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On the minimal solution for quasilinear degenerate elliptic equation and its blow-up

By

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Abstract

In this article we shall investigate the minimal and the extremal solutions of quasilinear elliptic equation with a positive nonlinear term in the right hand side. More precisely we shall study the boundary value problem

$$\begin{cases} L_p(u) = \lambda f(u), & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases}$$

where λ is a nonnegative parameter, Ω is a domain of R^N and $L_p(\cdot)(p > 1)$ is the p-Laplace operator defined by $L_p(\cdot) = -\operatorname{div}(|\nabla \cdot|^{p-2}\nabla \cdot))$. We assume that $f(t)$ is increasing on $[0, \infty)$ and strictly convex with $f(0) > 0$. Under some additional conditions, we first establish the existence of the minimal solution u_λ and the extremal solution u^* to this equation and study their behaviors in connection with the linearized operator given by $L'_p(u)(\cdot) = -\operatorname{div}\left(|\nabla u|^{p-2}(\nabla \cdot + (p-2)\frac{(\nabla u, \nabla \cdot)}{|\nabla u|^2}\nabla u)\right)$. The minimal solution $u_\lambda \in C^{1,\sigma}(\overline{\Omega})$ is defined as the smallest solution among all possible classical solutions, and the extremal solution is defined as an increasing limit of u_λ in $W_0^{1,p}(\Omega)$ as $\lambda \to \lambda^*$ (the extremal value). Though $L'_p(u_\lambda)(\cdot)$ is, roughly speaking, a degenerate elliptic operator, it is shown that $L'_p(u_\lambda)(\cdot)$ has a compact inverse from $L^2(\Omega)$ to itself if u_λ is minimal. Moreover the self-adjoint operator $L'_p(u_\lambda)(\cdot) - \lambda f'(u_\lambda)$ on $L^2(\Omega)$ has a positive first eigenvalue if λ is sufficiently small and a nonnegative first eigenvalue for any $\lambda \in (0, \lambda^*)$. Finally in Section 10 we give the characterizations of the extremal solution which are essentially depend upon the value of p and the topology of Ω (see Theorem 10.1 and subsequent Propositions). When Ω is a ball, we investigate these problems rather precisely using the weighted Hardy type inequality with a sharp missing term.

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1. Introduction

Let N be a positive integer and let Ω be a bounded open set of $\mathbb{R}^N$ whose boundary $\partial\Omega$ is of class C^2. In connection with combustion theory and other applications, we are interested in the study of positive solutions of the quasilinear elliptic boundary value problem

$$(1.1) \qquad \begin{cases} L_p(u) = \lambda f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $L_p(\cdot)$ is the p-Laplace operator defined by $L_p(\cdot) = -\operatorname{div}(|\nabla \cdot|^{p-2}\nabla \cdot))$. Here $p > 1$, λ is a nonnegative parameter and the nonlinearity f is, roughly speaking, continuously differentiable, positive, increasing and strictly convex on $[0, +\infty)$ (see also the condition (2.2)). Typical examples are $f(t) = e^t$ and $(1+t)^q$ for $q > p-1$. When $p = 2$, it is known that there is a finite number λ^* such that (1.1) has a classical positive solution $u \in C^2(\overline{\Omega})$ if $0 < \lambda < \lambda^*$. On the other hand no solution exists, even in the weak sense, for $\lambda > \lambda^*$. This value λ^* is often called the extremal value and solutions for this extremal value are called extremal solutions. It has been a very interesting problem to find and study the properties of these extremal solutions.

In this paper we shall study similar problems for the quasilinear operator $L_p(u)$ $(p > 1)$. In Section 2 we explain our general setting and prepare results concerned with p-Laplace operator, which are basic in the present paper. The minimal solution $u_\lambda \in C^{1,\sigma}(\overline{\Omega})$ $(0 < \sigma < 1)$ is defined by Definition 2.3 as the smallest solution among all possible classical solutions, and then the extremal solution is introduced as an increasing limit of u_λ as $\lambda \to \lambda^*$ (the extremal value). For the precise definition, see Definition 1.1 below (see also Definition 1.2). Under some additional conditions, we first establish the existence of the minimal solutions to (1.1) and study their behaviors in connection with the linearized operator defined by

$$(1.2) \qquad L'_p(u)(\cdot) = -\operatorname{div}\left(|\nabla u|^{p-2}\left(\nabla \cdot + (p-2)\frac{(\nabla u, \nabla \cdot)}{|\nabla u|^2}\nabla u\right)\right).$$

Since $L_p(u)$ is not always differentiable at any point $u \in W_0^{1,p}(\Omega)$ in the sense of Frechet, we shall employ the directional derivatives at the minimal solution u_λ. More precisely we introduce in Section 3 a Hilbert space $V_{\lambda,p}(\Omega)$ and an admissible class of directions $\tilde{V}_{\lambda,p}(\Omega) \subset V_{\lambda,p}(\Omega)$ which depend essentially upon u_λ. Then the operator $L_p(\cdot)$ becomes differentiable at u_λ in the direction to $\tilde{V}_{\lambda,p}(\Omega)$ (see Proposition 3.1).

Although $L'_p(u_\lambda)(\cdot)$ is, roughly speaking, a degenerate elliptic operator, it will be shown in Section 4 that $L'_p(u_\lambda)(\cdot)$ has a compact inverse from $L^2(\Omega)$ to itself. This crucial property is based on the compactness of the imbedding; $V_{\lambda,p}(\Omega) \longrightarrow L^2(\Omega)$ for $\lambda \in (0, \lambda^*)$ (see Proposition 4.2). It is also shown that $L'_p(u_\lambda) - \lambda f'(u_\lambda)$ is extended as a self-adjoint operator on $L^2(\Omega)$ by virtue of a coercive quadratic form on $V_{\lambda,p}(\Omega) \times V_{\lambda,p}(\Omega)$ defined in Section 3. Then the positivity of the first eigenvalue of $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ will be proved in case that λ is sufficiently small. From this fact we will study the behaviors of u_λ and its left derivative v_λ near $\lambda = 0$ in Scetions 6 and 7. We shall also prove in Sections 8 and 9 the nonnegativity of the first eigenvalue of $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ under the assumption (AC) below on the first eigenfunction.

In order to describe the main results of this paper, here we prepare the precise definition of the extremal value λ^*.

Definition 1.1 (Extremal value λ^*)**.** The extremal value λ^* is defined as the supremum of μ such that:

(a) For any $\lambda \in (0, \mu]$ there exists the minimal solution u_λ of (1.1).

(b) The following Hardy type inequality is valid:

$$\int_\Omega |\nabla u_\lambda|^{p-2} \left(|\nabla \varphi|^2 + (p-2) \frac{(\nabla u_\lambda, \nabla \varphi)^2}{|\nabla u_\lambda|^2} \right) dx \geq \lambda \int_\Omega f'(u_\lambda) \varphi^2 \, dx \tag{1.3}$$

for any $\varphi \in V_{\lambda,p}(\Omega)$. Here $V_{\lambda,p}(\Omega)$ is defined by

$$V_{\lambda,p}(\Omega) = \{\varphi \in M(\Omega) : ||\varphi||_{V_{\lambda,p}} < +\infty, \varphi = 0 \text{ on } \partial\Omega\}, \tag{1.4}$$

where

$$||\varphi||_{V_{\lambda,p}} = \left(\int_\Omega |\nabla u_\lambda(x)|^{p-2} |\nabla \varphi|^2 \, dx \right)^{\frac{1}{2}} \tag{1.5}$$

and by $M(\Omega)$ we denote the set of all measurable functions on Ω.

Remark 1.1. (1) For the definition of the minimal solution, see Definition 2.3.

(2) The validity of the Hardy type inequality (1.3) is equivalent to the nonnegativity of the first eigenvalue of $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ for $\lambda \in (0, \lambda^*)$ as usual (see Theorem 5.1 in Section 5 and Theorem 8.1 in Section 8).

(3) In the definition of $V_{\lambda,p}(\Omega)$, the condition $\varphi = 0$ on $\partial\Omega$ is taken in a sense of trace. Since ∇u_λ does not vanish near the boundary, $\varphi \in V_{\lambda,p}(\Omega)$ is differentiable there in a weak sense. We will see $V_{\lambda,p}(\Omega) \subset L^2(\Omega)$ as Corollary 4.3 in Section 4.

We also define

Definition 1.2 (Extremal solution)**.** The solution for the extremal value λ^* is called the extremal solution.

Combining all results among these sections, we first obtain

Theorem 1.1. *Assume that $1 < p < +\infty$ and f satisfies the conditions (2.1) and (2.2). Then we have the followings*:

(1) *The extremal value λ^* is positive. Moreover the first eigenvalue of $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ is positive provided that λ is suficiently small.*

(2) *Let* $u_\lambda \in C^1(\overline{\Omega})$ *be the minimal solution of* (1.1) *for* $\lambda \in (0, \lambda^*)$. *We have as* $\lambda \to \lambda^*$ *a finite limit a.e.*

$$u^*(x) = \lim_{\lambda \to \lambda^*} u_\lambda(x). \tag{1.6}$$

Moreover $u^* \in W_0^{1,p}(\Omega)$ *and* u^* *is a weak energy solution of* (1.1) *with* $\lambda = \lambda^*$.

Remark 1.2. (1) As for the definition of the weak energy solution, see Definition 2.1.

(2) This result will be proved in two theorems in Sections 2 and 5 (See also Section 8). More precisely the assertion (1) will be proved as Theorem 5.1 in Section 5 using the results in Section 4. The assertion (2) will be proved in Section 2 as Theorems 2.1 admitting the assertion (1).

As for the smooth dependency of u_λ on λ we shall show in Sections 6 and 7 the following:

Theorem 1.2. *Assume that* $p \in [2, \infty)$ *and* $\lambda \in (0, \lambda^*)$. *Then the following statements are equivalent:*

(1) *The self-adjoint operator* $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ *on* $L^2(\Omega)$ *has a positive first eigenvalue.*

(2) u_λ *is left differentiable at* λ *in* $V_{\lambda,p}(\Omega)$. *Moreover the left derivative* $v_\lambda \in V_{\lambda,p}(\Omega)$ *satisfies the boundary value problem*

$$\begin{cases} L_p'(u_\lambda)v_\lambda - \lambda f'(u_\lambda)v_\lambda = f(u_\lambda) & \text{in} \quad \Omega, \\ v_\lambda = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.7}$$

Remark 1.3. If the minimal solution u_λ is continuous on λ for each $x \in \Omega$ and weakly continuous as a $W_0^{1,p}(\Omega)$-valued function, then u_λ becomes differentiable and the derivative of u_λ satisfies (1.7) under the condition (1). Later we shall give an example in which these assumptions are satisfied. See Proposition 12.1 in Section 12.

Since $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ has a discrete spectrum, we can define the first eigenfunction as follows.

Definition 1.3 (First eigenfunction $\hat{\varphi}^\lambda$)**.** Let u_λ be the minimal solution of (1.1) for $\lambda \in (0, \lambda^*)$. By $\hat{\varphi}^\lambda$ we denote the first eigenfunction of the self-adjoint operator $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ on $L^2(\Omega)$, which is nonnegative and unique up to a multiplication by constants.

We also define

Definition 1.4 (Accessibility Condition)**.** The first eigenfunction $\hat{\varphi}^\lambda$ is said to satisfy the accessibility condition (AC) if for any $\varepsilon > 0$ there exists a nonnegative $\varphi \in \tilde{V}_{\lambda,p}(\Omega)$ such that

$$L_p'(u_\lambda)(\varphi - \hat{\varphi}^\lambda) + |\varphi - \hat{\varphi}^\lambda| \le \varepsilon \max(\hat{\varphi}^\lambda, dist(x, \partial\Omega)) \quad \text{in} \quad \Omega. \tag{1.8}$$

Here $\tilde{V}_{\lambda,p}(\Omega)$ is given by Definition (3.6) in Section 3.

In the next we establish the nonnegativity of the first eigenvalue of $L'_p(u_\lambda) - \lambda f'(u_\lambda)$ for any minimal solution u_λ.

Theorem 1.3. *Assume that* $1 < p < +\infty$ *and* f *satisfies the conditions* (2.1) *and* (2.2)*. Let* $u_\lambda \in C^1(\overline{\Omega})$ *be the minimal solution of* (1.1) *for some* $\lambda > 0$*. In addition we assume that the first eigenfunction* $\hat{\varphi}^\lambda$ *satisfies the accessiblity condition* (AC)*. Then the first eigenvalue of* $L'_p(u_\lambda) - \lambda f'(u_\lambda)$ *is nonnegative.*

Remark 1.4. (1) This will be established as Theorem 8.1 through a chain of Propositions and the proof will be finished in Section 9.

(2) If Ω is radially symmetric, then the minimal solution becomes radial. In this case the accessiblity condition (AC) on the first eigenfunction $\hat{\varphi}^\lambda$ is satisfied (see Proposition 12.2 in Section 12). In fact, assuming that Ω is a unit ball, φ can be constructed by truncating the eigenfunction $\hat{\varphi}^\lambda$ smoothly in a small neighborhood of the origin. Then, near the origin $L'_p(u_\lambda)\varphi$ vanishes but $L'_p(u_\lambda)\hat{\varphi}^\lambda$ is nonnegative by virture of the positivity of the first eigenvalue of $L'_p(u_\lambda)$. Therefore it is not difficult to check (1.8).

In Section 10 we shall give characterizations of the extremal solutions which are essentially depending upon the value of p and the topology of Ω. To this end we introduce a singular energy solution.

Definition 1.5 (Singular energy solution)**.** If a weak energy solution u is not bounded, u is said to be singular.

First we prove a non-existence of weak energy solutions for any $\lambda > \lambda^*$.

Theorem 1.4. *Let* $u^* = u_{\lambda^*}$ *be a singular extremal solution. Assume that* $f(t)$ *satisfies the growth condition* (GC) *in addition to* (2.2)*. Then there is no weak energy solution to* (1.1) *provided that* $\lambda > \lambda^*$*. Here the growth condition* (GC) *is defined by Definition* 10.1 *in Section* 10*.*

Remark 1.5. In Section 12 we give two examples of singular energy solutions assuming that $\Omega \equiv B = \{x \in \mathbb{R}^N : |x| < 1\}$ (a unit ball). See Lemma 12.2.

Then we shall give characterizations for the extremal solutions according to the range of p in terms of propositions. Roughly speaking, if $p \geq 2$, then it is necessary to satisfy the Hardy type inequality (1.3) with $\lambda = \lambda^*$ for $u^* \in W_0^{1,p}(\Omega)$ to be the extremal. More precisely we have

Proposition 1.1. *Assume that* $p \geq 2$*. Let* u^* *be the extremal solution. Then we have*

$$\int_\Omega |\nabla u^*|^{p-2} \left(|\nabla\varphi|^2 + (p-2)\frac{(\nabla u^*, \nabla\varphi)^2}{|\nabla u^*|^2} \right) dx \geq \lambda^* \int_\Omega f'(u^*)\varphi^2 \, dx \tag{1.9}$$

for any $\varphi \in V_{\lambda^*,p}(\Omega)$.

Remark 1.6. If $1 < p < 2$, then (1.9) is also necessary under additional conditions (10.20). See Proposition 10.2.

Conversely assume that $u \in W_0^{1,p}(\Omega)$ is singular (unbounded) and satisfies the Hardy type inequality (1.3) for some $\lambda > 0$ with u in place of u_λ. If $1 < p \leq 2$, then we can show $\lambda = \lambda^*$ and $u = u^*$ under additional conditions. Namely we have

Proposition 1.2. *Assume that $1 < p \leq 2$ and the nonlinearlity $f(t)$ satisfies the growth condition (GC) in addition to* (2.2)*. For $\lambda > 0$, let u_λ be the minimal solution or possibly the extremal solution. Let $u \in W_0^{1,p}(\Omega)$ be a singular weak energy solution of* (1.1) *such that*

$$\int_\Omega |\nabla u|^{p-2} \left(|\nabla \varphi|^2 + (p-2) \frac{(\nabla u, \nabla \varphi)^2}{|\nabla u|^2} \right) dx \geq \lambda \int_\Omega f'(u) \varphi^2 \, dx \tag{1.10}$$

for any $\varphi \in V_{\lambda,p}(\Omega)$. Moreover, if $1 < p < 2$, then we assume that

$$|\nabla u| \geq |\nabla u_\lambda| \qquad a.e. \ in \quad \Omega. \tag{1.11}$$

Then we have $\lambda = \lambda^*$ and $u = u_\lambda = u^*$

Remark 1.7. If $p > 2$, we have somewhat weaker result. See Proposition 10.4.

When Ω is a ball, in Section 12 we investigate these problems rather precisely by using **the weighted Hardy type inequality with a sharp missing term** established in Section 11. The extremal solutions are determined in most cases and the continuity of the minimal solution u_λ on λ is also shown in the case that $1 < p \leq 2$. For the precise organization of this paper, see the table of contents below.

Contents

2. Preliminaries

Let Ω be a bounded domain of $\mathbb{R}^N$ having C^2 class boundary. Let p satisfy $1 < p < +\infty$. Let $f(t)$ satisfy the following conditions throughout this paper:

$$\begin{cases} f(t) \in C^1([0,+\infty)), \\ f(t) \text{ is } \textbf{increasing} \text{ and } \textbf{strictly convex} \text{ with } f(0) > 0. \end{cases} \tag{2.1}$$

Moreover, $f(t)$ satisfies

$$\liminf_{t\to\infty} \frac{f'(t)t}{f(t)} > p-1. \tag{2.2}$$

Now we consider the boundary value problem:

$$\begin{cases} L_p(u) = \lambda f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.3}$$

where $L_p(\cdot)$ is the p-Laplace operator defined by $L_p(\cdot) = -\operatorname{div}(|\nabla\cdot|^{p-2}\nabla\cdot))$.

First we define a weak energy solution and a classical solution of the problem (2.3). To do so we need more notations. Let l be an arbitrary nonnegative integer. By $W^{l,p}(\Omega)$ we denote the space of all functions on Ω, whose generalized derivatives $\partial^\gamma u$ of order $\leq l$ satisfy

$$||u||_{W^{l,p}(\Omega)} = \sum_{|\gamma|\leq l} \left(\int_\Omega |\partial^\gamma u(x)|^p\, dx\right)^{1/p} < +\infty. \tag{2.4}$$

By $W_0^{l,p}(\Omega)$ we denote the completion of $C_0^\infty(\Omega)$ with respect to the norm defined by (2.4). Conventionally we set $L^p(\Omega) = W^{0,p}(\Omega)$.

Definition 2.1 (Weak energy solution of (2.3) in $W_0^{1,p}(\Omega)$)**.** By $\delta(x) = dist(x, \partial\Omega)$ we denote the distance to the boundary from x. A function $u \in W_0^{1,p}(\Omega)$ is called a weak energy solution of (2.3) if $f(u)$ satisfies

$$dist(x,\partial\Omega)\cdot f(u) \in L^1(\Omega) \tag{2.5}$$

and u satisfies (2.3) in the following weak sense:

$$\int_\Omega (|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi - \lambda f(u)\varphi)\, dx = 0 \tag{2.6}$$

for all $\varphi \in C^1(\overline{\Omega})$ with $\varphi = 0$ on $\partial\Omega$.

Definition 2.2 (Classical solution of (2.3) in $C^1(\overline{\Omega})$). A weak energy solution u of (2.3) is called a classical solution if it belongs to $C^1(\overline{\Omega})$.

From the standard elliptic regularity theory it follows that bounded weak energy solutions for this problem belong to Hölder space $C^{1,\sigma}(\overline{\Omega})$ for some $\sigma \in (0,1)$. Therefore a bounded weak energy solution of (2.3) becomes a classical solution. More precisely we have

Lemma 2.1. *Let g be a continuous function on $\mathbb{R}$. Let $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a weak solution of*

$$L_p(u) = g(u). \tag{2.7}$$

Then there exist $C > 0$ and $\sigma \in (0,1)$ such that

$$\begin{cases} |\nabla u(x)| \le C & \text{for any} \quad x \in \Omega, \\ |\nabla u(x) - \nabla u(y)| \le C|x-y|^\sigma & \text{for any} \quad (x,y) \in \Omega \times \Omega. \end{cases} \tag{2.8}$$

For the proof see [10; Theorems 1,2] and [14; Theorem 5.1] for example. See also [5] and [12].

We also recall the following elementary lemmas 2.2 and 2.3 for the sake of self-containedness.

Lemma 2.2 (Weak comparison principle). *Let $p' = \frac{p}{p-1}$. Assume that $f, g \in L^{p'}(\Omega)$ satisfy $0 \le f \le g$ a.e.. Moreover assume that $u, v \in W_0^{1,p}(\Omega)$ satisfy*

$$\begin{cases} L_p(u) = f & \text{in} \quad \Omega, \\ L_p(v) = g & \text{in} \quad \Omega. \end{cases} \tag{2.9}$$

Then $u \le v$ a.e. in Ω.

Proof. We may assume that u and v are smooth. Then it suffices to set in Definition 2.1 of weak energy solution $\varphi = \max(0, u-v)$ and $f(u) \equiv 0$. In fact we see $\varphi = 0$ from the monotonicity of the p-Laplace operator. □

Lemma 2.3. *For any $p \in (1, +\infty)$ we have*

$$(|X|^{p-2}X - |Y|^{p-2}Y) \cdot (X-Y) \ge C_p|X-Y|^2(|X|+|Y|)^{p-2}. \tag{2.10}$$

In particular if $p \ge 2$

$$(|X|^{p-2}X - |Y|^{p-2}Y) \cdot (X-Y) \ge C_p|X-Y|^p, \tag{2.11}$$

where X and Y are arbitrary points in $\mathbb{R}^N$ and C_p is a positive number independent of each (X,Y).

The proof is omitted (see [4] for example). The next is known as strong maximum principle. For the proof see [13; J. L. Vazquez].

Lemma 2.4 (Strong maximum principle). *Let $u \in C^1(\Omega)$ be such that $L_p(u) \in L^2_{loc}(\Omega)$, $u \geq 0$, a.e. in Ω, $-L_p(u) \leq \beta(u)$ a.e. in Ω with $\beta : [0, \infty) \to \mathbb{R}$ continuous, nondecreasing, $\beta(0) = 0$ and either $\beta(s) = 0$ for some $s > 0$ or $\beta(s) > 0$ for all $s > 0$ but $\int_0^1 (\beta(s)s)^{-\frac{1}{p}} \, ds = \infty$.*

Then if u does not vanish identically on Ω, it is positive everywhere in Ω.

Moreover, if $u \in C^1(\Omega \cup \{x_0\})$ for an $x_0 \in \partial\Omega$ that satisfies an interior sphere condition and $u(x_0) = 0$, then

$$\frac{\partial u}{\partial \nu} > 0, \tag{2.12}$$

where ν is an interior normal at x_0.

We can show that there exists a classical solution to (2.3) for sufficiently small $\lambda > 0$. In fact we can construct so-called supersolution and subsolution. Then from the standard method of nonlinear iteration, we can show the existence of a classical solution for a small $\lambda > 0$. In this way we have

Lemma 2.5. *Under these assumptions, there exist a supersolution and a subsolution for a sufficiently small $\lambda > 0$. Moreover there exists at least one classical solution u of* (2.3) *if λ is sufficiently small.*

Proof. Since 0 is a subsolution, it suffices to construct a supersolution for a sufficiently small $\lambda > 0$. To this end we consider the Dirichlet boundary value problem given by $L_p(v) = 1$ in $\Omega\,; v = 0$ on $\partial\Omega$. From the theory of monotone operator and elliptic regularity, we see that there is a unique classical solution v. Since v is nonnegative by the maximum principle, $L_p(v) \geq \lambda f(v)$ holds for a sufficiently small $\lambda > 0$. This proves the assertion. □

By virtue of this, we are able to define the so-called minimal solution u_λ.

Definition 2.3 (Minimal solution). The minimal solution $u_\lambda \in C^1(\overline{\Omega})$ is defined as the smallest solution among all possible classical solutions.

The existence of the minimal solution follows from a standard argument of monotone iteration (for the sake of self-containedness we give a short proof).

