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On the inclusion of some Lorentz spaces

By

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Abstract

Let (X, Σ, μ) be a measure space. It is well known that $l^p(X) \subseteq l^q(X)$ whenever $0 < p \leq q \leq \infty$. Subramanian [12] characterized all positive measures μ on (X, Σ) for which $L^p(\mu) \subseteq L^q(\mu)$ whenever $0 < p \leq q \leq \infty$ and Romero [10] completed and improved some results of Subramanian [12]. Miamee [6] considered the more general inclusion $L^p(\mu) \subseteq L^q(\nu)$ where μ and ν are two measures on (X, Σ).

Let $L(p_1, q_1)(X, \mu)$ and $L(p_2, q_2)(X, \nu)$ be two Lorentz spaces,where $0 < p_1, p_2 < \infty$ and $0 < q_1, q_2 \leq \infty$. In this work we generalized these results to the Lorentz spaces and investigated that under what conditions $L(p_1, q_1)(X, \mu) \subseteq L(p_2, q_2)(X, \nu)$ for two different measures μ and ν on (X, Σ).

1. Introduction

Let (X, Σ, μ)be a measure space and let f be a measurable function on X.For each $y > 0$ let

$$\lambda_f(y) = \mu\{x \in X : f(x) > y\}. \tag{1}$$

The function λ_f is called the distribution function of f. The rearrangement of f is defined by

$$f^*(t) = \inf \{y > 0 : \lambda_f(y) \leq t\} = \sup \{y > 0 : \lambda_f(y) > t\}, \quad t > 0,$$

where $\inf \phi = +\infty$. Also the average function of f is defined by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds, t > 0. \tag{2}$$

Note that $\lambda_f(\cdot), f^*(\cdot)$ and $f^{**}(\cdot)$ are non-increasing and right continuous on $(0, \infty)$, [2]. For $p, q \in (0, \infty)$ we define

$$\|f\|^*_{p,q} = \|f\|^*_{p,q,\mu} = \left(\frac{q}{p} \int_0^\infty [f^*(t)]^q \, .t^{\frac{q}{p}-1} dt \right)^{\frac{1}{q}}$$

$$\|f\|_{p,q} = \|f\|_{p,q,\mu} = \left(\frac{q}{p} \int_0^\infty [f^{**}(t)]^q \, .t^{\frac{q}{p}-1} dt \right)^{\frac{1}{q}}. \tag{3}$$

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If $0 < p, q = \infty$ we also define

$$\|f\|^*_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}}.f^*(t) \text{ and } \|f\|_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}}.f^{**}(t). \tag{4}$$

For $0 < p < \infty$ and $0 < q \le \infty$, the Lorentz space denoted by $L(p,q)(X,\mu)$ (or shortly $L(p,q)$) is defined to be the vector space of all (equivalence classes of) measurable functions f on X such that $\|f\|^*_{p,q} < \infty$. We know that $\|f\|_p = \|f\|^*_{p,p}$ and so $L^p(\mu) = L(p,p)(X,\mu)$ where $L^p(\mu)$ is the Lebesgue space. Also $L(p,q_1) \subset L(p,q_2)$ for $q_1 \le q_2$. In particular

$$L(p,q_1) \subset L^p(\mu) \subset L(p,q_2) \subset L(p,\infty)$$

for $0 < q_1 \le p \le q_2 \le \infty$ ([2]). It is also known that if $1 < p < \infty$ and $1 \le q \le \infty$ then

$$\|f\|^*_{p,q} \le \|f\|_{p,q} \le \frac{p}{p-1}\|f\|^*_{p,q} \tag{5}$$

for each $f \in L(p,q)(X,\mu)$ ([11]). Moreover $\|f\|_{p,q}$ is a complete norm on $L(p,q)(X,\mu)$.

2. Main results

In this section we will accept that (X,Σ) is a measurable space and all measures are defined on the $\sigma-$algebra Σ. Also if two measures μ and ν are absolutely continuous with respect to each other (i.e $\mu << \nu$ and $\nu << \mu$) then we denote it by the symbol $\mu \approx \nu$.

