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One dimensional wave equations in domain with quasiperiodically moving boundaries and quasiperiodic dynamical systems

Dedicated to Professor Mitsuru Ikawa on his 60th birthday

By

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Abstract

We shall deal with IBVP for a linear one-dimensional wave equation in domain with time-quasiperiodically oscillating boundaries. We shall show that for any given initial data and almost all boundary data, every solution is quasiperiodic in t, provided that the basic frequencies of time-quasiperiodic data of IBVP satisfy the number-theoretic Diophantine conditions. In order to solve this problem, we shall show the reduction theorem of one-dimensional quasiperiodic dynamical systems. To prove the reduction theorem, we shall define upper and lower rotation numbers of dynamical systems and apply the rapidly iteration method to the related dynamical system defined by the boundary functions. Also we shall construct a class of time-quasiperiodic boundary data of IBVP and the basic frequencies such that IBVP has quasiperiodic solutions that are the superposition of the sequentially time-unbounded forward and backward waves.

1. Introduction

Let D be a noncylindrical domain in two dimensional (x,t)-plane with time-quasiperiodic boundaries

$$a_1(t) < x < a_2(t), \ \ t \in R^1.$$

Here the given functions $a_i(t)$, $i = 1, 2$, are quasiperiodic functions. For the definition of quasiperiodic functions, see Section 2.

Consider IBVP for a linear nonhomogeneous wave equation:

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) = h(x,t), \quad (x,t) \in D, \tag{1.1}$$

$$u(a_1(t),t) = r_1(t), \quad u(a_2(t),t) = r_2(t), \quad t \in R^1, \tag{1.2}$$

$$u(x,0) = \phi(x), \ \ \partial_t u(x,0) = \psi(x), \ \ x \in [a_1(0), a_2(0)]. \tag{1.3}$$

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Here $r_i(t)$, $i = 1, 2$, and $h(x,t)$ are quasiperiodic functions in t. We assume that

$$|a_i'(t)| < 1 \quad \text{for all } t \in R^1 \tag{1.4}$$

in order to deal with non-shock waves.

This IBVP describes some physical phenomena like the motions of the string with time-quasiperiodically oscillating ends ([Ya1], [Ya3]), the problem of one-dimensional optical resonator with a quasiperiodically moving wall ([L-P], [P-L-V]) and so on.

In this paper we shall investigate the behavior of the solutions of IBVP (1.1)–(1.3). For example, let us consider the motions of a vibrating string with quasiperiodically moving end points. Since the ends of the string quasiperiodically vibrate and the outer force works to the string in a time-quasiperiodic way, in general we may expect that

(C) *every solution of* IBVP (1.1)–(1.3) *is quasiperiodic in* t.

However, J. Cooper [C] showed that even in a simpler IBVP,

$$\begin{gathered} \partial_t^2 u(x,t) - \partial_x^2 u(x,t) = 0, \quad (x,t) \in \{0 < x < a(t), t \in R^1\}, \\ u(0,t) = u(a(t),t) = 0, \quad t \in R^1, \\ u(x,0) = \phi(x), \;\; \partial_t u(x,0) = \psi(x), \;\; x \in [0, a(0)], \end{gathered} \tag{1.5}$$

where $a(t)$ is periodic but not quasiperiodic, the energy of each solution is unbounded in t, and as a matter of course, the solution is not quasiperiodic under some conditions on $a(t)$. Also in [Ya4], nonhomogeneous IBVP is considered in a cylindrical domain

$$\begin{gathered} \partial_t^2 u(x,t) - \partial_x^2 u(x,t) = h(x,t), \quad (x,t) \in (0,\pi) \times R^1, \\ u(0,t) = u(\pi,t) = 0, \quad t \in R^1, \\ u(x,0) = \phi(x), \;\; \partial_t u(x,0) = \psi(x), \;\; x \in [0,\pi], \end{gathered}$$

and it is shown that there exists a family of time-quasiperiodic functions $h(x,t)$ for which all the solutions of IBVP are unbounded in t. Such elements h have the property that the Diophantine order of its basic frequencies is large and the differentiability with respect to t and x is small. On the other hand, we can deduce from [Ya7] that in the fixed ends case every solution of the above IBVP is time-quasiperiodic if the differentiability of h is larger than the Diophantine order of the basic frequencies of h. Thus the Diophantine condition is necessary in order that the solutions of IBVP (1.1)–(1.3) are time-quasiperiodic.

