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An exceptional example of twistor spaces of four-dimensional almost Hermitian manifolds

By

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Abstract

A four-dimensional non-integrable almost Hermitian manifold whose second twistor space is a complex manifold is given.

Introduction

It is known that the integrable twistor space of almost Hermitian manifolds defined by O'Brian and Rawnsley in [4] are very restrictive in higher dimensions. Namely, if the k-th twistor space of an n-dimensional almost Hermitian manifold X is integrable, X should be a Bochner-Kähler manifold, for general pair (n, k).

In the case $n = 3$, the six-sphere with the well-known G_2-invariant non-integrable almost complex structure gives an exceptional example. But, in the case $n = 4$, the existence of an exceptional example is an open problem. Note that the proof of the non-existence theorem of four-dimensional non-integrable example in [4] has a mistake. (See [3] Theorem 1.4, for detail.)

In this paper, we give a four-dimensional non-integrable almost Hermitian manifold whose second twistor space is integrable. The base space is the product of the six-sphere S^6 with the standard G_2-invariant non-integrable almost Hermitian structure and the hyperbolic plane $\mathbb{H}$ with the standard Kähler structure. Since $S^6 \times \mathbb{H}$ is conformally flat, there exists an immersion to S^8 called the developing map. This is shown to be an embedding and the image is $S^8 \setminus S^1$. In this paper, we first construct an almost complex structure on $S^8 \setminus S^1$ and prove that its second twistor space is integrable. Then, by using that its automorphism group is isomorphic to $G_2 \times \mathrm{PSL}(2, \mathbb{R})$, we conclude that the developing map is an isomorphism between almost complex manifolds.

1. Construction of an almost complex structure on $S^8 \setminus S^1$

In this section, we construct an almost complex structure on the eight-dimensional Euclidean space excluding a closed circle, which can be extended to a point at infinity.

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Let $Z = \mathrm{SO}(8)/\mathrm{U}(4)$ be a parameter space of complex structures on the real vector space $\mathbb{R}^8$ compatible with the standard orientation and the metric. The space Z is naturally embedded to $\mathbf{P}(\Delta^+)$ as the space of projectivized pure spinors, where Δ^+ is the positive half spin module.

We use notation given in [1], [2]. Let $(\theta_I)_{I\subset\{1,2,3,4\}}$ be a basis of the spin module Δ and Z^I be the corresponding homogeneous coordinate on $\mathbf{P}(\Delta)$. Hence $(Z^\emptyset, Z^{12}, Z^{13}, Z^{14}, Z^{23}, Z^{24}, Z^{34}, Z^{1234})$ is a system of homogeneous coordinates on $\mathbf{P}(\Delta^+)$. The space of projectivized pure spinors Z is defined by an equation

$$Z^\emptyset Z^{1234} - Z^{12}Z^{34} + Z^{13}Z^{24} - Z^{14}Z^{23} = 0. \tag{1.1}$$

Let $(x^1, \dots, x^8)$ be the standard coordinates on the Euclidean space $\mathbb{R}^8$. For simplicity we introduce complex valued functions $\xi^i = x^i + \sqrt{-1}x^{4+i}$, $i = 1, 2, 3, 4$. Note that these will not be holomorphic functions, since the almost complex structure we construct is not the standard one.

Now, we define an almost complex structure on $\mathbb{R}^8 \setminus S^1$ by constructing a section to the trivial Z bundle. Put

$$\alpha(\xi) = \theta_\emptyset + \sqrt{-1}(\bar{\xi^1}\theta_{23} - \bar{\xi^2}\theta_{13} + \bar{\xi^3}\theta_{12} + \xi^4\theta_{1234}),$$
$$\beta(\xi) = \theta_{1234} + \sqrt{-1}(\xi^1\theta_{14} + \xi^2\theta_{24} + \xi^3\theta_{34} - \bar{\xi^4}\theta_\emptyset).$$