Lemma 2.1. *Let $0 < p_1, p_2 < \infty$ and $0 < q_1, q_2 \le \infty$. Then the inclusion $L(p_1,q_1)(X,\mu) \subseteq L(p_2,q_2)(X,\nu)$ holds in the sense of equivalence classes if and only if $\mu \approx \nu$ and $L(p_1,q_1)(X,\mu) \subseteq L(p_2,q_2)(X,\nu)$ in the sense of individual functions.*

Proof. Assume that $L(p_1,q_1)(X,\mu) \subseteq L(p_2,q_2)(X,\nu)$holds in the sense of equivalent classes. Let $f \in L(p_1,q_1)(X,\mu)$ be any individual function. This implies $f \in L(p_1,q_1)(X,\mu)$ in the sense of equivalent classes thus $f \in L(p_2,q_2)(X,\nu)$ in the sense of equivalent classes by the assumption. Hence we have $f \in L(p_2,q_2)(X,\nu)$ in the sense of individual functions. This shows that

$$L(p_1,q_1)(X,\mu) \subseteq L(p_2,q_2)(X,\nu)$$

in the sense of individual functions. Now take any $E \in \Sigma$ with $\mu(E) = 0$. If χ_E is the characteristic function of E then $\chi_E = 0$ $\mu-$almost everywhere. Also the rearrangement of χ_E is

$$\chi^*_E(t) = \begin{cases} 1, & 0 < t < \mu(E), \\ 0, & t \ge \mu(E). \end{cases} \tag{6}$$

If $p_1, q_1 \in (0, \infty)$ we obtain

(7)

$$\begin{aligned}
\|\chi_E\|^*_{p_1,q_1} &= \left(\frac{q_1}{p_1} \int_0^\infty \left[t^{\frac{1}{p_1}} . \chi_E^*(t) \right]^{q_1} . \frac{dt}{t} \right)^{\frac{1}{q_1}} \\
&= \left(\frac{q_1}{p_1} \int_0^{\mu(E)} \left[t^{\frac{1}{p_1}} . \chi_E^*(t) \right]^{q_1} . \frac{dt}{t} \right)^{\frac{1}{q_1}} + \left(\frac{q_1}{p_1} \int_{\mu(E)}^\infty \left[t^{\frac{1}{p_1}} . \chi_E^*(t) \right]^{q_1} . \frac{dt}{t} \right)^{\frac{1}{q_1}} \\
&= \left(\frac{q_1}{p_1} \int_0^{\mu(E)} \left[t^{\frac{1}{p_1}} . \right]^{q_1} . \frac{dt}{t} \right)^{\frac{1}{q_1}} = \left(\frac{q_1}{p_1} \int_0^{\mu(E)} t^{\frac{q_1}{p_1}-1} . dt \right)^{\frac{1}{q_1}} \\
&= \left(\mu(E)^{\frac{q_1}{p_1}} \right)^{\frac{1}{q_1}} = \mu(E)^{\frac{1}{p_1}} = 0.
\end{aligned}$$

Also for the case $0 < p_1 < \infty$ and $q_1 = \infty$ we have

$$\|\chi_E\|^*_{p_1,\infty} = \sup_{t>0} t^{\frac{1}{p_1}} . \chi_E^*(t) = \mu(E) = 0. \tag{8}$$

Then we have $\chi_E \in L(p_1, q_1)(X, \mu)$ for $0 < p_1 < \infty$ and $0 < q_1 \leq \infty$.Thus χ_E is in the equivalent classes of $0 \in L(p_1, q_1)(X, \mu)$. Since the equivalence classes of 0 (with respect to μ) is also an element of $L(p_2, q_2)(X, \nu)$ by the hypothesis, then χ_E is in the equivalent classes of $0 \in L(p_2, q_2)(X, \nu)$ with respect to ν. That means $\nu(E) = 0$. Thus $\nu << \mu$. Similarly one can prove that $\mu << \nu$.