The aim of this paper is to show that our above conjecture (C) is true for perturbed type boundary functions a_i and *almost all* given functions r_i, h. Namely, we shall give general conditions under which every solution of IBVP (1.1)–(1.3) is quasiperiodic in t. We shall clarify that *IBVP* (1.1)–(1.3) *has the structure of one-dimensional quasiperiodic dynamical systems.*

In the previous papers [Ya1], [Ya2], [Ya3] we were concerned with homogeneous wave equations and periodic boundary functions *i.e.,* $h(x,t)$ vanishes identically, and $a_i(t)$ and $r_i(t)$ $(i = 1, 2)$ are periodic. The quasiperiodic or periodic properties of the solutions of IBVP (1.1)–(1.3) with $h(x,t) \equiv 0$ were studied in detail. It was made clear that a composed function

$$A = A_1^{-1} \circ A_2, \quad A_i = (I + a_i) \circ (I - a_i)^{-1}, \quad i = 1, 2, \tag{1.6}$$

is an one-dimensional periodic dynamical system, where I is the identity, f^{-1} means the inverse of f and $f \circ g$ means the composition of f and g, and that A and *its rotation number* are the essential notions to describe periodic and quasiperiodic property of the solutions of IBVP and BVP with periodically oscillating boundaries. To show the results we applied the reduction theorem by Herman-Yoccoz ([H], [Yoc]) to A under the Diophantine condition on the rotation number of A. Our results on the quasiperiodic properties of the solutions of BVP (1.5) are more exact expression of qualitative property of the solutions of Dittrich-Duclos-Gonzalez [D-D-G]. [D-D-G] also studied the time-unboundedness of the solutions in the energy norm.

In general, even if $a_i(t)$ are not necessarily periodic, the mapping A is well-defined for $a_i(t)$ satisfying $a_1(t) < a_2(t)$ and $|a_i'(t)| < 1$. It also has the following geometrical definition. In the space-time plane ((x,t)-plane) we consider characteristic curves (polygons) reflected by two curves $C_1 : x = a_1(t)$ and $C_2 : x = a_2(t)$. Let $X = (0, t_0)$ be any point on the t-axis. Consider a straight line W through X whose gradient is $+1$. Let P_1 be the intersection point of the line W and the curve C_2. Let L be a reflected characteristics through P_1. Let reflected points of L by C_2 and C_1 be P_i and Q_i $(i = 1, 2, \dots)$ (resp.) in order as the ray goes to the direction of the plus infinity of t starting at P_1. We denote the intersection point of the line Q_1P_2 and t-axis by $(0, t_1)$. Then A is defined by a mapping $A : t_0 \in R^1 \to t_1 \in R^1$.

In the above works [Ya1], [Ya2], [Ya3], both $a_i(t)$, $i = 1, 2$, are periodic and the ratio of the periods is *a rational number*. In this case $A - I$ is clearly a periodic function. It is well-known [H] that for periodic $A - I$ the rotation number of A,

$$\lim_{n\to\infty} \frac{(A^n - I)(x)}{n},$$

exists for each $x \in R^1$ and is independent of x, where A^n means the n-th iterate of A. In [Ya1] and [Ya3] the case where both $a_i(t)$, $i = 1, 2$, are 1-periodic and $r_i(t)$ are α_i-periodic was considered. It was shown that if the periods of $a_i(t)$, $r_i(t)$ and the rotation number ω of A satisfy some Diophantine approximation inequality from number theory, then every solution is quasiperiodic in t with basic periods $(\omega, \alpha_1, \alpha_2, 1)$. In [Ya2] it was shown that the properties of A and the reflected characteristics determine the periodicity of the solutions of IBVP (1.1)–(1.3) with $a_1(t) = r_1(t) = r_2(t) = h(x,t) \equiv 0$. From the point of view of the spectrum, Cooper and Koch [C-K] studied in detail the spectrum of evolution operator U of BVP (1.5) defined in $H_0^1(0,1) \times L^2(0,1)$ by

$$U : (\phi, \psi) \to (u(\cdot, T), \partial_t u(\cdot, T)),$$

where T is the period of $a(t)$, and showed that the spectrum is the unit circle in the complex plane if the rotation number is irrational.

