J. Math. Kyoto Univ. (JMKYAZ)
44-1 (2004), 99–118

Calabi–Yau threefolds with infinitely many divisorial contractions

By

Hokuto UEHARA*

Abstract

We study Calabi–Yau 3-folds with infinitely many divisorial contractions. We also suggest a method to describe Calabi–Yau 3-folds with the infinite automorphism group.

0. Introduction

A smooth complex projective n-dimensional variety X is a Calabi–Yau n-fold (C–Y n-fold) if $K_X = 0$ and $h^1(\mathcal{O}_X) = 0$. If the Abundance Conjecture and the Minimal Model Conjecture are true, a $\mathbb{Q}$-factorial terminal n-fold Y with Kodaira dimension $\kappa(Y) = 0$ is always birationally equivalent to a $\mathbb{Q}$-factorial terminal n-fold X with $K_X \equiv 0$ ([6], [10]). We can regard C–Y n-folds as special cases of this. As is well-known, for a smooth K3 surface S, the nef cone $\overline{\mathcal{A}}(S)$ is rational polyhedral if and only if $\operatorname{Aut} S$ is finite ([22]). Moreover if a K3 surface S with infinite $\operatorname{Aut} S$ contains a -2-curve, then S contains infinitely many -2-curves ([12]). In the same way, the Morrison Cone Conjecture (2.1) states that for a C–Y 3-fold X the nef cone $\overline{\mathcal{A}}(X)$ is rational polyhedral if and only if $\operatorname{Aut} X$ is finite. By analogy with K3 surfaces and C–Y 3-folds, if a C–Y 3-fold X with infinite $\operatorname{Aut} X$ admits a divisorial contraction, it is highly likely that it admits infinitely many such. In addition to this, a C–Y 3-fold always admits a birational contraction when its Picard number is more than 13 ([2]). In this context, it seems worthwhile to study C–Y 3-folds with infinitely many divisorial contractions. One of the aim of this article is to give a characterization of C–Y 3-folds which admit infinitely many divisorial contractions (see Theorem 0.3. See also Theorem 3.6 and Remark 3.8 for the precise statement).

Another aim of this article is to suggest a method to describe C–Y 3-folds X with infinite $\operatorname{Aut} X$. If we have such X, then $\overline{\mathcal{A}}(X) \cap c_2^{\perp} \neq \{0\}$ (Remark 2.3), where $c_2 (= c_2(X))$ is the second Chern class of X. If $\overline{\mathcal{A}}(X) \cap c_2^{\perp}$ contains the class of a rational divisor, it is likely (cf. Conjecture 1.2) that some multiple of

---

2000 *Mathematics Subject Classification(s).* Primary: 14J30, 14J32, Secondary: 14E30
Received February 17, 2003

*Research Fellow of the Japan Society for the Promotion of Science.

the divisor determines a nontrivial contraction $\varphi : X \to Y$ satisfying $\varphi^* H \cdot c_2 = 0$ for an ample divisor H on Y. We call such a contraction *c_2-contraction.* In this context our first task to describe C–Y 3-folds with infinite $\operatorname{Aut} X$ is to:

(i) describe C–Y 3-folds X with infinite $\operatorname{Aut} X$ such that X does not admit any nontrivial c_2-contractions.

I guess such X has the small Picard number greater than 2. Secondly we should:

(ii) classify C–Y 3-folds which admit a nontrivial c_2-contraction.

Presumably we can do this because we have the remarkable classification of C–Y 3-folds X admitting a c_2-contraction $\varphi : X \to Y$ in the case $\dim Y \geq 2$ by K. Oguiso (cf. [20] or Theorem 3.3). Next we should:

(iii) determine which C-Y 3-folds in the list obtained by (ii) have infinite $\operatorname{Aut} X$.

If we carry out these, we can describe all C–Y 3-folds with infinite $\operatorname{Aut} X$.

In Section 1, we prove several lemmas for the latter use. Let $\tilde{I}(= \tilde{I}_X)$ be the index of the set $\{\varphi_i\}_{i \in \tilde{I}}$ of all possible divisorial contractions on a C–Y 3-fold X and let us denote the exceptional divisor of φ_i by E_i. The most important lemma in Section 1 is:

Lemma 0.1 (= Proposition 1.10 + Remark 1.5)**.** *Let J be an infinite subset of $\tilde{I}$. Then there exist $1, 2, 3 \in J$ such that $E_1 + E_2 + E_3$ is nef.*

We use this lemma in Section 3 to construct a nontrivial c_2-contraction on C–Y 3-folds with infinitely many divisorial contractions.

In Section 2, we give a partial answer to the following conjecture. Put $\overline{\mathcal{A}}(X)_\epsilon := \{x \in \overline{\mathcal{A}}(X) \mid c_2 \cdot x \geq \epsilon H^2 \cdot x\}$ for an ample divisor H on X and let ϵ be a positive real number.

Conjecture 0.2 (=Conjecture 2.6)**.** *Let X be a C–Y 3-fold.*

(i) *Let $\varphi : X \to Y$ be a contraction such that $\varphi^*\overline{\mathcal{A}}(Y) \subset \overline{\mathcal{A}}(X)_\epsilon$. Then the cardinality of the set of such φ is finite.*

(ii) *Let $\varphi : X \to Y$ be a contraction such that $\varphi^*\overline{\mathcal{A}}(Y) \subset \overline{\mathcal{A}}(X)_\epsilon$. Then $\overline{\mathcal{A}}(Y)$ is rational polyhedral.*

If $\operatorname{Aut} X$ is infinite, then $\overline{\mathcal{A}}(X)$ is not rational polyhedral (Remark 2.3). Hence Conjecture 0.2 means the shape of $\overline{\mathcal{A}}(X)$ is complicated near $\overline{\mathcal{A}}(X) \cap c_2^\perp$. We expect this "complexity" produces a rational point on $\overline{\mathcal{A}}(X) \cap c_2^\perp \backslash \{0\}$.

In Section 3, we consider C–Y 3-folds with infinitely many divisorial contractions. Define $\tilde{I}_{c_2 * 0} := \{i \in \tilde{I} \mid E_i \cdot c_2 * 0\}$, where $*$ is $<, =$ or $>$. The main result of Section 3 is:

Theorem 0.3 (See Theorem 3.6 for the precise statement)**.** *Assume that $\tilde{I}_{c_2=0}$ is infinite for a C–Y 3-fold X. Then there exist a K3 surface S*

containing infinitely many smooth rational curves, an elliptic curve E and a finite Gorenstein automorphism group G of $S \times E$ such that X is birational to $(S \times E)/G$.

In the proof of Theorem 0.3 we use Lemma 0.1 to prove the existence of a nontrivial c_2-contraction on X and we use the Oguiso's classification to determine the structure of X. Hence Theorem 0.3 is regarded as a realization of the method to describe C–Y 3-folds with infinite $\operatorname{Aut} X$ we mention above.

Finally, in Section 4 we construct C-Y 3-folds with $|\tilde{I}_{c_2=0}| = \infty$. In passing, we show that the set $\tilde{I}_{c_2<0}$ is always finite in Corollary 1.11 and Remark 1.5. I do not know any examples of C-Y 3-folds with $|\tilde{I}_{c_2>0}| = \infty$.

Notation and Convention

(i) When a normal projective variety X over $\mathbb{C}$ has at most rational Gorenstein singularities and it satisfies $h^1(\mathcal{O}_X) = 0$ and $K_X = 0$, we call it a C–Y model. X always means a C–Y 3-fold and a C–Y model means a 3-dimensional C–Y model throughout this paper unless we specify otherwise.

(ii) For a n-dimensional projective variety X, let $\mathcal{A}(X)$ denote the cone generated by ample divisors in $N^1(X)$ and $\mathcal{A}^e(X)$ denotes the effective nef cone, namely, the cone generated by nef effective divisors in $N^1(X)$. Let us denote the cone $\{x \in N^1(X) \,|\, x^n = 0\}$ by $\mathcal{W}$. Suppose the symbol $*$ denotes $>, \geq$ etc. For a real divisor D on X and a constant c, set $D_{*c} := \{z \in N_1(X) \,|\, (D \cdot z) * c\} \cup \{0\}$. Moreover $[D]$ denotes the element in $N^1(X)$ corresponding to D. For a real 1-cycle z, define the subspace z_{*c} of $N^1(X)$ and the class $[z] \in N_1(X)$ in the similar way. Define $\overline{NE}(X)_{D*0} := \overline{NE}(X) \cap D_{*0}$.

(iii) For a C–Y 3-fold X, we can regard the second Chern class $c_2(X)$ as a linear form on $H^2(X, \mathbb{Z})$. We often abbreviate it by c_2 in this article. As is well-known, $c_2 \cdot x \geq 0$ for all $x \in \overline{\mathcal{A}}(X)$ by Y. Miyaoka ([13]). We define $\overline{\mathcal{A}}(X)_\epsilon := \overline{\mathcal{A}}(X) \cap (c_2 - \epsilon H^2)_{\geq 0}$ for a fixed ample divisor H and a positive real number ϵ.

(iv) We use the terminology *terminal*, *canonical*, *klt* (Kawamata log terminal), *lc* (log canonical) and *plt* (purely log terminal) for a log pair (X, Δ) in the sense in [10], but we always assume that Δ is effective in these definitions. Klt is same as log terminal in [6]. We also use the terminology *semismooth* in the sense in [9].