The proof of the other side is clear. □

Theorem 2.2. *Let $0 < p_1, p_2 < \infty$ and $0 < q_1, q_2 \leq \infty$.Then the inclusion*

$$L(p_1, q_1)(X, \mu) \subset L(p_2, q_2)(X, \nu)$$

holds in the sense of equivalence classes if and only if $\mu \approx \nu$ and there exists $C > 0$ such that

$$\|f\|^*_{p_2 q_{2,\nu}} \leq C \|f\|^*_{p_1 q_{1,\mu}}$$

for all $f \in L(p_1, q_1)(X, \mu)$).

Proof. Assume that $L(p_1, q_1)(X, \mu) \subseteq L(p_2, q_2)(X, \nu)$ holds in the sense of equivalent classes. Define the unit operator $I(f) = f$ from $L(p_1, q_1)(X, \mu)$ into $L(p_2, q_2)(X, \nu)$. We shall show that I is closed. Let (f_n) be a sequence such that $f_n \to f$ in $L(p_1, q_1)(X, \mu)$ and $I(f_n) = f_n \to g$ in $L(p_2, q_2)(X, \nu)$. It is known that

$$\|f\|^*_{p_1,\infty} \leq \|f\|^*_{p_1,q_1} \tag{9}$$

and

(10) $$\|f\|^*_{p_1\infty}=\sup_{t>0} t^{\frac{1}{p_1}}.f^*(t)=\sup_{y>0} y\ (\lambda_f(y))^{\frac{1}{p_1}}.$$

Let $\varepsilon>0$ be given. Since $f_n\to f$ in $L(p_1,q_1)(X,\mu)$, there exists $n_0\in N$ such that

(11) $$y(\lambda_{f_n-f})^{\frac{1}{p_1}}\le \|f_n-f\|^*_{p_1,q_1}<\varepsilon^{\frac{1}{p_1}}y$$

for all $n\ge n_0$. This implies $(\lambda_{f_n-f})<\varepsilon$ for all $n\ge n_0$. Then (f_n) converges to f in measure (with respect to μ). Hence there is a subsequence $(f_{n_i})\subset(f_n)$ such that (f_{n_i}) pointwise converges to f , $\mu-$ almost everywhere $(a.e)$. Also since (f_n) converges to g in $L(p_2,q_2)(X,\nu)$ then it is easy to prove that (f_{n_i}) converges to g in $L(p_2,q_2)(X,\nu)$. Then (f_{n_i}) converges to g in measure (with respect to ν). Therefore one can find a subsequence $(f_{n_{i_k}})\subset(f_{n_i})$ such that $(f_{n_{i_k}})$ converges to g pointwise $\nu-a.e.$ Let M be the set of the points such that $(f_{n_{i_k}})$ doesn't converge to g pointwise. Hence $\nu(M)=0$.Since by the assumption $L(p_1,q_1)(G,\mu)\subseteq L(p_2,q_2)(G,\nu)$ in the sense of equivalent classes then $\mu\approx\nu$ by Lemma 2.1. Thus $\nu(M)=\mu(M)=0$. Hence $(f_{n_{i_k}})$ converges to the function g pointwise $\mu-$a.e. Finally using the inequality

(12) $$|f(x)-g(x)|\le\left|f(x)-f_{n_{i_k}}(x)\right|+\left|f_{n_{i_k}}(x)-g(x)\right|$$

one can prove that $f=g$ $\mu-$a.e. Also it is clear that $f=g$ $\nu-$a.e. That means the unit function I is closed. Hence by the closed graph theorem there exists $C>0$ such that

$$\|f\|^*_{p_2,q_2,\nu}\le C.\|f\|^*_{p_1,q_1,\mu}$$

for all $f\in L(p_1,q_1)(G,\mu)$.

The proof of the other direction is easy.