All these results are essentially due to the *periodicity* of $A - I$ that assures the existence of the rotation number ([Ya1], [Ya2], [Ya3], [C-K], [D-D-G]) and the periodic reflected characteristics ([Ya2]). However, for example, if the periods of $a_1(t)$ and $a_2(t)$ have *an irrational ratio*, $A - I$ is not periodic but quasiperiodic with two basic periods. Then the situation seems to be much more complicated. In this paper, we shall be interested in such cases. We shall treat IBVP (1.1)–(1.3) under more general condition that all $a_i(t)$ and $r_i(t)$, $i = 1, 2$, are quasiperiodic functions. In this case naturally $A - I$ is quasiperiodic.

In order to treat the case that $A - I$ is quasiperiodic, we shall generally consider one dimensional quasiperiodic dynamical system (DS) defined by monotone increasing mapping $f(x) = (I + g)(x)$ with quasiperiodic term $g(x)$. It is not known that for the quasiperiodic DS the rotation number exists for all x. As a matter of fact, in this paper we shall see that the existence of the rotation number is not necessary to show the reduction theorem for quasiperiodic DS. Instead, we shall introduce a more weak notion *upper and lower rotation number of f at every point x* (see Section 2)

$$\limsup_{n\to\infty} \frac{(A^n - I)(x)}{n}, \quad \liminf_{n\to\infty} \frac{(A^n - I)(x)}{n}.$$

Clearly this upper (lower) rotation number is a generalization of the rotation number. The upper (lower) rotation numbers have several important properties like as *semi-invariant property under conjugation* that are used to show the reduction theorem for quasiperiodic DS (Section 3 and Section 8).

Roughly speaking, the following holds: Assume that an upper (lower) rotation number ω of f and the basic frequencies of the quasiperiodic term g satisfy the Diophantine condition. Then the nearly affine mapping f written in the form $x + \omega + q(x)$ is conjugate to an affine mapping $x + \omega$, provided that q is small enough. As a consequence, it will be shown that under the same Diophantine condition the rotation number of f exists and coinsides with the upper (lower) rotation number (Corollary of the Reduction Theorem in Section 3). The main tool we shall use here to show the above reduction theorem is the rapidly convergent iteration method based on the Newton iteration method [S-M], instead of the Herman-Yoccoz theory [H], [Yoc] used in case of the periodic $A-I$ in [Ya1], [Ya3]. Then we shall show that under the Diophantine conditions on an upper (lower) rotation number of A and the basic periods of a_i, r_i, h, every solution of IBVP (1.1)–(1.3) is quasiperiodic in t and x. In this case the rotation number of A exists and coincides with the upper (lower) rotation number of A. The reduction problem is also treated by [P-L-V], [L-P] in the quite different point of view.

Note that our results are obtained for $a_i(t)$ with the small perturbation forms, different from those of [Ya1]–[Ya3].

All above works [Ya1]–[Ya3] dealt with homogeneous string equations. In order to treat a nonhomogeneous equation (1.1) with $h(x, t) \not\equiv 0$, we shall introduce the useful domain transformation of the noncylindrical domain $\bar{D}$ to a cylindrical domain $[0, \omega/2] \times R^1$, where ω is the upper (lower) rotation number of A. It is remarkable that different from other domain transformations which change the noncylindrical domain to a cylindrical domain, our transformation preserves the d'Alembert operator and does not produce any lower order differential operators. This will be constructed by using the conjugate function of the Reduction Theorem. In case where $a_1(t)$ vanishes identically and $a_2(t)$ is periodic, in [Ya-Yos] we have already constructed such d'Alembertian-preserving transformation of a time-periodic one-sided noncylindrical domain onto the cylindrical domain. And using this transformation, we treated IBVP for a nonhomogeneous string equation ([Ya-Yos]). In [Ya5] the above domain transformation is generalized to a time-periodic both sided noncylindrical domain, and IBVP for a 3-dimensional radially symmetric wave equation is studied. Also [Ya6] treated the periodic solutions of nonlinear string equation with periodic nonlinear term, using the domain transformation.