By (1.1), the spinor of type $\mu\alpha(\xi) + \lambda\beta(\xi)$ is pure or zero if and only if

$$\sqrt{-1}\xi^4\mu^2 + (1 + |\xi^1|^2 + |\xi^2|^2 + |\xi^3|^2 + |\xi^4|^2)\mu\lambda - \sqrt{-1}\bar{\xi^4}\lambda^2 = 0 \tag{1.2}$$

Since its discriminant

$$(1 + |\xi^1|^2 + |\xi^2|^2 + |\xi^3|^2 + |\xi^4|^2)^2 - 4|\xi^4|^2$$

is a non-negative real number, we can distinguish two solutions $\gamma_+(\xi)$ and $\gamma_-(\xi)$, unique up to multiplication by non-vanishing scalar functions. Since the space spanned by $\alpha(\xi)$ and $\beta(\xi)$ is invariant by the anti-linear map inducing the standard involution on Z, these pure spinors define mutually conjugate almost complex structures if they do not vanish. The section $\gamma_\pm(\xi)$ vanishes if and only if (ξ^i) is a point of $S^1 = \{(\xi^i) \mid \xi^1 = \xi^2 = \xi^3 = 0, |\xi^4| = 1\}$, where the subspace spanned by $\alpha(\xi)$ and $\beta(\xi)$ collapses to one-dimensional.

Now we have two almost complex structures on $\mathbb{R}^8 \setminus S^1$ by the pure spinors $\gamma_\pm(\xi)$. It is easy to verify that they can be smoothly extended to almost complex structures around the point at infinity.

Throughout this paper, we consider $S^8 \setminus S^1$ as an almost Hermitian manifold by one of the above almost complex structures. Note that the choice of an almost complex structure is irrelevant, since there is an involution on $\mathbb{R}^8 \setminus S^1$ which exchanges $\gamma_+(\xi)$ and $\gamma_-(\xi)$.

Let $S^6 = \{(\xi) \in \mathbb{R}^8 \mid \xi^4 = 0\} \cup \{\infty\}$. On this submanifold, $\alpha(\xi)$ is the pure spinor defining the standard non-integrable almost complex structures on S^6. Hence it is an almost complex submanifold of $S^8 \setminus S^1$. Furthermore, we shall show later that the almost complex manifold $S^8 \setminus S^1$ is isomorphic to $S^6 \times \mathbb{H}$, where $\mathbb{H}$ is the hyperbolic plane with the standard Kähler structure.

2. Twistor space of $S^8 \setminus S^1$

The almost complex structures on $S^8 \setminus S^1$ constructed in the previous section gives an important example in twistor theory of almost Hermitian manifolds.

Let X be an n-dimensional almost Hermitian manifold. For an integer $k = 1, 2, \ldots, n-1$, let $Z_k(X)$ be the k-th twistor space of X defined in [4]. As in [3], $Z_k(X)$ can be considered as a submanifold of the total twistor space $Z(X) \cup Z_-(X)$, where $Z_-(X)$ is the twistor space of X with the opposite orientation. In other words, $Z_k(X)$ is the subbundle of $Z(X) \cup Z_-(X)$ with fibers $Z_k = (Z \cup Z_-) \cap \mathbf{P}(\Delta^k)$, where $\Delta = \bigoplus_{l=0}^{n} \Delta^l$ is the irreducible decomposition as $\mathrm{U}(n)$-modules. Assume that $Z_k(X)$ is an almost complex submanifold. Then

Theorem 2.1 ([4], [3]). *If n is greater than 4 and $k = 1, 2, \ldots, n-1$, or (n, k) is $(4, 1)$ or $(4, 3)$, then $Z_k(X)$ is a complex manifold if and only if X is a Bochner-Kähler manifold.*

The almost complex structure on $S^8 \setminus S^1$ gives an exceptional example to the above theorem, that is, the almost complex structure of the second twistor space $Z_2(S^8 \setminus S^1)$ is integrable.

Let Δ' be the spin module of SPIN(10), and Z' be the space of projectivized pure spinors in $\mathbf{P}(\Delta'^+)$. The twistor space of S^8 can be identified with Z'. We consider $Z(S^8 \setminus S^1)$ as an open submanifold of Z' and gives explicitly the defining equations of $Z_2(S^8 \setminus S^1)$.