If $0<p_1,p_2<\infty$ and $q_1=q_2=\infty$ then the proof is clear from (10). □

Lemma 2.3. *Let $0<p<\infty$,$0\le q\le\infty$ and $f\in L(p,q)(X,\mu)$ be a real valued measurable function. If there exists $M>0$ such that $\nu(E)\le M\mu(E)$ for all $E\in\Sigma$ then we have the inequality*

$$\|f\|^*_{p,q,\nu}\le M^{\frac{1}{p}}\|f\|^*_{p,q,\mu}.$$

Proof. Since $f\in L(p,q)(X,\mu)$ is a measurable real valued function then

(13) $$E_y=\{x\in X: f(x)>y\}\in\Sigma$$

for all real number y. If we set $k=M\mu$, it easy to see that k is a measure. Denote by $\nu(E_y)=\lambda^{\nu}_f(y)$ and $k(E_y)=\lambda^k_f(y)$. We also denote the rearrangements of f with respect to the measures k and ν by $f^{*,k}$ and $f^{*,\nu}$ respectively. Let A and B be such that

(14) $$A=\left\{y>0:\lambda^{\nu}_f(y)\le t\right\},$$
$$B=\left\{y>0:\lambda^{k}_f(y)\le t\right\}.$$

Since $\nu(E_y) \le M\mu(E_y) = k(E_y)$ we have $\lambda_f^{\nu}(y) \le \lambda_f^{k}(y)$. Thus we obtain $B \subseteq A$ and

$$f^{*,k}(t) = \inf_y B \ge \inf_y A = f^{*,\nu}(t). \tag{15}$$

This implies

$$\left(\frac{q}{p}\int_0^\infty t^{\frac{q}{p}-1}[f^{*,\nu}(t)]^q dt\right)^{\frac{1}{q}} \le \left(\frac{q}{p}\int_0^\infty t^{\frac{q}{p}-1}[f^{*,k}(t)]^q dt\right)^{\frac{1}{q}} \tag{16}$$

and

$$\|f\|^*_{p,q,\nu} \le \|f\|^*_{p,q,k}.$$

Also we write

$$\begin{aligned} \left\{y>0 : \lambda_f^k(y) \le t\right\} &= \{y>0 : k(E_y) \le t\} \\ &= \{y>0 : M\mu(E_y) \le t\} = \left\{y>0 : \mu(E_y) \le \frac{t}{M}\right\} \end{aligned} \tag{17}$$

and

$$f^{*,k}(t) = f^{*,\mu}\left(\frac{t}{M}\right). \tag{18}$$

Combining (15) and (18) we find

$$f^{*,k}(t) = f^{*,\mu}\left(\frac{t}{M}\right) > f^{*,\nu}(t).$$

This implies

$$\begin{aligned} \|f\|^*_{p,q,k} &= \left(\frac{q}{p}\int_0^\infty \left[f^{*,k}(t)\right]^q . t^{\frac{q}{p}-1} dt\right)^{\frac{1}{q}} \\ &= \left(\frac{q}{p}\int_0^\infty \left[f^{*,\mu}\left(\frac{t}{M}\right)\right]^q . t^{\frac{q}{p}-1} dt\right)^{\frac{1}{q}} = M^{\frac{1}{p}} \|f\|^*_{p,q,\mu} \end{aligned} \tag{19}$$

for all $f \in L(p,q)(X,\mu)$, where $k = M\mu$. Consequently we have

$$\|f\|^*_{p,q,\nu} \le \|f\|^*_{p,1,k} = M^{\frac{1}{p}} \|f\|^*_{p,q,\mu}. \tag{20}$$

□

Proposition 2.4. *Let $0 < p < \infty$ and $0 \le q \le \infty$. The following statements are equivalent*:

(1) $L(p,q)(X,\mu) \subseteq L(p,q)(X,\nu)$.

(2) $\mu \approx \nu$ *and there exists* $M > 0$ *such that* $\nu(E) \le M\mu(E)$ *for all* $E \in \Sigma$.

(3) $L^1(\mu) \subseteq L^1(\nu)$.