This paper is organized as follows. In Section 2 we shall introduce the upper and lower rotation number of quasiperiodic DS and investigate its properties, and in Section 3 we shall show the reduction theorem that plays an essential role to show our results on IBVP and also the existence of the rotation numbers of quasiperiodic DS. In Section 4 we shall state our main theorem. The theorem shall be proved in Sections 5 and 6. First IBVP for the homogeneous wave equation will be dealt with in Section 5, and second the boundary value problem (BVP) will be treated in Section 6. In Sections 5 and 6 the similar methods in [Ya1], [Ya3], [Ya-Yos], [Ya5] will be used. In Section 7 we shall deal with quasiperiodic solution of (1.1)–(1.3) that are represented by the superposition of the sequentially time-unbounded waves. We shall construct $r_i(t)$ for which every solution has such property. In Section 8 we shall prove the Reduction Theorem stated in Section 3.

2. Upper and lower rotation numbers of one-dimensional DS with quasiperiodic terms

In this section we shall introduce *upper* (*lower*) *rotation number* and investigate several properties. They play an essential role to prove the reduction theorem and seem to be interesting from the point of view of quasiperiodic DS.

First we recall the definition of quasiperiodic functions. A function $g(t)$, $t \in R^1$, is called *quasiperiodic with basic frequencies* $\beta = (\beta_1, \ldots, \beta_m) \in R^m$ (*briefly* $2\pi/\beta$*-q.p.*) if there exists a continuous function $\hat{g}(\theta), \theta = (\theta_1, \ldots, \theta_m) \in R^m$, that is 2π-periodic in each θ_i such that $g(t) = \hat{g}(\beta t)$ holds. $\hat{g}(\theta)$ is called a corresponding function of g and $2\pi/\beta = (2\pi/\beta_1, \ldots, 2\pi/\beta_m)$ is called the basic periods of g. Without loss of generality, basic frequencies $\beta_1, \ldots, \beta_m$ of any q.p. functions are always assumed to be rationally independent.

We consider a monotone increasing mapping of R^1 to R^1

$$f(x) = x + g(x), \tag{2.1}$$

where $g(x)$ is a $2\pi/\beta$-q.p. function. We denote the set of such functions f by

D_β.

Remark 1. Clearly D_β is a group with respect to the operation of the composition of functions.

H. Poincaré first introduced a rotation number for any monotone increasing mapping f with continuous periodic g:

$$\rho = \rho(f) = \lim_{n\to\infty} \frac{(f^n - I)(x)}{n},$$

where f^n is the n-th composed iterate of f. In this periodic DS case it is well-known [H] that the above limit ρ always exists, is independent of x and has a property of the conjugacy invariant *i.e.,* $\rho(\phi\circ f\circ\phi^{-1}) = \rho(f)$ for every periodic and continuous DS ϕ.

Now we introduce an upper and lower rotation number that are generalizations of the rotation number. These numbers will play an important role in showing the reduction theorem. In case of periodic g the rotation number played the same role ([Yoc]). We define *an upper* (*lower*) *rotation number* $\bar{\rho}(f)(x)$ $(\underline{\rho}(f)(x))$ *of* $f \in D_\beta$ *at* $x \in R^1$ as follows:

$$\begin{aligned} \bar{\rho}(f)(x) &= \limsup_{n\to\infty} \frac{(f^n - I)(x)}{n}, \\ \underline{\rho}(f)(x) &= \liminf_{n\to\infty} \frac{(f^n - I)(x)}{n}. \end{aligned} \tag{2.2}$$