Lemma 2.6. *For a sufficiently small $\lambda \geq 0$, there exists the minimal solution $u_\lambda \in C^1(\overline{\Omega})$ uniquely.*

Proof. From the previous lemma we have at least one classical solution u for a small $\lambda > 0$. If we have another classical solution v for the same λ, we set $w = \min(u, v)$ and let $\tilde{u} \in W_0^{1,p}(\Omega)$ be a solution of the boundary value problem below.

$$\begin{cases} L_p(\tilde{u}) = \lambda f(w) & \text{in} \quad \Omega, \\ \tilde{u} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.13}$$

Since $f(w) \in L^\infty$, we see $\tilde{u} \in C^{1,\sigma}(\overline{\Omega})$ for some $\sigma \in (0, 1)$. We claim that $\tilde{u} \leq w$ in Ω. Since $L_p(u) = \lambda f(u) \geq \lambda f(w)$ in Ω in the sense of distribution, it follows from the weak comparison principle Lemma 2.2 that $\tilde{u} \le u$. In a similar way we have $\tilde{u} \le v$ and this proves the claim. Then we have

$$\begin{cases} L_p(\tilde{u}) = \lambda f(w) \ge \lambda f(\tilde{u}) & \text{in} \quad \Omega, \\ \tilde{u} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.14}$$

This implies that $\tilde{u}$ is a supersolution of the equation $L_p(u) = \lambda f(u)$ with Dirichlet boundary condition. Hence from a standard monotone iteration argument we see the existence of classical solution $\tilde{w}$ such that $0 < \tilde{w} \le \tilde{u} \le \min(u, v)$ in Ω. Therefore as a decreasing limit, using an argument of weak compactness in $W_0^{1,p}(\Omega)$, there is a unique minimal solution. □

More precisely we have the following.

Lemma 2.7. *For a sufficiently small $\lambda_0 > 0$ there exists the minimal solutions u_λ for any $\lambda \in [0, \lambda_0]$ such that:*
(1) *$u_\lambda \in C^{1,\sigma}(\overline{\Omega})$ for some $\sigma \in (0, 1)$.*
(2) *For $\lambda > 0$, $u_\lambda > 0$ in Ω and $u_\lambda = 0$ on $\partial\Omega$. If $\lambda = 0$, then $u_0 \equiv 0$.*
(3) *u_λ is a strongly increasing and left continuous function on λ for each $x \in \Omega$.*
(4) *The mapping: $[0, \lambda_0] \ni \lambda \longrightarrow u_\lambda \in W_0^{1,p}(B)$ is weakly left continuous.*

Proof. The existence of the minimal solution u_λ for a small $\lambda > 0$ follows from Lemma 2.6. Therefore the assertions (1) and (2) follow from Lemma 2.1 and the classical maximum principle respectively. Let $u = u_\lambda \in C^{1,\sigma}(\overline{\Omega})$ be the minimal solution. For any positive number m we set $v = mu$. Then

$$L_p(v) = m^{p-1}\lambda f(v/m).$$

If $0 < m < 1$, then $L_p(v) \ge m^{p-1}\lambda f(v)$. Hence v is a supersolution (on the other hand, if $m > 1$, v becomes a subsolution). From the comparison principle we see $u_\lambda \ge \frac{1}{m} u_{\lambda m^{p-1}}$ provided $0 < m < 1$. Now we put $m = (1-\varepsilon)^{\frac{1}{p-1}}$ for $\varepsilon \in (0, 1)$. Then we see for sufficiently small $\varepsilon > 0$

$$u_\lambda \ge (1-\varepsilon)^{\frac{-1}{p-1}} u_{\lambda(1-\varepsilon)} > u_{\lambda(1-\varepsilon)}.$$

This clearly implies the strict monotonicity of u_λ w.r.t. λ. Let u_{λ_0} be the minimal solution for $\lambda = \lambda_0$. Since u_λ is increasing, the limit $\lim_{\lambda < \lambda_0, \lambda \to \lambda_0} u_\lambda = u_{\lambda_0 - 0} \le u_{\lambda_0}$ exists. Moreover we can show u_λ also converges weakly to some element in $W_0^{1,p}(\Omega)$. Hence $u_{\lambda_0 - 0}$ becomes a weak solution of (2.3). Then it follows from the minimality that $u_{\lambda_0 - 0} = u_{\lambda_0}$ in $C^{1,\sigma}(\Omega) \cap W_0^{1,p}(\Omega)$. This proves the left continuity of u_λ at λ_0. □

Remark 2.1. (1) When $1 < p \le 2$ and Ω is a ball in $\mathbb{R}^N$, then under some additional conditions, the family of minimal solutions are right continuous as well. See Proposition12.1. Later in Section 6 we shall give a result on the (left) differentiability of u_λ w.r.t. λ.

(2) u_λ is smooth on on open set where $|\nabla u_\lambda|$ does not vanish. Because u_λ satisfies uniformly elliptic equation of the second order. See (9.1).

In the rest of this subsection we shall establish the assertion (2). in Theorem 1.1 admitting the assertion (1), namely

Theorem 2.1. *Assume that $\lambda^* > 0$ and f satisfies the conditions* (2.1) *and* (2.2). *Let $u_\lambda \in C^1(\overline{\Omega})$ be the minimal solution of* (1.1) *for $\lambda \in (0, \lambda^*)$.*

Then we have as $\lambda \to \lambda^*$ a finite limit a.e.

$$u^*(x) = \lim_{\lambda \to \lambda^*} u_\lambda(x). \tag{2.15}$$

Moreover $u^* \in W_0^{1,p}(\Omega)$ and u^* is a weak energy solution of (2.3) *with $\lambda = \lambda^*$.*

Remark 2.2. (1) The positivity of λ^* will be proved as Theorem 5.1 in Section 5 using the results in Section 4.

(2) As we defined in Definition 1.2 in Section 1, u^* is called the extremal solution for the extremal value λ^*. The extremal solution u^* can be classical or singular.

Proof of Theorem 2.1. From the definition of $V_{\lambda,p}(\Omega)$, we see $u_\lambda \in V_{\lambda,p}(\Omega)$. By the assumption we have

$$(p-1)\int_\Omega |\nabla u_\lambda|^p \, dx \geq \lambda \int_\Omega f'(u_\lambda) u_\lambda^2 \, dx. \tag{2.16}$$

Since u_λ is a minimal solution of (2.3), we have

$$\int_\Omega |\nabla u_\lambda|^p \, dx = \int_\Omega f(u_\lambda) u_\lambda \, dx. \tag{2.17}$$

From the condition (2.2), for any $\varepsilon > 0$ there is a positive number $C_\varepsilon > 0$ such that

$$(p - 1 + \varepsilon) f(t) t \leq f'(t) t^2 + C_\varepsilon. \tag{2.18}$$

Hence

$$\int_\Omega f'(u_\lambda) u_\lambda^2 \, dx \leq \frac{p-1}{p-1+\varepsilon} \int_\Omega f'(u_\lambda) u_\lambda^2 \, dx + C'_\varepsilon. \tag{2.19}$$

Here C'_ε is a positive number independent of each $\lambda < \lambda^*$. Then, for some positive number C

$$\int_\Omega |\nabla u_\lambda|^p \, dx = \lambda \int_\Omega f(u_\lambda) u_\lambda \, dx \leq C \quad \text{and} \quad \int_\Omega f'(u_\lambda) u_\lambda^2 \, dx \leq C, \tag{2.20}$$

and so u_λ is uniformly bounded in $W_0^{1,p}(\Omega)$ for $\lambda < \lambda^*$. Therefore $\{u_\lambda\}$ contains a weakly convergent subsequence in $W_0^{1,p}(\Omega)$. Since u_λ is increasing in λ, the limit $u^* = \lim_{\lambda \to \lambda^*} u_\lambda$ uniquely exists a.e. and clearly $u^* \in W_0^{1,p}(\Omega)$ becomes a weak energy solution of (2.3). □

3. Differentiability of $L_p(u_\lambda)$

In this section we shall study differentiability of $L_p(u)$ at $u = u_\lambda$ assuming $\lambda^* > 0$ and $\lambda \in (0, \lambda^*)$. First we introduce a linearized operator of $L_p(\cdot)$.

Definition 3.1. For $u \in C^1(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$ set

$$L_p'(u)\varphi = -\operatorname{div}\left(|\nabla u|^{p-2}\left(\nabla\varphi + (p-2)\frac{(\nabla u, \nabla\varphi)}{|\nabla u|^2}\nabla u\right)\right).$$

If $p > 2$, this is a degenerate elliptic operator, and if $1 < p < 2$, this is elliptic but coefficients are unbounded in general. We introduce a dual form as usual:

Definition 3.2. Let $p \geq 2$. We set for any test functions $\varphi, \psi \in C_0^\infty(\Omega)$,

$$\begin{aligned} &\langle L_p'(u)\varphi, \psi\rangle_{[C_0^\infty]' \times C_0^\infty} \\ &\equiv \int_\Omega |\nabla u|^{p-2}\left((\nabla\varphi, \nabla\psi) + (p-2)\frac{(\nabla u, \nabla\varphi)}{|\nabla u|^2}(\nabla u, \nabla\psi)\right) dx. \end{aligned} \tag{3.1}$$

By this dual form $L_p'(u)\varphi$ ($u \in C^1(\Omega)$) is clearly defined as a distribution on $C_0^\infty(\Omega)$ provided $p \geq 2$. In order to define the linearized operator $L_p'(u)$ for every $p > 1$, we prepare admissible function spaces as follows. Assume that u_λ is the minimal solution of (2.3) for $\lambda \in (0, \lambda^*)$. Then $u_\lambda \in W_0^{1,p}(\Omega)$ is positive and differentiable in Ω. Moreover ∇u_λ does not vanish near the boundary.

Definition 3.3. Set

$$\begin{cases} ||\varphi||_{V_{\lambda,p}} = \left(\int_\Omega |\nabla u_\lambda(x)|^{p-2}|\nabla\varphi|^2\, dx\right)^{\frac{1}{2}}, \\ (\varphi, \psi)_{V_{\lambda,p} \times V_{\lambda,p}} = \int_\Omega |\nabla u_\lambda(x)|^{p-2}(\nabla\varphi, \nabla\psi)\, dx, \end{cases} \tag{3.2}$$

and

$$V_{\lambda,p}(\Omega) = \{\varphi \in M(\Omega) : ||\varphi||_{V_{\lambda,p}} < +\infty, \varphi = 0 \text{ on } \partial\Omega\}. \tag{3.3}$$

Here by $M(\Omega)$ we denote the set of all measurable functions on Ω.

Remark 3.1. $V_{\lambda,p}(\Omega)$ becomes a Hilbert space for all $\lambda \in (0, \lambda^*)$ and $p \in (1, +\infty)$ with the inner-product $(\cdot, \cdot)_{V_{\lambda,p} \times V_{\lambda,p}}$. See also the remark just after Definition 1.1 in Section 1.

Now we define $L_p'(u_\lambda)\varphi$ for $\varphi \in V_{\lambda,p}(\Omega)$ as an element in the dual space $[V_{\lambda,p}(\Omega)]'$.

Definition 3.4. For $\varphi, \psi \in V_{\lambda,p}$,

$$\begin{aligned} &\langle L_p'(u_\lambda)\varphi, \psi\rangle_{V_{\lambda,p}' \times V_{\lambda,p}} \equiv \\ &\int_\Omega \left(|\nabla u_\lambda(x)|^{p-2}\left((\nabla\varphi, \nabla\psi) + (p-2)\frac{(\nabla u_\lambda, \nabla\varphi)(\nabla u_\lambda, \nabla\psi)}{|\nabla u_\lambda(x)|^2}\right)\right) dx. \end{aligned} \tag{3.4}$$

Putting $\varphi = \psi$ we also have From the definition we easily see

Lemma 3.1. *For any $\varphi, \psi \in V_{\lambda,p}(\Omega)$, it holds that*

$$
\begin{aligned}
(3.5)\qquad |\langle L_p'(u_\lambda)\varphi, \psi\rangle_{V_{\lambda,p}'\times V_{\lambda,p}}| &\le C_1 \int_\Omega |\nabla u_\lambda|^{p-2}|\nabla\varphi||\nabla\psi|\,dx \\
&\le C_1||\varphi||_{V_{\lambda,p}}||\psi||_{V_{\lambda,p}}, \\
(3.6)\qquad |\langle L_p'(u_\lambda)\varphi, \varphi\rangle_{V_{\lambda,p}'\times V_{\lambda,p}}| &\ge C_2 \int_\Omega |\nabla u_\lambda(x)|^{p-2}|\nabla\varphi|^2\,dx \\
&= C_2||\varphi||^2_{V_{\lambda,p}}.
\end{aligned}
$$

Here $C_1 = \max(p-1, 3-p)$ and $C_2 = \min(1, p-1)$.

From (3.6) we see the coercivity of the operator $L_p'(u_\lambda)$ on $V_{\lambda,p}(\Omega)$. In addition we see

Lemma 3.2. *Let u_λ be the minimal solution of* (2.3). *Then the linearized operator $L_p'(u_\lambda)$ maps $V_{\lambda,p}(\Omega)$ continuously into $[V_{\lambda,p}(\Omega)]'$.*

Remark 3.2. Later we see that $L_p'(u_\lambda)$ is surjective. When $p \ge 2$, $C_0^\infty(\Omega)$ is densely contained in $V_{\lambda,p}(\Omega)$. Hence $L_p'(u_\lambda)\varphi$ for $\varphi \in V_{\lambda,p}(\Omega)$ coincides with a distribution as usual. But in case that $1 < p < 2$, $C_0^\infty(\Omega)$ is not generally dense in $V_{\lambda,p}(\Omega)$, because $|\nabla u_\lambda|^{p-2}$ does not belong to $L^1_{loc}(\Omega)$ in general.

By $F_{\lambda,p}$ we denote the closed set of all points on which $|\nabla u_\lambda(x)|$ vanishes.

Definition 3.5.

$$
(3.7)\qquad F_{\lambda,p} = \{x \in \Omega : |\nabla u_\lambda(x)| = 0\}.
$$

Later we see that $F_{\lambda,p}$ is a discrete set. Namely

Lemma 3.3. *For any $\lambda \in (0, \lambda^*)$ and $p \in (1, +\infty)$ $F_{\lambda,p}$ is discrete.*

Definition 3.6. By $\tilde V_{\lambda,p}(\Omega)$ we donete a set of all functions ψ such that:

$$
(3.8)\qquad \begin{cases} \psi \in C^\infty(\Omega) \cap C^2(\overline{\Omega}), \\ \psi = 0 \quad \text{on } \partial\Omega, \\ |\nabla\psi| \equiv 0 \quad \text{on some neighborhood of } F_{\lambda,p}, \end{cases}
$$

and by $W_{\lambda,p}(\Omega)$ we denote the completion of $\tilde V_{\lambda,p}(\Omega)$ with respect to the norm $||\cdot||_{V_{\lambda,p}}$, namely,

$$
(3.9)\qquad W_{\lambda,p}(\Omega) = \text{the completion w.r.t } ||\cdot||_{V_{\lambda,p}} \text{ of } \tilde V_{\lambda,p}(\Omega).
$$

Most of the followings are direct consequences from the definition:

Lemma 3.4. *Assume that $0 < \lambda < \lambda^*$. Then the followings are valid: If $p \geq 2$, then*

$$\tilde{V}_{\lambda,p}(\Omega) \subset W_0^{1,p}(\Omega) \subset V_{\lambda,p}(\Omega). \tag{3.10}$$

If $1 < p < 2$, then

$$\tilde{V}_{\lambda,p}(\Omega) \subset V_{\lambda,p}(\Omega) \subset W_0^{1,p}(\Omega). \tag{3.11}$$

Moreover for any $p \in (1, +\infty)$,

$$\tilde{V}_{\lambda,p}(\Omega) \subset W_{\lambda,p}(\Omega) \subset V_{\lambda,p}(\Omega). \tag{3.12}$$

Proof. First we assume that $p \geq 2$. It suffices to show the inclusion $W_0^{1,p}(\Omega) \subset V_{\lambda,p}(\Omega)$. For $\varphi \in W_0^{1,p}(\Omega)$ we have

$$||\varphi||^2_{V_{\lambda,p}} = \int_\Omega |\nabla u_\lambda(x)|^{p-2} |\nabla \varphi|^2 \, dx \leq ||u_\lambda||^{p-2}_{W_0^{1,p}} ||\varphi||^2_{W_0^{1,p}}. \tag{3.13}$$

This proves the assertion.

We proceed to the case that $1 < p < 2$. It suffices to show the last inclusion, and this follows from the usual imbedding inequality. Since there is a positive number $C(\lambda)$ such that $|\nabla u_\lambda| \leq C(\lambda)$, we immediately get

$$|\nabla u_\lambda|^{p-2} \geq C(\lambda)^{p-2}. \tag{3.14}$$

Hence we have

$$||\varphi||_{W_0^{1,p}} \leq C ||\varphi||_{W_0^{1,2}} \leq C' ||\varphi||_{V_{\lambda,p}}, \tag{3.15}$$

and this implies the desired inclusion. $\square$

Remark 3.3. Since u_λ is finite if $\lambda < \lambda^*$, it follows from Lemma 4.1 in Section 4 that

$$\int_\Omega \lambda f'(u_\lambda) \varphi^2 \, dx \leq C \int_\Omega |\nabla u_\lambda(x)|^{p-2} |\nabla \varphi|^2,$$

where C is a positive number independent of each $\varphi \in V_{\lambda,p}(\Omega)$.

Corollary 3.1. *Assume that $0 < \lambda < \lambda^*$. If $p \geq 2$, then*

$$[V_{\lambda,p}(\Omega)]' \subset [W_0^{1,p}(\Omega)]' \subset [C_0^\infty(\Omega)]'. \tag{3.16}$$

If $1 < p < 2$, then

$$[W_0^{1,p}(\Omega)]' \subset [V_{\lambda,p}(\Omega)]' \subset [W_{\lambda,p}(\Omega)]'. \tag{3.17}$$

Here by X' we denote the dual space of X.

Now we define the differentiability of the non-linear operator $L_p(\cdot)$. We have to note that $L_p(\cdot)$ and $L_p'(u_\lambda)$ are defined on $W_0^{1,p}(\Omega)$ and $V_{\lambda,p}(\Omega)$ respectively. If the operator $L_p(\cdot)$ is differentiable at the point u_λ, then the derivative coincides with the linearized operator $L_p'(u_\lambda)$ as usual.

Definition 3.7 (Differentiability in $V_{\lambda,p}(\Omega)$)**.** Let $p \in (1, +\infty)$ and let u_λ be the minimal solution for $\lambda \in (0, \lambda^*)$. Let S be a subset of $V_{\lambda,p}(\Omega)$. Then $L_p(\cdot)$ is said to be differentiable at u_λ in the direction to S in $V_{\lambda,p}(\Omega)$, if for any $\varphi \in S$ it holds that as $t \to 0$

$$\frac{1}{t}(L_p(u_\lambda + t\varphi) - L_p(u_\lambda) - tL_p'(u_\lambda)\varphi) = o(1) \quad \text{in} \quad [V_{\lambda,p}(\Omega)]'. \tag{3.18}$$

In addition if S is dense in $V_{\lambda,p}(\Omega)$ (if $S = V_{\lambda,p}(\Omega)$), then $L_p(\cdot)$ is said to be differentiable at u_λ in $V_{\lambda,p}(\Omega)$a.e. (in $V_{\lambda,p}(\Omega)$) respectively.

Remark 3.4. The condition (3.18) means that for any $\varphi \in S \subset V_{\lambda,p}(\Omega)$ and $\psi \in V_{\lambda,p}(\Omega)$

$$\lim_{t\to 0}\left|\left\langle \frac{1}{t}(L_p(u_\lambda + t\varphi) - L_p(u_\lambda) - tL_p'(u_\lambda)\varphi), \psi \right\rangle_{V_{\lambda,p}' \times V_{\lambda,p}}\right| = 0.$$

Lemma 3.5. *Let u_λ be the minimal solution for $\lambda \in (0, \lambda^*)$ and let φ be any element of $\tilde{V}_{\lambda,p}(\Omega)$. Then it holds that $L_p(u_\lambda + t\varphi) \in [V_{\lambda,p}(\Omega)]'$ for a sufficiently small $t \geq 0$.*

Proof. First we recall that $u_\lambda \in W_0^{1,p}(\Omega)$ for any $\lambda \in (0, \lambda^*)$. Take a $\psi \in V_{\lambda,p}(\Omega)$. By Definition 3.4, it suffices to show

$$\left|\int_\Omega |\nabla(u_\lambda + t\varphi)|^{p-2}(\nabla(u_\lambda + t\varphi), \nabla\psi)\, dx\right| < +\infty. \tag{3.19}$$

Let us set $F_\eta = \{x \in \Omega : dist(x, F_{\lambda,p}) \leq \eta\}$. For a small $\eta > 0$ we can assume $\nabla\varphi \equiv 0$ in F_η. Then we have for some constant $C > 0$ depending on η

$$C^{-1} \leq |\nabla u_\lambda| \leq C \quad \text{in } F_\eta^c\,; (\text{the complement of } F_\eta). \tag{3.20}$$

Therefore $|\nabla(u_\lambda + t\varphi)|$ does not vanish in F_η^c provided that t is small. Since $\nabla\varphi$ vanishes in F_η, there is a positive number C' such that we have for a sufficiently small t

$$\begin{aligned} |\langle L_p(u_\lambda + t\varphi), \psi\rangle_{[V_{\lambda,p}(\Omega)]' \times V_{\lambda,p}(\Omega)}| &\leq C' \int_\Omega |\nabla u_\lambda|^{p-2}|(\nabla u_\lambda, \nabla\psi)|\, dx \\ &\leq C'||u_\lambda||_{V_{\lambda,p}(\Omega)}||\psi||_{V_{\lambda,p}(\Omega)} = C'||u_\lambda||^{\frac{p}{2}}_{W_0^{1,p}(\Omega)}||\psi||_{V_{\lambda,p}(\Omega)}. \end{aligned} \tag{3.21}$$

Hence $L_p(u_\lambda + t\varphi)$ can be extended to be a continuous linear functional on $V_{\lambda,p}(\Omega)$. □

After all we can show a basic result on the differentiability of $L_p(\cdot)$.

Proposition 3.1. *Let u_λ be the minimal solution for $\lambda \in (0, \lambda^*)$. For $p \in (1, +\infty)$, $L_p(\cdot)$ is differentiable at u_λ in the direction to $\tilde{V}_{\lambda,p}(\Omega)$.*

Proof. For any $\varphi \in \tilde{V}_{\lambda,p}(\Omega)$ and $\psi \in V_{\lambda,p}(\Omega)$, we have

$$\begin{aligned}
&\frac{1}{t}\langle L_p(u_\lambda + t\varphi)) - L_p(u_\lambda) - tL'_p(u_\lambda)\varphi, \psi\rangle_{V'_{\lambda,p}\times V_{\lambda,p}} \\
&= \frac{1}{t}\int_0^t \langle (L'_p(u_\lambda + s\varphi) - L'_p(u_\lambda))\varphi, \psi\rangle_{V'_{\lambda,p}\times V_{\lambda,p}}\, ds \\
&= \int_0^1 \langle (L'_p(u_\lambda + t\rho\varphi) - L'_p(u_\lambda))\varphi, \psi\rangle_{V'_{\lambda,p}\times V_{\lambda,p}}\, d\rho.
\end{aligned} \tag{3.22}$$

Since $supp|\nabla\varphi| \cap F_{\lambda,p} = \phi$, we are able to assume that for some $\eta > 0$

$$supp|\nabla\varphi| \subset (F_{\lambda,p})^c_\eta \equiv \{x \in \Omega : dist(x, F_{\lambda,p}) \geq \eta\}.$$

Then we have for a sufficiently small t

$$\begin{aligned}
&|\langle L'_p(u_\lambda + t\varphi)\varphi, \psi\rangle_{V'_{\lambda,p}\times V_{\lambda,p}}| \\
&\leq C\int_{(F_{\lambda,p})^c_\eta} |\nabla(u_\lambda + t\varphi)|^{p-2}|\nabla\varphi||\nabla\psi|\, dx \\
&\leq C'||\varphi||_{V_{\lambda,p}(\Omega)}||\psi||_{V_{\lambda,p}(\Omega)}.
\end{aligned} \tag{3.23}$$

Therefore the assertion follows from Lebesgue's convergence theorem. □

Remark 3.5. From this $L_p(\cdot)$ is differentiable at least in the direction to $\tilde{V}_{\lambda,p}(\Omega)$. But in certain cases $\tilde{V}_{\lambda,p}(\Omega)$ becomes dense in $V_{\lambda,p}(\Omega)$. For example, if $F_{\lambda,p}$ consists of finitely many points, then clearly $\tilde{V}_{\lambda,p}(\Omega)$ becomes dense in $V_{\lambda,p}(\Omega)$. In fact, $\varphi \in V_{\lambda,p}(\Omega)$ can be approximated by a sequence of regularlized step functions of $\tilde{V}_{\lambda,p}(\Omega)$.

