $\varphi = \Delta\varphi = 0$ on ∂B.

If $u \in H^4(B)$ and u is an energy solution of (7.3), then u satisfies the boundary conditions $u = \Delta u = 0$.

Let $u \in H^4(B)$ be an energy solution of (7.3), and we set $-\Delta u = v$. Then we see $v \in H^2(B) \cap H_0^1(B)$ solves

$$\begin{cases} -\Delta v = \lambda f(u,r) & \text{in} \quad B, \\ v = 0 & \text{on} \quad \partial B. \end{cases} \tag{7.7}$$

From the maximum principle for the second order elliptic equation, we see v is nonnegative. As a result we have $u \geq 0$, since $u \in H^2(B) \cap H_0^1(B)$ solves

$$\begin{cases} -\Delta u = v & \text{in} \quad B, \\ u = 0 & \text{on} \quad \partial B. \end{cases} \tag{7.8}$$

In other words, the maximum principle works in this boundary value problem even if the operator is of the fourth order. Therefore we can show that there exists a solution to (7.3) for sufficiently small $\lambda > 0$. In fact we can construct so-called supersolution and subsolution of (7.3) as follows.

Lemma 7.1. *There exist a supersolution and a subsolution of* (7.3) *for a sufficiently small* $\lambda > 0$*. Moreover there exists at least one classical solution* u *of* (7.3)*.*

Proof. Let λ_0 and φ_0 be the first eigenvalue and nonnegative eigenfunction of the operator Δ^2 under the boundary conditions $\varphi_0 = \Delta\varphi_0 = 0$ on ∂B respectively. For $\epsilon > 0$, we set $\psi(x) = \varphi_0(x) + \epsilon(r^2 - 1)^4$. Then we see $\psi = \Delta\psi = 0$ on ∂B. Since $\varphi_0 > 0$ on B and $\Delta^2(r^2-1)^4 > 0$ on ∂B, it holds that $\Delta^2\psi = \lambda_0\varphi_0 + \epsilon\Delta^2(r^2-1)^4 > 0$ on $\overline{B}$ for a sufficiently small $\epsilon > 0$. Therefore for a small $\lambda > 0$ we see

$$\Delta^2\psi \geq \lambda f(\psi, r). \tag{7.9}$$

Then ψ becomes a supersolution. As a subsolution it suffices to take $u = 0$. Then from the method of nonlinear iteration, we can show the existence of a classical solution. □

By virtue of this, we can define the mimimal solution $u_\lambda \in C^4(\overline{B})$ which is minimal among all possible solutions. Then we define the extremal value λ^* as a upper bound of λ for which the minimal solution exists. The family of such solutions depends smoothly and monotonically on λ. Then the following property is well known.

Lemma 7.2. *Minimal solutions are stable. More precisely, the linearized operator*

$$L_\lambda\varphi = \Delta^2\varphi - \lambda f'(u_\lambda, r)\varphi \tag{7.10}$$

has a positive first eigenvalue for all $0 < \lambda < \lambda^*$*.*

We also have

Lemma 7.3. *As* $\lambda \uparrow \lambda^*$*, a finite limit* a.e. $u^*(x) = \lim_{\lambda\uparrow\lambda^*} u_\lambda(x)$ *exists, where* u^* *is a weak solution of* (7.3) *with* $\lambda = \lambda^*$*.*

Proof. It follows from (7.2) that there is C such that $f(u, r) \geq (2\mu_1/\lambda^*)u - C$, for all $u \geq 0$. Here μ_1 is the first eigenvalue of $(-\Delta)^2$ in $H^2(B)$ with boundary conditions $u = \Delta u = 0$ on ∂B, and let φ_1 be a corresponding eigenfunction. Multiplying (7.3) by φ_1, we obtain

$$\lambda \int_B f(u_\lambda, r)\varphi_1 \, dx = \mu_1 \int_B u_\lambda \varphi_1 \, dx \leq \frac{\lambda^*}{2} \int_B \big(f(u_\lambda, r) + C\big)\varphi_1 \, dx. \tag{7.11}$$