(v) The term *contraction* means a surjective morphism between normal projective varieties with connected fibers and thus contractions consist of the fiber space case and the birational contraction case. Let $I_X (= I)$ be the index of the set $\{\varphi_i \colon X \to Y_i\}_{i \in I}$ of all possible birational contractions of type III on a C–Y 3-fold X (see Definition 1.1 for this terminology). For $i \in I$, let E_i be the exceptional divisor of φ_i, C_i the irreducible curve $\varphi_i(E_i)$ and F_i a general fiber of $\varphi_i|_{E_i} \colon E_i \to C_i$. It is known that $E_i \cdot F_i = -2$. Furthermore let us denote by V_i the image of the closed cone of curves $\overline{NE}(E_i)$ under the natural map $N_1(E_i) \to N_1(X)$. We know that V_i is a 2-dimensional cone (see Fact (iii)) generated by the rays $\mathbb{R}_{\geq 0}[F_i]$ and $\mathbb{R}_{\geq 0}[v_i]$, where v_i is a real 1-cycle.

(vi) We denote the biregular (respectively, birational) automorphism group of a variety X by $\mathrm{Aut}\, X$ (respectively, $\mathrm{Bir}\, X$).

(vii) If V is given as $V_{\mathbb{Q}} \otimes \mathbb{R}$ for some $\mathbb{Q}$-vector space $V_{\mathbb{Q}}$, a *rational polyhedral* cone is a closed cone generated by a finite set of rational points. A cone $\mathcal{C}$ is *locally rational polyhedral at a point* x if there is a neighborhood U of x and a rational polyhedral cone $\mathcal{D}$ such that $\mathcal{C} \cap U = \mathcal{D} \cap U$. Let $\mathcal{E}$ be a open cone in V. We say that a cone $\mathcal{C}$ is *locally rational polyhedral in* $\mathcal{E}$ if $\mathcal{C}$ is a rational polyhedral at every point in $\mathcal{E}$.

Acknowledgement. Part of this paper was written while I was at the University of Warwick from April 2000 to March 2001 and had a short visit at the University of Bayreuth. I would like to thank both universities (in particular Professors M. Reid and Th. Peternell) for providing a pleasant environment. I wish to thank Professors Y. Kawamata and Masa-hiko Saito for pertinent comments. I am especially grateful to Professors K. Oguiso and B. Szendröi for their generosity with their time and expertise. Last but not least, I would like to express gratitude to my friends, K. Inui, M. Fukushima and D. Ryder for their warm encouragement.

1. Divisorial contractions on C–Y 3-folds

We say that a birational contraction $\varphi\colon X \to Y$ between normal projective varieties is *primitive* if $\rho(X/Y) = 1$. We classify a primitive birational contraction on a $\mathbb{Q}$-factorial C–Y model according to the dimensions of its exceptional set and its image.

Definition 1.1. We say that a primitive birational contraction on a (3-dimensional) C–Y model is *of type I* if it contracts only finitely many curves, *of type II* if it contracts an irreducible surface to a single point and *of type III* if it contracts an irreducible surface to a curve. Hence a primitive birational contraction is, so called, a small (respectively, divisorial) contraction if it is of type I (respectively, type II or III). Every birational contractions on a $\mathbb{Q}$-factorial C–Y model is one of types I, II and III.

Let $\varphi\colon X \to Y$ be a birational contraction on a n-dimensional C–Y model X. Let H, H' denote ample divisors on X, Y respectively. Since $\Delta := -H + m\varphi^* H'$ is effective for sufficiently large m, the pair $(X, \epsilon\Delta)$ defines a log variety with klt singularities for $0 < \epsilon \ll 1$. Therefore we can regard φ as a $K_X + \epsilon\Delta$-extremal face contraction and so we may apply theory of the log Minimal Model Program (log MMP) to study φ. All of the following facts come from theory of the log MMP ([6], [10]).

Fact

(i) Since $-(K_X + \epsilon\Delta)$ is φ-ample, the cone $\overline{NE}(X/Y)$ is rational polyhedral by the cone theorem.

(ii) Since every extremal *face* contraction can be decomposed into extremal *ray* contractions, we can write $\varphi = \psi_m \circ \cdots \circ \psi_1$, where ψ_i is a primitive contraction and $m = \rho(X/Y)$. A contraction φ corresponds to a codimension m face Δ_m of $\overline{\mathcal{A}}(X)$, not entirely contained in $\mathcal{W}$, which is just the image of $\overline{\mathcal{A}}(Y)$ under the injection $\varphi^*\colon N^1(Y) \to N^1(X)$. Thus a decomposition of φ corresponds to a sequence of faces $\Delta_0 := \overline{\mathcal{A}}(X) > \Delta_1 > \cdots > \Delta_m$, where Δ_i is a codimension 1 face of Δ_{i+1}.

(iii) Since the image of $\varphi^*\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$ coincides with

$$\{D \in \operatorname{Pic}(X) \mid D \cdot z = 0 \text{ for all } z \in (\varphi^* H')^\perp \cap \overline{NE}(X)\}$$

and since X is a C–Y model, Y is also a C–Y model. We also obtain an exact sequence

$$0 \to N_1(X/Y) \to N_1(X) \to N_1(Y) \to 0.$$

Assume that $\dim X = 3$. Pick $i \in I$. By the exact sequence above, we know that V_i is a 2-dimensional cone in $N_1(X)$.

(iv) Let X be a C–Y 3-fold and L an *effective* nef divisor on it. Since $(X, \epsilon L)$ is a klt pair for $0 < \epsilon \ll 1$ and $K_X + \epsilon L$ is nef, we know that L is semi-ample by the log abundance theorem ([7], see also [17]).

Conjecture 1.2. *Let X be a C–Y 3-fold and L a nef divisor on it. Then L is semi-ample.*

If $L \cdot c_2 > 0$, we can show that L is effective ([25]). So in this case, Conjecture 1.2 is true.

(v) By the cone theorem for klt pairs, the nef cone $\overline{\mathcal{A}}(X)$ is locally rational polyhedral inside the cone $\mathcal{W}$. See [4], [5] and [25] for the proof.

In passing, for a C–Y 3-fold X and an effective divisor Δ on it such that the pair (X, Δ) has at most klt singularities, if every $K_X + \Delta$-extremal ray corresponds to a divisorial contraction, the number of $K_X + \Delta$-extremal rays is finite by the observation in Fact (iii). On the other hand, the pair of the C–Y 3-fold X constructed by C. Schoen (cf. [15]) and some effective divisor Δ on X gives an example where $\overline{NE}(X)_{K_X+\Delta<0}$ contains infinitely many extremal rays corresponding to contractions of type I ([15]). This supplies a negative answer for the problem stated in [6, 4-2-5], i.e. for a klt pair (X, Δ) with $\kappa(X, K_X + \Delta) \geq 0$, is the number of $K_X + \Delta$-extremal rays finite? But I still feel (4-2-5) ibid. is affirmative when Δ is trivial.

We have the following result by V. V. Nikulin [16, p. 282].

Proposition 1.3. *The sets $I^1 := \{i \in I \mid E_i$ is an exceptional divisor of two different divisorial contractions$\}$ and $I^2 := \{i \in I \mid$ there exists $j \in I$ such that* either *$E_i \cdot F_j > 0$ and $E_j \cdot F_i = 0$* or *$E_j \cdot F_i > 0$ and $E_i \cdot F_j = 0\}$ are finite.*

Lemma 1.4. *Let X be a $\mathbb{Q}$-factorial C–Y model with its Picard number ρ. Define $K_i := \{j \in I \mid E_i \cap E_j \neq \emptyset\}$ for $i \in I$.*

(i) *Assume $J \subset I$. If $|J| \geq \rho$, there exist i, $j \in J$ such that $E_i \cap E_j$ is not empty.*

(ii) *There is no subset* $J \subset I$ *such that* J *satisfies the following property* $(*)$.
$(*)$ *Assume that we have* $1, \dots, n \in J$ *such that* $i \in J \setminus \bigcup_{k=1}^{i-1} K_k$ *for all* $i \leq n$. *Then* $J \setminus \bigcup_{k=1}^{n} K_k \neq \emptyset$.

(iii) *Assume* $J \subset I$ *such that* $|J| = \infty$. *Then there exists* $i \in J$ *such that* $|K_i \cap J| = \infty$. *In particular, there exists an infinite subset* $J' \subset J$ *such that* $E_i \cap E_j$ *is not empty for all* $i, j \in J'$.

Proof. (i) Assume that we have elements $1, \dots, \rho \in J$ such that $E_i \cap E_j$ is empty for all $i \neq j$. Then there exists a nontrivial relation $\Sigma_{k=1}^{\rho} a_k E_k + a_0 H \equiv 0$ for $a_k \in \mathbb{R}$ and some ample divisor H. Then because $E_i \cdot F_j = 0$ if and only if $i \neq j$, the numbers $a_k \cdot a_0 > 0$ for all k. This is absurd, since $(\Sigma a_k E_k + a_0 H) \cdot H^2 \neq 0$.