The following is clear. If $\bar{\rho}(f)(x) = \underline{\rho}(f)(x)$ holds, then the rotation number $\rho(f)(x)$ exists and vice versa, and $\bar{\rho}(f)(x) = \underline{\rho}(f)(x) = \rho(f)(x)$ holds. Note that

$$f^n(x) = x + g(x) + g\circ f(x) + \cdots + g\circ f^{n-1}(x), \tag{2.3}$$

where $F\circ G$ means the composition of F and G, *i.e.,* $F\circ G(x) = F(G(x))$. Since $g(x)$ is q.p. hence bounded in R^1, it follows from (2.3) that the above superior (inferior) limit always exists for each $x \in R^1$. It is clear that

$$\inf_{x\in R^1} g(x) \le \underline{\rho}(f)(x) \le \bar{\rho}(f)(x) \le \sup_{x\in R^1} g(x).$$

It is natural to ask whether for any given q.p. g the above rotation number exists and is independent of x. As will be seen from Corollary of Reduction Theorem in Section 3, the limit exists and is independent of x for every mapping (2.1) with small perturbation term under suitable number-theoretic condition (the Diophantine condition) on an upper (lower) rotation number of f and the basic periods of g.

From now on we shall deal with only upper rotation numbers. The corresponding results hold for the lower rotation numbers.

We have a generalization of the conjugacy invariant of the rotation number.

Proposition 2.1. *Consider a mapping $f \in D_\beta$. For any point $x \in R^1$ and any $H \in D_\beta$ there exists a point $y \in R^1$ such that*

$$\bar{\rho}(H^{-1} \circ f \circ H)(y) = \bar{\rho}(f)(x). \tag{2.4}$$

Especially, if the mapping f has a rotation number $\rho(f)$ independent of x, then

$$\rho(H^{-1} \circ f \circ H) = \rho(f) \tag{2.5}$$

holds.

Proof. We set $f_H = H^{-1} \circ f \circ H$ and $H^{-1} = I + \tilde{h}$, where $\tilde{h}(x)$ is $(2\pi/\beta)$-q.p.. For a given x we set $y = H^{-1}(x)$. Then since $f_H^n(y) = H^{-1} \circ f^n \circ H(y)$ holds, we have

$$\begin{aligned}
\frac{(f_H^n - I)(y)}{n} &= \frac{H^{-1} \circ f^n(x) - H^{-1}(x)}{n} \\
&= \frac{(I + \tilde{h}) \circ f^n(x) - (I + \tilde{h})(x)}{n} \\
&= \frac{f^n(x) - x}{n} + \frac{\tilde{h} \circ f^n(x) - \tilde{h}(x)}{n}.
\end{aligned}$$

Since $\tilde{h}$ is a bounded function, the second term tends to 0 as n tends to ∞. Hence (2.4) holds. (2.5) is shown from the above identity and the relation $y = H^{-1}(x)$. In fact, since by the assumption the right hand side of (2.5) is constant for all x, the left hand side is constant for any fixed $H \in D_\beta$ and $y = H^{-1}(x)$. Since H is a surjection of R^1 to R^1, $\bar{\rho}(H^{-1} \circ f \circ H)(y)$ is constant for all y. □

Remark 2. For any real number ω there exist $f_\omega \in D_\beta$ that has the rotation number ω.

In fact, we denote $\phi^{-1} \circ R_\omega \circ \phi$ by $f_{\phi,\omega}$ for any $\phi \in D_\beta$, where $R_\omega(x) = x + \omega$. Clearly R_ω has the rotation number ω. Hence by Proposition 2.1 $f_{\phi,\omega}$ has the rotation number ω.

For brevity, for any $x_0 \in R^1$ denote $\bar{\rho}(f)(x_0)$ by $\omega = \omega(x_0)$. Then we rewrite f in the following form

$$f(x) = x + \omega + q(x). \tag{2.6}$$

The following property will be effectively used to show the reduction theorem.

Proposition 2.2. *For any mapping $f \in D_\beta$ of the form* (2.6)

$$\inf_{x \in R^1} |\, q(x)| = 0$$

holds i.e., for any positive ε there exists a point $z = z_\varepsilon \in R^1$ such that $|q(z)| \le \varepsilon$ holds.