Proof. (1) ⇒ (2). By Theorem 2.2, there exists $C > 0$ such that

$$\|f\|_{p,q,v}^{*} \leq C\|f\|_{p,q,\mu}^{*} \tag{21}$$

for all $f \in L(p,q)(X,\mu)$. It follows from (7) in Lemma 2.1, and from (21) that

$$(\nu(E))^{\frac{1}{p}} \leq C.\,(\mu(E))^{\frac{1}{p}}\,,$$

and hence

$$\nu(E) \leq M(\mu(E)), \tag{22}$$

where $M = C^p$.

(2) ⇒ (1). It is known that the set S of simple functions are dense in $L(p,q)(X,\mu)$ ([3]). Define the unit function I from S into $L(p,q)(X,\nu)$. By Lemma 2.3, we have the inequality

$$\|f\|_{p,q,\nu}^{*} \leq C\|f\|_{p,q,\mu}^{*} \tag{23}$$

for all $f \in S$. Thus I is continuous from S into $L(p,q)(X,\nu)$. Then I is continuously extended to the space $(L\,(p,q)\,(X,\mu)$. Thus we have

$$\|f\|_{p,q,\nu}^{*} \leq C\|f\|_{p,q,\mu}^{*}$$

for all $f \in (L\,(p,q)\,(X,\mu)$. That means $L(p,q)(X,\mu) \subseteq L(p,q)(X,\nu)$.

(2) ⇒ (3). It is known that $L^1(\mu) = L(1,1)(X,\mu)$ and $L(1,1)(X,\nu) = L^1(\nu)$.Take any simple function $h(x) = \sum_{k=1}^{N} a_k.\chi_{E_k}(x)$ in $L^1(\mu)$ with E_i and E_j disjoint if $i \neq j$. Using (22) we have

$$\begin{aligned} \|h\|_{1,1,\nu}^{*} = \|h\|_{L^1(\nu)} &= \sum_{k=1}^{N} |a_k|\,\nu(E_k) \leq M \sum_{k=1}^{N} |a_k|\,\mu(E_k) \\ &= M.\,\|h\|_{L^1(\mu)} = M\,\|h\|_{1,1,\mu}^{*} < \infty. \end{aligned} \tag{24}$$

Hence h is a simple function in $L^1(\nu)$. Now let any $f \in L^1(\mu)$ be given. Since the set of simple functions is dense in $L^1(\mu)$ then there exists a sequence $(f_n) \subset L^1(\mu)$ of simple functions such that $f_n \to f$ in $L^1(\mu)$. Since (f_n) is a Cauchy sequence in $L^1(\mu)$ then (f_n) is also a Cauchy sequence in $L^1(\nu)$ from (24) and converges to a function g in $L^1(\nu)$. Using the subsequence argument similar as in the proof of Theorem 2.2. one can show that $f = g$. Thus $f \in L^1(\nu)$ and we have $L^1(\mu) \subseteq L^1(\nu)$.

The proof of (3) ⇒ (2) is easy from Theorem 2.2.

(3) ⇒ (1). Let $f \in L(p,q)(X,\mu)$ be given. Since $\chi_{(0,\infty)}.t^{\frac{q}{p}-1}.\,[f^*(t)]^q \in L^1(\mu)$ and $L^1(\mu) \subseteq L^1(\nu)$ we have $\chi_{(0,\infty)}.t^{\frac{q}{p}-1}.\,[f^*(t)]^q \in L^1(\nu)$. This implies $f \in L(p,q)(X,\nu)$ and we have $L(p,q)(X,\mu) \subseteq L(p,q)(X,\nu)$.

This completes the proof. □

Proposition 2.5. *Let* $p_1, p_2, q_{1,} q_2$ *be real numbers with* $0 < q_1 \leq p_1 < p_2 \leq q_2 < \infty$. *The following statements are equivalent:*

(1) $L(p_1, q_1)(X, \mu) \subseteq L(p_2, q_2)(X, \mu)$.

(2) *There exists a constant* $m > 0$ *such that* $\mu(E) \geq m$ *for every* $\mu-$*non-null set* $E \in \Sigma$.