(ii) If J satisfies $(*)$ then we have $1, \dots, \rho \in J$ such that $k \notin \bigcup_{i=1}^{k-1} K_i$ for all $k \leq \rho$. This contradicts (i).

(iii) Assume that $K_i \cap J$ is finite for all $i \in J$. By $|J| = \infty$, J satisfies $(*)$ in (ii). The second statement follows from the first one. □

Remark 1.5. Every exceptional divisor of a birational contraction of type II does not meet each other. Therefore the number of contractions of type II is finite by the same proof of (i) above.

Lemma 1.6. *For general* $i \in I$ *(namely, all but a finite number of* $i \in I$*)* $\overline{NE}(X) = \overline{NE}(X)_{E_i \geq 0} + \mathbb{R}_{\geq 0}[F_i]$.

Proof. It is enough to check the finiteness of $J := I \setminus (I^1 \cup I^2 \cup \{i \in I \mid \overline{NE}(X) = \overline{NE}(X)_{E_i \geq 0} + \mathbb{R}_{\geq 0}[F_i]\})$. For $i \in J$, not only $\mathbb{R}_{\geq 0}[F_i]$ but also $\mathbb{R}_{\geq 0}[v_i]$ is a $K_X + \epsilon E_i$-extremal ray. Then $\mathbb{R}_{\geq 0}[v_i]$ determines a birational contraction of type I. If J is infinite, there exists an infinite subset $J' \subset J$ such that $E_i \cap E_j$ is not empty for all $i, j \in J'$ by Lemma 1.4. Then $\mathbb{R}_{\geq 0}[v_i] = \mathbb{R}_{\geq 0}[v_j]$ for all $i, j \in J'$. Let $\varphi \colon X \to Y$ be the associated contraction of type I and H a general hyperplane section on Y, and define $l_i := \varphi(E_i)|_H$ for $i \in J'$. Then since $l_i \cdot l_j = 0$ on H if and only if $i \neq j$, the l_i's are linearly independent in $N_1(H)$. This is absurd. □

Pick $i \in I$. Define $t_i = \min\{t \in \mathbb{R} \mid E_i + tH \text{ is nef}\}$, where H is a fixed ample divisor on X. $\{t_i\}$ denotes the round up of t_i.

Lemma 1.7. $t_i \leq 4$ *for all* $i \in I$.

Proof. If E_i is normal, E_i has at most RDP. By the inversion of adjunction, (X, E_i) has at most plt singularities. If E_i is non-normal, E_i is semi-smooth ([27]). Then we use the inversion of adjunction again and know (X, E_i) has at most lc singularities. In both cases, we can apply the rationality theorem ([6]) for the klt pairs $(X, (1-\epsilon)E_i)$ for sufficiently small positive rational numbers ϵ and we obtain the statement. □

Lemma 1.8. *Let* $J \subset I$ *and let* H *be an ample divisor on* X. *Assume that there exist an integer* N *and* $z \in \overline{NE}(X)$ *such that* $z \cdot E_i \leq N$ *for all* $i \in J$.

(i) *Let* ϵ *be a positive real number. Then the set* $J_\epsilon(z) := \{i \in J \mid \varphi_i^* \overline{\mathcal{A}}(Y_i) \subset (z - \epsilon H^2)_{\geq 0}\}$ *is finite.*

(ii) *If* z *is in the interior of* $\overline{NE}(X)$, J *is finite.*

Proof. (i) By Lemma 1.6, we may assume that $E_i + t_i H \in \varphi_i^* \overline{\mathcal{A}}(Y_i)$ for all $i \in J_\epsilon(z)$. Then we get $(E_i + \{t_i\}H) \cdot (z - \epsilon H^2) \geq (\{t_i\} - t_i) H \cdot (z - \epsilon H^2) \geq 0$ and $(E_i + \{t_i\}H) \cdot z \leq N + 4H \cdot z =: c$. Thus $E_i + \{t_i\}H \in (z - \epsilon H^2)_{\geq 0} \cap z_{\leq c} \cap \overline{\mathcal{A}}(X)$. Since $(z - \epsilon H^2)_{\geq 0} \cap z_{\leq c} \cap \overline{\mathcal{A}}(X)$ is a compact set, J_ϵ is finite.

(ii) This is the special case of (i). □

Let D be a prime divisor on X. By the Serre duality for a Cohen-Macaulay surface D,

$$\chi(\mathcal{O}_D) = \chi(\omega_D) = \chi(\mathcal{O}_D(D)) = \chi(\mathcal{O}_X(D)).$$

Combining this equality with the Riemann-Roch theorem for a C–Y 3-fold X, we obtain:

Lemma 1.9. *For a prime divisor* D *on* X, *we have*

$$\chi(\mathcal{O}_D) = (1/6)D^3 + (1/12)D \cdot c_2.$$

The following proposition is a key to prove Theorem 3.6.

Proposition 1.10. *Let* J *be an infinite subset of* I. *Then there exist* $1, 2, 3 \in J$ *such that* $E_1 + E_2 + E_3$ *is nef.*

Proof. We may assume that $\overline{NE}(X) = \overline{NE}(X)_{E_i \geq 0} + \mathbb{R}_{\geq 0}[F_i]$ for all $i \in J$ by Lemma 1.6 and that $E_i \cdot F_j > 0$ for all different $i, j \in J$ by Proposition 1.3 and Lemma 1.4 (iii). Pick $1, 2, 3 \in J$. Then $(E_1 + E_2 + E_3) \cdot F_i \geq 0$ for $i = 1, 2, 3$. Thus $E_1 + E_2 + E_3$ is nef. □

Note that the nef divisor $E_1 + E_2 + E_3$ is semi-ample by Fact (iv). By Proposition 1.10, the set $\{i \in I \mid E_i \cdot z < 0\}$ is finite for a pseudo-effective element $z \in N_1(X)$, i.e. $z \cdot x \geq 0$ for all $x \in \overline{\mathcal{A}}(X)$.

Corollary 1.11. *The sets* $I_{c_2 < 0} := \{i \in I \,|E_i \cdot c_2 < 0\}$, $\{i \in I \mid E_i$ *is a Hirzebruch surface*$\}$ *and* $I_{dP} := \{i \in I \mid E_i$ *is a generalized del Pezzo surface*$\}$ *are finite.*

Proof. Because c_2 is pseudo-effective on minimal model 3-folds by [13], the set $I_{c_2<0}$ is finite. For $i \in I$ such that E_i is a Hirzebruch surface, $E_i \cdot c_2 = -4$ by Lemma 1.9. Next suppose that I_{dP} is infinite. By Proposition 1.3 and Lemma 1.4 (iii), we may assume that $E_i \cdot F_j > 0$ for all different $i, j \in I_{dP}$. Then there exists a real 1-cycle v such that $\mathbb{R}_{\geq 0}[v] = \mathbb{R}_{\geq 0}[v_i]$ for all $i \in I_{dP}$. This is absurd, since $E_i \cdot v < 0$ for all $i \in I_{dP}$. □

2. The second Chern class and the nef cone

Let us remember the following conjecture of D. Morrison concerning the finiteness properties of the nef cones ([14], [5]). We refer to 2.1 as the Morrison Cone Conjecture.

Conjecture 2.1. *Let X be a C–Y n-fold. The number of the* $\operatorname{Aut} X$*-equivalence classes of faces of the effective nef cone $\mathcal{A}^e(X)$ corresponding to birational contractions or fiber space structures is finite. Moreover, there exists a rational polyhedral cone Π which is a fundamental domain for the action of* $\operatorname{Aut} X$ *on $\mathcal{A}^e(X)$ in the sense that*

(i) $\mathcal{A}^e(X) = \bigcup_{\alpha \in \operatorname{Aut} X} \alpha_* \Pi$,

(ii) $\operatorname{Int} \Pi \cap \alpha_* \operatorname{Int} \Pi = \emptyset$ *unless* $\alpha_* = id$.

Let H be a nef and big divisor on a (3-dimensional) C-Y model Y. Set $\operatorname{Aut}(Y, H) := \{\alpha \in \operatorname{Aut} Y \mid \alpha_* H \equiv H\}$.

Lemma 2.2. *Let Y, H be as above. Then the group* $\operatorname{Aut}(Y, H)$ *is finite.*

Proof. Let $\varphi\colon Y \to Z$ be the birational contraction defined by the free complete linear system mH for sufficiently large integer m. Take an element of $\operatorname{Aut}(Y, H)$. Then it descends to an element of $\operatorname{Aut}(Z, H')$, where H' is an ample divisor on Z such that $\varphi^* H' = mH$. On the other hand, the natural map $\operatorname{Bir} Y \to \operatorname{Bir} Z$ is injective, hence it is enough to prove the finiteness of $\operatorname{Aut}(Z, H')$. Grothendieck proved that $\operatorname{Aut}(Z, H')$ is a projective scheme, in particular, it has finitely many components. On the other hand, because $H^0(Y, T_Z) = 0$ by Corollary 8.6 [3], $\operatorname{Aut} Z$ is discrete and thus $\operatorname{Aut}(Z, H')$ is finite. □

Remark 2.3. If c_2 is positive on $\overline{\mathcal{A}}(X) \backslash \{0\}$ or if $\overline{\mathcal{A}}(X)$ is rational polyhedral, then since we can find an ample divisor H such that $\operatorname{Aut} X = \operatorname{Aut}(X, H)$, $\operatorname{Aut} X$ is finite ([26]). Consequently if the Morrison Cone Conjecture is true for C–Y 3-folds X, $\overline{\mathcal{A}}(X)$ is rational polyhedral if and only if $\operatorname{Aut} X$ is finite.