In order to give a nontrivial example in which $\tilde{V}_{\lambda,p}(\Omega)$ becomes dense in $V_{\lambda,p}(\Omega)$, we introduce a modified relative 2 capacity as follows:

Definition 3.8. Let $p \in (1, +\infty)$ and let u_λ be the minimal solution of (2.3). Let us set for any compact set F in Ω

$$\begin{aligned}
&Cap(F, |\nabla u_\lambda|^{p-2}) \\
&= \inf\left[\int_\Omega |\nabla u_\lambda|^{p-2}|\nabla\varphi|^2\, dx : \varphi \in C_0^\infty(\Omega), \varphi \geq 1 \text{ on } F\right].
\end{aligned} \tag{3.24}$$

Then we can show

Proposition 3.2. *If $Cap(F_{\lambda,p}, |\nabla u_\lambda|^{p-2}) = 0$, then $W_{\lambda,p}(\Omega) = V_{\lambda,p}(\Omega)$ holds.*

From this we have

Corollary 3.2. *If $Cap(F_{\lambda,p}, |\nabla u_\lambda|^{p-2}) = 0$, then $L_p(\cdot)$ is differentiable at u_λ in $V_{\lambda,p}(\Omega)$ a.e..*

Proof. Assume that $W_{\lambda,p}(\Omega)$; the completion of $\tilde{V}_{\lambda,p}(\Omega)$ does not coincide with $V_{\lambda,p}(\Omega)$, that is, $W_{\lambda,p}(\Omega) \subset V_{\lambda,p}(\Omega)$. Then we have a $\varphi \in V_{\lambda,p}(\Omega)$ such that φ is not identically zero and

$$\int_\Omega |\nabla u_\lambda|^{p-2}(\nabla\varphi, \nabla\psi)\,dx = 0 \qquad \text{for any } \psi \in W_{\lambda,p}(\Omega).$$

Moreover for some ball B such as $B \cap F_{\lambda,p} = \phi$ we have

$$\int_B |\nabla u_\lambda|^{p-2}|\nabla\varphi|^2\,dx > 0.$$

Since $L^\infty(\Omega)$ is dense in $V_{\lambda,p}(\Omega)$, we may assume that $\varphi \in L^\infty(\Omega)$. Now put $\psi = \varphi f^2$ with $f \in C^\infty(\Omega)$ vanishing on $F_{\lambda,p}$. Since $\psi \in W_{\lambda,p}(\Omega)$, we have

$$\int_\Omega |\nabla u_\lambda|^{p-2}(\nabla\varphi, \nabla(\varphi f^2))\,dx = 0. \tag{3.25}$$

Note that

$$|\nabla(\varphi f)|^2 - (\nabla\varphi, \nabla(\varphi f^2)) = \varphi^2|\nabla f|^2. \tag{3.26}$$

Then

$$\int_\Omega |\nabla u_\lambda|^{p-2}|\nabla(\varphi f)|^2\,dx = \int_\Omega |\nabla u_\lambda|^{p-2}|\nabla f|^2\varphi^2\,dx. \tag{3.27}$$

Since $Cap(F_{\lambda,p}, |\nabla u_\lambda|^{p-2}) = 0$, for any $\varepsilon > 0$ there is some $g \in C_0^\infty(\Omega)$ such that $g \geq 1$ on a neighborhood of $F_{\lambda,p}$ and $\int_\Omega |\nabla u_\lambda|^{p-2}|\nabla g|^2\,dx < \varepsilon$. Putting $f = 1 - g$ we have

$$\int_\Omega |\nabla u_\lambda|^{p-2}|\nabla(\varphi f)|^2\,dx < \varepsilon \sup_{x\in\Omega} |\varphi|^2. \tag{3.28}$$

Since we can assume $f = 1$ on B,

$$\int_B |\nabla u_\lambda|^{p-2}|\nabla\varphi|^2\,dx = 0.$$

Therefore $\varphi = 0$ in $V_{\lambda,p}(\Omega)$, and this is a contradiction. □

In the case that $p \geq 2$, we have $W_0^{1,p}(\Omega) \subset V_{\lambda,p}(\Omega)$. But we can not take $W_0^{1,p}(\Omega)$ as S in Definition 3.7. Because $L_p(u_\lambda + t\varphi)$ with $\varphi \in W_0^{1,p}(\Omega)$ does not belong to $[V_{\lambda,p}(\Omega)]'$ but to $[W_0^{1,p}(\Omega)]'$ in general. But $L_p'(u_\lambda)$ is continuous from $W_0^{1,p}(\Omega)$ to its dual $[W_0^{1,p}(\Omega)]'$, hence we can give an alternative definition of differentiability of $L_p(\cdot)$ in $[W_0^{1,p}(\Omega)]'$ as follows.

Definition 3.9 (Differentiability in $W_0^{1,p}(\Omega)$)**.** Assume $p \in [2,+\infty)$. Let u_λ be the minimal solution for $\lambda \in (0,\lambda^*)$. $L_p(\cdot)$ is said to be differentiable at u_λ in $W_0^{1,p}(\Omega)$, if for any $\varphi \in W_0^{1,p}(\Omega)$ it holds that as $t \to 0$

$$\frac{1}{t}(L_p(u_\lambda + t\varphi) - L_p(u_\lambda) - tL_p'(u_\lambda)\varphi) = o(1), \quad \text{in} \quad [W_0^{1,p}(\Omega)]'. \tag{3.29}$$

Remark 3.6. The condition (3.29) means that for any $\psi \in W_0^{1,p}(\Omega)$

$$\lim_{t\to 0}\left|\left\langle \frac{1}{t}(L_p(u_\lambda + t\varphi) - L_p(u_\lambda) - tL_p'(u_\lambda)\varphi), \psi \right\rangle_{[W_0^{1,p}]' \times W_0^{1,p}}\right| = 0.$$

Then we have

Proposition 3.3. *Let u_λ be the minimal solution for $\lambda \in (0,\lambda^*)$. If $p \in [2,+\infty)$, then $L_p(\cdot)$ is differentiable at u_λ in the direction to $W_0^{1,p}(\Omega)$.*

Proof. For any $\varphi \in W_0^{1,p}(\Omega)$ and $\psi \in W_0^{1,p}(\Omega)$ we have in a similar way as before

$$\begin{aligned}&\frac{1}{t}\langle L_p(u_\lambda + t\varphi)) - L_p(u_\lambda) - tL_p'(u_\lambda)\varphi, \psi\rangle_{[W_0^{1,p}]' \times W_0^{1,p}} \\ &= \int_0^1 \langle (L_p'(u_\lambda + t\rho\varphi) - L_p'(u_\lambda))\varphi, \psi\rangle_{[W_0^{1,p}]' \times W_0^{1,p}}\, d\rho.\end{aligned} \tag{3.30}$$

It is easy to see that

$$\begin{aligned}&\left|\langle L_p'(u_\lambda + t\varphi)\varphi, \psi\rangle_{[W_0^{1,p}]' \times W_0^{1,p}}\right| \\ &\le C\int_\Omega |\nabla(u_\lambda + t\varphi)|^{p-2}|\nabla\varphi||\nabla\psi|\, dx \\ &\le C||u_\lambda + t\varphi||_{W_0^{1,p}(\Omega)}^{p-2}||\varphi||_{W_0^{1,p}(\Omega)}||\psi||_{W_0^{1,p}(\Omega)}.\end{aligned} \tag{3.31}$$

Hence it follows from Lebesgue's convergence theorem that

$$\lim_{t\to 0}\left|\left\langle \frac{1}{t}(L_p(u_\lambda + t\varphi) - L_p(u_\lambda) - L_p'(u_\lambda)\varphi), \psi \right\rangle_{[W_0^{1,p}]' \times W_0^{1,p}}\right| = 0.$$

□

4. The linearized operator $L_p'(u_\lambda)$

In this section we shall collect fundamental results concerned with the linearized operator $L_p'(u_\lambda)$, which are rather basic in the present paper.

Let u_λ be the minimal solution of (2.3). Then $u_\lambda \in C^{1,\sigma}(\overline{\Omega})$ for some $\sigma \in (0,1)$ satisfies in the distribution sense that

$$\begin{cases} L_p(u_\lambda) \equiv -\operatorname{div}(|\nabla u_\lambda|^{p-2}\nabla u_\lambda) = \lambda f(u_\lambda) & \text{in} \quad \Omega, \\ u_\lambda = 0 \qquad \text{on} \quad \partial\Omega, \end{cases} \tag{4.1}$$

and also

$$\int_\Omega |\nabla u_\lambda|^p = \lambda \int_\Omega f(u_\lambda) u_\lambda \, dx. \tag{4.2}$$

Then we can show

Lemma 4.1. *Let ε be any number in $(0, \lambda^*)$. Then for any $\lambda \in (0, \lambda^* - \varepsilon)$ and $p \in (1, \infty)$, the following inequalities are valid for any $\varphi \in C_0^1(\Omega)$:*

$$\int_\Omega |\nabla u_\lambda|^{p-1} |\nabla \varphi| \, dx \geq C_\varepsilon \lambda \int_\Omega |\varphi| \, dx, \tag{4.3}$$

$$\int_\Omega |\nabla u_\lambda|^{2(p-1)} |\nabla \varphi|^2 \, dx \geq C_\varepsilon \lambda^2 \int_\Omega \varphi^2 \, dx, \tag{4.4}$$

$$\int_\Omega |\nabla u_\lambda|^{p-2} |\nabla \varphi|^2 \geq C_\varepsilon \lambda^2 \int_\Omega \varphi^2 \, dx. \tag{4.5}$$

Here C_ε is a positive number independent of each φ and λ.

Proof. From (4.1) we have for any $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega |\nabla u_\lambda|^{p-2} (\nabla u_\lambda, \nabla \varphi) \, dx = \lambda \int_\Omega f(u_\lambda) \varphi \, dx. \tag{4.6}$$

Noting that $f(u_\lambda) > 0$ and φ can be expressed as $\varphi = \varphi_+ - \varphi_-$ for $\varphi_+ = \max(\varphi, 0)$, we have

$$\int_\Omega |\nabla u_\lambda|^{p-1} |\nabla \varphi| \, dx \geq \lambda \int_\Omega f(u_\lambda) |\varphi| \, dx. \tag{4.7}$$

Since u_λ is a classical solution and $C_0^\infty(\Omega)$ is dense in $C_0^1(\Omega)$, the assertion follows. The second inequality can be obtained from the first one by replacing φ for φ^2 and by using Schwartz inequality. Now we note that for some positive number C, $\int_\Omega |\nabla u_\lambda|^p |\varphi|^2 \, dx \leq C \int_\Omega |\varphi|^2 \, dx$. Then the last one is also obtained by Schwartz inequality and the equality (4.3) with replacing φ by φ^2. □

Let us recall $F_{\lambda,p} = \{x \in \Omega : |\nabla u_\lambda| = 0\}$. Then we see

Corollary 4.1. *$F_{\lambda,p}$ is a discrete set in Ω (i.e., $F_{\lambda,p}$ has no interior point.)*

Proof. Assume that $F_{\lambda,p}$ contains an open ball B.Then for any $\varphi \in C_0^1(B)$ we get $\int_B f(u_\lambda) \varphi \, dx = 0$, but this contradicts to the positivity of $f(u_\lambda)$. □

Corollary 4.2. *Let u_λ be the minimal solution for $\lambda \in (0, \lambda^*]$. Then $L_p'(u_\lambda)$ is a continuous, one to one mapping from $V_{\lambda,p}$ onto the dual space $[V_{\lambda,p}]'$. Hence $L_p'(u_\lambda)$ is invertible.*

Proof. It follows from Lemma 3.1 and (4.5) that $L'_p(u_\lambda)$ is one to one. In fact, if $L'_p(u_\lambda)\varphi = 0$ for some $\varphi \in V_{\lambda,p}(\Omega)$, then $C\int_\Omega \varphi^2\,dx \le \int_\Omega |\nabla u_\lambda|^{p-2}|\nabla\varphi|^2\,dx = 0$ for some $C > 0$. Hence $\varphi \equiv 0$. Since $L'_p(u_\lambda)$ is symmetric in $V_{\lambda,p}(\Omega)$, we see the surjectivity. Therefore it is invertible. □

Since $L^2(\Omega) \subset [V_{\lambda,p}]'$, from this we immediately have the following:

Corollary 4.3. *$V_{\lambda,p}(\Omega)$ is dense in $L^2(\Omega)$.*

Definition 4.1. $\varphi \in L^2(\Omega)$ is said to belong to $D(L'_p(u_\lambda))$ if and only if $\varphi \in V_{\lambda,p}(\Omega)$ and we have for some $f \in L^2(\Omega)$

$$\langle L'_p(u_\lambda)\varphi, \psi\rangle_{V'_{\lambda,p}\times V_{\lambda,p}} = \int_\Omega f\psi\,dx \qquad (\forall\psi \in V_{\lambda,p}(\Omega)). \tag{4.8}$$

Here we note that since $L'_p(u_\lambda)$ is invertible in $V_{\lambda,p}(\Omega)$, f is uniquely determined.

By virtue of the nondegenerate quadratic form $\langle L'_p(u_\lambda)\varphi, \psi\rangle_{V'_{\lambda,p}\times V_{\lambda,p}}$, the operator $L'_p(u_\lambda)$ is naturally extended to an operator on $L^2(\Omega)$ with its domain being $D(L'_p(u_\lambda))$, which is still denoted by $L'_p(u_\lambda)$ for the sake of simplicity. Namely,

Definition 4.2. For $\varphi \in D(L'_p(u_\lambda))$ we define $L'_p(u_\lambda)\varphi$ by

$$L'_p(u_\lambda)\varphi = f.$$

Then we can show

Proposition 4.1. *The extended operator $L'_p(u_\lambda) : D(L'_p(u_\lambda)) \longrightarrow L^2(\Omega)$ is one to one and surjective. Moreover $D(L'_p(u_\lambda))$ is dense in $V_{\lambda,p}(\Omega)$ and $L'_p(u_\lambda)$ is a self-adjoint operator on $L^2(\Omega)$.*

Proof. Assume that $L'_p(u_\lambda)\varphi = 0$ for some $\varphi \in V_{\lambda,p}(\Omega)$. Then $\langle L'_p(u_\lambda)\varphi, \varphi\rangle_{V'_{\lambda,p}\times V_{\lambda,p}} = 0$. Hence $\varphi = 0$. We show the suejectivity in the next. For $f \in L^2(\Omega)$ we consider a functional $F(\varphi) = \int_\Omega f\varphi\,dx$ on $V_{\lambda,p}(\Omega)$. From (4.5) $F(\varphi)$ is continuous on $V_{\lambda,p}(\Omega)$. Therefore by Riesz's representation theorem there is a $\psi \in V_{\lambda,p}(\Omega)$ such that $F(\varphi) = \langle L'_p(u_\lambda)\varphi, \psi\rangle_{V'_{\lambda,p}\times V_{\lambda,p}}$. Since$L'_p(u_\lambda)$ is symmetric, $L'_p(u_\lambda)\psi = f$ holds. From the surjectivity of $L'_p(u_\lambda)$, we see that $D(L'_p(u_\lambda))$ is densely contained in $V_{\lambda,p}(\Omega)$. Since the rest of proof is rather standard, we omit the detail. □

Definition 4.3. By $I_{V\to L^2}$ we denote the imbedding operator from $V_{\lambda,p}(\Omega)$into $L^2(\Omega)$ defined by

$$I_{V\to L^2} : \varphi \in V_{\lambda,p}(\Omega) \longrightarrow \varphi \in L^2(\Omega). \tag{4.9}$$

Then we can show

Proposition 4.2. *$I_{V\to L^2}$ is compact, namely, the space $V_{\lambda,p}(\Omega)$ is compactly imbedded into $L^2(\Omega)$.*

Since $L^2(\Omega) \subset [V_{\lambda,p}(\Omega)]'$, one can restrict the operator $(L_p'(u_\lambda))^{-1}$ on $L^2(\Omega)$ to obtain a continuous operator $(L_p'(u_\lambda))^{-1}|_{L^2}$: $L^2(\Omega) \to V_{\lambda,p}(\Omega)$. Then it holds that

Corollary 4.4. *The operator $M_{\lambda,p} \equiv I_{V\to L^2} \circ (L_p'(u_\lambda))^{-1}|_{L^2}$ is compact from $L^2(\Omega)$ into $L^2(\Omega)$.*

When $p \in (1,2]$, we have $|\nabla u_\lambda|^{p-2} \geq C > 0$ for some constant C. Therefore in this case, the imbedding operator $I_{V\to L^2}$ is clearly compact. Because $V_{\lambda,p}(\Omega)$ is imbedded into $W_0^{1,2}(\Omega)$ and $W_0^{1,2}(\Omega)$ is compact in $L^2(\Omega)$ by virtue of Sobolev imbedding theorem. Therefore we assume $p > 2$ from now on. Auxilially we define

Definition 4.4.

$$L^2(\Omega; |\nabla u_\lambda|^{p-2}) = \left\{ \varphi \in M(\Omega) : \int_\Omega |\nabla u_\lambda|^{p-2} \varphi^2 \, dx < +\infty \right\} \tag{4.10}$$

Then we see

Lemma 4.2. *$V_{\lambda,p}(\Omega)$ is compactly imbedded into $L^2(\Omega; |\nabla u_\lambda|^{p-2})$.*

Proof. Let us set $\delta(x) = |\nabla u_\lambda|$ and $F = \{x \in \Omega : \delta(x) = 0\}$ for simplicity. By F_η we denote a tubular neighborhood of F given by

$$F_\eta = \{x \in \Omega : dist(x, F) < \eta\} \qquad \text{for any} \quad \eta > 0. \tag{4.11}$$

For any $\eta > 0$ and $\varphi \in V_{\lambda,p}(\Omega)$,

$$\int_{F_\eta} \varphi^2 |\nabla u_\lambda|^{p-2} \, dx \leq \sup_{x \in F_\eta} \delta^{p-2} \int_{F_\eta} \varphi^2 \, dx \leq C \sup_{x \in F_\eta} \delta^{p-2} ||\varphi||^2_{V_{\lambda,p}(\Omega)}. \tag{4.12}$$

Since the imbedding $V_{\lambda,p}(F_\eta^c) \to L^2(F_\eta^c)$ is clearly compact, where F_η^c is the complement of F_η, the assertion follows from this inequality. □

Proof of Proposition 4.2. We make use of a uniformly locally finite open cover of F by balls $\{B_j\}$ and a partition of unity $\{\varphi_j\}$ such that

$$F_\eta \subset \bigcup_{j=1}^\infty B_j, \quad diam(B_j) = \eta, \quad 0 \leq \varphi_j \in C_0^\infty(B_j),$$

$$\operatorname{supp} \varphi \subset F_{2\eta}, |\nabla \varphi| \leq C\eta^{-1}, \sum_j^\infty \varphi_j = 1 \quad \text{on } F_\eta.$$

Then we have

$$
\begin{aligned}
(4.13)\quad \int_{F_\eta} \varphi^2\,dx &= \int_{F_\eta}\left(\sum_j \varphi_j\varphi\right)^2 dx \\
&\le C\sum_j \int_{F_\eta} (\varphi_j\varphi)^2\,dx \le C\sum_j \int_{F_\eta} \delta^{2(p-1)}|\nabla(\varphi_j\varphi)|^2\,dx \\
&\le C\sum_j \left(\int_{F_{2\eta}} \delta^{2(p-1)}|\nabla\varphi|^2\varphi_j^2\,dx + \int_{F_{2\eta}} \delta^{2(p-1)}|\nabla\varphi_j|^2\varphi^2\,dx\right) \\
&\le C\sup_{F_{2\eta}}\delta^p\left(\int_\Omega \delta^{p-2}|\nabla\varphi|^2\,dx + \eta^{-2}\int_\Omega \delta^{p-2}\varphi^2\,dx\right).
\end{aligned}
$$

Here C is a positive number independent of $\eta > 0$. Then for any $\varepsilon > 0$ there are some $\eta > 0$ and $C_\varepsilon > 0$ such that

$$
\begin{aligned}
\int_{F_\eta} \varphi^2\,dx &\le \varepsilon\int_\Omega \delta^{p-2}|\nabla\varphi|^2\,dx + C_\varepsilon\int_\Omega \delta^{p-2}\varphi^2\,dx \\
&\le \varepsilon||\varphi||^2_{V_{\lambda,p}(\Omega)} + C_\varepsilon||\varphi||^2_{L^2(\Omega;\delta^{p-2})}.
\end{aligned}
$$

Since the second norm in the right hand side is compact with respect to the first one, the assertion follows. $\square$

In the subsequent we give an isoperimetric inequality which may be of independent interest. Though it is rather standard, we give a short proof for the sake of self-containedness.

Lemma 4.3. *For any open subset $M \subset \Omega$ of $C^{1,1}$ class we have*

$$
(4.14)\qquad \int_{\partial M} |\nabla u_\lambda|^{p-1}\,dH^{N-1}(x) \ge C meas[M].
$$

Here $H^{N-1}(x)$ is th $N-1$-dimensional Hausdorff measure.

Proof. Let M be an open subset in Ω with a $C^{1,1}$ boundary such that the closure of M is contained in Ω. We construct approximative characteristic function φ_ε's of M for sufficiently small $\varepsilon > 0$ as follows. Let us set $M_\varepsilon = \{x \in M; dist(x, \partial M) < \varepsilon\}$ and

$$
(4.15)\qquad \varphi_\varepsilon(x) = \begin{cases} 1, & x \in M \setminus M_\varepsilon, \\ dist(x,\partial M)/\varepsilon, & x \in M_\varepsilon, \\ 0, & x \in M^c = \Omega \setminus M. \end{cases}
$$

Since φ's belong to the space $C_0^{1,1}(\Omega)$, we have

$$
(4.16)\qquad \int_\Omega |\nabla u_\lambda|^{p-1}|\nabla\varphi_\varepsilon|\,dx \ge C\int_\Omega |\varphi_\varepsilon|\,dx.
$$

Then we have, letting $\varepsilon \to 0$, $\lim_{\varepsilon\to 0}\int_\Omega |\varphi_\varepsilon|\,dx = meas[M]$. Since ∂M_ε is $C^{1,1}$ manifolds for sufficiently small $\varepsilon > 0$, we also have

$$
\begin{aligned}
(4.17)\qquad &\limsup_{\varepsilon\to 0}\int_\Omega |\nabla u_\lambda|^{p-1}|\nabla\varphi_\varepsilon|\,dx \\
&= \limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon dt\int_{\partial M_t}|\nabla u_\lambda|^{p-1}\,dH^{N-1}(x) \\
&= \int_{\partial M}|\nabla u_\lambda|^{p-1}\,dH^{N-1}(x).
\end{aligned}
$$

Here $H^{N-1}(x)$ is th $N-1$-dimensional Hausdorff measure. Then we have the desired inequality for any smooth open subset $M \subset \Omega$. Since $F_{\lambda,p}$ is approximated by a sequence of M_j of class $C^{1,1}$, the assertion holds. □

Then we immediately have

Corollary 4.5. *If $F_{\lambda,p}$ is smooth, then $meas[F_{\lambda,p}] = 0$.*

Proof. If $F_{\lambda,p}$ is of class $C^{1,1}$, then the assertion follows from the previous inequality. □

5. Positivity of $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ for a small λ

Since $L_p'(u_\lambda)$ has a compact inverse from $L^2(\Omega)$ to itself, the operator $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ has discrete spectrums. In particular, there exist the first eigenvalue and the corresponding first eigenfunction $\hat\varphi^\lambda$ (recall Definition 1.3). In case that $p = 2$ this operator has a positive first eigenvalue as long as a bounded minimal solution exists. In this section we shall establish somewhat weaker result that the operator $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ has a positive first eigenvalue provided that λ is sufficiently small. This fact is closely connected with the validity of the Hardy type inequalities. By virtue of the Hermite form on $V_{\lambda,p}(\Omega)\times V_{\lambda,p}(\Omega)$ given by Definition 3.4 and by the imbedding theorem, there exists a unique self-adjoint operator on $L^2(\Omega)$ with its domain being $D(L_p'(u_\lambda))$, which we again denote by $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ for simplicity.

Theorem 5.1. *If λ is sufficiently small, then the self-adjoint operator $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ on $L^2(\Omega)$ has a positive first eigenvalue.*

In other words, there is a positive number $\mu > 0$ such that we have

$$
\begin{aligned}
(5.1)\qquad &\int_\Omega |\nabla u_\lambda|^{p-2}\left(|\nabla\varphi|^2 + (p-2)\frac{(\nabla u_\lambda,\nabla\varphi)^2}{|\nabla u_\lambda|^2}\right)dx \\
&\ge \lambda\int_\Omega f'(u_\lambda)\varphi^2\,dx + \mu\int_\Omega \varphi^2\,dx
\end{aligned}
$$

for any $\varphi \in V_{\lambda,p}(\Omega)$.