Proof. When $q(x)$ takes both positive and negative values, then by the continuity of $q(x)$ there exists z such that $q(z) = 0$ so that the conclusion is clear. Hence we suppose $q(x) > 0$ for all $x \in R^1$. Assume that the conclusion does not hold. Then there exists a positive constant c such that $q(x) \geq c$ for all $x \in R^1$. Since from (2.3)

$$\frac{f^n(x) - x}{n} = \frac{q(x) + q \circ f(x) + \cdots + q \circ f^{n-1}(x)}{n} + \omega$$

holds and the upper rotation number of f at x_0 is equal to ω, we have

$$\limsup_{n \to \infty} \frac{q(x_0) + q \circ f(x_0) + \cdots + q \circ f^{n-1}(x_0)}{n} = 0.$$

On the other hand, from the above assumption, we have

$$\limsup_{n \to \infty} \frac{q(x_0) + q \circ f(x_0) + \cdots + q \circ f^{n-1}(x_0)}{n} \geq c > 0.$$

This leads to a contradiction. In case of negative q we can show contradiction in the similar way. □

3. Reduction problem of nearly affine mapping with quasiperiodic perturbation

In this section we shall formulate the Reduction Theorem of a quasiperiodic DS with small perturbation.

Consider a mapping $Q \in D_\beta$ of the form (2.1):

$$Q(x) = x + g(x).$$

Here $g(x)$ is a $2\pi/\beta$-q.p. function. We shall assume that the corresponding function $\hat{g}(\theta)$ is a real analytic function defined in a strip $|\Im\theta_i| \leq r$, $i = 1, \ldots, m$. Here $\Im z$ is the imaginery part of $z \in C^1$, where C^1 is one dimendional complex plane. Let a_0 be a point in R^1 and denote the upper rotation number $\bar{\rho}(Q)(a_0)$ by $\omega = \omega(a_0)$. Then we rewrite Q as a nearly affine mapping from $\{x \in C^1 : |\Im x| \leq \tilde{r}\}$ to C^1 in the following form:

$$x_1 = Q(x) = x + \omega + q(x). \tag{3.1}$$

Clearly $q(x)$ is a $2\pi/\beta$-q.p. function whose corresponding function $\hat{q}(\theta)$ is a real analytic function defined in a strip $|\Im\theta_i| \leq r, i = 1, \ldots, m$.

Our reduction problem is the following: *By a suitable transformation of the variable x to ξ*

$$x = H(\xi) = \xi + h(\xi), \tag{3.2}$$

where h is a $2\pi/\beta$-q.p. function and $\hat{h}(\theta)$ is a real analytic function, reduce (3.1) *to an affine mapping*

$$\xi_1 = H^{-1} \circ Q \circ H(\xi) = R(\xi) = \xi + \omega. \tag{3.3}$$

We introduce notation and definitions. Let C^m be the m-dimensional complex Euclidean space. For $\theta = (\theta_1, \ldots, \theta_m) \in C^m$ we set $|\Im\theta| = \max_{1\le j\le m} |\Im\theta_i|$. Let Π_r and $\hat{\Pi}_r$ be sets $\{\theta \in C^m : |\Im\theta| \le r\}$ and $\{\theta \in C^m : |\Im\theta| < r\}$ (resp.). Let $f(\theta)$ be 2π-periodic in each θ_i and real analytic in Π_r. Set $\partial_\theta f(\theta) = (\partial_{\theta_1} f(\theta), \ldots, \partial_{\theta_m} f(\theta))$. We define the norms

$$|f|_r = \max_{\theta\in\Pi_r} |f(\theta)|, \quad |\partial_\theta f|_r = \max_{1\le j\le m} \max_{\theta\in\Pi_r} |\partial_{\theta_j} f(\theta)|.$$

For $\beta = (\beta_1, \cdots, \beta_m)$ set $\|\beta\| = \max_{1\le i\le m} |\beta_i|$. Let $F(x)$ and $G(x)$ be any $2\pi/\beta$-q.p. functions whose corresponding functions $\hat{F}(\theta)$ and $\hat{G}(\theta)$ are real analytic in Π_r. Then clearly one has

$$|\hat{F}\,\hat{G}|_r \le |\hat{F}|_r\,|\hat{G}|_r, \quad |(dF/dx)\hat{}\,|_r \le m\|\beta\|\,|\partial_\theta \hat{F}|_r.$$

Now we shall formulate the reduction theorem. Consider a mapping Q of the form (3.1).