We study birational contractions of type III whose exceptional divisors are non-normal. If the Morrison Cone Conjecture is true, we can bound the numbers E_i^3 and $E_i \cdot c_2$ for $i \in I$. In fact, for non-normal exceptional divisors E_i we can prove (without assuming the Morrison Cone Conjecture):

Proposition 2.4. *$7 - 7h^{1,2}(X) \leq E_i^3 \leq 7$ and $-2 \leq E_i \cdot c_2 \leq 6h^{1,2}(X) - 2$ for all $i \in I$ such that E_i is non-normal.*

Proof. Fix $i \in I$ such that E_i is non-normal and let E, C denote E_i, C_i respectively. Since E is non-normal, E is semi-smooth and $C_0 := \operatorname{Sing}(E)$ is an irreducible smooth curve, which gives a section of $E \to C$ ([27]). Let $\psi : Z \to X$, E' and D be the blowup along C_0, the strict transform of E on Z and the exceptional divisor of ψ respectively. Let us also define $p := \psi|_{E'}$ and $C_0' := p^{-1}(C_0)$ with the reduced structure. By local calculation, we can check easily that p gives the normalization of E and that D and E' meet transversally, in particular, $D|_{E'} = C_0'$. Let $E' \to C' \to C$ be the Stein factorization of the morphism $E' \to E \to C$, then we know that E' is a $\mathbb{P}^1$-bundle over a smooth curve C' and $C' \to C$ is a double cover. We know from these facts that C_0' is a section of the $\mathbb{P}^1$-bundle.

Let F be a ruling of the Hirzebruch surface D over C_0. Because $\psi^*E|_D \cdot F = 0$, $\psi^*E|_D$ is numerically proportional to F on D and so $0 = (\psi^*E)^2 \cdot D$. Then we have

$$0 = {E'}^2 \cdot D + 4E' \cdot D^2 + 4D^3.$$

Furthermore because of $K_Z = D$ and the adjunction formula, we obtain

$$8(1 - g(C')) = K_{E'}^2 = D^2 \cdot E' + 2D \cdot {E'}^2 + {E'}^3,$$

$$2g(C') - 2 = (K_{E'} + C_0') \cdot C_0' = 2D^2 \cdot E' + {E'}^2 \cdot D$$

and

$$8(1 - g(C)) = K_D^2 = 4D^3.$$

By these equalities, we get

$$E^3 = (E' + 2D)^3 = 7 - 3g(C') - 4g(C).$$

By the fact that $g(C') \leq h^{1,2}(X)$ ([1]), we get the bound of E^3. On the other hand, because every fiber of $\varphi|_E \colon E \to C$ is a conic we have $R^i\varphi_*\mathcal{O}_E = 0$ for $i > 0$. Thus we know $\chi(\mathcal{O}_E) = \chi(\mathcal{O}_C)$ and therefore

$$E \cdot c_2 = 12\chi(\mathcal{O}_E) - 2E^3 = 6g(C') - 4g(C) - 2$$

by Lemma 1.9. We use $g(C') \leq h^{1,2}(X)$ again to obtain the bound of $E \cdot c_2$. □

Remark 2.5. We use the notation in the proof above. It seems worthwhile to restate the following formulae, that is, $E^3 = 7 - 3g(C') - 4g(C)$ and $E \cdot c_2 = 6g(C') - 4g(C) - 2$.

Conjecture 2.6 (cf. [26, Problem 3])**.**
(i) *Let $\varphi : X \to Y$ be a contraction such that $\varphi^*\overline{\mathcal{A}}(Y) \subset \overline{\mathcal{A}}(X)_\epsilon$. Then the cardinality of the set of such φ is finite.*
(ii) *Let $\varphi : X \to Y$ be a contraction such that $\varphi^*\overline{\mathcal{A}}(Y) \subset \overline{\mathcal{A}}(X)_\epsilon$. Then $\overline{\mathcal{A}}(Y)$ is rational polyhedral.*

If Aut X is finite, the Morrison Cone Conjecture implies that the nef cone $\overline{\mathcal{A}}(X)$ is rational polyhedral. Hence obviously Conjecture 2.6 is true for such X (modulo the Morrison Cone Conjecture). If Aut X is infinite, then by Conjecture 2.6 we can expect the shape of the nef cone $\overline{\mathcal{A}}(X)$ is complicated near $\overline{\mathcal{A}}(X) \cap c_2^\perp$ (see also the argument after Problem 3.10).

If we have a bound of the number $E_i \cdot c_2$ for $i \in I$, Conjecture 2.6 (i) is affirmative in the case when φ is a birational contraction of type III, due to Lemma 1.8 (i).

Theorem 2.7. *Conjecture* 2.6 (i) *is affirmative in the following cases*:
(i) *φ is a fiber space* ([19])*.*
(ii) *φ is a birational contraction of type* III *whose exceptional divisor is non-normal.*

Theorem 2.8. *Conjecture* 2.6 (ii) *is affirmative in the following cases*:

(i) φ *is a fiber space.*

(ii) *Assume that the Morrison Cone Conjecture holds true and* φ *is a birational contraction.*

Proof. (i) We may assume $\rho(Y) \geq 2$ so in particular $\dim Y = 2$. By our assumption and Theorem 2.7 (i) we know that Y admits at most finitely many contractions. By Theorem 3.1 in [17] there exists a nonzero effective divisor $\Delta = \sum a_i D_i$ ($a_i > 0$, D_i a prime divisor) such that (Y, Δ) is a klt pair and $K_Y + \Delta \equiv 0$. Let $R = \mathbb{R}_{\geq 0}[z]$ be a *geometrically extremal ray* of the cone $\overline{NE}(Y)$, where z is a real 1-cycle (by the definition of a *geometrically extremal ray*, if $z_1 + z_2 \in R$ for $z_1, z_2 \in \overline{NE}(Y)$ we have $z_1, z_2 \in R$. Of course an extremal ray in the Minimal Model theory is geometrically extremal). Note that R is a K_Y-extremal ray if $K_Y \cdot z < 0$, and R is a $K_Y + \Delta + \epsilon D_i$-extremal ray for some i and $0 < \epsilon \ll 1$ if $K_Y \cdot z > 0$. Now we prove that $\overline{\mathcal{A}}(Y)$ is rational polyhedral by the induction for $\rho(Y)$. Denote the set of the geometrically extremal rays R with $R \subset K_Y^{\perp}$ by $\mathcal{S}$. If $\mathcal{S} = \emptyset$ we have a contraction $f : Y \to Z$ for any geometrically extremal rays R such that f contracts only R. So the proof is done by Theorem 2.7 (i). Hence we may assume $\mathcal{S} \neq \emptyset$. Pick $R(= \mathbb{R}_{\geq 0}[z]) \in \mathcal{S}$. It is enough to show that we can take the real 1-cycle z as a rational one and $\mathcal{S}$ is a finite set. Since the cone $\overline{NE}(Y)$ is generated by the finitely many K_Y-extremal rays and the subcone $\overline{NE}(Y)_{K_Y \leq 0}$, there exists a contraction $f(= f_R) : Y \to Z$ associated to a K_Y-extremal ray such that $\mathbb{R}_{\geq 0}[z] + \mathbb{R}_{\geq 0}[F] = (f^*L)^{\perp} \cap \overline{NE}(Y)$, where F is a curve contracted by f and L is a nef $\mathbb{R}$-divisor on Z. We can check that f_*R is a geometrically extremal ray of the cone $\overline{NE}(Z)$ by using the exact sequence $0 \to \langle [F] \rangle_{\mathbb{R}} \to N_1(Y) \to N_1(Z) \to 0$. Hence by the induction hypothesis (the finiteness of geometrically extremal rays of $\overline{NE}(Z)$), there exists only finitely many $R_1 \in \mathcal{S}$ such that $f_R = f_{R_1}$ (here note that $f_*R_1 = f_*R_2$ implies $R_1 = R_2$ for $R_1, R_2 \in \mathcal{S}$). Moreover since we may assume that f_*z is a rational 1-cycle by the induction hypothesis (the rationality of the geometrically extremal rays of $\overline{NE}(Z)$), combining the short exact sequence above with the fact $K_Y \cdot z = 0$ and $K_Y \cdot F \in \mathbb{Q}_{<0}$, we can conclude that we may take z as a rational 1-cycle. Use Theorem 2.7 (i) again, we have that the set $\{f_R\}_{R \in \mathcal{S}}$ is finite and in particular $\mathcal{S}$ is finite. This completes the proof.