Remark 5.1. In the Hardy type inequality (5.1), we can replace the last term by $\mu'\int_\Omega f'(u_\lambda)\varphi^2\,dx$, where $\mu' > 0$. Because u_λ is bounded if λ is small.

Proof of Theorem 5.1. We choose and fix a small $\varepsilon_0 > 0$. Let us set

$$u_\lambda = \lambda^{\frac{1}{p-1}} w_\lambda. \tag{5.2}$$

Then $w_\lambda \in C^{1,\sigma}(\overline{\Omega})$ satisfies

$$\begin{cases} L_p(w_\lambda) = f(u_\lambda) & \text{in} \quad \Omega, \\ w_\lambda = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{5.3}$$

Hence for any $\lambda \in [0, \varepsilon_0]$ there is a positive number C such that

$$||w_\lambda||_{L^\infty(\Omega)} + ||\nabla w_\lambda||_{L^\infty(\Omega)} \le C < +\infty. \tag{5.4}$$

Therefore there is a subsequence, which is denoted by $\{w_\lambda\}$ again, such that $\lim_{\lambda\to+0} w_\lambda = w_0 \in W_0^{1,p}(\Omega)$ exists (a.e.) and w_0 becomes a unique solution of limiting equation:

$$\begin{cases} L_p(w_0) = f(0) & \text{in} \quad \Omega, \\ w_0 = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{5.5}$$

Definition 5.1. Let us set

$$||\varphi||_{Z_{\lambda,p}(\Omega)} = \left(\int_\Omega |\nabla w_\lambda(x)|^{p-2} |\nabla \varphi|^2 \, dx \right)^{\frac{1}{2}}, \tag{5.6}$$

and

$$Z_{\lambda,p}(\Omega) = \{\varphi \in M(\Omega) : ||\varphi||_{Z_{\lambda,p}} < +\infty, \varphi = 0 \text{ on } \partial\Omega\}. \tag{5.7}$$

Here by $M(\Omega)$ we denote the set of all measurable functions on Ω.

Then $Z_{\lambda,p}(\Omega)$ becomes a Hilbert space as before. It follows from the same argument as in Corollary 4.2 in Section 4 that the linearized operators $L'_p(w_\lambda)$ $(0 \le \lambda < \lambda^*) : Z_{\lambda,p}(\Omega) \to [Z_{\lambda,p}(\Omega)]'$ are invertible. Moreover the imbedding operators $\varphi \in Z_{\lambda,p}(\Omega) \to \varphi \in L^2(\Omega)$ are compact (see Proposition 4.2). Hence the first eigenvalues of the self-adjoint operators $L'_p(w_\lambda)$ on $L^2(\Omega)$ are positive. Namely $\lambda \in [0, \lambda^*)$ we see

$$\langle L'_p(w_\lambda)\varphi, \varphi \rangle_{[Z_{\lambda,p}(\Omega)]' \times Z_{\lambda,p}(\Omega)} \ge C_\lambda \int_\Omega \varphi^2 \, dx \qquad \text{for all} \quad \varphi \in Z_{\lambda,p}(\Omega), \tag{5.8}$$

where C_λ is a positive number independent of each φ.

Now we assume that there is a sequence λ_j such that $\lambda_j \to 0$ as $(j \to \infty)$ and for each λ_j there is a non-trivial $\psi_j \in V_{\lambda_j,p}(\Omega)$ satisfying

$$\begin{cases} (L'_p(u_{\lambda_j}) - \lambda_j f'(u_{\lambda_j}))\psi_j = 0, \\ ||\psi_j||_{V_{\lambda_j,p}(\Omega)} = 1 \qquad (j = 1, 2, 3, \ldots). \end{cases} \tag{5.9}$$

Put

$$\psi_j = \varphi_j \lambda_j^{-\frac{p-2}{2(p-1)}}, \tag{5.10}$$

then we have

$$\begin{cases} (L_p'(w_{\lambda_j}) - \lambda_j^{\frac{1}{p-1}} f'(u_{\lambda_j}))\varphi_j = 0, \\ ||\varphi_j||_{Z_{\lambda_j,p}(\Omega)} = 1 \qquad (j = 1, 2, 3, \ldots). \end{cases} \tag{5.11}$$

Then

$$\langle L_p'(w_{\lambda_j})\varphi_j, \varphi_j \rangle_{[Z_{\lambda_j,p}(\Omega)]' \times Z_{\lambda_j,p}(\Omega)} = \lambda_j^{\frac{1}{p-1}} \int_\Omega f'(u_{\lambda_j})\varphi_j^2 \, dx. \tag{5.12}$$

For some positive numbers C_1 and C_2 we have

$$\begin{cases} C_1 ||\varphi_j||^2_{Z_{\lambda_j,p}(\Omega)} \leq \langle L_p'(w_{\lambda_j})\varphi_j, \varphi_j \rangle_{[Z_{\lambda_j,p}(\Omega)]' \times Z_{\lambda_j,p}(\Omega)}, \\ C_2 \int_\Omega f'(u_{\lambda_j})\varphi_j^2 \, dx \leq ||\varphi_j||^2_{Z_{\lambda_j,p}(\Omega)}. \end{cases} \tag{5.13}$$

Therefore we see

$$\lim_{j \to +\infty} \langle L_p'(w_{\lambda_j})\varphi_j, \varphi_j \rangle_{[Z_{\lambda_j,p}(\Omega)]' \times Z_{\lambda_j,p}(\Omega)} = 0. \tag{5.14}$$

Since $||\varphi_j||_{Z_{\lambda,p}(\Omega)} = 1 (j = 1, 2, 3, \ldots)$ holds, this is a contradiction. □

6. Differentiability of the minimal solution on λ

When u_λ is left differentiable with respect to λ, by v_λ we denote the left derivative of u_λ, namely

Definition 6.1 (A left derivative of u_λ)**.** Let us set

$$v_\lambda = \lim_{\mu \to \lambda, \mu < \lambda} \frac{u_\lambda - u_\mu}{\lambda - \mu}. \tag{6.1}$$

We shall establish the following:

Theorem 6.1. *Assume that $p \in [2, \infty)$ and assume that $\tilde{V}_{\lambda,p}(\Omega)$ is dense in $V_{\lambda,p}(\Omega)$ for a fixed $\lambda \in (0, \lambda^*)$. Then the followings are equivalent:*

(1) *The self-adjoint operator $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ on $L^2(\Omega)$ has a positive first eigenvalue.*

(2) *u_λ is left differentiable at λ in $V_{\lambda,p}(\Omega)$. Moreover the left derivative $v_\lambda \in V_{\lambda,p}(\Omega)$ satisfies the boundary value problem*

$$\begin{cases} L_p'(u_\lambda)v_\lambda - \lambda f'(u_\lambda)v_\lambda = f(u_\lambda) & \text{in} \quad \Omega, \\ v_\lambda = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{6.2}$$

Remark 6.1. (1) If the minimal solution u_λ is weakly continuous on λ as a $W_0^{1,p}(\Omega)$-valued function, then u_λ becomes differentiable and the derivative of u_λ satisfies (6.2) under the condition (1).

(2) It follows from Theorem 5.1 that the boundary value problem (6.2) has a unique solution in $V_{\lambda,p}(\Omega)$ for a sufficiently small $\lambda > 0$.

Proof of Theorem 6.1. Assume that $0 < \mu < \lambda < \lambda^*$. Then we see

$$\begin{aligned} L_p(u_\lambda) - L_p(u_\mu) &= \lambda f(u_\lambda) - \mu f(u_\mu) \\ &= (\lambda - \mu) f(u_\lambda) + \mu(f(u_\lambda) - f(u_\mu)) \\ &= (\lambda - \mu) f(u_\lambda) + \mu f'(\xi)(u_\lambda - u_\mu), \end{aligned}$$

where ξ is a quantity satisfying $u_\mu < \xi < u_\lambda$. We set

$$v_{\mu,\lambda} = \frac{u_\lambda - u_\mu}{\lambda - \mu} > 0. \tag{6.3}$$

Proof of the implication (1) $\rightarrow$ (2).

First step. Assume that $||v_{\mu,\lambda}||_{V_{\lambda,p}(\Omega)} \le C < +\infty$ for some positive number C. It follows from the compactness of the imbedding operator $I_{V\to L^2}$: $V_{\lambda,p}(\Omega) \to L^2(\Omega)$ that there are some $v_\lambda \in V_{\lambda,p}(\Omega)$ and a subsequence of $\{v_{\mu,\lambda}\}$, which is again denoted by $\{v_{\mu,\lambda}\}$ for simplicity, such that as $\mu \to \lambda\,(\mu < \lambda)$

$$\begin{cases} v_{\mu,\lambda} \to v_\lambda \text{ weakly in } V_{\lambda,p}(\Omega), \\ v_{\mu,\lambda} \to v_\lambda \text{ strongly in } L^2(\Omega). \end{cases} \tag{6.4}$$

Note that

$$\begin{aligned} L_p(u_\lambda) - L_p(u_\mu) &= \int_{-1}^0 \frac{d}{dt}(L_p(u_\lambda + t(u_\lambda - u_\mu)))\,dt \\ &= \int_{-1}^0 L_p'(X(t))(u_\lambda - u_\mu)\,dt, \end{aligned} \tag{6.5}$$

where $X(t) = u_\lambda + t(u_\lambda - u_\mu)$. Thus we get for any $\psi \in \tilde{V}_{\lambda,p}(\Omega)$

$$\begin{aligned} &\left\langle \int_{-1}^0 L_p'(X(t))\, v_{\mu,\lambda}\,dt, \psi \right\rangle_{[V_{\lambda,p}(\Omega)]' \times V_{\lambda,p}(\Omega)} \\ &\qquad = \langle f(u_\lambda) + \mu f'(\xi) v_{\mu,\lambda}, \psi \rangle_{[V_{\lambda,p}(\Omega)]' \times V_{\lambda,p}(\Omega)}, \end{aligned} \tag{6.6}$$

where ξ is a quantity satisfying $u_\mu < \xi < u_\lambda$. We can show

Lemma 6.1. *Assume the same assumptions in Theorem* 6.1 *and* (6.4). *Then* $v_\lambda \in V_{\lambda,p}(\Omega)$ *satisfies*

$$\begin{cases} L_p'(u_\lambda)v_\lambda - \lambda f'(u_\lambda)v_\lambda = f(u_\lambda) & in \quad \Omega, \\ v_\lambda = 0 & on \quad \partial\Omega. \end{cases} \tag{6.7}$$

Proof. In (6.6), it is easy to see that

$$\lim_{\mu\to\lambda}\langle f(u_\lambda)+\mu f'(\xi)v_{\mu,\lambda},\psi\rangle_{[V_{\lambda,p}(\Omega)]'\times V_{\lambda,p}(\Omega)}, \tag{6.8}$$
$$=\langle f(u_\lambda)+\lambda f'(u_\lambda)v_\lambda,\psi\rangle_{[V_{\lambda,p}(\Omega)]'\times V_{\lambda,p}(\Omega)},$$

$$\lim_{\mu\to\lambda}\langle L_p'(u_\lambda)v_{\mu,\lambda},\psi\rangle_{[V_{\lambda,p}(\Omega)]'\times V_{\lambda,p}(\Omega)}=\langle L_p'(u_\lambda)v_\lambda,\psi\rangle_{[V_{\lambda,p}(\Omega)]'\times V_{\lambda,p}(\Omega)}. \tag{6.9}$$

We also note that

$$\begin{aligned}&\left\langle\int_{-1}^0 L_p'(X(t))v_{\mu,\lambda}\,dt,\psi\right\rangle_{[V_{\lambda,p}(\Omega)]'\times V_{\lambda,p}(\Omega)}\\&=\int_{-1}^0\int_\Omega|\nabla X(t)|^{p-2}(\nabla v_{\mu,\lambda},\nabla\psi)\,dx\,dt\\&+(p-2)\int_{-1}^0\int_\Omega|\nabla X(t)|^{p-4}(\nabla X(t),\nabla v_{\mu,\lambda})(\nabla X(t),\nabla\psi)\,dx\,dt.\end{aligned} \tag{6.10}$$

Therefore if $p=2$ then the assertion is clear. Hence we assume that $p>2$ from now on. Here we employ the following elementary inequalities: For any $\varepsilon>0$ there is a positive number C_ε such that for any $\zeta,\eta,a,b\in\mathbb{R}^N$ and any $t\in[0,1]$

$$\begin{aligned}&||\zeta(t)|^{p-4}(\zeta(t),a)(\zeta(t),b)-|\zeta|^{p-4}(\zeta,a)(\zeta,b)|\\&\qquad\le(\varepsilon|\zeta|^{p-2}+C_\varepsilon|\zeta-\eta|^{p-2})|a||b|,\end{aligned} \tag{6.11}$$

where $\zeta(t)=\zeta+t(\zeta-\eta)$. Thus it suffices to control error terms of the next type;

$$I=\int_\Omega|\nabla(u_\mu-u_\lambda)|^{p-2}|\nabla v_{\mu,\lambda}||\nabla\psi|\,dx. \tag{6.12}$$

Noting that $\psi\in\tilde V_{\lambda,p}(\Omega)\subset W_0^{1,p}(\Omega)$ we have

$$\begin{aligned}|I|&=\frac{1}{\lambda-\mu}\int_\Omega|\nabla(u_\lambda-u_\mu)|^{p-1}|\nabla\psi|\,dx\\&\le\frac{1}{\lambda-\mu}\left(\int_\Omega|\nabla\psi|^p\,dx\right)^{\frac1p}\left(\int_\Omega|\nabla(u_\lambda-u_\mu)|^p\,dx\right)^{1-\frac1p}\\&\le\frac{C}{\lambda-\mu}||\psi||_{W_0^{1,p}(\Omega)}||u_\lambda-u_\mu||^{p-1}_{W_0^{1,p}(\Omega)}.\end{aligned} \tag{6.13}$$

Here

$$\begin{aligned}&||u_\lambda-u_\mu||^p_{W_0^{1,p}(\Omega)}\\&\le C\int_\Omega(|\nabla u_\lambda|^{p-2}\nabla u_\lambda-|\nabla u_\mu|^{p-2}\nabla u_\mu,\nabla(u_\lambda-u_\mu))\,dx\\&=C\int_\Omega(\lambda f(u_\lambda)-\mu f(u_\mu))(u_\lambda-u_\mu)\,dx\\&=C(\lambda-\mu)\int_\Omega f(u_\lambda)(u_\lambda-u_\mu)\,dx+C\mu\int_\Omega(u_\lambda-u_\mu)^2f'(u_\lambda)\,dx.\end{aligned} \tag{6.14}$$

Hence we get

(6.15)

$$
\begin{aligned}
|I| &\le \frac{C}{\lambda-\mu}||\psi||_{W_0^{1,p}(\Omega)}||u_\lambda-u_\mu||^{p-1}_{W_0^{1,p}(\Omega)}\\
&\le C(\lambda-\mu)^{1-\frac{2}{p}}||\psi||_{W_0^{1,p}(\Omega)}\left(\int_\Omega v_{\mu,\lambda}f(u_\lambda)\,dx+\lambda\int_\Omega v_{\mu,\lambda}^2 f'(u_\lambda)\,dx\right)^{1-\frac{1}{p}}\\
&\le C'(\lambda-\mu)^{1-\frac{2}{p}}||\psi||_{W_0^{1,p}(\Omega)}(||v_{\mu,\lambda}||^{2-\frac{2}{p}}_{L^2(\Omega)}+||v_{\mu,\lambda}||^{1-\frac{1}{p}}_{L^1(\Omega)})\\
&= O((\lambda-\mu)^{1-\frac{2}{p}}).
\end{aligned}
$$

Since $\tilde V_{\lambda,p}(\Omega)$ is densely contained in $W_0^{1,p}(\Omega)\subset V_{\lambda,p}(\Omega)$, v_λ satisfies the desired equation in the weak sense. This proves the lemma. □

Second step. Assume that $\{v_{\mu,\lambda}\}$ is unbounded in $V_{\lambda,p}(\Omega)$. Then there are sequences $\{\mu_j\}$ and $\{v_{\mu_j,\lambda}\}$ such that $\mu_j\to\lambda$ and $||v_{\mu_j,\lambda}||_{V_{\lambda,p}(\Omega)}\to+\infty$ as $j\to+\infty$. In other words

$$
\frac{\lambda-\mu_j}{||u_\lambda-u_{\mu_j}||_{V_{\lambda,p}(\Omega)}}\to 0 \quad \text{as } j\to+\infty. \tag{6.16}
$$

Now we set

$$
\delta_{j,\lambda}=\frac{u_\lambda-u_{\mu_j}}{||u_\lambda-u_{\mu_j}||_{V_{\lambda,p}(\Omega)}} \quad (||\delta_{j,\lambda}||_{V_{\lambda,p}(\Omega)}=1). \tag{6.17}
$$

Since $\{\delta_{j,\lambda}\}$ is bounded in $V_{\lambda,p}(\Omega)$, we are able to assume that for some $\delta_\lambda\in V_{\lambda,p}(\Omega)$

$$
\begin{cases}\delta_{j,\lambda}\to\delta_\lambda \text{ weakly in } V_{\lambda,p}(\Omega)\\ \delta_{j,\lambda}\to\delta_\lambda \text{ strongly in } L^2(\Omega)\end{cases} \quad \text{as } j\to+\infty. \tag{6.18}
$$

As before we see that $\delta_{j,\lambda}$ satisfies for any $\psi\in\tilde V_{\lambda,p}(\Omega)$

$$
\begin{aligned}
&\left\langle\int_{-1}^0 L_p'(X_j(t))\delta_{j,\lambda}\,dt,\psi\right\rangle_{[V_{\lambda,p}(\Omega)]'\times V_{\lambda,p}(\Omega)}\\
&\quad=\left\langle f(u_\lambda)\frac{\lambda-\mu_j}{||u_\lambda-u_{\mu_j}||_{V_{\lambda,p}(\Omega)}}+\mu_j f'(\xi)\delta_{j,\lambda},\psi\right\rangle_{[V_{\lambda,p}(\Omega)]'\times V_{\lambda,p}(\Omega)},
\end{aligned}
\tag{6.19}
$$

where $X_j(t)=u_{\mu_j}+t(u_\lambda-u_{\mu_j})$ and ξ is a quantity satisfying $u_{\mu_j}<\xi<u_\lambda$. Then we can show

Lemma 6.2. *Assume the same assumptions in Theorem* 6.1 *and* (6.18). *Then* $\delta_\lambda\in V_{\lambda,p}(\Omega)$ *does not vanish identically and satisfies*

$$
\begin{cases}L_p'(u_\lambda)\delta_\lambda-\lambda f'(u_\lambda)\delta_\lambda=0 & in\quad \Omega,\\ \delta_\lambda=0 & on\quad \partial\Omega.\end{cases} \tag{6.20}
$$

Admitting this for the moment we finish the proof of the implication (1) $\to$(2). From this δ_λ becomes a non trivial first eigenfunction corresponding to the eigenvalue 0. But this contradicts to (1). $\square$

Proof of Lemma 6.2. As before we immediately see that

$$\lim_{\mu\to\lambda}\left\langle f(u_\lambda)\frac{\lambda-\mu_j}{||u_\lambda-u_{\mu_j}||_{V_{\lambda,p}(\Omega)}}+\mu_j f'(\xi)\delta_{j,\lambda},\psi\right\rangle_{[V_{\lambda,p}(\Omega)]'\times V_{\lambda,p}(\Omega)} \\ =\langle\lambda f'(u_\lambda)\delta_\lambda,\psi\rangle_{[V_{\lambda,p}(\Omega)]'\times V_{\lambda,p}(\Omega)} \tag{6.21}$$

and

$$\lim_{\mu\to\lambda}\langle L_p'(u_\lambda)\delta_{j,\lambda},\psi\rangle_{[V_{\lambda,p}(\Omega)]'\times V_{\lambda,p}(\Omega)}=\langle L_p'(u_\lambda)\delta_\lambda,\psi\rangle_{[V_{\lambda,p}(\Omega)]'\times V_{\lambda,p}(\Omega)}.$$

Therefore it suffices to show that

$$\lim_{j\to+\infty}\int_\Omega|\nabla(u_\lambda-u_{\mu_j})|^{p-2}|\nabla\delta_{j,\lambda}||\nabla\psi|\,dx=0, \tag{6.22}$$

and this follows from the inequality (6.14). In fact, from the definition of $\delta_{j,\lambda}$

$$J=\int_\Omega|\nabla(u_\lambda-u_{\mu_j})|^{p-2}|\nabla\delta_{j,\lambda}||\nabla\psi|\,dx \\ \le\frac{1}{||u_\lambda-u_{\mu_j}||_{V_{\lambda,p}(\Omega)}}||\psi||_{W_0^{1,p}(\Omega)}||u_\lambda-u_{\mu_j}||^{p-1}_{W_0^{1,p}(\Omega)} \tag{6.23}$$

and from the inequality (6.14)

$$||u_\lambda-u_{\mu_j}||^p_{W_0^{1,p}(\Omega)}||u_\lambda-u_{\mu_j}||^{-2}_{V_{\lambda,p}(\Omega)} \\ \le\frac{(\lambda-\mu_j)}{||u_\lambda-u_{\mu_j}||_{V_{\lambda,p}(\Omega)}}\int_\Omega f(u_\lambda)\delta_{j,\lambda}\,dx+C\mu_j\int_\Omega\delta_{j,\lambda}^2f'(u_\lambda)\,dx \\ \le C<+\infty. \tag{6.24}$$

Here we used the strong convergence of $\{\delta_{j,\lambda}\}$ in $L^2(\Omega)$. Then there is a constant $C>0$ such that

$$J\le C||u_\lambda-u_{\mu_j}||^{1-\frac{2}{p}}_{V_{\lambda,p}(\Omega)}. \tag{6.25}$$

Since

$$||u_\lambda-u_{\mu_j}||^2_{V_{\lambda,p}(\Omega)}\le||u_\lambda||^{p-2}_{W_0^{1,p}(\Omega)}||u_\lambda-u_{\mu_j}||^2_{W_0^{1,p}(\Omega)}\to0\quad\text{as } j\to+\infty, \tag{6.26}$$

the assertion (6.22) is proved.

Now δ_λ becomes a nonnegative weak solution of (6.20), so it is sufficient to show that δ_λ is not trivial. But we have for some number $C>0$

$$
\begin{aligned}
(6.27)\qquad & |\langle L_p(u_\lambda) - L_p(u_{\mu_j}), u_\lambda - u_{\mu_j}\rangle| \\
&\geq C\int_\Omega (|\nabla u_{\mu_j}| + |\nabla u_\lambda|)^{p-2}|\nabla(u_\lambda - u_{\mu_j})|^2\,dx \\
&\geq C||u_\lambda - u_{\mu_j}||^2_{V_{\lambda,p}(\Omega)}.
\end{aligned}
$$

Hence

$$
(6.28)\qquad |\langle L_p(u_\lambda) - L_p(u_{\mu_j}), \delta_{j,\lambda}\rangle| \geq C'||u_\lambda - u_{\mu_j}||_{V_{\lambda,p}(\Omega)}.
$$

On the other hand from (6.14)
we get

$$
\begin{aligned}
(6.29)\ & |\langle L_p(u_\lambda) - L_p(u_{\mu_j}), \delta_{j,\lambda}\rangle| \\
&\leq \left(o(1)\int_\Omega f(u_\lambda)|\delta_{j,\lambda}|\,dx + C\lambda\int_\Omega \delta_{j,\lambda}^2 f'(u_\lambda)\,dx\right)||u_\lambda - u_{\mu_j}||_{V_{\lambda,p}(\Omega)}.
\end{aligned}
$$

Since f is strictly convex and increasing, we get

$$
(6.30)\qquad 0 < C \leq \int_\Omega \delta_\lambda^2\,dx.
$$

This clearly implies the assertion. □

The proof of implication (2) → (1). Let $v_\lambda \in V_{\lambda,p}(\Omega)$ be a unique solution of the boundary value problem

$$
(6.31)\qquad \begin{cases} L'_p(u_\lambda)v_\lambda - \lambda f'(u_\lambda)v_\lambda = f(u_\lambda) & \text{in}\quad \Omega, \\ v_\lambda = 0 & \text{on}\quad \partial\Omega. \end{cases}
$$

Assume that $L'_p(u_\lambda) - \lambda f'(u_\lambda)$ has zero eigenvalue and let $\varphi \in V_{\lambda,p}(\Omega)$ be a corresponding eigenfunction. We can assume $\varphi > 0$ in Ω. Then

$$
\begin{aligned}
(6.32)\qquad \langle f(u_\lambda), \varphi\rangle &= \langle L'_p(u_\lambda)v_\lambda - \lambda f'(u_\lambda)v_\lambda, \varphi\rangle \\
&= \langle v_\lambda, L'_p(u_\lambda)\varphi - \lambda f'(u_\lambda)\varphi\rangle \\
&= 0.
\end{aligned}
$$

Since $f(u_\lambda)$ is positive, we reach to a contradiction. □

From this we can show a somewhat weak result in the case that $1 < p < 2$:

Corollary 6.1. *Assume that the same assumptions as in the previous theorem* 6.1. *Moreover assume that* $1 < p < 2$ *and there is a positive number* $\eta_0 < \min(\lambda, \lambda^* - \lambda)$ *such that for any* $\mu \in (\lambda - \eta_0, \lambda + \eta_0)$ *we have*

$$
(6.33)\qquad |\nabla u_\lambda(x) - \nabla u_\mu(x)| \leq \frac{1}{2}|\nabla u_\lambda(x)| \qquad in \quad \overline{\Omega}.
$$

Then the same conclusion holds.