We assume the following number-theoretic condition.

(**C**) There exists a point $a_0 \in R^1$ such that $\omega = \omega(a_0) = \bar{\rho}(Q)(a_0)$ and $\beta = (\beta_1, \ldots, \beta_m)$ satisfy the following Diophantine condition: There exists a positive constant C_0 depending on β such that

$$|(k, \beta) + \pi l/\omega| > \frac{C_0}{|k|^{m+1}}$$

holds for all $k \in Z^m \setminus \{0\}$ and all $l \in Z$.

Reduction Theorem. *Consider a mapping* (3.1) *with* $\omega = \bar{\rho}(Q)(a_0)$, *where* $q(x)$ *is a* $2\pi/\beta$*-q.p. function with* $\hat{q}(\theta)$ *real analytic in* $\hat{\Pi}_r$ *and continuous in* Π_r. *Assume that* (C) *holds. Then there exists a positive constant* M^0 *dependent on* C_0, r *such that if* $|\hat{q}|_r \le M^0$ *holds, then the mapping* (3.1) *is reduced to the affine mapping* (3.3) *by a transformation* (3.2) *with a* $2\pi/\beta$*-q.p. term* $h(\xi)$ *with* $\hat{h}(\theta)$ *real analytic in* $\hat{\Pi}_{r/2}$.

The proof of this theorem is done in Section 7.

As a corollary of the reduction theorem we have the existence of a rotation number of a mapping Q independent of x.

Corollary. *Consider a mapping* (3.1) *with* $\omega = \bar{\rho}(Q)(a_0)$. *Under the same assumptions of the reduction theorem the mapping* (3.1) *has a rotation number independent of* $x \in R^1$. *In other words,* $\rho(Q) = \bar{\rho}(Q)(x) = \omega(x)$ *holds for any* $x \in R^1$.

Proof of Corollary. By the above theorem the mapping Q is reduced to the affine mapping by a transformation $H \in D_\beta$:

$$H^{-1} \circ Q \circ H(\xi) = \xi + \omega.$$

Since the rotation number of $\xi+\omega$ is ω, from Proposition 2.1 we have $\rho(Q)=\omega$ independent of x. □

4. Quasiperiodic solutions of IBVP (1.1)–(1.3)

We shall deal with IBVP (1.1)–(1.3) with the following assumptions on the boundary functions $a_i(t)$, the boundary values $r_i(t)$, $i=1,2$, the nonhomogeneous term $h(x,t)$ and the initial values $\phi(x),\psi(x)$.

(C1) $a_i(t)$, $i=1,2$, are η-q.p. functions, where $\eta\in R^m$. $\hat{a}_i(\theta)$ are real analytic, and satisfy $0<\inf_{\theta\in R^m}\hat{a}_2(\theta)-\sup_{\theta\in R^m}\hat{a}_1(\theta))$ and $|\hat{a'}_i(\theta)|<1$ for $\theta\in R^m$. $a_i(t)$ satisfy $a_i'(0)=a_i''(0)=0$, $i=1,2$, $a_1(0)=0$.

(C2) $r_i(t)$, $i=1,2$, are α_i-q.p. functions, where $\alpha_i\in R^{m_i}$ (resp.). $\hat{r}_i(\theta)$ are C^∞-function satisfying $\int_{T^{m_i}}\hat{r}_i(\theta)d\theta=0$. $r_i(t)$ satisfy $r_i(0)=r_i'(0)=r_i''(0)=0$, $i=1,2$.