(ii) We may assume that φ is primitive. Put $\mathrm{B}_\Delta := \{\alpha \in \operatorname{Aut} X \mid \alpha_*\Delta \subset \varphi^*\mathcal{A}^e(Y)\}$ for a codimension 1 face Δ of Π and $\mathrm{B} := \coprod_{\Delta \subset \Pi} \mathrm{B}_\Delta$, where Δ runs through every codimension 1 face of Π. Then we have

$$\varphi^*\overline{\mathcal{A}}(Y) = \overline{\varphi^*\mathcal{A}^e(Y)} = \overline{\bigcup_{\alpha \in \mathrm{B}} (\alpha_*\Pi \cap \varphi^*\mathcal{A}^e(Y))}.$$

Here we take the closure in the relative topology of the real vector subspace $\langle \varphi^*\mathcal{A}^e(Y) \rangle \subset N^1(X)$. Hence it is enough to prove that B_Δ is a finite set for every Δ. Fix a codimension 1 face Δ such that $\mathrm{B}_\Delta \neq \emptyset$. Replace Π with $\alpha_*\Pi$ for some $\alpha \in \operatorname{Aut}(X)$ if necessary, then we may assume that $\Delta \subset \varphi^*\mathcal{A}^e(Y)$. First we look for classes of ample divisors on Y on which φ_*c_2 takes minimum value and whose pull back on X belongs to Δ. Since $\varphi^*\overline{\mathcal{A}}(Y) \subset c_{2>0}$, there are only finitely many such and by adding these together and pulling it back on X, we get a nef and big divisor H on X. Of course $[H] \in \Delta$ by the definition. Note that the set $\{[\alpha_* H]\}_{\alpha \in \mathrm{B}_\Delta}$ is finite and so put this by $\{[\alpha_{1*}H], \dots, [\alpha_{n*}H]\}$, where $\alpha_i \in \mathrm{B}_\Delta$. It is straightforward to see that $\mathrm{B}_\Delta = \coprod_{i=1}^n \alpha_i \cdot \mathrm{Aut}(X, H)$. Therefore we know that B_Δ is a finite set by Lemma 2.2 and the proof is done. □

3. The structure of certain C–Y 3-folds with infinitely many divisorial contractions

The main results of this section are Theorem 3.6 and Corollary 3.9. We use the following notation and terminology.

(i) Let X be a normal projective variety such that $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X$. We denote by ω_X a generator of $H^0(X, \mathcal{O}_X(K_X))$. A finite automorphism group G is called *Gorenstein* if $g^*\omega_X = \omega_X$ for all $g \in G$.

(ii) Suppose we have a faithful finite group action G on a variety X. Put $X^g := \{x \in X \mid g(x) = x\}$ for $g \in G$; $\quad X^{[G]} := \bigcup_{g \in G \setminus \{1\}} X^g$.

(iii) Put $\zeta_n := \exp(2\pi i/n)$, the primitive n-th root of unity in $\mathbb{C}$. Denote by E_ζ the elliptic curve whose period is ζ in the upper half plane. Let us recall the following pairs of an Abelian 3-fold and its specific Gorenstein automorphism group: the pair (A_3, g_3), where A_3 is the triple product of E_{ζ_3} and g_3 is its automorphism $\mathrm{diag}(\zeta_3, \zeta_3, \zeta_3)$ and the pair (A_7, g_7) is the Jacobian 3-fold of the Klein quartic curve $C = (x_0x_1^3 + x_1x_2^3 + x_2x_0^3 = 0) \subset \mathbb{P}^2$ and g_7 is the automorphism of A_7 induced by the automorphism of C given by $[x_0 : x_1 : x_2] \mapsto [\zeta_7 x_0 : \zeta_7^2 x_1 : \zeta_7^4 x_2]$. We call (A_3, g_3) a *Calabi pair* and (A_7, g_7) a *Klein pair.*

Definition 3.1. Let W be a normal projective surface over $\mathbb{C}$ with at most klt singularities. We call W a *log Enriques surface* if $h^1(\mathcal{O}_W) = 0$, $mK_W = 0$ for some positive integer m. We call the integer $I(W) := \min\{m \in \mathbb{Z}_{>0} \mid mK_W = 0\}$ the global canonical index of W.

We construct C–Y 3-folds with infinitely many birational contractions from certain log Enriques surfaces in Section 4.

Definition 3.2. Let $\varphi : X \to Y$ be a contraction from a C–Y 3-fold X and a divisor L on X the pull back of an ample divisor on Y. We call φ a *c_2-contraction* if $L \cdot c_2 = 0$. For example, a fibration $\varphi : X \to \mathbb{P}^1$ is a c_2-contraction if and only if the general fiber is an Abelian surface. Moreover for an elliptic fibration $\varphi : X \to W$, it is a c_2-contraction if and only if W is a log Enriques surface by [17] (we do not have to assume there that X is simply connected). There exists a unique c_2-contraction $\varphi_0 : X \to Y_0$ such that every c_2-contraction $\varphi : X \to Y$ on X factors through φ_0 (see [20, Lemma-Definition (4.1)]). We call φ_0 the *maximal c_2-contraction.*

We have the beautiful classification of C–Y 3-folds which admit either a birational c_2-contraction or an elliptic c_2-contraction, due to K. Oguiso (see [20]).

It plays an important role to prove Theorem 3.6. The following result is coarser than the Oguiso's original classification.

Theorem 3.3 (Oguiso)**.**
(i) *Let* $\varphi : X \to Y$ *be a non-isomorphic birational* c_2*-contraction. Then* φ *is isomorphic to either one of the following*:

(a) *The unique crepant resolution* $\Phi_7 : X_7 \to \bar{X}_7 := A_7/\langle g_7\rangle$ *of* $\bar{X}_7$*, where* (A_7, g_7) *is the Klein pair.*

(b) *The unique crepant resolution* $\Phi_3 : X_3 \to \bar{X}_3 := A_3/\langle g_3\rangle$ *of* $\bar{X}_3$*, where* (A_3, g_3) *is the Calabi pair.*

(c) *The unique crepant resolution* $\Phi_{3,i} : X_{3,i} \to \bar{X}_{3,i}$ *of* $\bar{X}_{3,i}$*,* $(i = 1, 2)$*, where* $\bar{X}_{3,i}$ *is an étale quotient of* $\bar{X}_3$*.*

(ii) *Let* $\varphi : X \to W$ *be an elliptic* c_2*-contraction. Then* φ *is isomorphic to either one of the following*:

(a) *One of the relatively minimal models over* W_3 *of*

$$p_{12} : X_3 \xrightarrow{\Phi_3} \overline{X}_3 \xrightarrow{\overline{p}} W_3,$$

where $\Phi_3 : X_3 \to \bar{X}_3$ *is as above and* $\overline{p}$ *is an elliptic fibration on* $\overline{X}_3$*.*

(b) *An elliptic fiber space structure on an étale quotient of an Abelian 3-fold.*

(c) *One of the relatively minimal models over* $W_{3,1}$ *of*

$$\kappa_{3,1} : X_{3,1} \xrightarrow{\Phi_{3,1}} \overline{X}_{3,1} \xrightarrow{\overline{\kappa}} W_{3,1},$$

where $\Phi_{3,1} : X_{3,1} \to \bar{X}_{3,1}$ *is as above and* $\overline{\kappa}$ *is an elliptic fibration on* $\overline{X}_{3,1}$*.*

(d) *One of the relatively minimal models over* S/G *of*

$$\psi : Y \xrightarrow{\nu} (S \times E)/G \xrightarrow{\mu} S/G,$$

where S *is a normal* $K3$ *surface (namely its minimal resolution is a smooth* $K3$ *surface),* E *is an elliptic curve,* G *is a finite Gorenstein automorphism group of* $S \times E$ *whose element is of the form* $(g_S, g_E) \in \operatorname{Aut} S \times \operatorname{Aut} E$ *and* ν *is a crepant resolution of* $(S\times E)/G$*. Slightly more precisely,* G *is of the form* $G = H \rtimes \langle a\rangle$*, where* H *is a commutative group consisting of elements like* $h = (h_S, h_E)$ *such that* $ord(h_S) = ord(h_E) = ord(h)$ *and* h_E *is a translation, furthermore the generator* a *of* $\langle a\rangle$ *is the element of the form* $(a_S, \zeta_{I(W)}^{-1})$ *such that* $a_S^*\omega_S = \zeta_{I(W)}\omega_S$*. Moreover* $I(W) \in \{2, 3, 4, 6\}$*.*

For a contraction $\varphi : X \to Y$ on a C–Y 3-fold X, we define $M(\varphi) := \{i \in I \mid E_i \cdot C = 0$ for all curves C such that $\varphi(C)$ is a point$\}$.

Lemma 3.4.

(i) *Let $\varphi : X \to Y$ be a primitive birational contraction on a C–Y 3-fold X. Denote the extremal ray corresponding to φ by R. Then the set*

$$L(\varphi) := \{i \in I \mid R \subset V_i \text{ and } \varphi(E_i) \text{ is a } \mathbb{Q}\text{-Cartier divisor on } Y\}$$

is finite.

(ii) *Let $\varphi : X \to Y$ be a (not necessarily primitive) birational contraction on a C–Y 3-fold X. The set*

$$\begin{aligned}\overline{M(\varphi)} &:= \{i \in M(\varphi) \mid E_i \cap \operatorname{Exc}(\varphi) \neq \emptyset\}\\ &= \{i \in I \mid E_i \cap \operatorname{Exc}(\varphi) \neq \emptyset \text{ and } E_i = 0 \text{ in } N^1(X/Y)\}\end{aligned}$$

is finite.