Letting $\lambda \uparrow \lambda^*$, we get

$$\lim_{\lambda \uparrow \lambda^*} \int_B f(u_\lambda, r)\varphi_1 \, dx < \infty. \tag{7.12}$$

Let ψ satisfy $\Delta^2 \psi = 1$ in B with $\psi = \Delta\psi = 0$ on ∂B. Multiplying now (7.3) by ψ, we obtain for some positive number C

$$\int_B u_\lambda \, dx = \lambda \int_B f(u_\lambda, r)\psi \, dx \leq C\lambda \int_B f(u_\lambda, r)\varphi_1 \, dx. \tag{7.13}$$

Hence u_λ is bounded in $L^1(B)$. Since u_λ is increasing on λ, it follows that u_λ has a limit $u^* \in L^1(B)$ and that $\delta(x)f(u_\lambda, r)$ converges to $\delta(x)f(u^*, r) \in L^1(B)$. Then it follows that u^* is a weak solution of (7.3) with $\lambda = \lambda^*$. □

Remark 7.2. From these lemmas, it holds that for any $\lambda \in (0, \lambda^*]$

$$\int_B \lambda f'(u_\lambda, r)\varphi^2 \, dx \leq \int_B |\Delta\varphi|^2 \, dx, \qquad \varphi \in C_0^2(B). \tag{7.14}$$

The limit u^* can be classical or singular. If u^* is classical, then it is clear from the implicit function theorem that the linearized operator

$$L_{\lambda^*}\varphi = \Delta^2\varphi - f'(u^*, r)\varphi \tag{7.15}$$

has zero first eigenvalue.

The following characterizes singular solutions to some extent:

Proposition 7.1. *Assume that $u \in H^2(B) \cap H_0^1(B)$ is an unbounded weak solution of (7.3) for some $\lambda > 0$. Assume that*

$$\lambda \int_B f'(u, r)\varphi^2 \, dx \leq \int_B |\Delta\varphi|^2 \, dx \tag{7.16}$$

for all $\varphi \in C_0^2(B)$. Then $\lambda \geq \lambda^*$.

Conversely, if $\lambda = \lambda^*$ and $u = u^*$, then (7.16) holds.

Remark 7.3. In the first assertion, we can not conclude $\lambda = \lambda^*$ so far. But in the examples below we have $\lambda = \lambda^*$ and we can determine exactly the singular extremal solutions with somewhat more consideration.

Proof. First we assume that u is a unbounded energy solution satisfying (7.16). Assume that $\lambda < \lambda^*$. Then we have

$$\begin{aligned}\lambda \int_B f'(u,r)(u-u_\lambda)^2\,dx &\le \int_B |\Delta(u-u_\lambda)|^2\,dx \\ &= \lambda \int_B (f(u,r)-f(u_\lambda,r))(u-u_\lambda)\,dx.\end{aligned} \tag{7.17}$$

Hence we have

$$\lambda \int_B \big(f(u,r)-f(u_\lambda,r)-\lambda f'(u,r)(u-u_\lambda)\big)(u-u_\lambda)\,dx \ge 0. \tag{7.18}$$

Since $f(\cdot,r)$ is convex the integrand is nonpositive, so that the inequality is only possible if

$$f(u,r) = f(u_\lambda,r) + f'(u,r)(u-u_\lambda) \quad \text{a.e. in } B. \tag{7.19}$$

Since f is strictly convex, we see that $u = u_\lambda$, hence u is the minimal solution, which is a contradiction. Hence $\lambda \ge \lambda^*$ holds.