Remark 6.2. The assumption on ∇u_λ means not only the invariance of the set $F_{\mu,p}$ with respect to $\mu \in (\lambda - \eta_0, \lambda + \eta_0)$ but also that of the vanishing order of $|\nabla u_\mu|$ on $F_{\mu,p}$. Later we shall give an example in which these assumptions are satisfied. See Lemma 12.1 in Section 12.

Proof of Corollary 6.1. Again we put $\zeta = \nabla u_\lambda$, $\eta = \nabla u_\mu$ and $\zeta(t) = \nabla u_\lambda + t\nabla(u_\mu - u_\lambda)$ with $t \in [-1, 0]$. Then we prepare the elementary inequality.

Lemma 6.3. *Assume the same assumptions as in the corollary. Then for any* $a, b \in \mathbb{R}^N$

$$(6.34)\qquad \begin{aligned} ||\zeta(t)|^{p-4}(\zeta(t), a)(\zeta(t), b) - |\zeta|^{p-4}(\zeta, a)(\zeta, b)| \\ \le (2-p)2^{3-p}|\zeta|^{p-3}|t||\zeta - \eta||a||b| \end{aligned}$$

Proof. For $t \in [-1.0]$

$$(6.35)\qquad |\nabla u_\lambda + t(\nabla u_\lambda - \nabla u_\mu)| \ge |\nabla u_\lambda| - |\nabla u_\lambda - \nabla u_\mu| \ge \frac{1}{2}|\nabla u_\lambda|.$$

Hence we see $\frac{1}{2}|\zeta| \le |\zeta(t)| \le \frac{3}{2}|\zeta|$. The required inequality easily follows from the integral representation of the left-hand side. □

End of the proof of Corollary 6.1. We put

$$(6.36)\qquad J = \int_\Omega |\nabla u_\lambda|^{p-3}|\nabla(u_\lambda - u_\mu)||\nabla v_{\mu,\lambda}||\nabla\psi|\,dx, \quad \psi \in \tilde{V}_{\lambda,p}(\Omega).$$

From Lemma 6.3 it suffices to show

$$(6.37)\qquad \lim_{\mu\to\lambda,\mu<\lambda} J = 0.$$

Since $v_{\mu,\lambda}$ is bounded in $\tilde{V}_{\lambda,p}(\Omega)$, for some positive number C

$$(6.38)\qquad J^2 \le C\int_\Omega |\nabla u_\lambda|^{p-2}|\nabla v_{\mu,\lambda}|^2\,dx \cdot \int_\Omega |\nabla u_\lambda|^{p-3}|\nabla(u_\lambda - u_\mu)||\nabla\psi|\,dx.$$

Noting that $\nabla\psi$ vanishes in a neighborhood of $F_{\lambda,p}$, we can apply Lebesgue's convergence theorem to obtain (6.37) and

$$(6.39)\quad \langle L'_p(u_\lambda)v_\lambda, \psi\rangle_{[V_{\lambda,p}(\Omega)]'\times V_{\lambda,p}(\Omega)} = \langle f(u_\lambda) + \lambda f'(u_\lambda)v_\lambda, \psi\rangle_{[V_{\lambda,p}(\Omega)]'\times V_{\lambda,p}(\Omega)}$$

for any $\psi \in \tilde{V}_{\lambda,p}(\Omega)$. So we see that v_λ is a weak solution. The rest of the proof will be done in the same line as before. □

Remark 6.3. In Section 2 we showed $u_\lambda \ge \frac{1}{m}u_{\lambda m^{p-1}}$ provided $0 < m < 1$. From this we see $\lambda^{\frac{1}{p-1}}u_\mu \le \mu^{\frac{1}{p-1}}u_\lambda$ provided $0 < \mu \le \lambda < \lambda^*$. Then we immediately have

$$(6.40)\qquad \frac{1}{p-1}u_\lambda \le \lambda v_\lambda, \qquad \text{if } v_\lambda \text{ exists}.$$

Hence if $u^* = u_{\lambda^*}$ is singular, then $v_{\lambda^*} = \lim_{\lambda\to\lambda^*} v_\lambda$ is also singular. Later we shall give an example of a singular v_λ in a ball. See Lemma 12.4 in Section 12.

7. Behaviors of u_λ and v_λ near $\lambda = 0$

In this subsection we shall discuss about the behaviors of u_λ and v_λ near $\lambda = 0$. Here by v_λ we denote the unique solution in $V_{\lambda,p}(\Omega)$ of

$$\begin{cases} L_p'(u_\lambda)v_\lambda - \lambda f'(u_\lambda)v_\lambda = f(u_\lambda) & \text{in} \quad \Omega, \\ v_\lambda = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{7.1}$$

for a sufficiently small $\lambda > 0$. As we proved in the previous section, this v_λ coincides with the left derivative of u_λ under certain conditions. Let w_0 be the unique solution of

$$L_p(w_0) = f(0) \quad \text{in} \quad \Omega\,; \quad w_0 = 0 \quad \text{on} \quad \partial\Omega. \tag{7.2}$$

From the maximum principle we see $w_0 > 0$ in Ω, and its normal derivative $\frac{dw_0}{dn} \neq 0$ on $\partial\Omega$. Then we can show

Lemma 7.1. *Let $u_\lambda \in C^{1,\sigma}(\overline{\Omega})$ be the minimal solution for $\lambda \in [0, \lambda^*)$. Then for any $\varepsilon_0 \in (0, \lambda^*)$ there is a positive number C shct that for any $\lambda \in [0, \varepsilon_0]$:*

(1) $\int_\Omega |\nabla u_\lambda|^q \, dx \leq C\lambda^{\frac{q}{p-1}}$ *for any* $q \geq 0$.

(2) $|\nabla u_\lambda| \leq C\lambda^{\frac{1}{p-1}}$.

(3) $\lambda^{\frac{1}{p-1}} w_0 \leq u_\lambda \leq C\lambda^{\frac{1}{p-1}}$.

Here C is independent of each $x \in \Omega$.

Proof. Let w_λ be the same function as in the proof of Theorem 5.1, that is, $w_\lambda = \lambda^{-\frac{1}{p-1}} u_\lambda$. We recall that $||w_\lambda||_{L^\infty(\Omega)} + ||\nabla w_\lambda||_{L^\infty(\Omega)} < +\infty$ $(0 \leq \lambda \leq \varepsilon_0)$. Then we see

$$||u_\lambda||_{L^\infty(\Omega)} \leq C\lambda^{\frac{1}{p-1}}, \quad ||\nabla u_\lambda||_{L^\infty(\Omega)} \leq C\lambda^{\frac{1}{p-1}}. \tag{7.3}$$

This proves the assertion (1) and (2). Since $L_p(w_\lambda) = f(u_\lambda) \geq f(0) > 0$, w_0 becomes subsolution of the same equation. Therefore we see for some number $C > 0$

$$\lambda^{\frac{1}{p-1}} w_0 \leq u_\lambda \leq C\lambda^{\frac{1}{p-1}}. \tag{7.4}$$

Thus we see the assertion (3). □

In order to describe the behavior of v_λ, we consider the next boundary value problem:

$$\begin{cases} L_p'(w_\lambda)\psi_\lambda = f(0) & \text{in} \quad \Omega, \\ \psi_\lambda = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{7.5}$$

Then we see $\psi_\lambda \in V_{\lambda,p}(\Omega)$ and $\frac{d\psi_\lambda}{dn} \neq 0$ near the boundary. From the definition of v_λ and w_λ, we have

$$\lambda^{\frac{p-2}{p-1}} L_p'(w_\lambda)v_\lambda = \lambda v_\lambda f'(u_\lambda) + f(u_\lambda) \geq f(0) > 0. \tag{7.6}$$

Hence

$$L_p'(w_\lambda)v_\lambda \geq f(0)\lambda^{-\frac{p-2}{p-1}}. \tag{7.7}$$

Therefore $\lambda^{-\frac{p-2}{p-1}}\psi_\lambda$ becomes a subsolution of (7.1), and so we see

$$v_\lambda \geq \lambda^{-\frac{p-2}{p-1}}\psi_\lambda \qquad \text{(near the boundary)}. \tag{7.8}$$

More precisely we have

Lemma 7.2. *Let $v_\lambda \in V_{\lambda,p}(\Omega)$ satisfy (7.1) for $\lambda \in [0,\lambda^*)$. Then for any $\varepsilon_0 \in (0,\lambda^*)$ there is a positive number C such that we have:*
If $p \geq 2$, then for any $\lambda \in [0,\varepsilon_0]$,
(1) $\int_\Omega v_\lambda\, dx \geq C\lambda^{-\frac{p-2}{p-1}}$.
(2) $\int_\Omega |\nabla v_\lambda|\, dx \geq C\lambda^{-\frac{p-2}{p-1}}$.
If $1 < p < 2$, then for any $\lambda \in [0,\varepsilon_0]$,
(3) $\int_\Omega v_\lambda\, dx \leq C\lambda^{\frac{2-p}{p-1}}$.
(4) $\int_\Omega |\nabla v_\lambda|^2\, dx \leq C\lambda^{2\frac{2-p}{p-1}}$.
Here C is independent of each $x \in \Omega$.

Proof. First we assume that $p > 2$. The assertion (1) follows from (7.8). Using u_λ as a test function we have

$$\begin{aligned}\int_\Omega (f(u_\lambda) + \lambda v_\lambda f'(u_\lambda))u_\lambda &= \langle L_p'(u_\lambda)v_\lambda, u_\lambda\rangle_{[V_{\lambda,p}(\Omega)]'\times V_{\lambda,p}(\Omega)} \\ &\leq C\int_\Omega |\nabla u_\lambda|^{p-1}|\nabla v_\lambda|\, dx \leq C'\lambda\int_\Omega |\nabla v_\lambda|\, dx\end{aligned} \tag{7.9}$$

and

$$\int_\Omega (f(u_\lambda) + \lambda v_\lambda f'(u_\lambda))u_\lambda\, dx \geq f(0)\int_\Omega u_\lambda\, dx \geq C\lambda^{\frac{1}{p-1}}. \tag{7.10}$$

Then we see

$$\int_\Omega |\nabla v_\lambda|\, dx \geq C\lambda^{-\frac{p-2}{p-1}}. \tag{7.11}$$

This proves the assertion(2). Now we assume that $1 < p < 2$. Using v_λ as a test function, we have

$$\begin{aligned}&\int_\Omega (f(u_\lambda) + \lambda v_\lambda f'(u_\lambda))v_\lambda = \langle L_p'(u_\lambda)v_\lambda, v_\lambda\rangle_{[V_{\lambda,p}(\Omega)]'\times V_{\lambda,p}(\Omega)} \\ &\geq C\int_\Omega |\nabla u_\lambda|^{p-2}|\nabla v_\lambda|^2\, dx \geq C'\lambda^{\frac{p-2}{p-1}}\int_\Omega |\nabla v_\lambda|^2\, dx.\end{aligned} \tag{7.12}$$

On the other hand we see by the Poincaré inequality

$$\begin{aligned}\int_\Omega (f(u_\lambda) + \lambda v_\lambda f'(u_\lambda))v_\lambda\, dx &\leq C\int_\Omega (\lambda v_\lambda + 1)v_\lambda\, dx \\ &\leq C'\lambda\int_\Omega |\nabla v_\lambda|^2\, dx + C\int_\Omega v_\lambda\, dx.\end{aligned} \tag{7.13}$$

Hence, we can choose ε_0 so that for all $\lambda \in [0, \varepsilon_0]$ we have

$$\lambda^{\frac{p-2}{p-1}} \int_\Omega |\nabla v_\lambda|^2\, dx \le C \int_\Omega v_\lambda\, dx. \tag{7.14}$$

Here we note the following elementary inequalities:

$$\left(\int_\Omega |\nabla v_\lambda|^2\, dx\right)^{\frac{1}{2}} \left(\int_\Omega |v_\lambda|^{\frac{2N}{N-2}}\, dx\right)^{\frac{N-2}{2N}} \le C \int_\Omega |\nabla v_\lambda|^2\, dx, \tag{7.15}$$

$$\int_\Omega v_\lambda\, dx \le C \left(\int_\Omega |v_\lambda|^{\frac{2N}{N-2}}\, dx\right)^{\frac{N-2}{2N}}. \tag{7.16}$$

Then we get the assersion (4)

$$\left(\int_\Omega |\nabla v_\lambda|^2\, dx\right)^{\frac{1}{2}} \le C\lambda^{\frac{2-p}{p-1}}. \tag{7.17}$$

Combining this with (7.15) and (7.16) the assertion (3) is proved. □

8. Nonnegativity of $L_p'(u_\lambda) - \lambda f'(u_\lambda)$

In Section 5 we have showed the positivity of the first eigenvalue of $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ for a sufficiently small $\lambda > 0$. In this section we shall prove that the operator $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ has a nonnegative first eigenvalue for any $\lambda > 0$ under the accessibility condition (AC). This fact is equivalent to the validity of the Hardy type inequalities. We recall the definition of the first eigenfunction $\hat{\varphi}^\lambda$ in Definition 1.3.

Theorem 8.1. *Let u_λ be the minimal solution for $\lambda \in (0, \lambda^*)$ and let $\hat{\varphi}^\lambda$ be the first eigenfunction of the self-adjoint operator $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ on $L^2(\Omega)$. Assume that $\hat{\varphi}^\lambda$ satisfies the accessibility condition (AC) defined by Definition* 1.4.

Then the first eigenvalue of $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ is nonnegative. In other words, we have

$$\int_\Omega |\nabla u_\lambda|^{p-2} \left(|\nabla \varphi|^2 + (p-2)\frac{(\nabla u_\lambda, \nabla \varphi)^2}{|\nabla u_\lambda|^2}\right) dx \ge \lambda \int_\Omega f'(u_\lambda)\varphi^2\, dx \tag{8.1}$$

for any $\varphi \in V_{\lambda,p}(\Omega)$.

Remark 8.1. (1) The proof of this will be done in a chain of Propositions and will be finished in Section 9 finally.

(2) In case that Ω is radially symmetric, the minimal solution becomes radial by the minimality. Then $F_{\lambda,p}$ consists of a single point, and so $\tilde{V}_{\lambda,p}(\Omega)$ is dense in $V_{\lambda,p}(\Omega)$. See Remark 3.5 in Section 3. Moreover the first eigenfunction $\hat{\varphi}^\lambda$ also becomes radial from the uniqueness up to a multiplication by constants, hence the accessibility condition (AC) is easily verified in such a case. See Proposition 12.2 in Section 12.

We start with defining auxiliary function spaces.

Definition 8.1. By $C^0([0,T], \tilde{V}_{\lambda,p}(\Omega))$ for $T > 0$ we denote a space of all functions $\psi_t(x)$ such that $\psi_t(x) \in \tilde{V}_{\lambda,p}(\Omega)$ for each $t \in [0,T]$ and continuous in t as $\tilde{V}_{\lambda,p}(\Omega)$ -valued functions, where the norm is given by

$$||\psi_t||_{C^0([0,T],\tilde{V}_{\lambda,p}(\Omega))} = \sup_{t\in(0,T)} ||\psi_t(\cdot)||_{\tilde{V}_{\lambda,p}(\Omega)}. \tag{8.2}$$

We also define $C^0([0,T], V_{\lambda,p}(\Omega))$, $C^0([0,T], W_{\lambda,p}(\Omega))$ and $C^0([0,T], W_0^{1,p}(\Omega))$ in a similar way.

For $\psi_t \in C^0([0,T], \tilde{V}_{\lambda,p}(\Omega))$, let us set

$$g_t(x;\psi_t) = -\frac{1}{t}(L_p(u_\lambda - t\psi_t) - L_p(u_\lambda)). \tag{8.3}$$

Then it follows from Lemma 3.5 that $g_t(x;\psi_t) \in [V_{\lambda,p}(\Omega)]'$, if t is sufficiently small. We consider the equation for each $t > 0$

$$\begin{cases} L'_p(u_\lambda)\varphi_t(x) = g_t(x;\psi_t) & x \in \Omega, \\ \varphi_t = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{8.4}$$

From Corollary 4.2 we have a unique solution $\varphi_t \in V_{\lambda,p}(\Omega)$ for each small $t > 0$. Moreover we have

Proposition 8.1. *Assume $\psi_t \in C^0([0,T], \tilde{V}_{\lambda,p}(\Omega))$. Then for a sufficiently small number $T_0 > 0$ there exists a unique $\varphi_t \in C^0([0,T_0], \tilde{V}_{\lambda,p}(\Omega))$ such that*

$$\begin{cases} L'_p(u_\lambda)\varphi_t(x) = g_t(x;\psi_t) & \text{in} \quad \Omega \times [0,T_0], \\ \varphi_0 = \psi_0 & \text{in} \quad \Omega. \end{cases} \tag{8.5}$$

Moreover there is a positive number C such that

$$||\varphi_t - \varphi_0||_{V_{\lambda,p}(\Omega)} \le C||\psi_t - \psi_0||_{V_{\lambda,p}(\Omega)} + o(1)||\psi_0||_{V_{\lambda,p}(\Omega)}, \tag{8.6}$$

$$||\psi_t - \psi_0||_{V_{\lambda,p}(\Omega)} \le C||\varphi_t - \varphi_0||_{V_{\lambda,p}(\Omega)} + o(1)||\varphi_0||_{V_{\lambda,p}(\Omega)}, \tag{8.7}$$

where $o(1)$ denotes a quantity which goes to 0 as $t \to 0$.

Proof. Since $L'_p(u_\lambda)$ is invertible, we have

$$\varphi_t(x) = [L'_p(u_\lambda)]^{-1} g_t(x;\psi_t) \in C^0([0,T], \tilde{V}_{\lambda,p}(\Omega)). \tag{8.8}$$

In fact $g_t(\cdot;\psi_t)$ vanishes on some neighborhood D of $F_{\lambda,p}$, hence $\langle g_t, \xi\rangle_{V'_{\lambda,p}\times V_{\lambda,p}} = 0$ for any $\xi \in V_{\lambda,p}(D)$. From the coercivity of $L'_p(u_\lambda)$ we see $\nabla\varphi_t = 0$ in D. Moreover φ_t is smooth as a solution of uniformly elliptic equation. Therefore we see $\varphi_t \in \tilde{V}_{\lambda,p}(\Omega)$. In the next we prove that $\varphi_0 = \psi_0$. From the differentiability of $L_p(u_\lambda)$ at u_λ in $\tilde{V}_{\lambda,p}(\Omega)$, we claim that

$$g_t(x;\psi_t) \to L_p'(u_\lambda)\psi_0 \text{ in } [V_{\lambda,p}(\Omega)]' \quad \text{as } t \to 0. \tag{8.9}$$

If ψ_t is independent of t, this holds by definition. For a general ψ_t, the assertion follows from the estimate below: For any $\xi \in V_{\lambda,p}(\Omega)$

$$\begin{aligned}
&|\langle L_p(u_\lambda - t\psi_t) - L_p(u_\lambda - t\psi_0), \xi\rangle_{[V_{\lambda,p}(\Omega)]' \times V_{\lambda,p}(\Omega)}| \\
&\leq t\int_\Omega |\nabla\xi|(|\nabla(u_\lambda - t\psi_t)|^{p-2} + |\nabla(u_\lambda - t\psi_0)|^{p-2})|\nabla(\psi_t - \psi_0)|\, dx \\
&\leq t||\psi_t - \psi_0||_{V_{\lambda,p}(\Omega)}||\xi||_{V_{\lambda,p}(\Omega)} \qquad \text{as } t \to +0.
\end{aligned}$$

Here we used the fact $\nabla\psi_t$ vanishes near $F_{\lambda,p}$ uniformly in $t \in [0,T]$. Then,

$$||L_p(u_\lambda - t\psi_t) - L_p(u_\lambda - t\psi_0)||_{[V_{\lambda,p}(\Omega)]'} \leq t||\psi_t - \psi_0||_{V_{\lambda,p}(\Omega)}.$$

Hence

(8.10)

$$\begin{aligned}
&||g_t(x;\psi_t) - L_p'(u_\lambda)\psi_0||_{[V_{\lambda,p}(\Omega)]'} \\
&\leq \frac{1}{t}||L_p(u_\lambda - t\psi_t) - L_p(u_\lambda - t\psi_0)||_{[V_{\lambda,p}(\Omega)]'} + ||g_t(x;\psi_0) - L_p'(u_\lambda)\psi_0||_{[V_{\lambda,p}(\Omega)]'} \\
&\leq ||\psi_t - \psi_0||_{V_{\lambda,p}(\Omega)} + o(1)||\psi_0||_{V_{\lambda,p}(\Omega)}.
\end{aligned}$$

This proves the claim (8.9). Noting that

$$\varphi_t - \psi_0 = (L_p'(u_\lambda))^{-1}(g_t - L_p'(u_\lambda)\psi_0) \quad \text{ in } V_{\lambda,p}(\Omega), \tag{8.11}$$

we have

$$\lim_{t\to 0} ||\varphi_t - \psi_0||_{V_{\lambda,p}(\Omega)} = 0. \tag{8.12}$$

Hence we see $\varphi_0 = \psi_0$. The inequality (8.6) immediately follows from (8.9), (8.10) and (8.11). From (8.11) we also have

$$||g_t(x;\psi_t) - L_p'(u_\lambda)\psi_0||_{[V_{\lambda,p}(\Omega)]'} \leq C||\varphi_t - \varphi_0||_{V_{\lambda,p}(\Omega)} \tag{8.13}$$

and then, for a sufficiently small $t > 0$

$$\begin{aligned}
&||\psi_t - \psi_0||^2_{V_{\lambda,p}(\Omega)} \\
&\leq C\int_\Omega (|\nabla(u_\lambda - t\psi_t)|^{p-2} + |\nabla(u_\lambda - t\psi_0)|^{p-2})|\nabla(\psi_t - \psi_0)|^2\, dx \\
&\leq Ct^{-1}|\langle L_p(u_\lambda - t\psi_t) - L_p(u_\lambda - t\psi_0), \psi_t - \psi_0\rangle_{[V_{\lambda,p}(\Omega)]' \times V_{\lambda,p}(\Omega)}| \\
&= C|\langle g_t(x;\psi_t) - g_t(x;\psi_0), \psi_t - \psi_0\rangle_{[V_{\lambda,p}(\Omega)]' \times V_{\lambda,p}(\Omega)}| \\
&\leq C(||\varphi_t - \varphi_0||_{V_{\lambda,p}(\Omega)} + o(1)||\varphi_0||_{V_{\lambda,p}(\Omega)})||\psi_t - \psi_0||_{V_{\lambda,p}(\Omega)},
\end{aligned}$$

where C is a positive number. Thus we have the desired inequality (8.7). □

Conversely we consider the following boundary value problem.

$$\begin{cases} L_p(\eta_t(x)) = L_p(u_\lambda) - tL_p'(u_\lambda)\varphi & \text{in} \quad \Omega, \\ \eta_t = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{8.14}$$

where $\varphi \in \tilde{V}_{\lambda,p}(\Omega)$ is given and t is a small nonnegative parameter.