Let $(Z^I)_{I \subset \{0,1,2,3,4\}}$ be the system of homogeneous coordinates of $\mathbf{P}(\Delta')$. We have two systems of *functions* on $Z(\mathbb{R}^8)$, namely

$$(\xi^1, \ldots, \xi^4, Z^I; I \not\ni 0)$$

and

$$(Z^{0I}, Z^I; I \not\ni 0),$$

where, strictly speaking, Z^I's are sections to the hyperplane bundle. The transform between these two systems of functions is given as follows:

$$Z^{0I} = \sqrt{-1} \left(\sum_{a \in I} \xi^a Z^{aI} + \sum_{a \notin 0I} \bar{\xi^a} Z^{aI} \right), \quad I \subset \{1, 2, 3, 4\},$$

where we use index notation in [2] (see [2] proof of Lemma 2.9 for detail).

Theorem 2.2. *The second twistor space $Z_2(S^8 \setminus S^1)$ of $S^8 \setminus S^1$ is a complex submanifold of $Z(S^8 \setminus S^1)$ defined by equations*:

$$Z^{\emptyset} + Z^{0123} = 0,$$
$$Z^{1234} - Z^{04} = 0.$$

Proof. Since $\Delta^+ = \Delta^0 \oplus \Delta^2 \oplus \Delta^4$ is orthogonal decomposition and $\Delta^0 \oplus \Delta^4$ is spanned by $\alpha(\xi)$ and $\beta(\xi)$, Δ^2 is cut out by the equations:

$$Z^{\emptyset} - \sqrt{-1}(\xi^1 Z^{23} - \xi^2 Z^{13} + \xi^3 Z^{12} + \bar{\xi^4} Z^{1234}) = Z^{\emptyset} + Z^{0123} = 0,$$
$$Z^{1234} - \sqrt{-1}(\bar{\xi^1} Z^{14} + \bar{\xi^2} Z^{24} + \bar{\xi^3} Z^{34} - \xi^4 Z^{\emptyset}) = Z^{1234} - Z^{04} = 0.$$

□

Remark 1. The defining equations of $Z_2(S^8 \setminus S^1)$ define a smooth subvariety of Z'. Since equations above are equal on the fibers over S^1, the fiber over a point of S^1 is a five-dimensional nonsingular complex hyperquadric.

3. Decomposition of $S^8 \setminus S^1$

Let H be the conformal transformation group of $S^8 \setminus S^1$. By Liouville's theorem, H is a subgroup of the conformal transformation group of S^8.

Let g be an element of H. Let H_1 be the one-dimensional group of rotation around S^1. Fix a point ∞ on S^1. We can assume that S^1 is the geodesic circle of S^8. Then $-\infty$: the antipodal point of ∞ is also a point of S^1. There is an element $g_1 \in H_1$ such that $g_1 g$ fixes ∞. By the stereographic projection, $S^8 \setminus \{\infty\}$ can be identified with $\mathbb{R}^8$. Then the origin of $\mathbb{R}^8$ corresponds to $-\infty$. Let L be the image of $S^1 \setminus \{\infty\}$, which is a line through the origin of $\mathbb{R}^8$. Let H_2 be the one-dimensional group of translation on $\mathbb{R}^8$ along L, which is also a subgroup of H by Liouville's theorem. Then, there is an element $g_2 \in H_2$ such that $g_2 g_1 g$ fixes ∞ and $-\infty$. Let H_3 be the one-dimensional group of dilatation on $\mathbb{R}^8$. Let S^6 be the unit sphere of the orthogonal complement of L. Let H_4 be the subgroup of H whose elements fix points on S^1 and act on S^6 as isometries. Note that H_4 is naturally identified with $\mathrm{O}(7)$. Let σ be the reflection with respect to the hyperplane through the origin whose normal vector is parallel to L. Then, there are elements $g_3 \in H_3$ and $g_4 \in H_4$ such that $g_4 g_3 g_2 g_1 g$ is either the identity or σ.