Now we assume that $\lambda = \lambda^*$ and $u = u^*$. Then (7.16) clearly holds. In fact u^* is a monotone limit of a sequence of minimal solutions $\{u_\lambda\}$. The assertion follows from the monotone convergence theorem. □

Remark 7.4. If $f(u,r)$ satisfies

$$\liminf_{t\to\infty} \frac{f'(t,r)t}{f(t,r)} > 1 \qquad (\text{uniformly in } r \in [0,1]), \tag{7.20}$$

then any extremal solution u^* lies in the energy class (cf. Section 3 in [3]).

Now we consider the concrete example for which we can apply our refined Hardy inequalities. For $1 < p < \infty$ and $r = |x|$, we adopt as the nonlinearity $f(u,r)$ the following f_p and f_e, that is,

$$\begin{cases} f_p(u,r) = (1+u+Q_p(r))^p, \\ f_e(u,r) = e^{u+Q_e(r)}. \end{cases} \tag{7.21}$$

Here

$$\begin{cases} Q_p(r) = \beta(1-r^2), \\ \lambda_N(p) = \alpha(\alpha-2)(N+\alpha-2)(N+\alpha-4), \\ \alpha = -\dfrac{4}{p-1}, \quad \beta = \dfrac{2(N-2)}{N(p-1)^2}\left(p - \dfrac{N+2}{N-2}\right). \end{cases} \tag{7.22}$$

We define the function U_p as follows:

$$U_p(r) = r^\alpha - 1 - Q_p(r), \qquad \alpha = -\frac{4}{p-1}. \tag{7.23}$$

Under these notations, we have the following.

Lemma 7.4. *Assume that $\lambda = \lambda_N(p)$ and $f = f_p$. Then it holds that:*
(1) *If $p > N/(N-4)$, then U_p is a weak solution of* (7.3).
(2) *If $p > (N+4)/(N-4)$, then U_p is a singular energy solution of* (7.3).
(3) *If $p > N/(N-8)$, then $U_p \in H^4(B)$.*

Now we define

$$H(p) = p\lambda_N(p). \tag{7.24}$$

Since it holds that

$$\lim_{p\to+\infty} H(p) = 8(N-2)(N-4), \tag{7.25}$$

we see $\lim_{p\to+\infty} H(p) < (N(N-4)/4)^2$ (the best constant of the Hardy inequality) if and only if $N \geq 13$. For $N > 4$ we also note that $H(N-4)/(N+4) > (N(N-4)/4)^2$ and that $H(p)$ is monotonously decreasing for $p \geq (N-4)/(N+4)$. Then the results of Section 2 (Theorem 2.1 and the related proposition) allow us to study the singular energy solutions. First we have

Theorem 7.1 (Polynomial case). *Assume that $N \geq 13$.*

(1) *There exists a number $p^* \in ((N+4)/(N-4), \infty)$ such that U_p is a singular extremal solution with $\lambda^* = \lambda_N(p)$ for any $p \geq p^*$.*

(2) *If $p \in ((N+4)/(N-4), p^*)$, the U_p is not a singular extremal solution and $\lambda_N(p) < \lambda^*$. Here p^* is the same number in* (1).

(3) *If $p \in (4/(N-4), (N+4)/(N-4)]$, U_p is not an energy solution but a weak solution. Therefore U_p is not singular extremal and $\lambda_N(p) < \lambda^*$.*

Proof. It suffices to show the assertion (1). From the arguement just before this theorem, $p^* \in ((N+4)/(N-4), \infty)$ is geven as the unique solution of the equation $H(p) = (N(N-4)/4)^2$. Since U_p is singular and satisfies (7.16) with $u = U_p$ in this case, from Propositon 7.1 it follows that $\lambda_N(p) \geq \lambda^*$. Hence we have only to show $\lambda_N(p) \leq \lambda^*$. This follows from the same arguement in [2] (Theorem 3) replacing Lemma 4 for the next one. □