(iii) *Suppose that we have the following diagram:*

$$\begin{array}{ccc} X & \overset{\Phi}{\dashrightarrow} & Y \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & W & \end{array},$$

where φ, ψ are contractions on C–Y 3-folds X, Y and Φ is a birational map over W. Then for general $i \in M(\varphi)$, E_i is contained in the isomorphic locus of Φ. In particular, $|M(\varphi)| = \infty$ is equivalent to $|M(\psi)| = \infty$.

Proof. (i) Assume that $L(\varphi)$ is infinite. We can take $1, 2 \in L(\varphi)$ such that $E_1 \cap E_2 \neq \emptyset$. Since $R \subset V_1 \cap V_2$, the class of 1-cycle $[E_1 \cdot E_2]$ belongs to R and so $\dim \varphi(E_1 \cap E_2) = 0$. Hence $\dim \varphi(E_1) \cap \varphi(E_2) = 0$. This is a contradiction because $\varphi(E_1)$ and $\varphi(E_2)$ are $\mathbb{Q}$-Cartier divisors.

(ii) Let $R_1, \ldots, R_n$ be the generators of the cone $\overline{NE}(X/Y)$, namely extremal rays, and consider that ψ_k is the extremal contraction corresponding to R_k. It is enough to check that $\overline{M(\varphi)} \subset \bigcup_{k=1}^{n} L(\psi_k)$. Pick $0 \in \overline{M(\varphi)}$. Then there exist an integer k and an irreducible curve C such that $C \subset E_0$ and $[C] \in R_k$. Thus $R_k \subset V_0$. Now since $\psi_k(E_i)$ is a Cartier divisor for $i \in M(\varphi)$, we obtain the statement.

(iii) Note that Φ is a composition of flops over W. Apply (ii) for each flopping contraction, then we obtain the statement. □

Lemma 3.5. *We use the notation in Theorem* 3.3. *Neither X_7, X_3, $X_{3,1}$ nor $X_{3,2}$ admits infinitely many contractions of type* III.

Proof. Let Φ_3 be the unique crepant resolution of $\bar{X}_3$. Φ_3 is a composition of birational contractions of type II (cf. [18]). Pick $i \in I_{X_3}$, if any. Then $\Phi_3(E_i) \cap \operatorname{Sing} \bar{X}_3 \neq \emptyset$ because $\bar{X}_3$ is a quotient of an Abelian 3-fold. Since $\operatorname{Sing} \bar{X}_3 = \Phi_3(\operatorname{Exc}(\Phi_3))$, we have $E_i \cap \operatorname{Exc}(\Phi) \neq \emptyset$, which implies $i \in L(\psi)$ for some contraction ψ of type II. Hence if I_{X_3} is infinite, there exists a birational contraction ψ of type II on X_3 such that $L(\psi)$ is infinite. This is absurd. In the cases of $X_{3,1}$ and $X_{3,2}$, the same proof as above works, since $\bar{X}_{3,1}$,

$\bar{X}_{3,2}$ are étale quotients of $\bar{X}_3$. Next let Φ_7 be the unique crepant resolution of $\bar{X}_7$. Then $\mathrm{Exc}(\Phi_7) = E_1 \cup E_2 \cup E_3$, each E_j is a Hirzebruch surface of degree 2 and these divisors are crossing normally each other along the negative sections (cf. [18]) (thus $v_a \in \mathbb{R}_{\geq 0}[F_b]$, $v_b \in \mathbb{R}_{\geq 0}[F_c]$, $v_c \in \mathbb{R}_{\geq 0}[F_a]$ for some a, b, c with $\{a, b, c\} = \{1, 2, 3\}$). Because $\bar{X}_7$ is a quotient of an Abelian 3-fold, $E_i \cap (E_1 \cup E_2 \cup E_3) \neq \emptyset$ for all $i \in I_{X_7}$. Furthermore if E_i intersects E_a and if $v_i \notin \mathbb{R}_{\geq 0}[F_b]$, $v_i \in \mathbb{R}_{\geq 0}[F_a]$, since $v_i \in V_a \cap E_b^{\perp}$. So in this case E_i intersects E_a and E_c, does not intersect E_b. By this way, we know that every E_i intersects precisely two of E_1, E_2 and E_3. Assuming that I_{X_7} is infinite, we can find a divisorial contraction ψ which contracts either E_1, E_2 or E_3, such that $L(\psi)$ is infinite. So we obtain a contradiction. □

Theorem 3.6. *Assume that* $I_{c_2=0}(= I_{X,c_2=0}) := \{i \in I_X \mid E_i \cdot c_2 = 0\}$ *is infinite. Then the following hold.*

(i) *We have an elliptic* c_2*-contraction* $\varphi : X \to W$ *and* φ *fits in the case of* (ii)(*d*) *in Theorem* 3.3*, that is, we have the following diagram:*

$$\begin{array}{ccccc} X & & \overset{\Phi}{\dashrightarrow} & & Y \\ & \searrow^{\varphi} & & \swarrow^{\psi} & \downarrow^{\nu} \\ & & W \cong S/G & \xleftarrow[\mu]{} & (S \times E)/G, \end{array}$$

where Y*,* S*,* E*,* G*,* ψ*,* ν *and* μ *are given there. Let* $r : S \times E \to (S \times E)/G$ *be the quotient morphism. Then the normal K3 surface* S *contains infinitely many smooth rational curves* $\{l\}$ *such that*

(a) $r(l \times E) \cap \mathrm{Sing}(S \times E)/G = \emptyset$*, and*

(b) $\bigcup_{g \in G} g \cdot l$ *is contractible at the same time by a birational contraction on* S*.*

(ii) *Let* Φ *denote the birational map between* X *and* Y *over* W *in* (i)*. Then for general* $i \in I_{c_2=0}$*,* E_i *is contained in the isomorphic locus of the birational map* $\nu \circ \Phi$ *and* $E_i = r(l \times E)$ *under this isomorphism for some smooth rational curve* l *on* S *satisfying* (*a*) *and* (*b*) *in* (i)*.*

Proof. (i) Let us denote by $\varphi : X \to W$ the maximal c_2-contraction (a priori W may be a point).

Claim 3.7. *For a general* $i \in I_{c_2=0}$*,* $i \in M(\varphi)$*.*

Proof. If not, by Proposition 1.10 we can take $1, 2, 3 \in I_{c_2=0} \backslash M(\varphi)$ such that some multiple of $E_1 + E_2 + E_3$ determines a c_2-contraction, which factors through φ. By the choice of $1, 2, 3$, there exists one of the elements $1, 2, 3$, say 1, and there exists an irreducible curve C on X such that $\varphi(C)$ is a point and $E_1 \cdot C > 0$. By the proof of 1.10 we can pick $4, 5 \in I_{c_2=0} \backslash M(\varphi)$, different from $1, 2, 3$, such that some multiple of $E_1 + E_4 + E_5$ determines a c_2-contraction, which factors through φ. Thus there exists one of the elements $4, 5$, say 4, such that $E_4 \cdot C < 0$. By the same procedure, we have infinitely many elements $i \in I_{c_2=0} \backslash M(\varphi)$ such that $E_i \cdot C < 0$. This is a contradiction with 1.10. □

When $\dim W = 1$, at most finitely many E_i ($i \in I$) are contracted to a point on W by φ, so $M(\varphi)$ is finite. Hence we have $\dim W \geq 2$. If φ is isomorphic, $\overline{\mathcal{A}}(X) \subset c_2^{\perp}$ and in particular $c_2 = 0$. In this case, X is an étale quotient of an Abelian 3-fold by [8] and it never admits birational contractions. Combining Theorem 3.3 with Lemma 3.4 (iii) and Lemma 3.5, we know that φ fits in the case (ii)(d) of 3.3 and $|M(\psi)| = \infty$. Furthermore $|M(\psi)| = \infty$ implies that the set $\{i \in I_{(S\times E)/G} \,|\, E_i \cap \mathrm{Sing}(S \times E)/G = \emptyset\}$ is infinite by 3.4 (ii). Here we use the equality $\mathrm{Sing}(S \times E)/G = \nu(\mathrm{Exc}(\nu))$. Note that every primitive birational contraction on $S \times E$ is the form as $f \times id_E$, where f is a contraction of a single smooth rational curve on S. Thus we have the conditions (a) and (b).

(ii) This follows from 3.4 (ii) and 3.4 (iii). □

Remark 3.8.

(i) Assume that Theorem 3.6 (i) holds. Then we have an infinite set $\{i \in M(\mu) \,|\, E_i \cap \mathrm{Sing}(S \times E)/G = \emptyset\}$. Using Lemma 3.4 (iii), we know that $I_{X,c_2=0}$ is infinite. Namely 3.6 (i) is a characterization of C–Y 3-folds X with $|I_{X,c_2=0}| = \infty$.

(ii) Because $(\mathrm{Sing}\, S \times E) \cup (S \times E)^{[G]} = r^{-1}\,\mathrm{Sing}(S \times E)/G$ by the purity of branch locus, the condition (a) in 3.6(i) is equivalent to the condition

$$(a)' \quad (l \times E) \cap ((\mathrm{Sing}\, S \times E) \cup (S \times E)^{[G]}) = \emptyset.$$

Corollary 3.9. *The set $I_{c_2=0}$ is finite up to* $\mathrm{Aut}\, X$.