From the theory of monotone operator we see that there is a unique solution $\eta_t \in W_0^{1,p}(\Omega)$ for each $t \geq 0$. Here we note that $L_p(u_\lambda) \in [W_0^{1,p}(\Omega)]'$ and $L_p'(u_\lambda)\varphi$ is smooth. Since $\eta_0 = u_\lambda$ for $t = 0$, we can put $\eta_t(x) = u_\lambda - t\psi_t(x)$ formally. From Lemma 2.1 we see η_t and $\psi_t \in C^{1,\sigma}(\overline{\Omega})$ for some $\sigma \in (0, 1]$. Then we can show the following:

Proposition 8.2. *Let $\varphi \in \tilde{V}_{\lambda,p}(\Omega)$ satisfy $|\nabla\varphi| = 0$ on $F_\varepsilon = \{x \in \Omega : dist(x, F_{\lambda,p}) < \varepsilon\}$ for some $\varepsilon > 0$. Then there is a unique solution η_t of* (8.14) *for a small $T > 0$ such that $\eta_t = u_\lambda - t\psi_t$ for $\psi_t \in C^0([0,T], \tilde{V}_{\lambda,p}(\Omega))$ and*

$$\begin{cases} |\nabla\psi_t| = 0 \quad \text{on } F_\varepsilon, \\ \lim_{t\to 0} ||\psi_t - \varphi||_{V_{\lambda,p}(\Omega)} = 0. \end{cases} \tag{8.15}$$

Proof. Without the loss of generality we assume that F_ε is smooth. Since $\varphi \in C^\infty(\Omega)$ and φ is a constant on F_ε, we see $\partial^\alpha\varphi = 0$ on ∂F_ε for any multi-index $\alpha \neq 0$. First we claim that $\nabla\psi_t = 0$ on F_ε. We consider the boundary value problem

$$\begin{cases} L_p(\eta_t^1) = L_p(u_\lambda) & \text{in} \quad F_\varepsilon, \\ \eta_t^1 = u_\lambda & \text{on} \quad \partial F_\varepsilon. \end{cases} \tag{8.16}$$

This clearly has the unique solution $\eta_t^1 = u_\lambda$. In the complement of F_ε, the problem

$$\begin{cases} L_p(\eta_t^2) = L_p(u_\lambda) - tL_p'(u_\lambda)\varphi & \text{in} \quad (F_\varepsilon)^c, \\ \eta_t^2 = u_\lambda & \text{on} \quad \partial F_\varepsilon. \end{cases} \tag{8.17}$$

has a unique solution in $W_0^{1,p}(\Omega)$ as well. Note that both u_λ and η_t^2 become smooth in a small neighborhood of ∂F_ε, and the right hand side also equals a smooth function $L_p(u_\lambda)$ there. Hence it is easy to see that these solutions satisfy the compatibility conditions on ∂F_ε, that is,

$$\eta_t^1 = \eta_t^2, \qquad |\nabla\eta_t^1|^{p-2}\frac{d\eta_t^1}{dn} = |\nabla\eta_t^2|^{p-2}\frac{d\eta_t^2}{dn}, \tag{8.18}$$

where n is a unit outer normal to ∂F_ε. Then the function η_t defined by

$$\eta_t = \begin{cases} \eta_t^1 & \text{in } F_\varepsilon, \\ \eta_t^2 & \text{in } (F_\varepsilon)^c \end{cases} \tag{8.19}$$

becomes a weak solution of (8.14) in $W_0^{1,p}(\Omega)$. From the uniqueness of the solution of boundary value problem in $W_0^{1,p}(\Omega)$ it follows that $\nabla\psi_t$ vanishes on F_ε. This proves the claim. In the next we claim that $\psi_t \in \tilde{V}_{\lambda,p}(\Omega)$. This follows from the next estimate:

$$\begin{aligned}
||\psi_t||^2_{V_{\lambda,p}(\Omega)} &= \int_\Omega |\nabla u_\lambda|^{p-2}|\nabla\psi_t|^2\,dx \\
&\le C\int_\Omega (|\nabla(u_\lambda - t\psi_t)| + |\nabla u_\lambda|)^{p-2}|\nabla\psi_t|^2\,dx \\
&\le C\frac{1}{t}|\langle L_p(u_\lambda - t\psi_t) - L_p(u_\lambda), \psi_t\rangle| \\
&\le C|\langle L_p'(u_\lambda)\varphi, \psi_t\rangle| = C||\varphi||_{V_{\lambda,p}(\Omega)}||\psi_t||_{V_{\lambda,p}(\Omega)}.
\end{aligned}$$

So that we have

$$||\psi_t||_{V_{\lambda,p}(\Omega)} \le C||\varphi||_{V_{\lambda,p}(\Omega)}. \tag{8.20}$$

The last statement also follows from (8.7) in Proposition 8.1 by putting $\varphi_t \equiv \varphi$. □

Since the right hand side of (8.14) is positive for a sufficiently small $t \ge 0$, it follows from Lemma 2.4 that $\nabla\eta_t = \nabla(u_\lambda - t\psi_t)$ does not vanish near the boundary $\partial\Omega$. Note that η_t satisfies the elliptic equation with smooth coefficients:

$$-\left(\Delta\eta_t + (p-2)\frac{\partial_j\eta_t\partial_k\eta_t}{|\nabla\eta_t|^2}\partial^2_{i,j}\eta_t\right) = \frac{L_p(u_\lambda) - tL_p'(u_\lambda)\varphi}{|\nabla\eta_t|^{p-2}} \quad \text{in} \quad \Omega_\rho. \tag{8.21}$$

Here ρ is a small positive number and $\Omega_\rho = \{x \in \Omega : dist(x, \partial\Omega) < \rho\}$. Therefore η_t and ψ_t are smooth as well as u_λ near the boundary $\partial\Omega$ for a sufficiently small $t > 0$. Moreover we can show the following strong convergence.

Proposition 8.3. *Assume that $\varphi \in \tilde{V}_{\lambda,p}(\Omega)$. Let η_t be a unique solution (8.14) for a small $T > 0$ such that $\eta_t = u_\lambda - t\psi_t$ for $\psi_t \in C^0([0,T], V_{\lambda,p}(\Omega))$. Then there is a small number $\rho > 0$ such that*

$$\lim_{t\to 0} ||\psi_t - \varphi||_{C^1(\overline{\Omega_\rho})=0}. \tag{8.22}$$

Here $||\cdot||_{C^1(\overline{\Omega_\rho})}$ is defined by

$$||u||_{C^1(\overline{\Omega_\rho})} = \sup_{x\in\Omega_\rho}(|u(x)| + |\nabla u(x)|). \tag{8.23}$$

This will be proved in the next section. Admitting this for the present we establish Theorem 8.1 in the rest of this section.

Proof of Theorem 8.1. Assume that the self-adjoint operator $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ on $L^2(\Omega)$ has a negative first eigenvalue μ and a corresponding first eigenfunction $\hat{\varphi}^\lambda \in V_{\lambda,p}(\Omega)$ which is positive except on $F_{\lambda,p}$. Namely

$$L_p'(u_\lambda)\hat{\varphi}^\lambda - \lambda f'(u_\lambda)\hat{\varphi}^\lambda = \mu\hat{\varphi}^\lambda \quad (\mu < 0, \hat{\varphi}^\lambda \in V_{\lambda,p}(\Omega)). \tag{8.24}$$

From the accessibility condition (AC) we claim that

Lemma 8.1. *There exist positive numbers ρ and C_0, a negative number ν, a positive $\varphi \in \tilde{V}_{\lambda,p}(\Omega)$ and a nonnegative $\xi \in \tilde{V}_{\lambda,p}(\Omega) \cap C_0^\infty(\Omega)$ such that*

$$\begin{cases} L'_p(u_\lambda)\varphi - \lambda f'(u_\lambda)\varphi \le \nu(\varphi + \xi) \quad in \quad \Omega, \\ |\nabla\varphi| \ge C_0 \quad in \quad \Omega_\rho = \{x \in \Omega : dist(x, \partial\Omega) < \rho\}. \end{cases} \tag{8.25}$$

Proof. It follows from the accessiblity condition (AC) that a nonnegative $\hat{\varphi}^\lambda \in D(L'_p(u_\lambda))$ is approximated by elements in $\tilde{V}_{\lambda,p}(\Omega)$ in the following way: For any $\varepsilon > 0$ there exists a nonnegative $\varphi \in \tilde{V}_{\lambda,p}(\Omega)$ such that

$$L'_p(u_\lambda)(\varphi - \hat{\varphi}^\lambda) + |\varphi - \hat{\varphi}^\lambda| \le \varepsilon \max(\hat{\varphi}^\lambda, dist(x, \partial\Omega)) \quad \text{in} \quad \Omega. \tag{8.26}$$

Note that $\tilde{V}_{\lambda,p}(\Omega) \subset D(L'_p(u_\lambda)) \subset V_{\lambda,p}(\Omega) \subset L^2(\Omega)$. (See also Corollary 4.3 in Section 4.) Note that u_λ is of class C^2 in the complement of any neighborhood of $F_{\lambda,p}$ as a solution of uniformly elliptic equation. Moreover $|\nabla u_\lambda|$ does not vanish near $\partial\Omega$. Therefore $|\nabla\hat{\varphi}^\lambda| > 0$ near $\partial\Omega$, and so we have $C_0 \cdot dist(x, \partial\Omega)) \le \hat{\varphi}^\lambda$ near $\partial\Omega$ for some constant $C_0 > 0$. Now we show that $|\nabla\varphi|$ does not vanish in Ω_ρ if ε and ρ are sufficiently small. In fact we immediately see from (8.26) $\hat{\varphi}^\lambda = \hat{\varphi}^\lambda - \varphi + \varphi \le \varepsilon C_0^{-1}\hat{\varphi}^\lambda + \varphi$, hence $(1 - \varepsilon C_0^{-1})\hat{\varphi}^\lambda \le \varphi$. Then we have

$$0 < (1 - \varepsilon C_0^{-1})\frac{d\hat{\varphi}^\lambda}{dn} \le \frac{d\varphi}{dn} \qquad \text{near the boundary } \partial\Omega, \tag{8.27}$$

where $0 < \varepsilon < C_0$ and n is an interior normal to $\partial\Omega$.

Temporally we assume $\hat{\varphi}^\lambda > 0$ in Ω. Then we have $C_1 \cdot dist(x, \partial\Omega)) \le \hat{\varphi}^\lambda$ in the whole Ω for some constant $C_1 > 0$. From (8.24) and (8.26) we see

$$\begin{aligned} &L'_p(u_\lambda)\varphi - \lambda f'(u_\lambda)\varphi \\ &\qquad = (L'_p(u_\lambda) - \lambda f'(u_\lambda))(\varphi - \hat{\varphi}^\lambda) + \mu\varphi^\lambda \\ &\qquad \le \varepsilon \max(\hat{\varphi}^\lambda, dist(x, \partial\Omega)) + \mu\varphi^\lambda \\ &\qquad \le (\mu + \varepsilon \max(1, C_1^{-1}))\varphi^\lambda. \end{aligned} \tag{8.28}$$

Therefore the claim is clear for a small $\varepsilon > 0$.

In the next we remove the assumption of positivity on $\hat{\varphi}^\lambda$. Choose and fix a nonnegative $\xi \in \tilde{V}_{\lambda,p}(\Omega) \cap C_0^\infty(\Omega)$ which will be specified later. Then

$$\begin{aligned} (L'_p(u_\lambda)&-\lambda f'(u_\lambda))(\hat{\varphi}^\lambda + \xi) \\ &= \mu\hat{\varphi}^\lambda + (L'_p(u_\lambda) - \lambda f'(u_\lambda))\xi \\ &< \mu\hat{\varphi}^\lambda - \lambda f'(0)\xi + L'_p(u_\lambda)\xi \\ &< -\Lambda(\hat{\varphi}^\lambda + \xi) + L'_p(u_\lambda)\xi, \end{aligned} \tag{8.29}$$

where $\Lambda = \min(-\mu, \lambda f'(0)) > 0$. Note that for any $\xi \in \tilde{V}_{\lambda,p}(\Omega)$, $L'_p(u_\lambda)\xi$ is smooth in Ω and vanishes on some neighborhood of $F_{\lambda,p}$. For any $\varepsilon \in (0, \Lambda)$ we choose a nonnegative $\xi \in \tilde{V}_{\lambda,p}(\Omega) \cap C_0^\infty(\Omega)$ so that we have $L'_p(u_\lambda)\xi \le \epsilon\hat{\varphi}^\lambda$. Since $|\nabla\hat{\varphi}^\lambda| > 0$ near the boundary, this is possible. After all we have

$$\begin{cases} (L'_p(u_\lambda) - \lambda f'(u_\lambda))(\hat{\varphi}^\lambda + \xi) < (\epsilon - \Lambda)(\hat{\varphi}^\lambda + \xi), \\ \hat{\varphi}^\lambda + \xi > 0 \quad \text{in} \quad \Omega. \end{cases} \quad . \tag{8.30}$$

Hence by replacing $\hat{\varphi}^\lambda$ by $\hat{\varphi}^\lambda + \xi$ if necessary, the same conclusion (8.25) holds. $\square$

Proposition 8.4. *Assume that $\varphi \in \tilde{V}_{\lambda,p}(\Omega)$ is positive in Ω and satisfies* (8.25) *with a negative number ν. Then there are a small $T > 0$ and an $\eta_t = u_\lambda - t\psi_t$ for $\psi_t \in C^0([0,T], \tilde{V}_{\lambda,p}(\Omega))$ such that*:

(1) *η_t satisfies* (8.14) *for any $t \in [0,T]$, that is*

$$\begin{cases} L_p(\eta_t(x)) = L_p(u_\lambda) - tL'_p(u_\lambda)\varphi & in \quad \Omega, \\ \eta_t = 0 & on \quad \partial\Omega. \end{cases}$$

(2) *$\psi_t \in C^{1,\sigma}(\overline{\Omega})$ for some $\sigma \in (0,1]$ satisfies*

$$\begin{cases} \lim_{t\to 0} ||\psi_t - \varphi||_{V_{\lambda,p}(\Omega)} = 0, \\ \lim_{t\to 0} ||\psi_t - \varphi||_{C^1(\overline{\Omega_\rho})=0} \text{ for some } \rho > 0. \end{cases} \tag{8.31}$$

(3) *$\eta_t \le u_\lambda$ holds in Ω for any $t \in [0,T]$. Moreover there exists a set having positive measure on which $\eta_t < u_\lambda$ holds.*

(4) *For each $t \in [0,T]$ there are some point $x_t \in \Omega$ and a positive number r_t such that $L_p(\eta_t) \le \lambda f(\eta_t)$ in $B_{r_t}(x_t)$ in the distribution sense, that is,*

$$\langle L_p(\eta_t), \xi\rangle_{[C_\infty]'\times C_\infty} \le \lambda \int_\Omega f(\eta_t)\xi\, dx \tag{8.32}$$

for any $\xi \in C_0^\infty(B_{r_t}(x_t))$ satisfying $\xi \ge 0$.

Proof. The assertions (1) and (2) follow from Propositions 8.2 and 8.3 respectively. If T is sufficiently small, then ψ_t is nonnegative for any $t \in [0,T]$. Hence the assertion (3) holds. We proceed to the proof of (4). Assume that for some $t \in [0,T]$, $L_p(\eta_t) > \lambda f(\eta_t)$ in Ω. Then η_t becomes a supersolution. Since 0 is a subsolution, it follows from the standard argument of monotone iteration that we get at least one solution w satisfying $0 < w < \eta_t \le u_\lambda$ in Ω. But this contradicts to the minimality of u_λ. $\square$

End of proof of Theorem 8.1. From Lemma 8.1 and (4) in Proposition 8.4, for each $t \in [0,T]$ there exist some point $x_t \in \Omega$ and a positive number r_t such that we have

$$\begin{cases} L'_p(u_\lambda)\varphi - \lambda f'(u_\lambda)\varphi \le \nu(\hat{\varphi}^\lambda + \xi) & \text{in} \quad \Omega, \\ L_p(u_\lambda) - tL'_p(u_\lambda)\varphi \le \lambda f(u_\lambda - t\psi_t) & \text{in} \quad B_{r_t}(x_t). \end{cases} \tag{8.33}$$

We note that for any $t \in [0,T]$

$$f(u_\lambda - t\psi_t) - f(u_\lambda) + tf'(u_\lambda)\psi_t = o(t)|\psi_t|. \tag{8.34}$$

Hence we have

$$0 \le \lambda f'(u_\lambda)(\varphi - \psi_t) + \nu(\hat{\varphi}^\lambda + \xi) + o(1)|\psi_t| \quad \text{in} \quad B_{r_t}(x_t). \tag{8.35}$$

Here we remark that (8.33) and this inequality have to be valid in the sence of pointwise, since each term is continuous. Since Ω is bounded, we can assume that $\lim_{t\to+0} x_t = x^0 \in \overline{\Omega}$ by choosing subsequence. Letting $t \to +0$, we get the inequality $0 \le \nu(\hat{\varphi}^\lambda + \xi)(x^0)$. If $x^0 \in \Omega$, then $(\hat{\varphi}^\lambda + \xi)(x^0) > 0$, hence this immediately leads us to a contradiction. So we proceed to the case that $x^0 \in \partial\Omega$. Let us prepare the following:

Lemma 8.2. *Assume that $x_t \in \Omega \to x^0 \in \partial\Omega$ as $t \to +0$. Then*

$$\lim_{t\to+0} \frac{\varphi(x_t) - \psi_t(x_t)}{\varphi(x_t)} = 0. \tag{8.36}$$

If we admit this, dividing the both side of (8.35) and letting $t \to +0$ we have $0 \le \nu$, and again we reach to a contradiction. This proves Theorem 8.1. □

Proof of Lemma 8.2. Since $\partial\Omega$ is of class C^2, we see that φ and ψ_t are of class $C^2(\partial\Omega) \cap C^\infty(\Omega_\rho)$. Using a suitable diffeomorphism, we can suppose that $x^0 = 0$, and Ω is a half space $\mathbb{R}^N_+ = \{x = (x', x_N) \in \mathbb{R}^N : x' \in \mathbb{R}^{N-1}, x_N > 0\}$. Since φ vanishes on a plane $\{x = (x', x_N) \in \mathbb{R}^N : x' \in \mathbb{R}^{N-1}, x_N = 0\}$, we have for $x_N < \rho$

$$\varphi(x', x_N) = \int_0^{x_N} \partial_s\varphi(x', s)\, ds = x_N \int_0^1 \partial_s\varphi(x', tx_N)\, dt. \tag{8.37}$$

In a similar way

$$\psi_t(x) = x_N \int_0^1 \partial_s\varphi(x', tx_N)\, dt. \tag{8.38}$$

Hence

$$\frac{\varphi(x_t) - \psi_t(x_t)}{\varphi(x_t)} = \frac{\displaystyle\int_0^1 (\partial_s\varphi - \partial_s\psi_t)(x', t(x_t)_N)\, dt}{\displaystyle\int_0^1 \partial_s\varphi(x', t(x_t)_N)\, dt}. \tag{8.39}$$

There is a positive number C such that $|\partial_s\varphi| \ge C$ for any $x \in \Omega_\rho$. Since ψ_t converges φ in $C^1(\overline{\Omega})$, the right hand side of (8.39) goes to 0 as $t \to 0$. □

9. Proof of Proposition 8.3

In this section we establish Proposition 8.3 which was stated in Section 8.

Proof of Proposition 8.3. First note that

$$L_p(u) = -|\nabla u|^{p-2}\left(\Delta u + (p-2)\frac{\partial_j u \partial_k u}{|\nabla u|^2}\partial^2_{i,j} u\right). \tag{9.1}$$

As was already seen just after the proof of Proposition 8.2, we can choose $T > 0$ and $\rho > 0$ so small that $\nabla\eta_t = \nabla u_\lambda - t\nabla\psi_t$, ∇u_λ and $\nabla\psi_t$ do not vanish in $\Omega_\rho = \{x \in \Omega : dist(x, \partial\Omega) < \rho\}$ for any $t \in [0, T]$. Then η_t and u_λ are smooth in Ω_ρ as solutions of uniformly elliptic equations with regular coefficients, and so ψ_t also can be assumed to be smooth in Ω_ρ for any $t \in [0, T]$. Let us set

$$-\sum_{j,k} A_{jk}(x,t)\partial^2_{jk}\eta_t = \frac{L_p(\eta_t)}{|\nabla\eta_t|^{p-2}}. \tag{9.2}$$

Here

$$A_{j,k}(x,t) = \delta_{j,k} + (p-2)\frac{\partial_j\eta_t\partial_k\eta_t}{|\nabla\eta_t|^2}. \tag{9.3}$$

By the definition of η_t we have

$$-\sum_{j,k} A_{jk}(x,t)\partial^2_{jk}\eta_t = \frac{L_p(u_\lambda) - tL'_p(u_\lambda)\varphi}{|\nabla\eta_t|^{p-2}} \in C^\infty(\Omega_\rho) \cap C^2(\overline{\Omega_\rho}). \tag{9.4}$$

Then ψ_t satisfies

$$t\sum_{j,k} A_{j,k}(x,t)\partial^2_{j,k}\psi_t = G_{\lambda,t}(x) \in C^\infty(\Omega_\rho) \cap C^2(\overline{\Omega_\rho}), \tag{9.5}$$

where

$$G_{\lambda,t}(x) = \sum_{j,k} A_{j,k}(x,t)\partial^2_{j,k}u_\lambda + \frac{L_p(u_\lambda) - tL'_p(u_\lambda)\varphi}{|\nabla\eta_t|^{p-2}}. \tag{9.6}$$

In a similar way we have

$$t\sum_{j,k} B_{j,k}(x,t)\partial^2_{j,k}\varphi = \tilde{G}_{\lambda,t}(x) \in C^\infty(\Omega_\rho) \cap C^2(\overline{\Omega_\rho}), \tag{9.7}$$

where

$$\begin{cases} \tilde{G}_{\lambda,t}(x) = \sum_{j,k} B_{j,k}(x,t)\partial^2_{j,k}u_\lambda + \dfrac{L_p(u_\lambda - t\varphi)}{|\nabla(u_\lambda - t\varphi)|^{p-2}}, \\ B_{j,k}(x,t) = \delta_{j,k} + (p-2)\dfrac{\partial_j(u_\lambda - t\varphi)\partial_k(u_\lambda - t\varphi)}{|\nabla(u_\lambda - t\varphi)|^2}. \end{cases} \tag{9.8}$$

From a mean value theorem for smooth functions and the differentiability of $L_p(\cdot)$ at u_λ in the direction to φ, there is a positive number C such that for any $t \in [0, T]$ and any $x \in \overline{\Omega_\rho}$

$$|G_{\lambda,t}(x) - \tilde{G}_{\lambda,t}(x)| \le Ct|\nabla(\psi_t - \varphi)| + o(t). \tag{9.9}$$

Here by $o(t)$ we denote a quantity satisfying $\frac{o(t)}{t} \to 0$ as $t \to 0$. Set

$$W_t = \psi_t - \varphi. \tag{9.10}$$

Then W_t satisfies

$$\sum_{j,k} A_{j,k}\partial^2_{j,k}W_t = H(x), \tag{9.11}$$

where

$$H(x) = \sum_{j,k}(B_{j,k} - A_{j,k})\partial^2_{j,k}\varphi + \frac{G_{\lambda,t}(x) - \tilde{G}_{\lambda,t}(x)}{t}. \tag{9.12}$$

It is easy to see that $H(x) \in C^\infty(\overline{\Omega_\rho})$ satisfies the estimate

$$|H(x)| \le C_1(|\nabla\psi_t - \nabla\varphi| + o(1)) = C_1(|\nabla W_t| + o(1)), \tag{9.13}$$

where C_1 is a positive number depending on $|\nabla\varphi|, |\nabla\psi_t|$ and $|\nabla^2\varphi|$.