For a positive small number ε set

$$\begin{cases} U_p^\varepsilon(r) = g(r) - 1 - Q_p^\varepsilon(r), \\ f_p^\varepsilon(u,r) = (1 + u + Q_p^\varepsilon(r))^p, \end{cases} \tag{7.26}$$

where

$$\begin{cases} g(r) = (\varepsilon + (1-\varepsilon)r^2)^{\frac{2}{1-p}}, \\ Q_p^\varepsilon(r) = \beta(\varepsilon)(1-r^2), \\ \beta(\varepsilon) = \dfrac{2(N-2)(1-\varepsilon)}{N(p-1)^2}\left(p + \dfrac{2\varepsilon(1+p)}{N-2} - \dfrac{N+2}{N-2}\right). \end{cases} \tag{7.27}$$

Lemma 7.5. *For any $\delta > 0$ there is a positive number ε_0 such that for any $\varepsilon \in (0, \varepsilon_0)$*

$$\begin{cases} \Delta^2 U_p^\varepsilon(r) \geq (1-\delta)\lambda_N(p) f_p(U_p^\varepsilon(r), r) & in \quad B, \\ U_p^\varepsilon = \Delta U_p^\varepsilon = 0 & on \quad \partial B. \end{cases} \tag{7.28}$$

Proof of Lemma. By a direct calculation we see

$$\begin{aligned} \Delta^2 g(r) = & \frac{8(1-\varepsilon)^2(p+1)(\varepsilon+(1-\varepsilon)r^2)^{-4+\frac{2}{1-p}}}{(p-1)^4} \\ & \times ((Np-N-4p)(Np-N-2p-2)(\varepsilon+(1-\varepsilon)r^2)^2 \\ & + 8\varepsilon p((Np-N-4p)r^2 + \varepsilon(N(p-1)(1-r^2)+4pr^2-p-1))) \\ \geq & \frac{8(1-\varepsilon)^2(p+1)(\varepsilon+(1-\varepsilon)r^2)^{-4+\frac{2}{1-p}}}{(p-1)^4} \\ & \times (Np-N-4p)(Np-N-2p-2)(\varepsilon+(1-\varepsilon)r^2)^2 \\ = & (1-\varepsilon)^2\lambda_N(p)g(r)^p \qquad \text{for} \quad r \in [0,1). \end{aligned}$$

Therefore we have

$$\begin{aligned} \Delta^2 U_p^\varepsilon(r) & \geq (1-\varepsilon)^2\lambda_N(p) f_p^\varepsilon(U_p^\varepsilon(r), r) \\ & = (1-\varepsilon)^2\lambda_N(p)(1+U_p^\varepsilon(r)+Q_p(r)+(Q_p^\varepsilon(r)-Q_p(r)))^p. \end{aligned}$$

Here we note that for some constant $C > 0$

$$|Q_p^\varepsilon(r) - Q_p(r)| = |(\beta(\varepsilon)-\beta)(1-r^2)| \leq C\varepsilon. \tag{7.29}$$

Hence for any $\varepsilon' \in (0,1)$ there is some $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$

$$\frac{(1+U_p^\varepsilon(r)+Q_p(r)+(Q_p^\varepsilon(r)-Q_p(r))^p}{(1+U_p^\varepsilon(r)+Q_p(r))^p} \geq 1-\varepsilon'.$$

After all we have

$$\Delta^2 U_p^\varepsilon(r) \geq (1-\varepsilon')(1-\varepsilon)^2\lambda_N(p) f_p(U_p^\varepsilon(r), r), \tag{7.30}$$

and this proves the desired inequality for a sufficiently small $\varepsilon_0 > 0$. □

End of the proof of Theorem. Assume that $\lambda_N(p) > \lambda^*$. Since U_p^ε becomes a bounded supersolution, we have a bounded solution for $\lambda = \lambda^*$ by a standard monotone iteration arguement. But this contradicts to the fact that λ^* is extremal. The uniqueness of the singular extremal also follows from the same arguement in the proof of the first assertion of Proposition 7.1. □