Proof. We may assume that $I_{c_2=0}$ is infinite. Now X is birational to $(S \times E)/G$ via $\nu \circ \Phi$ as in Theorem 3.6. Consider the minimal resolution $S' \to S$. We may assume that Y is obtained as a crepant resolution $\nu' : Y \to (S' \times E)/G$, that is, ν factors through ν'. The existence of ν' is guaranteed by [21]. By 3.6 (ii) and Claim 3.7, for general $i \in I_{c_2=0}$, E_i is contained in the isomorphic locus of $\nu' \circ \Phi$ and E_i is isomorphic to the image on $(S' \times E)/G$ of $l \times E$ for some smooth rational curve l on S'. On the other hand, the set $I_{(S'\times E)/G}$ is finite up to $\mathrm{Aut}(S' \times E)/G$ by Theorem (2.23) in [20] (note that the proof of Theorem (2.23) in [20] works even if G does not act on $S' \times E$ freely). Therefore the set $I_{c_2=0}$ is finite up to $\mathrm{Bir}\, X$. By the proof of Lemma (1.15) in [5], the set $I_{c_2=0}$ is finite up to $\mathrm{Aut}\, X$. □

As we mention in the Introduction, the following problem seems worthwhile to think about.

Problem 3.10. *Assume that* $\mathrm{Aut}\, X$ *is infinite and its Picard number* $\rho(X)$ *is sufficiently large. Then does* X *admit a nontrivial* c_2*-contraction?*

Conjecture 2.6 says that if $\mathrm{Aut}\, X$ is infinite the shape of $\overline{\mathcal{A}}(X)$ is complicated near $\overline{\mathcal{A}}(X) \cap c_2^{\perp}$. We expect that this "complexity" produces a rational point on $\overline{\mathcal{A}}(X) \cap c_2^{\perp} \setminus \{0\}$ and some multiple of the divisor corresponding to the rational point defines a c_2-contraction. In fact when we study the structure of C–Y 3-folds X with $|I_{c_2=0}| = \infty$ in Theorem 3.6, we showed the existence of an elliptic c_2-contraction on X by Proposition 1.10.

4. Construction of C–Y 3-folds with infinitely many birational contractions

The aim of this section is to give construction of C–Y 3-folds with infinitely many birational contractions of type I or III from certain log Enriques surfaces. First of all, given a log Enriques surface W with $I(W) \in \{2, 3, 4, 6\}$, we construct a C–Y 3-fold X with a c_2-contraction $\varphi : X \to W$. Let $q : S \to W$ be the global canonical cover and denote by $G = \langle a \rangle$ ($\cong \mathbb{Z}/I(W)\mathbb{Z}$) the Galois group of q. The S may be an Abelian surface in general but here we assume that S is a normal K3 surface (this assumption is satisfied, for example, if W contains a *contractible* smooth rational curve. Here a curve m on W is said *contractible* if it is contracted by a birational contraction and this is equivalent to $m^2 < 0$). Let E be an elliptic curve such that E has an automorphism of order $I(W)$ which fixes the origin. Suppose that the generator a of G satisfies that $a^*\omega_S = \zeta_{I(W)}\omega_S$. Then define the action of a on E as $a(x) = \zeta_{I(W)}^{-1}x$ for $x \in E$. Then G gives a Gorenstein action on $S \times E$. Take the minimal resolution $S' \to S$, then G acts on S' and we know that $(S' \times E)/G$ is a C–Y model. By [21] there exists a crepant resolution $\nu' : X \to (S' \times E)/G$. Of course this X is a C–Y 3-fold and $\varphi : X \to (S' \times E)/G \to (S \times E)/G \to S/G = W$ is an elliptic c_2-contraction.

For a log Enriques surface W, let us denote by Σ_W the locus of klt points on W which are neither RDP's nor smooth points.

Proposition 4.1. *Let* $\varphi : X \xrightarrow{\nu'} (S' \times E)/G \xrightarrow{\mu} S/G = W$ *be as is constructed from* W *above. Suppose that there exists a contractible smooth rational curve* m *on* W*.*

(i) *Assume that* $m \cap \Sigma_W = \emptyset$*. Then there exists a contraction of type* III *on* X *contracting a prime divisor* D_0 *such that* $\varphi(D_0) = m$*.*

(ii) *Assume that* $m \cap \Sigma_W \neq \emptyset$*. Then there exists a contraction of type* I *on* X *contracting an irreducible curve* m_0 *such that* $\varphi(m_0) = m$*.*

Proof. Let $r' : S' \times E \to (S' \times E)/G$ be the quotient morphism. Moreover let l be an irreducible component of $q^{-1}m$ and denote by l' the strict transform of l on S'. Put $D := r'(l' \times E)$. In the first case, because $l' \cap S'^{[G]} = \emptyset$ we know that $D \cap \operatorname{Sing}(S' \times E)/G = \emptyset$. Furthermore since m is contractible on W, $\bigcup_{g \in G} g \cdot l'$ is contractible on S' and in particular, D is contractible by a birational contraction of type III on $(S' \times E)/G$. Hence $\nu'^{-1}_* D$ gives a desired divisor D_0. In the second case, we have $(l \times E) \cap (S \times E)^{[G]} \neq \emptyset$ (we prove the contraposition of this in the proof of Proposition 4.4 below) and D is an exceptional divisor of a contraction of type III, since $\bigcup_{g \in G} g \cdot l'$ is contractible on S'. Moreover D contains a point $y \in r'((S' \times E)^{[G]})$ such that y is over a point in $m \cap \Sigma_W$ by the morphism μ. Note that $\dim(S' \times E)^{[G]} \cap (l' \times E) = 0$. Because the problem is local, we may assume that $\{y\} = (\operatorname{Sing}(S' \times E)/G) \cap D$. Let

$$X =: X_0 \xrightarrow{\psi_1} X_1 \cdots \xrightarrow{\psi_n} X_n := (S' \times E)/G$$

be a primitive decomposition of ν' and let us denote by m_n the unique irreducible curve passing through y, of the form $r'(l' \times \{z\})$, where z is a point in $E^{[G]}$. Suppose that D_i (resp. m_i) stands for the strict transform of D (resp. m_n) on X_i. Let V be an irreducible component of ${\nu'}^{-1}y$ such that $V \cap D_0 \neq \emptyset$. When $\dim V = 2$, we have $\dim V \cap D_0 = 1$. If every component V such that $V \cap D_0 \neq \emptyset$ is 1-dimensional, the equality ${\nu'}^{*}D \cdot V = 0$ implies that $V \subset D_0$, hence $\dim V \cap D_0 = 1$ (note that D_0 is not contractible any more by a divisorial contraction on X, since the dimension of the image of the map $N_1(D_0) \to N_1(X)$ is more than 2 (cf. Fact (iii))). Therefore there exists an integer $k \geq 1$ such that $\dim \psi_{k+1}^{-1} \cdots \psi_n^{-1} y \cap D_k = 0$ and $\dim \psi_k^{-1} \cdots \psi_n^{-1} y \cap D_{k-1} = 1$. The following claim comes from the general theory and we leave the proof to the reader, since it is an easy exercise.

Claim 4.2. *Let $f : X \to Y$, $g : Y \to Z$ be primitive birational contractions between C–Y models. Suppose that the strict transforms $f_*^{-1}l$ of all curves l contracted by g are numerically proportional. Then if g is of type* I (*resp. of type* III), *there exists a contraction f' of type* I (*resp. of type* III) *over Z such that $f_*^{-1}l$ are contracted by f'.*

We apply the claim repeatedly and then we have a contraction of type III on X_k, $\psi : X_k \to Z$, such that $\mathrm{Exc}(\psi) = D_k$. Let C_{k-1} be an irreducible curve on X_{k-1} such that $C_{k-1} \subset \psi_k^{-1} \cdots \psi_n^{-1} y \cap D_{k-1}$. Then we know that $\overline{NE}(X_{k-1}/Z)$ is generated by $\mathbb{R}_{\geq 0}[C_{k-1}]$ and $\mathbb{R}_{\geq 0}[m_{k-1}]$. The latter extremal ray determines a contraction of type I on X_{k-1} and using the claim again, we obtain a contraction of type I on X whose exceptional set consists of m_0. □

Consider a log Enriques surface W with $I(W) \in \{2, 3, 4, 6\}$ such that W contains infinitely many contractible smooth rational curves. Then by Proposition 4.1, we can construct a C–Y 3-fold X with infinitely many birational contractions of type I or type III.

Example 4.3.

(i) See the nice survey, [11], by S. Kondō and its references for the details of the following. Due to E. Horikawa we know that the moduli space $\mathcal{M}$ of Enriques surfaces is 10-dimensional. The moduli space $\mathcal{N}$ of Enriques surfaces which contains at least one smooth rational curve is an irreducible subvariety of codimension 1 in $\mathcal{M}$. Enriques surfaces whose automorphism group is finite are classified by S. Kondō and the moduli of them consists of seven families $\{\mathcal{F}_i\}_{i=1}^{7}$ and each family is at most 1-dimensional. On the other hand for Enriques surfaces W, $\mathrm{Aut}\, W$ is finite if and only if W contains at least one but at most finitely many smooth rational curves. Consequently there exists the 9-dimensional moduli space, $\mathcal{N} \setminus \bigcup_{i=1}^{7} \mathcal{F}_i$, whose elements are Enriques surfaces which contain infinitely many smooth rational curves.