Proof. (1) $\Rightarrow$ (2). By Theorem 2.2, there exists $C > 0$ such that $\|f\|_{p_2,q_2} \leq C\|f\|_{p_1,q_1}$ for all $f \in L(p_1, q_1)(X, \mu)$. Let $E \in \Sigma$ be a $\mu-$non-null set and $\mu(E) < \infty$. It follows from (7) as in the proof of Lemma 2.2, that

$$(\mu(E))^{\frac{1}{p_2}} \leq C.(\mu(E))^{\frac{1}{p_1}}. \tag{25}$$

Since $p_1 < p_2$ then $\frac{1}{p_1} - \frac{1}{p_2} > 0$. Thus we have

$$\frac{1}{C} \leq (\mu(E))^{\frac{1}{p_1} - \frac{1}{p_2}} = \mu(E)^{\frac{p_2 - p_1}{p_1 \cdot p_2}}. \tag{26}$$

If we set $m = C^{\frac{p_1 \cdot p_2}{p_1 - p_2}}$, we obtain $\mu(E) \geq m$.

(2) $\Rightarrow$ (1). Let $f \in L(p_1, q_1)(X, \mu)$. For every $n \in N$ we define

$$E_n = \{x \in X : |f(x)| > n\}. \tag{27}$$

Since $q_1 \leq p_1$ one writes $L(p_1, q_1)(X, \mu) \subseteq L(p_1, p_1)(X, \mu) = L^{p_1}(\mu)$ and there exists $K > 0$ such that

$$\|f\|_{p_1} \leq K.\|f\|_{p_1,q_1} \tag{28}$$

for all $f \in L(p_1, q_1)(X, \mu)$. It follows from (27) that

$$n^{p_1}.\mu(E_n) \leq \int\limits_{E_n} |f|^{p_1} d\mu \leq \int\limits_X |f|^{p_1} d\mu \leq \left(K\|f\|_{p_1,q_1}\right)^{p_1} < \infty \tag{29}$$

for all $n \in N$. By the hypothesis either $\mu(E_n) = 0$ or $\mu(E_n) \geq m$. Since the sequence (E_n) is a non-increasing and $\bigcap\limits_{n=1}^{\infty} E_n = \phi$, thus $\mu(E_n) \to 0$. Therefore there exists $n_0 \in N$ such that $|f(x)| \leq n_0$, $\mu-$a.e. for all $x \in X$. From formula (28) and the inequality

$$\int\limits_X |f|^{p_2} d\mu = \int\limits_X |f|^{p_1} |f|^{p_2 - p_1}.d\mu \leq n_0^{p_2 - p_1}.\int\limits_X |f|^{p_1} d\mu \tag{30}$$

we have $f \in L(p_2, p_2)(X, \mu)$. This implies $L(p_1, q_1)(X) \subseteq L(p_2, p_2)(X, \mu)$. Finally by the assumption $0 < q_1 \leq p_1 < p_2 \leq q_2 < \infty$ we obtain

$$L(p_1, q_1)(X, \mu) \subseteq L(p_1, p_1)(X) \subseteq L(p_2, p_2)(X, \mu) \subseteq L(p_2, q_2)(X, \mu).$$

$\square$

Proposition 2.6. *Let assume that* $0 < q_1 \leq q_2 \leq \infty$.

(1) *If* $\mu(X) < \infty$ *then* $L(p_1, q_1)\ (X,\ \mu) \subset L(p_2, q_2)(X, \mu)$ *whenever* $0 < p_1 < p_2 < \infty$ *if and only if any collection of disjoint measurable sets of positive measure is finite.*

(2) *If* $\mu(X) = \infty$ *then* $L(p_1, q_1)\ (X,\ \mu) \subset L(p_2, q_2)(X, \mu)$ *whenever* $0 < q_1 \leq p_1 < p_2 \leq q_2 < \infty$ *if and only if for any sequence* (E_n) *disjoint measurable sets of positive measure, the sequence* $(\mu(E_n))$ *is bounded away from zero.*