Thus we have shown that H is generated by H_i, $i = 1, 2, 3, 4$ and σ. More precisely, let K be the subgroup generated by H_1, H_2, H_3 and σ. Since elements of K are commutative with elements of H_4, we have $H \simeq K \times H_4$.

Let us take a coordinate system on $\mathbb{R}^8$ such that $L = \{(t, 0, \dots, 0)\}$ and put

$$S(t, r) = \{(-t, x_1, \dots, x_7) \mid x_1^2 + \cdots + x_7^2 = r^2\}.$$

Since

$$S^8 \setminus S^1 = \mathbb{R}^8 \setminus L = \coprod_{t+\sqrt{-1}r \in \mathbb{H}} S(t, r),$$

where $\mathbb{H}$ is the upper-half plane, we can define a map

$$\alpha : \mathbb{R}^8 \setminus L \to \mathbb{H}$$
$$x \mapsto t + \sqrt{-1}r \quad \text{such that } x \in S(t, r).$$

Let β be the map defined by

$$\beta : \mathbb{R}^8 \setminus L \to S^6$$
$$(t, x_1, \dots, x_7) \mapsto \frac{1}{\sqrt{x_1^2 + \cdots + x_7^2}} \begin{pmatrix} x_1 \\ \vdots \\ x_7 \end{pmatrix}$$

Then, they induce a diffeomorphism

$$\begin{aligned} S^8 \setminus S^1 = \mathbb{R}^8 \setminus L &\simeq \mathbb{H} \times S^6 \\ x &\mapsto (\alpha(x), \beta(x)) \end{aligned} \tag{3.1}$$

Thus we get a $K \times H_4$ action on $\mathbb{H} \times S^6$. Let g be a transform in K. We have $\beta(g(x)) = \beta(x)$ for every point x of $S^8 \setminus S^1$. For the $\mathbb{H}$ component, there is an isomorphism g' on $\mathbb{H}$ such that $\alpha(g(x)) = g'(\alpha(x))$. In a similar way, let g be a transform in H_4. We have $\alpha(g(x)) = \alpha(x)$ and $\beta(g(x)) = g(\beta(x))$, where the action of g on S^6 is the standard one as an element of $\mathrm{O}(7)$.

Hence we have shown that the $K \times H_4$ action on $\mathbb{H} \times S^6$ is induced by the action of K on $\mathbb{H}$ and the action of H_4 on S^6.

Let K_0 be the subgroup of K generated by H_1, H_2 and H_3. By the action on $\mathbb{H}$, K_0 is naturally identified with $\mathrm{PSL}(2, \mathbb{R})$: the orientation preserving conformal automorphism group of $\mathbb{H}$. The action of σ on $\mathbb{H}$ is the reflection with respect to the pure imaginary line.

Now we study the action of K on $S^8 \setminus S^1$.

Lemma 3.1. *K_0 is the subgroup of K which preserves the almost complex structure on $S^8 \setminus S^1$.*

Proof. Since transformations in σK_0 change orientation, it suffices to show that transformations in H_i, $i = 1, 2, 3$ preserve the almost complex structure.

A complex structure on the real vector space $\mathbb{R}^8$ decomposes the half spin module Δ^+ into Δ^0, Δ^2 and Δ^4. On the other hand, the subspace Δ^2 determines the complex structure up to a conjugate pair. In fact, the orthogonal complement of Δ^2 contains just two lines of pure spinors corresponding to the original complex structure and its conjugate. Furthermore, since Δ^2 can be characterized as the minimal subspace of Δ containing lines in $Z_2 = Z \cap P(\Delta^2)$, it suffices to show that elements of H_i, $i = 1, 2, 3$ preserve the twistor space $Z_2(S^8 \setminus S^1)$.

To compute the actions of H_i on $Z(S^8)$, we first give expressions as matrices in $\mathrm{O}(1, 9)$.