(ii) Let E_1, E_2 be elliptic curves which are not mutually isogenous and S' the Kummer surface associated to the Abelian surface $E_1 \times E_2$. Consider the involution a on S' induced by the involution $(x, y) \mapsto (x, -y)$ on $E_1 \times E_2$. Let $\{F_i\}_{i=1}^{4}$ (resp. $\{F_i'\}_{i=1}^{4}$) be the smooth rational curves on $E_1 \times E_2/(-1)$ which are the images of $\{x\} \times E_2$ (resp. $E_1 \times \{y\}$) by the natural map $E_1 \times E_2 \to E_1 \times E_2/(-1)$, where $x \in E_1$ (resp. $y \in E_2$) is a point of order 2. Then the fixed locus S'^a consists of the eight, disjoint smooth rational curves $f_*^{-1}F_i$, $f_*^{-1}F'_i$, where f is the minimal resolution of $E_1 \times E_2/(-1)$. Because the every generator of the Picard group of S' is fixed by the involution a, every smooth rational curve l' is also fixed, that is, $a \cdot l' = l'$. Contract the eight smooth rational curves $f_*^{-1}F_i$, $f_*^{-1}F'_i$ on S' and we get a normal K3 surface S with eight A_1-singularities. The group action of $\langle a \rangle$ on S' descends to the group action on S and let us use the same letter $\langle a \rangle$ for this action. Then we obtain a log Enriques surface $W := S/\langle a \rangle$ which contains infinitely many contractible smooth rational curves $\{m\}$ such that $m \cap \Sigma_W \neq \emptyset$. Here we use the fact that every Kummer surface has the infinite automorphism group and so in particular, it contains infinitely many smooth rational curves.

I do not know any example of *rational* log Enriques surface W which contains infinitely many smooth rational curves $\{m\}$ such that $m \cap \Sigma_W = \emptyset$.[*1]

The following statement is the converse of Proposition 4.1.

Proposition 4.4. *Suppose the conditions in Theorem* 3.6 (i) *hold. Then the log Enriques surface $W \cong S/G$ contains infinitely many contractible smooth rational curves $\{m\}$ such that $m = \varphi(E_i)$ and $m \cap \Sigma_W = \emptyset$.*

Proof. Because $G = H \rtimes \langle a \rangle$ as is in (ii)(d) in Theorem 3.3, we can decompose the quotient morphism $S \to W$ as follows:

$$S \xrightarrow{p} T := S/H \xrightarrow{q} S/G = T/\langle a \rangle \cong W\ .$$

Note that T is a normal K3 surface, for H is a Gorenstein group acting on S (and notice that H was trivial in the argument before Proposition 4.1). In particular, T has at most RDP's.

Claim 4.5. $l \cap S^{h \cdot a^i} = \emptyset$ *for all $h \in H$, all $i \neq 0$ modulo $I(W)$.*

Proof. The condition Remark 3.8$(a)'$ implies that $(l \times E) \cap (S \times E)^{[G]} = \emptyset$. Therefore if $E^{h \cdot a^i} \neq \emptyset$ for all $h \in H$, all $i \neq 0$ modulo $I(W)$, we know that $l \cap S^{h \cdot a^i} = \emptyset$. In fact this hypothesis is true, since the morphism $id_E - a^i$ on E is surjective. □

It is straightforward to see that

$$p^{-1}T^{a^i} = \bigcup_{h \in H} S^{h \cdot a^i} \text{ for all } i.$$

Thus we have $p(l) \cap T^{[\langle a \rangle]} = \emptyset$. On the other hand because $W \backslash q(T^{[\langle a \rangle]})$ has at most RDP's, $q \circ p(l) \cap \Sigma_W = \emptyset$. Since $q \circ p(l)$ is contractible by an extremal contraction on W, $q \circ p(l) \cong \mathbb{P}^1$. □

[*1] If a log Enriques surface W satisfies such conditions, the minimal resolution of W contains infinitely many -2 curves. I found an example of a smooth rational surface containing infinitely many -2 curves but unfortunately my surface is not the minimal resolution of log Enriques surface.

In summary, for a given C–Y 3-fold X with $|I_{c_2=0}| = \infty$ there exists an elliptic c_2-contraction $\varphi : X \to W$. Here W is a log Enriques surface with $I(W) \in \{2, 3, 4, 6\}$ which contains infinitely many smooth rational curves $\{m\}$ such that $m \cap \Sigma_W = \emptyset$ and $m = \varphi(E_i)$ for some $i \in I_{c_2=0}$. Conversely, for a given log Enriques surface W with $I(W) \in \{2, 3, 4, 6\}$ which contains infinitely many smooth rational curves $\{m\}$ such that $m \cap \Sigma_W = \emptyset$, there exists a C–Y 3-fold X with $|I_{c_2=0}| = \infty$ which admits an elliptic c_2-contraction $\varphi : X \to W$.

Research Institute
for Mathematical Sciences
Kyoto University
Kyoto 606-8502, Japan
e-mail: hokuto@kurims.kyoto-u.ac.jp

References

[1] M. Gross, *Primitive Calabi–Yau threefolds*, J. Differential Geom. **45** (1997), 288–318.

[2] D. R. Heath-Brown and P. M. H. Wilson, *Calabi–Yau threefolds with* $\rho > 13$, Math. Ann. **294** (1992), 49–57.

[3] Y. Kawamata, *Minimal models and the Kodaira dimension of algebraic fiber spaces*, J. Reine Angew. Math. **363** (1985), 1–46.

[4] ———, *Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces*, Ann. of Math. **127** (1988), 93–163.

[5] ———, *On the cone of divisors of Calabi–Yau fiber spaces*, Int. J. Math. **8** (1997), 665–687.

[6] Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem*, Adv. Stud. Pure Math. **10**, Alg. Geom. Sendai 1985 (T.Oda, ed.), Kinokuniya, Tokyo, 1987, 283–360.

[7] S. Keel, K. Matsuki and J. McKernan, *Log abundance theorem for threefolds*, Duke Math. J. **75** (1994), 99–119.

[8] S. Kobayashi, *Differential Geometry of Complex Vector Bundle*, Princeton University Press, 1987.

[9] J. Kollár et al., *Flips and Abundance for Algebraic Threefolds*, Astérisque **211**, Soc. Math. France, Paris, 1992.

[10] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Math. **134** (1998).

[11] S. Kondō, *Quadratic forms and K3 surfaces*, Enriques surfaces, Sugaku Expositions **6**-1 (1993), 53–72.

[12] S. J. Kovács, *The cone of curves of a K3 surface*, Math. Ann. **300** (1994), 681–691.

[13] Y. Miyaoka, *On the Kodaira dimension of minimal threefolds*, Math. Ann. **281** (1988), 325–332.

[14] D. Morrison, *Compactifications of moduli spaces inspired by mirror symmetry*, Astérisque **218**, Soc. Math. France, Paris 1993, 243–271.

[15] Y. Namikawa, *On the birational structure of certain Calabi–Yau* 3*-folds*, J. Math. Kyoto Univ. **31** (1991), 151–164.

[16] V. V. Nikulin, *The diagram method for 3-folds and its application to the Kähler cone and Picard number of Calabi–Yau* 3*-folds.* I, *Higher-dimensional complex varieties*, Proc. of Int. Conf. in Trento 1994, 1996, 261–328.

[17] K. Oguiso, *On algebraic fiber space structures on a Calabi–Yau* 3*-fold*, Int. J. Math. **4**, (1993), 439–465.

[18] ———, *On the complete classifications of Calabi–Yau three-folds of Type* III_0*, Higher Dimensional Complex Varieties*, Proc. of Int. Conf. in Trento 1994, 329–340, 1996.

[19] K. Oguiso and T. Peternell, *Calabi–Yau threefolds with positive second Chern class*, Comm. in Analysis and Geometry **6** (1998), 153–172.

[20] K. Oguiso and J. Sakurai, *Calabi–Yau threefolds of quotient type*, Asian J. Math. **5** (2001), 43–77.

[21] S. Roan, *Minimal resolution of Gorenstein Orbifolds in dimension three*, Topology **5** (1996), 489–508.

[22] H. Sterk, *Finiteness results for algebraic* $K3$ *surfaces*, Math. Z. **189** (1985), 507–513.

[23] N. I. Shepherd-Barron and P. M. H. Wilson, *Singular threefolds with numerically trivial first and second Chern class*, J. Algebraic Geom. **3** (1994), 265–281.

[24] P. M. H. Wilson, *Calabi–Yau manifolds with large Picard number*, Invent. Math. **98** (1989), 139–155.

[25] ———, *Minimal models of Calabi-Yau threefolds. Classification of algebraic varieties* (L'Aquila, 1992), Contemp. Math. **162**, Amer. Math. Soc., Providence, RI, 1994, 403–410.

[26] ———, *The role of* c_2 *in Calabi–Yau classification —a preliminary survey*, In Mirror Symmetry II, AMS/IP 1, 1997, 381–392.

[27] ———, *Symplectic deformations of Calabi–Yau threefolds*, J. Diff. Geom. **45** (1997), 611–637.