Remark 7.5. In the case that $N \geq 13$ and $p > p^*$, the linealized operator L_λ^p defined by

$$\begin{aligned} L_\lambda^p \varphi & = \Delta^2\varphi - \lambda f_p'(U_p, r)\varphi \\ & = \Delta^2\varphi - p\lambda\frac{\varphi}{r^4}. \end{aligned} \tag{7.31}$$

has a positive first eigenvalue $\mu(\lambda)$ for any $\lambda \in (0, \lambda_N(p)]$ corresponding to an eigenfunction $\varphi \in H^2(B) \cap H_0^1(B)$. In order to characterize the first eigenvalue we may consider the variational inequality

$$\begin{aligned}\int_B |\Delta\varphi|^2\,dx - \lambda_N(p)\int_B f_p'(U_p, r)\varphi^2\,dx \\ = \int_B \left(|\Delta\varphi|^2 - H(p)\frac{\varphi^2}{r^4}\right)dx \\ \geq \left(1 - \frac{16H(p)}{(N(N-4))^2}\right)\int_B |\Delta\varphi|^2\,dx.\end{aligned} \tag{7.32}$$

Therefore we see

$$\mu(\lambda_N(p)) \geq \left(1 - \frac{16H(p)}{(N(N-4))^2}\right)\mu_1, \tag{7.33}$$

where μ_1 is the first eigenvalue of Δ^2 with the boundary condition $\varphi = \Delta\varphi = 0$ on ∂B.

If $p = p^*$, then $H(p) = (N(N-4)/4)^2$ and $L^p_{\lambda_{N(p)}}$ does not have a first eigenfunction in $H^2(B) \cap H_0^1(B)$. However, the previous arguement gives a positive value for $\mu(\lambda_N(p))$ defined as a decreasing limit

$$\mu(\lambda_{N(p)}) = \lim_{\lambda \to \lambda_{N(p)}} \mu(\lambda) \geq \lambda_1 \frac{N(N-4)}{2} + \lambda_2. \tag{7.34}$$

Since $u_\lambda \leq U_p$ and $\lambda f_p'(u_\lambda, r) \leq H(p)(1/r^4)$, this is clear from Theorem 2.1.

Remark 7.6. We consider the case that $4 < N < 13$. Assume that $p > (N-4)/(N+4)$. Then U_p is not singular extremal, since the Hardy inequality (7.16) does not holds. In the next we assume that $p \leq (N-4)/(N+4)$. Then U_p is not an energy solution but a (singular) weak solution. Therefore we see that there exists a range of p where U_p is a weak solution and satisfies the Hardy inequality (7.16).

In the next we consider the limit of this problem as $p \to +\infty$. Let us set

$$\begin{cases} Q_e(r) = \dfrac{2(N-2)}{N}(1-r^2), \\ \lambda_N^e = 8(N-2)(N-4), \end{cases} \tag{7.35}$$

and we set

$$U_e = -4\log r - Q_e(r). \tag{7.36}$$

As $p \to +\infty$ we see that

$$\left(pQ_p(r), f_p\left(\frac{u}{p}, r\right), p\lambda_N(p), pU_p\right) \longrightarrow (Q_e(r), f_e(u,r), \lambda_N^e, U_e) \tag{7.37}$$

for any $r \in (0,1)$.

Therefore the boundary value problem (7.3) with $\lambda = \lambda_N^e$ and $f = f_e$ is considered as a formal limit of the previous one.

Lemma 7.6. *Assume that* $\lambda = \lambda_N^e$ *and* $f = f_e$*. Then it holds that*:
(1) *If* $N > 4$*,* U_e *is a singular energy solution of* (7.3)*.*
(2) *If* $N > 8$ *then* $U_e \in H^4(B)$*.*

Then we have the following:

Theorem 7.2 (Exponential case)**.**
(1) *If* $N \geq 13$*, then* U_e *is a singular extremal solution with* $\lambda^* = \lambda_N^e$*.*
(2) *If* $N < 13$*, then* U_e *is not a singular extremal solution and* $\lambda_N^e < \lambda^*$*.*