From (9.13) and L^2 energy estimate for uniformly elliptic equation (9.11) we get for $0 < \rho' < \rho$

$$\begin{aligned} ||W_t||_{W^{2,2}(\Omega_{\rho'})} &\le C(||H(x)||_{L^2(\Omega_\rho)} + ||W_t||_{W^{1,2}(\Omega_\rho)} \\ &\le C[||W_t||_{V_{\lambda,p}(\Omega)} + o(1)]. \end{aligned} \tag{9.14}$$

Here we used

$$\begin{cases} ||H(x)||_{L^2(\Omega_\rho)} \le C(||W_t||_{W^{1,2}(\Omega_\rho)} + o(1)), \\ ||W_t||_{W^{1,2}(\Omega_\rho)} \le C'||W_t||_{V_{\lambda,p}(\Omega)} \end{cases} \tag{9.15}$$

for some constants C and $C' > 0$ (note that ∇u_λ does not vanish near $\partial\Omega$). Hence we have from (8.31)

$$\lim_{t\to 0} ||W_t||_{W^{2,2}(\Omega_{\rho'})} = 0. \tag{9.16}$$

Now we differentiate the both side of (9.11) with respect to x_m to obtain

$$\sum_{j,k} A_{j,k}\partial^2_{j,k}\partial_m W_t = \partial_m H(x) - \sum_{j,k}\partial_m A_{j,k}\partial^2_{j,k}W_t. \tag{9.17}$$

Then $\partial_m H(x)$ and $\partial_m A_{j,k}$ satisfy

$$\begin{cases} ||\partial_m H(x)||_{L^2(\Omega_{\rho'})} \le C(||W_t||_{W^{2,2}(\Omega_{\rho'})} + o(1)), \\ ||\sum_{j,k}\partial_m A_{j,k}\partial^2_{j,k}W_t||_{L^2(\Omega_{\rho'})} \le C||W_t||_{W^{2,2}(\Omega_{\rho'})}, \end{cases} \tag{9.18}$$

where C is a positive number depending on $|\nabla^\alpha\varphi|, |\nabla^\alpha\psi_t|$ for $0 \le \alpha \le 3$, and $|\nabla^\alpha\varphi| = (\sum_{|\gamma|=\alpha}|\partial^\gamma\varphi|^2)^{\frac{1}{2}}$. Since m is any number, we have for $0 < \rho'' < \rho'$

$$\begin{aligned} ||W_t||_{W^{3,2}(\Omega_{\rho''})} &\le C(||\nabla H(x)||_{L^2(\Omega'_\rho)} + ||W_t||_{W^{2,2}(\Omega_{\rho'})} \\ &\le C[||W_t||_{V_{\lambda,p}(\Omega)} + o(1)] \to 0 \text{ as } t \to +0. \end{aligned} \tag{9.19}$$

Here the positive number C depends on ρ' and ρ''. Therefore we can show inductively that for any positive integer n and any $\rho' \in (0, \rho)$,

$$||W_t||_{W^{n,2}(\Omega_{\rho'})} \leq C(n, \rho, \rho')||W_t||_{V_{\lambda,p}(\Omega)} + o(1)] \to 0 \text{ as } t \to +0. \tag{9.20}$$

Here $C(n, \rho, \rho')$ is a positive number depending only on n, ρ and ρ'. After all, by Sobolev imbedding theorem we have

$$\lim_{t \to 0} ||\psi_t - \varphi||_{C^1(\overline{\Omega_\rho})=0}. \tag{9.21}$$

This proves the assertion. □

10. The extremal solution and its characterization

In this section we shall study the behaviors of u_λ and the operator $L'_p(u_\lambda)$ near $\lambda = \lambda^*$. As was seen before in Theorem 2.1 in Section 2, the extremal solution $u^* \in W_0^{1,p}(\Omega)$ always exists in our framework as a monotonically increasing limit of a sequence of classical solutions. Namely, $u^* = \lim_{\lambda \to \lambda^*} u_\lambda$ satisfies

$$\begin{cases} L_p(u^*) = \lambda^* f(u^*) & \text{in} \quad \Omega, \\ u^* = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{10.1}$$

But u^* can be classical or singular (that is to say, unbounded). In case that $p = 2$, it is known that there is no solution even in the weak sense for any $\lambda > \lambda^*$. We start to prove the counterpart to this fact.

Definition 10.1 (Growth Condition)**.** For $p > 1$, a function $f(t) \in C^1([0, \infty))$ is said to satisfy the growth condition (GC) if f is increasing, strictly convex with $f(0) > 0$ and

$$\frac{f'(t)}{f(t)^{\frac{p-2}{p-1}}} \quad \text{is nondecreasing on} \quad [0, \infty). \tag{10.2}$$

Remark 10.1. (1) If $1 < p \leq 2$, (10.2) is automatically satisfied for increasing convex functions with $f(0) > 0$.

(2) For example, e^t and $(1 + t)^q; q \geq p - 1$ satisfy (GC).

(3) If f is C^2 function, (10.2) follows from;

$$\frac{d}{dt}\left(\frac{f'(t)}{f(t)^{\frac{p-2}{p-1}}}\right) \geq 0 \qquad \text{for all } t \in [0, \infty), \tag{10.3}$$

or

$$f''(t)f(t) \geq \frac{p-2}{p-1}f'(t)^2 \qquad \text{for all } t \in [0, \infty). \tag{10.4}$$

If u_λ^* is singular, we can show the following. The idea of the proof is essentially due to [1: H. Brezis, Th. Cazenave, Y. Martel and A. Ramiandrisoa], see also [2].

Theorem 10.1. *Let u^* be the singular extremal solution. Assume that the nonlinearlity $f(t)$ satisfies the growth condition (GC) in addition to (2.2). Then there is no weak energy solution to (2.3) provided that $\lambda > \lambda^*$.*

We prepare two lemmas.

Lemma 10.1. *Let $u \in W_0^{1,p}(\Omega)$ be the weak energy solution of (2.3). Let $\Psi \in C^2(\mathbb{R})$ be concave, with Ψ' bounded and $\Psi(0) = 0$. Then $v = \Psi(u)$ satisfies*

$$L_p(v) \geq \lambda |\Psi'(u)|^{p-2}\Psi'(u)f(u) \tag{10.5}$$

in the sense that

$$\langle L_p(v), \varphi\rangle_{[W_0^{1,p}(\Omega)]' \times W_0^{1,p}(\Omega)} \geq \lambda \int_\Omega |\Psi'(u)|^{p-2}\Psi'(u)f(u)\varphi\, dx \tag{10.6}$$

for any $\varphi \in C_0^1(\Omega)$.

Proof. By a direct calculation we see

$$\begin{aligned} L_p(v) &= |\Psi'(u)|^{p-2}\Psi'(u)L_p(u) - (p-1)|\Psi'(u)|^{p-2}\Psi''(u)|\nabla u|^p \\ &\geq |\Psi'(u)|^{p-2}\Psi'(u)L_p(u) \\ &= \lambda f(u)|\Psi'(u)|^{p-2}\Psi'(u). \end{aligned} \tag{10.7}$$

This proves the assertion. □

For a given $\varepsilon \in (0, 1)$ we set

$$\tilde{f} = (1-\varepsilon)f. \tag{10.8}$$

Set for all $u \geq 0$

$$h(u) = \int_0^u \frac{ds}{f(s)^{\frac{1}{p-1}}} \quad \text{and} \quad \tilde{h}(u) = \int_0^u \frac{ds}{\tilde{f}(s)^{\frac{1}{p-1}}}, \tag{10.9}$$

then $\tilde{h}(u) = (1-\varepsilon)^{-\frac{1}{p-1}}h(u)$.

Lemma 10.2. *Assume that f satisfies (GC). Let us set for all $u \geq 0$*

$$\Psi(u) = \tilde{h}^{-1}(h(u)). \tag{10.10}$$

Then

(1) *$\Psi(0) = 0$ and $0 \leq \Psi(u) \leq u$ for all $u \geq 0$.*
(2) *If $h(+\infty) < +\infty$ and $\tilde{f} \neq f$, then $\Psi(+\infty) < +\infty$.*
(3) *Ψ is increasing, concave, and $\Psi' \leq 1$ for all $u \geq 0$.*

Proof. The assertions (1) and (2) are clear. We have

$$\Psi'(u) = \left(\frac{\tilde{f}(\Psi(u))}{f(u)} \right)^{\frac{1}{p-1}} \tag{10.11}$$

and

$$\Psi''(u) = \frac{1}{p-1} \left(\frac{\tilde{f}(\Psi(u))}{f(u)} \right)^{\frac{1}{p-1}} \left(\frac{\tilde{f}'(\Psi(u))}{\tilde{f}(\Psi(u))} \Psi'(u) - \frac{f'(u)}{f(u)} \right). \tag{10.12}$$

Set $\Phi(u) = \frac{\tilde{f}'(\Psi(u))}{\tilde{f}(\Psi(u))} \Psi'(u) - \frac{f'(u)}{f(u)}$. Then it suffices to show $\Phi \leq 0$, and this is equivalent to the inequality:

$$(1-\varepsilon)^{\frac{1}{p-1}} \frac{f'(\Psi(u))}{f(\Psi(u))^{\frac{p-2}{p-1}}} \leq \frac{f'(u)}{f(u)^{\frac{p-2}{p-1}}} \quad \text{for all } u \geq 0. \tag{10.13}$$

But this holds from the fact $\Psi(u) \leq u$. □

Proof of Theorem 10.1. Assume that there is a weak energy solution u of (2.3) for some $\lambda > \lambda^*$. We set $v = \Psi(u) = \tilde{h}^{-1}(h(u))$ for $\varepsilon \in (0, \min(1, \lambda - \lambda^*))$. Since f satisfies the condition (2.2), we see that h is bounded. Hence v is also bounded and satisfies

$$\begin{cases} L_p(v) \geq \lambda(1-\varepsilon) f(v) & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{10.14}$$

Then v is a bounded supersolution. Since 0 is a subsolution, by a standard monotone iteration argument, we see the existence of a classical solution for any $\mu < \lambda$. In particular for $\mu = \lambda^*$ there exists a classical solution. But this clearly contradicts to the minimality of u^*, which is singular . □

If $p \geq 2$, we can show the necessity of the Hardy type inequality for the extremal u^*.

Proposition 10.1. *Assume that $p \geq 2$. Let u^* be the extremal solution. Then we have*

$$\int_\Omega |\nabla u^*|^{p-2} \left(|\nabla \varphi|^2 + (p-2) \frac{(\nabla u^*, \nabla \varphi)^2}{|\nabla u^*|^2} \right) dx \geq \lambda^* \int_\Omega f'(u^*) \varphi^2 \, dx \tag{10.15}$$

for any $\varphi \in V_{\lambda^*,p}(\Omega)$.

Proof. For any $\mu < \lambda < \lambda^*$ it follows from Lemma 2.3 that

$$\begin{aligned} ||u_\lambda - u_\mu||^p_{W_0^{1,p}(\Omega)} &\leq C \int_\Omega (|\nabla u_\lambda|^{p-2} \nabla u_\lambda - |\nabla u_\mu|^{p-2} \nabla u_\mu, \nabla(u_\lambda - u_\mu)) \, dx \\ &= C \int_\Omega (\lambda f(u_\lambda) - \mu f(u_\mu))(u_\lambda - u_\mu) \, dx. \end{aligned}$$

Here C is a positive number independent of μ and λ. Then it follows from (2.20) in the proof of Theorem 2.1 that for almost all $x \in \Omega$

$$|(\lambda f(u_\lambda) - \mu f(u_\mu))(u_\lambda - u_\mu)| \leq 4\lambda^* f(u^*)u^* \quad \text{and} \quad f(u^*)u^* \in L^1(\Omega).$$

Since u_λ converges u^* as $\lambda \to \lambda^*$ monotonically, by Lebesgue's convergence theorem we have

$$\lim_{\lambda \to \lambda^*} u_\lambda = u^* \quad \text{in } W_0^{1,p}(\Omega). \tag{10.16}$$

In particular we get

$$\lim_{\lambda \to \lambda^*} |\nabla u_\lambda|^{p-2} = |\nabla u^*|^{p-2} \quad \text{in } L^1(\Omega). \tag{10.17}$$

Hence we immediately have for any $\varphi \in C_0^\infty(\Omega)$

$$\lim_{\lambda \to \lambda^*} \langle L_p'(u_\lambda)\varphi, \varphi\rangle_{[V_{\lambda,p}]' \times V_{\lambda,p}} = \langle L_p'(u^*)\varphi, \varphi\rangle_{[V_{\lambda^*,p}]' \times V_{\lambda^*,p}}. \tag{10.18}$$

Since u_λ is (strictly) increasing, by Fatou's lemma we have

(10.19)
$$\begin{aligned}\lambda^* \int_\Omega f'(u^*)\varphi^2\, dx &\leq \lim_{\lambda \to \lambda^* - 0} \lambda \int_\Omega f'(u_\lambda)\varphi^2\, dx \\ &\leq \lim_{\lambda \to \lambda^*} \langle L_p'(u_\lambda)\varphi, \varphi\rangle_{[V_{\lambda,p}]' \times V_{\lambda,p}} = \langle L_p'(u^*)\varphi, \varphi\rangle_{[V_{\lambda^*,p}]' \times V_{\lambda^*,p}}.\end{aligned}$$

Noting that $C_0^\infty(\Omega)$ is densely contained in $V_{\lambda^*,p}(\Omega)$, the desired Hardy type inequality follows. □

Remark 10.2. When $1 < p < 2$, we can not show the strong convergence of $|\nabla u_\lambda|^{p-2}$ in $L^1(\Omega)$ because of the negativity of exponent. But $|\nabla u_\lambda|$ ($\lambda \in (0, \lambda^*]$) vanishes only on each discrete set $F_{\lambda,p}$ ($\lambda \in (0, \lambda^*]$). Hence if $|\nabla u_\lambda|$ with λ being sufficiently close to λ^* is positive except for an arbitrary small neighborhood of $F_{\lambda^*,p}$, then $|\nabla u_\lambda|^{p-2}$ converges $|\nabla u^*|^{p-2}$ in $L^1_{loc}(\Omega \setminus F_{\lambda^*,p})$ as $\lambda \to \lambda^*$. Therefore we can show the following result in a similar way.

Proposition 10.2. *Assume that $1 < p < 2$. Let u^* be the extremal solution. Assume that there is a positive number ε_0 such that $\tilde{V}_{\lambda,p}(\Omega)$ is dense in $V_{\lambda,p}(\Omega)$ for any $\lambda \in (\lambda^* - \varepsilon_0, \lambda^*)$.*

Moreover assume that for any $\kappa > 0$ there is a positive number δ such that for any $\lambda \in (\lambda^* - \delta, \lambda^*]$

$$F_{\lambda,p} \subset (F_{\lambda^*,p})_\kappa = \{x \in \Omega : dist(x, F_{\lambda^*,p}) < \kappa\}. \tag{10.20}$$

Then we have

$$\int_\Omega |\nabla u^*|^{p-2} \left(|\nabla \varphi|^2 + (p-2)\frac{(\nabla u^*, \nabla \varphi)^2}{|\nabla u^*|^2} \right) dx \geq \lambda^* \int_\Omega f'(u^*)\varphi^2\, dx \tag{10.21}$$

for any $\varphi \in V_{\lambda^*,p}(\Omega)$.

Proof. From the remark just before this, we see

$$\lim_{\lambda\to\lambda^*-0} |\nabla u_\lambda|^{p-2} = |\nabla u^*|^{p-2} \text{ in } L^1_{loc}(\Omega\setminus F_{\lambda^*,p}). \tag{10.22}$$

Therefore for any $\varphi \in \tilde{V}_{\lambda,p}(\Omega)$ the Hardy type inequality (10.21) holds by the same argument as before. Since $\tilde{V}_{\lambda,p}(\Omega)$ is dense in $V_{\lambda,p}(\Omega)$, by an approximating argument this is valid for any $\varphi \in V_{\lambda^*,p}(\Omega)$. □

Remark 10.3. (1) If u^* is classical, the Hardy type inequality (10.21) holds under the assumption that the first eigenfunction of $L'_p(u^*) - \lambda f'(u^*)$ satisfies the accessibility condition (AC). The proof is same as that of Theorem 8.1.

(2) If Ω is a ball, then one can show $F_{\lambda,p} = \{0\}$ for all $\lambda > 0$. Hence the Hardy type inequality (10.21) holds in this case. See the example in Section 12.

Conversely we have

Proposition 10.3. *Assume that $1 < p \le 2$ and the nonlinearlity $f(t)$ satisfies the growth condition (GC) in addition to (2.2). For $\lambda > 0$, let u_λ be the minimal solution or possibly the extremal solution. Let $u \in W_0^{1,p}(\Omega)$ be a singular (unbounded) weak energy solution of (2.3) such that*

$$\int_\Omega |\nabla u|^{p-2}\left(|\nabla\varphi|^2 + (p-2)\frac{(\nabla u,\nabla\varphi)^2}{|\nabla u|^2}\right)dx \ge \lambda\int_\Omega f'(u)\varphi^2\,dx \tag{10.23}$$

for any $\varphi \in V_{\lambda,p}(\Omega)$. Moreover, if $1 < p < 2$, then we assume that

$$|\nabla u| \ge |\nabla u_\lambda| \qquad a.e.\ in \quad \Omega. \tag{10.24}$$

Then we have $\lambda = \lambda^*$ and $u = u^*$.

Proof. By Theorem 10.1, in order to see $\lambda = \lambda^*$ it suffices to show $\lambda \ge \lambda^*$. Assume that $\lambda < \lambda^*$. Then it follows from the strict convexity of f that

$$\begin{aligned}\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla u_\lambda|^{p-2}\nabla u_\lambda, \nabla(u-u_\lambda))\,dx \\ < \int_\Omega |\nabla u|^{p-2}\left(|\nabla(u-u_\lambda)|^2 + (p-2)\frac{(\nabla u,\nabla(u-u_\lambda))^2}{|\nabla u|^2}\right)dx.\end{aligned} \tag{10.25}$$

Note that the right hand side is finite from a Hölder inequality. Set $\nabla u = r\omega_1$ and $\nabla u_\lambda = \rho\omega_2$ for $\omega_1, \omega_2 \in S^{N-1}$ and set $A = \frac{\rho}{r}$. From the assumption we see $A \in [0,1]$ if $1 < p < 2$. Now we claim on the contrary that

$$\begin{aligned}(\omega_1 - A^{p-1}\omega_2, \omega_1 - A\omega_2) \\ \ge |\omega_1 - A\omega_2|^2 + (p-2)(\omega_1, \omega_1 - A\omega_2)^2\end{aligned} \tag{10.26}$$

for any $\omega_1, \omega_2 \in S^{N-1}$ and $A \in [0,1]$. Then we have, for $\beta = (\omega_1, \omega_2)$

$$A(A-\beta)(A^{p-2}-1) + (2-p)(A\beta - 1)^2 \geq 0. \tag{10.27}$$

Since $|\beta| \leq 1$ and $1 < p \leq 2$, it suffices to show

$$A(A-1)(A^{p-2}-1) + (2-p)(A-1)^2 \geq 0. \tag{10.28}$$

Now we can assume that $1 < p < 2$. Since $0 \leq A \leq 1$, this follows from the inequality below.

$$A^{p-1} - A + (2-p)(A-1) \leq 0. \tag{10.29}$$

Therefore the claim is proved and we see $\lambda = \lambda^*$. The uniqueness of energy solutions satisfying the Hardy inequality (10.23) is also clear from the same argument. □

Remark 10.4. When the domain Ω is a ball B, then the condition (10.24) is satisfied. See Lemma12.1 in §12. Therefore if $\Omega = B$, the Hardy type in equality 10.23 does not hold for a non-minimal classical solution of (2.3) with $0 < \lambda < \lambda^*$ for any $1 < p \leq 2$. From this fact one can show that the minimal solution u_λ is also right continuous on λ provided that $1 < p \leq 2$ and Ω is a ball. For the detailed see Proposition 12.1 in Section 12.

If $p > 2$, we can show the following instead, which seems rather weak but will be useful in Section 12 to determine the extremal in the case that Ω is a ball.

Proposition 10.4. *Assume that $p > 2$ and the nonlinearlity $f(t)$ satisfies the growth condition (GC) in addition to (2.2). For $\lambda > 0$, let u_λ be the minimal solution, or possibly the extremal solution. Let $u \in W_0^{1,p}(\Omega)$ be a singular (unbounded) weak energy solution of (2.3) such that for any $\varphi \in C_0^\infty(\Omega)$*

$$\int_\Omega |\nabla u|^{p-2} \left(|\nabla \varphi|^2 + (p-2) \frac{(\nabla u, \nabla \varphi)^2}{|\nabla u|^2} \right) dx \geq \lambda(p-1) \int_\Omega f'(u) \varphi^2 \, dx. \tag{10.30}$$

Moreover we assume one of the followings:

$$\begin{cases} 1. \quad \nabla u = \alpha \nabla u_\lambda \text{ for some } \alpha > 0 & a.e. \text{ in } \quad \Omega, \\ 2. \quad |\nabla u_\lambda|^2 \leq (\nabla u, \nabla u_\lambda) & a.e. \text{ in } \quad \Omega. \end{cases} \tag{10.31}$$

Then we have $\lambda = \lambda^*$ and $u = u^*$.

Proof. The proof is done in the same line of the previous one. Assume that $\lambda < \lambda^*$. Then we have

$$\begin{aligned} &(p-1) \int_\Omega (|\nabla u|^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda, \nabla(u - u_\lambda)) \, dx \\ &< \int_\Omega |\nabla u|^{p-2} \left(|\nabla(u - u_\lambda)|^2 + (p-2) \frac{(\nabla u, \nabla(u-u_\lambda))^2}{|\nabla u|^2} \right) dx. \end{aligned} \tag{10.32}$$

Note that the right hand side is finite from a Hölder inequality. Set $\nabla u = r\omega_1$ and $\nabla u_\lambda = \rho\omega_2$ for $\omega_1, \omega_2 \in S^{N-1}$ and set $A = \frac{\rho}{r}$. Now we claim on the contrary that

$$(p-1)(\omega_1 - A^{p-1}\omega_2, \omega_1 - A\omega_2) \geq |\omega_1 - A\omega_2|^2 + (p-2)(\omega_1, \omega_1 - A\omega_2)^2 \tag{10.33}$$

for $\omega_1, \omega_2 \in S^{N-1}$ and $A \in [0, +\infty)$. This is equivalent to

$$(p-1)(1 - A^{p-2})((\omega_1, \omega_2) - A) \geq (p-2)A((\omega_1, \omega_2)^2 - 1). \tag{10.34}$$

Therefore the claim is proved and we see $\lambda = \lambda^*$. The uniqueness of the energy solution satisfying the Hardy inequality (10.31) is also clear from the same argument. □

Remark 10.5. When the domain Ω is a ball B, then the condition (10.31) is satisfied. See Lemma 12.1 in Section 12.

If $p = 2$ we encounter the result in [2] due to H. Brezis and J.L. Vazquez, namely

Corollary 10.1. *Assume that $p = 2$ and that v is a singular energy solution of (2.3) for some $\lambda > 0$. Then the following two statements are equivalent with each other.*

(1) $\lambda = \lambda^*$ *and* $v = u^*$.

(2) *It holds that*

$$\int_\Omega |\nabla\varphi|^2\, dx \geq \lambda \int_\Omega f'(u_\lambda)\varphi^2\, dx \tag{10.35}$$

for any $\varphi \in V_{\lambda,2}(\Omega) = W_0^{1,2}(\Omega)$.

11. Weighted Hardy's inequality in a ball

In the next we state the results concerned with the weighted Hardy inequalities.

Theorem 11.1. *Suppose that a positive integer N and a real number α satisfy $N + \alpha > 2$. Then it holds that for any $u \in H_0^1(\Omega)$*

$$\int_\Omega |\nabla u|^2 |x|^\alpha\, dx \geq H(N, \nabla, \alpha) \int_\Omega |u|^2 |x|^{\alpha-2}\, dx + \lambda_1 \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}} \int_\Omega |u|^2 |x|^\alpha\, dx. \tag{11.1}$$

Here

$$H(N, \nabla, \alpha) = \left(\frac{N-2+\alpha}{2}\right)^2, \tag{11.2}$$

ω_N is a volume of N-dimensional unit ball, and λ_1 is the first eigenvalue of the Dirichlet problem given by:

$$\lambda_1 = \inf \left[\int_{B_1^2} |\nabla_2 v|^2 \, dx : v \in W_0^{1,2}(B_1^2), \int_{B_1^2} v^2 \, dx = 1 \right], \tag{11.3}$$

where by B_1^2 and ∇_2 we denote the two dimensional unit ball and the gradient.

Remark 11.1. When $\alpha = 0$, this result was initially established in [2;H. Brezis and J.L. Vazquez]. They also investigated in [2] fundamental properties of blow-up solutions of some nonlinear elliptic problems.

First we prepare an elementary lemma.

Lemma 11.1. *Let Ω be a domain of $\mathbb{R}^N$. Assume that $u \in C_0^\infty(\Omega)$ and $f \in C^2(\Omega)$. Then it holds that*

$$\int_\Omega |\nabla(uf)|^2 \, dx = \int_\Omega |\nabla u|^2 f \, dx - \frac{1}{2} \int_\Omega u^2(\Delta(f^2) - 2|\nabla f|^2) \, dx. \tag{11.4}$$

Proof. Integration by parts leads us to obtain (11.4). □

Using these formula we can easily show the assertion.