Proof. (1) Let $\mu(X) < \infty$ and $0 < p_1 < p_2 < \infty$. It is known that [3], $L(p_1, q_1)\ (X,\ \mu) \subset L(p_2, q_2)(X, \mu)$. If we get

$$r_1 = \min\{p_1, q_1\}, r_2 = \max\{p_2, q_2\}, \tag{31}$$

we obtain $r_1 \leq p_1 < p_2 \leq r_2$ and $r_1 \leq q_1 < q_2 \leq r_2$. Hence we have

$$L(r_1, r_1)(X, \mu) \subset L(p_1, q_1)(X, \mu) \subset L(p_2, q_2)(X, \mu) \subset L(r_2, r_2)(X, \mu). \tag{32}$$

Then for given any sequence (E_n) disjoint measurable sets of positive measure is finite by Proposition in [12]

The proof of the converse is clear again by Proposition in [12].

(2) Suppose $\mu(X) = \infty$. If a sequence (E_n) of disjoint measurable sets such that $\mu(E_n) > 0$ and the sequence (μE_n) is bounded away from zero then $L^{p_1}(\mu) \subseteq L^{p_2}(\mu)$ by Proposition in [12]. Thus

$$L^{p_1}(\mu) = L(p_1, p_1)(X, \mu) \subset L(p_2, p_2)(X, \mu) = L^{p_2}(\mu) \subset L(p_2, q_2)(X, \mu). \tag{33}$$

Since $q_1 \leq p_1 < p_2 \leq q_2$ then we have

$$\begin{aligned} L(p_1, q_1)(X, \mu) \subset L(p_1, p_1)(X, \mu) &= L^{p_1}(\mu) \subset L(p_2, p_2)(X, \mu) \\ &= L^{p_2}(X, \mu) \subset L(p_2, q_2)(X, \mu). \end{aligned} \tag{34}$$

Conversely assume that $L(p_1, q_1)(X, \mu) \subset L(p_2, q_2)(X, \mu)$ and (E_n) is collection of disjoint measurable sets of positive measure. If one applies the proof technic in (i) of this Proposition shows that the sequence $(\mu(E_n))$ is bounded away from zero by Proposition in [12]. □

Proposition 2.7. *Let* X *be a metrisable locally compact abelian group with Haar measure* μ *and* $\mu(X) = \infty$. *If* $0 \leq q_1 \leq p_1 < p_2 \leq q_2 < \infty$ *then the inclusion*

$$L(p_1, q_1)(X, \mu) \subseteq L(p_2, q_2)(X, \mu) \tag{35}$$

is not satisfied.

Proof. Let d be a metric on X and $(x_n)_{n \in N}$ be a sequence in X such that $d(x_i, x_j) \geq 2r$ for $i \neq j$, where $0 < r < 1$. Get the open balls $A_n = x_n + B(0, r^n), n \in N$. It is easy to see that $(A_n)_{n \in N}$ is a disjoint sequence. Since X is locally compact group then there exists compact subsets $E_n \subset$

A_n with $\mu(E_n) < \infty$ for all $n \in N$. Thus the sequence $(E_n)_{n\in N}$ is disjoint. Since $\lim\limits_{n\to\infty} \mu(A_n) = 0$ and $\mu(E_n) \le \mu(A_n)$ for all $n \in N$, then we obtain $\lim\limits_{n\to\infty} \mu(E_n) = 0$. Hence the inclusion $L(p_1, q_1)(X, \mu) \subseteq L(p_2, q_2)(X, \mu)$ does not satisfy by Proposition 2.6. □

Example: It is known that the Lebesgue measure of the set of real numbers $\mu(R) = \infty$. Define

$$A_n = n + \left(-\frac{1}{2^n}, \frac{1}{2^n}\right)$$

for all $n \in N$. The sequence of measurable sets $(A_n)_{n\in N}$ is disjoint and $\mu(A_n) = \frac{1}{2^{n-1}} > 0$ for all $n \in N$. But

$$\lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \frac{1}{2^{n-1}} = 0.$$

Hence if we take $X = R$ with the absolute value metric in the Proposition 2.7 we see that the inclusion (35) is not true.

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