Proof. As the proof in the polynomial case, it suffices to show that $\lambda_N^e \leq \lambda^*$. But this follows from the next elementary lemma as before. □

Set

$$Q_e^\varepsilon(r) = \frac{2(1-\varepsilon)(N+2\varepsilon-2)}{N}(1-r^2), \tag{7.38}$$

and set

$$U_e^\varepsilon = -2\log(\varepsilon + (1-\varepsilon)r^2) - Q_e^\varepsilon(r). \tag{7.39}$$

Lemma 7.7. *For any* $\delta > 0$ *there is a positive number* ε_0 *such that for any* $\varepsilon \in (0, \varepsilon_0)$

$$\begin{cases} \Delta^2 U_e^\varepsilon(r) \geq (1-\delta)\lambda_N^e f_p(U_e^\varepsilon(r), r) & \text{in} \quad B, \\ U_e^\varepsilon = \Delta U_e^\varepsilon = 0 & \text{on} \quad \partial B. \end{cases} \tag{7.40}$$

Proof of Lemma. For any $\varepsilon > 0$ we see

$$\begin{aligned} \Delta^2 U_e^\varepsilon &\geq (1-\varepsilon)^2 8(N-2)(N-4)e^{U_e^\varepsilon + Q_e^\varepsilon(r)} \\ &= (1-\varepsilon)^2 8(N-2)(N-4)e^{Q_e^\varepsilon(r) - Q_e(r)} e^{U_e^\varepsilon + Q_e(r)} \end{aligned}$$

Noting that $Q_e^\varepsilon(r) - Q_e(r) = -2\varepsilon(1-r^2)(N-4+2\varepsilon)/N < 0$, we have the desired estimate for a sufficiently small $\varepsilon_0 > 0$. □

Remark 7.7. In the case that $N \geq 13$, the linealized operator $L_{\lambda^*}^e$ defined by

$$\begin{aligned} L_{\lambda^*}^e \varphi &= \Delta^2 \varphi - \lambda_N^e f_e'(U_e, r)\varphi \\ &= \Delta^2 \varphi - \lambda_N^e \frac{\varphi}{r^4} \end{aligned} \tag{7.41}$$

has a positive first eigenvalue $\mu(\lambda_N^e)$ as before.

Department of Mathematical Science
Ibaraki University
Mito, Ibaraki 310-8512, Japan
e-mail: horiuchi@mito.ipc.ibaraki.ac.jp

References

[1] W. Allegretto, *Nonoscillation theory of elliptic equations of order* $2n$, Pacific J. Math. **64** (1976), 1–16.

[2] H. Brezis, Th. Cazenave, Y. Martel and A. Ramiandrisoa, *Blow up for* $u_t - \Delta u = g(u)$ *revisited*, Advancs in P.D.E. **1** (1996), 73–90.

[3] H. Brezis and J. L. Vazquez, *Blow-up solutins of some nonlinear elliptic problems*, Revista Matematica de la Univ. Comp. de Madrid **10**-2 (1997), 443–469.

[4] E. B. Davies and A. M. Hinz, *Explicit constants for Rellich inequalities in* $L^p(\Omega)$, Math. Z **227** (1998), 511–523.

[5] T. Horiuchi and P. Kumlin, *On the minimal solution for quasilinear degenerate elliptic equation and its blow-up*, preprint 2003, No. 36, Chalmers University of Technology, Göteborg Sweden.

[6] D. D. Joseph and T. S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rat. Mech. Anal. **49** (1973), 241–269.

[7] Y. Martel, *Uniqueness of weak extremal solutions for nonlinear elliptic problems*, Houston J. of Math. **23** (1997), 161–168.

[8] N. Shimakura, *Problèmes aux limites variatioannels du type elliptique*, Ann. scient. Éc. Norm. Sup. 4^e *serie* **2** (1969), 255–310.

[9] G. Talenti, *Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces*, Ann. Mat. Pura Appl. **120**-4 (1979), 160–184.