Proof of Theorem 11.1. From this the proof of Theorem 11.1 is reduced to the case $\alpha = 0$, which was established in [2]. In fact, for $f = |x|^{\frac{\alpha}{2}}$, we have

$$\begin{aligned} &\int_\Omega |\nabla u|^2 |x|^\alpha \, dx \\ &= \frac{\alpha(\alpha + 2N - 4)}{4} \int_\Omega |u|^2 |x|^{\alpha - 2} \, dx + \int_\Omega |\nabla(u|x|^{\frac{\alpha}{2}})|^2 \, dx. \end{aligned} \tag{11.5}$$

Here we note that the proof of Lemma 11.1 still works for this weight f, since $N + \alpha > 2$. Then we can apply the inequality (11.1) with a parameter α being 0, and we obtain

$$\begin{aligned} &\int_\Omega |\nabla(u|x|^{\frac{\alpha}{2}})|^2 \, dx \\ &\geq \frac{(N-2)^2}{4} \int_\Omega |u|^2 |x|^{\alpha - 2} \, dx + \lambda_1 \left(\frac{\omega_N}{|\Omega|} \right)^{\frac{2}{N}} \int_\Omega |u|^2 |x|^\alpha \, dx. \end{aligned} \tag{11.6}$$

The desired inequality follows from this and (11.5).

For the sake of the self-containedness, we give a proof of Theorem 11.1 in the case $\alpha = 0$. By the spherically symmetric decreasing rearragement, it suffices to show the inequality in the case that $\Omega = B$; a unit ball in $\mathbb{R}^N$ and

$u \in C_0^1(B)$ is radiall symmetric. Set $u = r^{-\beta} v$ for $u \in C_0^1(B)$ and $\beta = \frac{N-2}{2}$.

$$\begin{aligned}
&\int_B |\nabla u|^2\,dx - H(N,\nabla,0)\int_B \frac{u^2}{|x|^2}\,dx \\
&= N\omega_N\left(\int_0^1 |u'|^2 r^{N-1}\,dr - H(N,\nabla,0)\int_0^1 u^2 r^{N-3}\,dr\right) \\
&= N\omega_N\left(\int_0^1 |v'|^2 r\,dr\right) \geq \lambda_1 N\omega_N \int_0^1 v^2 r\,dr \\
&= \lambda_1 \int_B u^2\,dx
\end{aligned}$$

This proves the assertion. □

12. Examples in a unit ball of $\mathbb{R}^N$

In this subsection we shall apply our results to some examples. By B we denote a unit ball in $\mathbb{R}^N$. Let $u_\lambda \in W_0^{1,p}(B) \cap C_0^{1,\sigma}(\overline{B})$ for some $\sigma \in (0,1)$ be the minimal solution. Since B and the operator $L_p(\cdot)$ itself are radially symmetric, we see u_λ is also radial by the minimality. Then u_λ satisfies in a weak sense

$$\begin{cases} L_p(u) = -r^{1-N}\partial_r(r^{N-1}|\partial_r u_\lambda|^{p-2}\partial_r u_\lambda) = \lambda f(u_\lambda), & r \in (0,1), \\ u_\lambda(1) = 0. \end{cases} \tag{12.1}$$

From the symmetricity it also holds that

$$\partial_r u_\lambda(0) = 0. \tag{12.2}$$

By integrating (12.1) from 0 to r we get

$$-\lambda \int_0^r f(u_\lambda(r)) r^{N-1}\,dr = r^{N-1}|\partial_r u_\lambda|^{p-2}\partial_r u_\lambda. \tag{12.3}$$

Noting that $\partial_r u_\lambda < 0$ $(r > 0)$ we get

$$|\partial_r u_\lambda|^{p-1} = \lambda r \int_0^1 f(u_\lambda(rt)) t^{N-1}\,dt. \tag{12.4}$$

From this formula we have

Lemma 12.1. *Let $u_\lambda \in W_0^{1,p}(B) \cap C_0^{1,\sigma}(\overline{B})$ and $u^* \in W_0^{1,p}(B)$ be the minimal solution and the extremal solution respectively. Then $F_{\lambda,p} = F_{\lambda^*,p} = \{0\}$ and $|\partial_r u_\lambda|$ is increasing w.r.t. $\lambda \in [0,\lambda^*]$. In particular we have for any $\lambda \in [0,\lambda^*]$*

$$|\partial_r u_\lambda| \leq |\partial_r u^*| \qquad (0 < r \leq 1). \tag{12.5}$$

Proof. Since $f(\cdot)$ is convex and u_λ is monotone increasing w.r.t.λ, we see $|\partial_r u_\lambda|$ is also increasing w.r.t. λ. The rest of assertions are also clear. $\square$

Proposition 12.1. *Assume that $1 < p \leq 2$. Then, u_λ is strictly increasing and continuous on $\lambda \in [0, \lambda^*)$ for each $x \in B$. Moreover the mapping; $\lambda \longrightarrow u_\lambda \in W_0^{1,p}(B)$ is weakly continuous.*

Proof. Since u_λ is left continuous for each $x \in B$ and weakly left continuous as a $W_0^{1,p}(B)$-valued function, it suffices to show the right continuity. Note that $u_{\lambda_0+0} = \lim_{\lambda\to\lambda_0+0} u_\lambda$ uniquely exists in $W_0^{1,p}(B)$ as a decreasing limit and becomes a weak energy solution of (2.3) by a similar argument as in the proof of Lemma 2.6. We claim $u_{\lambda_0+0} = u_{\lambda_0}$ in $W_0^{1,p}(B)$. Since $F_{\lambda,p} \equiv \{0\}$, $\tilde{V}_{\lambda,p}(B)$ is dense in $V_{\lambda,p}(B)$. Hence by the same argument in the proof of Proposition 10.2, the Hardy type inequality

$$\int_\Omega |\nabla u_{\lambda_0+0}|^{p-2}\left(|\nabla\varphi|^2 + (p-2)\frac{(\nabla u_{\lambda_0+0}, \nabla\varphi)^2}{|\nabla u_{\lambda_0+0}|^2}\right) dx \geq \lambda_0 \int_\Omega f'(u_{\lambda_0+0})\varphi^2\, dx$$

holds for any $\varphi \in V_{\lambda_0,p}(B.)$(since $|\partial_r u_{\lambda_0}| \leq |\partial_r u_{\lambda_0+0}|$ holds, the left hand side is finite for any $\varphi \in V_{\lambda_0,p}(B)$). Then by the same argument in Proposition 10.3 we see $u_{\lambda_0+0} = u_{\lambda_0}$ in $W_0^{1,p}(B)$. This proves the claim. $\square$

In the next we consider the linearized operator at u_λ. Since u_λ is radial, we get for any $\varphi \in V_{\lambda,p}(B)$

(12.6)
$$L_p'(u_\lambda)\varphi = -\operatorname{div}\left(|\partial_r u_\lambda|^{p-2}\left(\nabla\varphi + (p-2)\frac{x}{r}\frac{d}{dr}\varphi\right)\right),$$

(12.7)
$$\langle L_p'(u_\lambda)\varphi, \varphi\rangle_{[V_{\lambda,p}]'\times V_{\lambda,p}} = \int_B \left(|\partial_r u_\lambda|^{p-2}\left(|\nabla\varphi|^2 + (p-2)\left|\frac{d}{dr}\varphi\right|^2\right)\right) dx.$$

In particular if $\varphi \in V_{\lambda,p}(B)$ is also radial, then

$$L_p'(u_\lambda)\varphi = -(p-1)r^{1-N}\partial_r(r^{N-1}|\partial_r u_\lambda|^{p-2}\partial_r\varphi), \tag{12.8}$$

$$\langle L_p'(u_\lambda)\varphi, \varphi\rangle_{[V_{\lambda,p}]'\times V_{\lambda,p}} = (p-1)\int_B |\partial_r u_\lambda|^{p-2}|\partial_r\varphi|^2\, dx. \tag{12.9}$$

Proposition 12.2. *Let $\hat{\varphi}^\lambda \in V_{\lambda,p}(B)$ be the first eigenfunction of the self-adjoint operator $L_p'(u_\lambda) - \lambda f'(u_\lambda)$. Then $\hat{\varphi}^\lambda$ is radial and satisfies the accessibility condition (AC).*

Proof. Since the first eigenfunction $\hat{\varphi}^\lambda \in V_{\lambda,p}(B)$ is unique up to a multiplication by constants, $\hat{\varphi}^\lambda$ becomes radial as well as u_λ. From (12.4) we have $|\partial_r u_\lambda(r)| = O(r^{\frac{1}{p-1}})$. Here by $O(r)$ we denote the quantity such that $\frac{O(r)}{r}$ remains bounded as $r \to +0$. Since $\hat{\varphi}^\lambda \in V_{\lambda,p}(B)$, we see

$$\int_B |x|^{\frac{p-2}{p-1}}|\nabla\hat{\varphi}^\lambda|^2\, dx < +\infty. \tag{12.10}$$

From now we assume that $N \geq 2$, because $\hat{\varphi}^\lambda(r)$ becomes continuous by Hölder inequality provided that $N = 1$. Then it follows from the imbedding theorem for weighted Sobolev spaces that for some positive number $C(N,p)$

$$\left(\int_B |\hat{\varphi}^\lambda|^{q(p)}\,dx\right)^{\frac{1}{q(p)}} \leq C(N,p)\left(\int_B |x|^{\frac{p-2}{p-1}}|\nabla\hat{\varphi}^\lambda|^2\,dx\right)^{\frac{1}{2}}, \tag{12.11}$$

where $q(p) > 2$ is given by the relation

$$q(p) = \begin{cases} \dfrac{2N(p-1)}{N(p-1)-p}, & N > 2, p \geq 2, \\ \dfrac{2N}{N-2}, & N > 2, 1 < p < 2, \\ q \quad (\text{ any positive number }), & N = 2. \end{cases} \tag{12.12}$$

For the proof of this, see [6; Thoerem 1]. Since $\hat{\varphi}^\lambda$ satisfies $L_p'(u_\lambda)\hat{\varphi}^\lambda = (\lambda f'(u_\lambda) + \mu)\hat{\varphi}^\lambda$ for some constant μ, it follows from a Moser's iteration argument that $\hat{\varphi}^\lambda$ is bounded in $\overline{B}$. Then by integrating this from 0 to r we get $\partial_r\hat{\varphi}^\lambda = O(r^{\frac{1}{p-1}})$. Hence $L'(u_\lambda)\hat{\varphi}^\lambda(r)$ is continuous in $[0,1]$ and

$$\hat{\varphi}^\lambda(r) = O(r^{1+\frac{1}{p-1}}) + \hat{\varphi}^\lambda(0) \qquad \text{as } r \to +0. \tag{12.13}$$

Since $\lambda f'(u_\lambda(0)) + \mu \geq \lambda f'(u_\lambda) + \mu$, $\lambda f'(u_\lambda(0)) + \mu$ has to be positive from the positivity of $L_p'(u_\lambda)$. Hence $L'(u_\lambda)\hat{\varphi}^\lambda(r)$ becomes nonnegative near the origin.

Then for any positive number ε it is possible to truncate $\hat{\varphi}^\lambda$ smoothly in a neighborhood of the origin so that we obtain $\varphi \in \tilde{V}_{\lambda,p}(B)$ satisfying $|\varphi - \hat{\varphi}^\lambda| \leq \varepsilon \max(\hat{\varphi}^\lambda, dist(x,\partial B))$ and $L_p'(u_\lambda)\varphi \leq L_p'(u_\lambda)\hat{\varphi}^\lambda + \varepsilon\max(\hat{\varphi}^\lambda, dist(x,\partial B))$. □

In the rest of this section we adopt as the nonlinearity $f(u)$ the following f_q and f_e;

$$\begin{cases} f_q(u) = (1+u)^q & (q > p-1), \\ f_e(u) = e^u. \end{cases} \tag{12.14}$$

Set

$$\begin{cases} \lambda_N(p,q) = \left(\dfrac{p}{q-p+1}\right)^{p-1}\left(N - \dfrac{pq}{q-p+1}\right), & q > p-1, \\ \lambda_N(p) = p^{p-1}(N-p). \end{cases} \tag{12.15}$$

We define the function $U_{p,q}$ as follows:

$$\begin{cases} U_{p,q}(r) = r^{-Q} - 1, \quad Q = \dfrac{p}{q-p+1}, \\ U_p(r) = -p\log r. \end{cases} \tag{12.16}$$

Under these notations, we have the following.

Lemma 12.2. *Under these notations, $U_p \in W_0^{1,p}(B)$ if $N > p$ and $U_{p,q} \in W_0^{1,p}(B)$ if $N > p(1+Q)$. Moreover they become singular energy solutions to the boundary value problems below respectively:*

$$\begin{cases} L_p(U_p) = \lambda_N(p)e^{U_p} & \text{in} \quad B \\ U_p = 0 & \text{on} \quad \partial B, \end{cases} \tag{12.17}$$

$$\begin{cases} L_p(U_{p,q}) = \lambda_N(p,q)(U_{p,q}+1)^q & \text{in} \quad B \\ U_{p,q} = 0 & \text{on} \quad \partial B. \end{cases} \tag{12.18}$$

As $q \to +\infty$ one can check that

$$(f_q(U_{p,q}(r)), q^{p-1}\lambda_N(p,q), qU_{p,q}(r)) \longrightarrow (f_e(U_p(r)), \lambda_N(p), U_p(r)) \tag{12.19}$$

for any $r \in (0,1)$. Therefore the boundary value problem (12.17) is considered as a formal limit of (12.18).

For these singular solutions, we can show the validity of weighed Hardy's inequalities introduced in the previous section.

Lemma 12.3. (1) *If $N \geq p\frac{p+3}{p-1}$ holds, then for any radial $\varphi \in C_0^\infty(B)$*

$$\langle L_p'(U_p)\varphi, \varphi\rangle_{[C_0^\infty(B)]' \times C_0^\infty(B)} \geq \lambda_N(p) \int_B e^{U_p}\varphi^2\, dx, \tag{12.20}$$

or equivalently we have

$$\int_B |\partial_r \varphi|^2 r^{2-p}\, dx \geq \frac{p(N-p)}{p-1} \int_B \varphi^2 r^{-p}\, dx. \tag{12.21}$$

(2) *Assume that*

$$\frac{qQ}{p-1}(N-qQ) \leq \frac{1}{4}(N-Q(q-1))^2. \tag{12.22}$$

Then, for any radial $\varphi \in C_0^\infty(B)$

$$\langle L_p'(U_{p,q})\varphi, \varphi\rangle_{[C_0^\infty(B)]' \times C_0^\infty(B)} \geq q\lambda_N(p,q) \int_B (1+U_{p,q})^{q-1}\varphi^2\, dx, \tag{12.23}$$

or equivalently we have

$$\int_B |\partial_r \varphi|^2 r^{-(p-2)(Q+1)}\, dx \geq \frac{qQ}{p-1}(N-qQ) \int_B \varphi^2 r^{-(p-2)(Q+1)-2}\, dx. \tag{12.24}$$

Here $Q = \frac{p}{q-p+1}$ and $r = |x|$.

Proof. The condition $N \geq p\frac{p+3}{p-1}$ is equivalent to $\frac{p(N-p)}{p-1} \leq \left(\frac{N-p}{2}\right)^2$; the best constant of weighted Hardy's inequality for $\alpha = 2-p$. Hence the assertion (1) holds. In a similar way we see that $\frac{qQ}{p-1}(N-qQ) \leq \frac{1}{4}(N-Q(q-1))^2$ is equivalent to $\frac{qQ}{p-1}(N-qQ) \leq \left(\frac{N-2}{2} - \frac{(p-2)(Q+1)}{2}\right)^2$; the best constant of Hardy's inequality for $\alpha = (2-p)(Q+1)$. This proves the assertion. $\square$

Remark 12.1. (1) In the assertion (2), $U_{p,q}$ is an energy solution if and only if $N-p>pQ$ holds. If $N-p>pQ$, then $N>qQ$ clearly holds. Therefore there is a range of q such that $U_{p,q}\notin W_0^{1,p}(B)$ but weighted Hardy's inequality holds.

(2) The both inequality (12.21) and (12.24) are valid for any $\varphi\in C_0^\infty(B)$ replacing $|\partial_r\varphi|$ by $|\nabla\varphi|$. In fact if $1<p\le 2$, then it suffices to note $|\partial_r\varphi|\le|\nabla\varphi|$. When $p>2$, this follows from the one dimensional Hardy inequality as well.

Assume that U_p is the singular extremal solution of (12.1) for $f=f_e$. Since $F_{\lambda,p}=\{0\}$ holds, $\tilde{V}_{\lambda,p}(\Omega)$ is densely contained in $V_{\lambda,p}(\Omega)$ for any $\lambda\in(0,\lambda^*)$. To see this fact it suffices to approximate an element in $V_{\lambda,p}(\Omega)$ by a step function near the origin. Then it follows from Propositions 10.1 and 10.2 that the inequality of Hardy type has to be valid. Therefore the condition $N\ge p\frac{p+3}{p-1}$ is necessary for U_p to be the singular extremal. If we restrict ourselves to the case that $1<p\le 2$, then we can show the converse.

Proposition 12.3 (Exponential case I)**.** *Assume that* $1<p\le 2$*. Then* U_p *is the singular extremal solution of* (12.1) *with* $f=f_e$*, if and only if* $N\ge p\frac{p+3}{p-1}$.

Proof. This follows from Proposition 10.3, Lemmas 12.1 and 12.3. □

In a similar way we have

Proposition 12.4 (Exponential case II)**.** *Assume that* $p>2$*. Then* U_p *is the singular extremal solution of* (12.1) *with* $f=f_e$*, if* $N\ge 5p$*.*

Proof. By the weighted Hardy inequality with the best constant, we see

$$\int_B|\nabla\varphi|^2r^{2-p}\,dx\ge\frac{(N-p)^2}{4}\int_B\varphi^2r^{-p}\,dx \tag{12.25}$$

for any $\varphi\in C_0^1(B)$. Since $5p\le N$, we see $\frac{(N-p)^2}{4}\ge p(N-p)$. Then we have

$$\int_B|\nabla U_p|^{p-2}|\nabla\varphi|^2\,dx\ge\lambda_N(p)\int_B e^{U_p}\varphi^2\,dx. \tag{12.26}$$

Therefore the assumption (10.30) in Proposition 10.4 is satisfied for $u=U_p$ and for a radial φ. There is the minimal (or possibly the extremal) solution u_λ which is radial and $0<u_\lambda\le U_p$ in B. From the integral representation (12.3), we also see $\partial_r u_\lambda<0$, and $\partial_r U_p<0$ as well. Hence from Proposition 10.4 we see $\lambda^*=\lambda_N(p)$ and $U_p=u_\lambda$. □

Proposition 12.5 (Polynomial case I)**.** *Assume that* $1<p\le 2$*. Then* $U_{p,q}$ *is the singular extremal solution of* (12.1) *with* $f=f_p$*, if and only if*

$$N\ge\frac{p(1+qQ)+2\sqrt{pqQ}}{p-1}. \tag{12.27}$$

Proof. It suffices to note that

$$\frac{qQ}{p-1}(N-qQ) \leq \frac{1}{4}(N-Q(q-1))^2 \text{ and } N-p > pQ$$

simply imply

$$N \geq \frac{p(1+qQ)+2\sqrt{pqQ}}{p-1}.$$

Also note that as $q \to +\infty$ this condition becomes $N \geq p\frac{p+3}{p-1}$. $\square$

In a similar way we have

Proposition 12.6 (Polynomial case II). *Assume that $p > 2$. Then $U_{p,q}$ is the singular extremal solution of* (12.1) *with $f = f_p$, if $N \geq Q(3q-1+2\sqrt{q(q-1)})$.*

Proof. By the weighted Hardy inequality with the best constant, we have

$$\int_B |\partial_r \varphi|^2 r^{-(p-2)(Q+1)}\, dx \geq \frac{(N-Q(q-1))^2}{4} \int_B \varphi^2 r^{-(p-2)(Q+1)-2}\, dx \tag{12.28}$$

for any $\varphi \in C_0^1(B)$. Here $Q = \frac{p}{q-p+1}$ and $r = |x|$. From the assumption we see $\frac{(N-Q(q-1))^2}{4} \geq qQ(N-qQ)$. Then we have

$$\int_B |\nabla U_{p,q}|^{p-2} |\nabla \varphi|^2\, dx \geq q\lambda_N(p) \int_B (U_{p,q}+1)^{q-1}\varphi^2\, dx. \tag{12.29}$$

Therefore the assumption (10.30) in Proposition 10.4 is satisfied for $u = U_{p,q}$ and for a radial φ. The rest of thr proof is completely same as that in Exponential case. Also note that $N > p(Q+1)$ is satisfied and as $q \to +\infty$ the condition becomes $N \geq 5p$. $\square$

Remark 12.2. (1) In case that $p > 2$, it is unknown if $U_p; 5p > N \geq p\frac{p+3}{p-1}$ $\left(U_{p,q};\ Q(3q-1+2\sqrt{q(q-1)}) > N \geq \frac{p(1+qQ)+2\sqrt{pqQ}}{p-1}\right)$ becomes the extremal or not.

(2) Assume that $1 < p \leq 2$. If $N > p\frac{p+3}{p-1}$, then the linealized operator

$$\begin{aligned} &L_p'(U_p) - \lambda_N(p)e^{U_p} \\ &= -p^{p-2}\left(\operatorname{div}\left(r^{2-p}\nabla \cdot + (p-2)\frac{x}{r}\frac{d}{dr}\cdot\right) - p(N-p)r^{-p}\right) \end{aligned} \tag{12.30}$$

has a positive first eigenvalue $\mu(\lambda_N(p))$.

If $N = p\frac{p+3}{p-1}$, then the linearized operator does not have a first eigenfunction in $W_0^{1,p}(B)$. However, the weighted Hardy inequality in the previous section gives a positive value for $\mu(\lambda_N(p))$ defined as

$$\mu(\lambda_{N(p)}) = \lim_{\lambda \to \lambda_{N(p)}} \mu(\lambda) = \lambda_1 p^{p-2}(p-1),$$

where λ_1 is defined by (11.3) and $\mu(\lambda)$ is the first eigenvalue for $L'_p(u_\lambda) - \lambda e^{u_\lambda}$. From Theorem 11.1 we see $\mu(\lambda_{N(p)}) \leq \lambda_1 p^{p-2}(p-1)$. Since $u_\lambda \leq U_p$ and $|\partial_r u_\lambda|^{p-2}$ is decreasing w.r.t. λ,

$$\int_B |\partial_r u_\lambda|^{p-2}|\nabla\varphi|^2\,dx - \lambda \int_B e^{u_\lambda}\varphi^2\,dx$$

is decreasing w.r.t. λ, and so the reverse inequality also holds.

(3) When $1 < p \leq 2$ and $N \geq \frac{p(1+qQ)+2\sqrt{pqQ}}{p-1}$, one can show similar results for the linearized operator of $L_p(\cdot)$ at $U_{p,q}$.

Lastly we study the behavior of v_λ as $\lambda \to \lambda^*$ assuming that $1 < p \leq 2$. For the sake of simplicity we treat the exponential case only. We recall that v_λ satisfies for $\lambda < \lambda_N(p) = \lambda^*$

$$\begin{cases} -(p-1)\partial_r(|\partial_r u_\lambda|^{p-2} r^{N-1}\partial_r v_\lambda) = r^{N-1}(\lambda v_\lambda + 1)e^{u_\lambda}, & r \in (0,1), \\ v_\lambda(1) = 0. \end{cases} \tag{12.31}$$

Assuming $N > p\frac{p+3}{p-1}$ and replacing (u_λ, λ) by the extremal pair $(U_p, \lambda_N(p))$, we study the equation

$$\begin{cases} -p^{p-2}(p-1)(\partial_r^2\tilde{v} + \frac{N-p+1}{r}\partial_r\tilde{v}) = r^{-2}(\lambda_N(p)\tilde{v} + 1), & r \in (0,1), \\ \tilde{v}(1) = 0. \end{cases} \tag{12.32}$$

By setting $\mu = \frac{\lambda_N(p)}{p^{p-2}(p-1)} = \frac{p(N-p)}{p-1}$, this has a unique solution in $V_{\lambda_N(p),p}(B)$ given by

$$\tilde{v} = \frac{1}{\lambda_N(p)}(1 - r^{-\frac{N-p}{2} + \frac{1}{2}\sqrt{(N-p)^2 - 4\mu}}). \tag{12.33}$$

Here we note that $(N-p)^2 - 4\mu = (N-p)\left(N - p\frac{p+3}{p-1}\right) > 0$.

Therefore we have the following:

Lemma 12.4. *Assume that $1 < p \leq 2$ and $N > p\frac{p+3}{p-1}$. Then*

$$\lim_{\lambda \to \lambda^*} v_\lambda = \tilde{v} \quad \text{in} \quad V_{\lambda_N(p),p}(B). \tag{12.34}$$

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