Let $(x_0, x_1, \dots, x_8)$ be the standard coordinate system of $\mathbb{R}^9$, and let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

be a matrix in $\mathrm{O}(1, 9)$, where a_{11}, a_{12}, a_{21} and a_{22} are submatrices of size $(1, 1)$, $(1, 9)$, $(9, 1)$ and $(9, 9)$, respectively. Then we define an action of A on S^8 by

$$x \mapsto \frac{1}{a_{11} + a_{12}x}(a_{21} + a_{22}x).$$

Let the circle S^1 and the point at infinity ∞ be:

$$S^1 = \{(x_0, x_1, 0, \dots, 0) \in S^8\},$$
$$\infty = (1, 0, \dots, 0).$$

Then the matrices corresponding to elements of H_i, $i = 1, 2, 3$ are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos t & -\sin t & 0 \\ 0 & \sin t & \cos t & 0 \\ 0 & 0 & 0 & 1_7 \end{pmatrix}, \qquad i = 1,$$
$$\frac{1}{2}\begin{pmatrix} 2+t^2 & -t^2 & 2t & 0 \\ t^2 & 2-t^2 & 2t & 0 \\ 2t & -2t & 2 & 0 \\ 0 & 0 & 0 & 2_7 \end{pmatrix}, \qquad i = 2,$$
$$\begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1_8 \end{pmatrix}, \qquad i = 3.$$

Hence, by changing indices ($0 \to 4$, $1 \to 8$) as chosen in previous sections, the infinitesimal transforms corresponding to the above one-parameter groups of transformations are given by the Clifford algebras:

$$\frac{1}{2} f_4 f_8, \quad -\frac{\sqrt{-1}}{2} f_0 f_8 - \frac{1}{2} f_4 f_8, \quad -\frac{\sqrt{-1}}{2} f_0 f_4,$$

where $(f_0, f_{0'}, f_1, \dots, f_8)$ is the basis of $\mathbb{R}^{10}$. Now it is easy to show that they preserve the space of defining equations: $\langle Z^{1234} - Z^{04}, Z^{\emptyset} + Z^{0123} \rangle$. □

To summarize, we have:

Theorem 3.1. *The diffeomorphism* (3.1) *is the almost complex conformal isomorphism. The automorphism group of* $S^8 \setminus S^1$ *as an almost complex conformal manifold is* $\mathrm{PSL}(2, \mathbb{R}) \times G_2$.

Proof. We first prove the second part of the theorem.

We have a $\mathrm{PSL}(2, \mathbb{R}) \times G_2$ action on $S^8 \setminus S^1$ as a restriction of the $K \times H_4$ action. We have proved in Lemma 3.1 that a transform in $\mathrm{PSL}(2, \mathbb{R})$ preserves the almost complex structure on $S^8 \setminus S^1$. A transform in G_2 also preserve the almost complex structure because $Z^{123} - Z^0$ or $Z^{\emptyset} + Z^{0123}$ is the defining equation of $Z_1(S^6)$ or $Z_2(S^6)$, respectively, and adding the index 4 to the first equation is compatible with the action of G_2.

On the other hand, let g be a conformal transform on $S^8 \setminus S^1$ which preserves the almost complex structure. There are $g_1 \in K$ and $g_2 \in H_4$ such that $g = g_1 g_2$. If g_1 is orientation preserving, that is, $g_1 \in \mathrm{PSL}(2, \mathbb{R})$, it also preserves the almost complex structure by Lemma 3.1. Hence $g_2 = g_1^{-1} g$ preserves the almost complex structure. Since it acts on the almost complex submanifold S^6, we have $g_2 \in G_2$. If g_1 is orientation reversing, $g_1\sigma$ preserves the almost complex structure by Lemma 3.1. Hence $\sigma g_2 = (g_1\sigma)^{-1}g$ preserves the almost complex structure. Since σ acts on S^6 as identity, g_2 should preserve the almost complex structure on S^6. This is impossible because g_2 should be orientation reversing.

Now the first part of the theorem can be proved by calculating the differential map at a point, because of the equivariant property of the diffeomorphism. □

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