(C3) $h(x,t)$ is a μ-q.p. function, where μ belongs to R^p. $\hat{h}(x,\theta)$ is of C^∞ in D, and the support of h is contained in the cylinder $W=(\sup_{t\in R^1}a_1(t),\inf_{t\in R^1}a_2(t))\times R^1$.

Remark 3. (1) $a_i'(0)=a_i''(0)=0$, $i=1,2$, $a_1(0)=0$ in (C1) and $r_i(0)=r_i'(0)=r_i''(0)=0$, $i=1,2$ in (C2) are compatibility conditions with the latter part of (C6) below.
(2) Note that

$$\sup_{t\in R^1}a_1(t)=\sup_{\theta\in R^m}\hat{a}_1(\theta),\quad \inf_{t\in R^1}a_2(t)=\inf_{\theta\in R^m}\hat{a}_2(\theta)$$

hold. For, the flow $\{\eta t: t\in R^1\}$ is dense in T^m from the Weyl Theorem.
(3) For the same reason as (2) it follows immediately that

$$|a_i'(t)|\le\sup|a_i'(t)|=\sup|\hat{a'}_i(\theta)|<1$$

for all $t\in R^1$.

By $|a_i'(t)|<1$ for $t\in R^1$ in (C1) the composed functions $A_i=(I+a_i)\circ(I-a_i)^{-1}$, $i=1,2$, are defined. We define $A(x)$ by $A_1^{-1}\circ A_2(x)$. Then $A(x)$ belongs to D_β. Let ω be an upper rotation number of A. Then $A(x)$ is represented by

$$A(x)=x+\omega+q(x). \tag{4.1}$$

Here $q(x)$ is an η-q.p. function. As is shown in the same way as in Lemma 5.2, the corresponding function $\hat{q}(\theta)$ of $q(x)$ is real analytic in $\{\theta\in C^m;|\Im\theta_i|\le r_0, i=1,\dots,m\}$ for some positive constant r_0.

Remark 4. Without loss of generality we can assume that the basic periods of $a_1(t)$ and $a_2(t)$ are the same. Otherwise, we take the maximal rationally independent set from the basic frequencies of $a_1(t)$ and $a_2(t)$ as the common basic frequencies of $a_1(t)$ and $a_2(t)$. We shall show that the qualitative property of the solutions is determined by A.

The following proposition is important. It assures the existence of infinitely many boundary functions $a_1(t)$, $a_2(t)$ that satisfy both the analytical condition (C1) and the number-theoretic conditions (C4) and (C5).

Proposition 4.1. *Let ω be any positive number. Then for any small ε there exist infinitely many real analytic $a_1(t)$ and $a_2(t)$ satisfying $\inf_t a_2(t) > \sup_t a_1(t)$ such that A has the rotation number ω and $q(x)$ satisfies $|\hat{q}|_r < \varepsilon$. The set of such functions $a_1(t)$, $a_2(t)$ has a continuum cardinal number.*

Proof. Let $R_\omega(x)$ be a constant rotation $x + \omega$. Let us take

$$A = \phi^{-1} \circ R_\omega \circ \phi$$

for any real analytic $\phi = I + \psi \in D_\beta$. Then clealy A is an element of D_β. It follows from Proposition 2.1 that the rotation number of A is equal to ω. Let us write A in the form (4.1)

$$A(x) = x + \omega + q(x),$$

where $q(x) = \psi(x) - \psi \circ A(x)$. Then we take ψ so small that $q(x) \in D_\beta$ satisfies $|\hat{q}|_r < \varepsilon$.

We shall show the existence of $a_i(t)$, $i = 1, 2$, satisfying $A(x) = A_1^{-1} \circ A_2(x)$ with $A_i = (I + a_i) \circ (I - a_i)^{-1}$, $i = 1, 2$. Simple calculations show that $a_i(t)$, $i = 1, 2$, satisfy

$$\begin{aligned} q(x) = {} & 2a_2 \circ (I - a_2)^{-1}(x) \\ & - 2a_1 \circ (I + a_1)^{-1} \circ (I + 2a_2 \circ (I - a_2)^{-1})(x) - \omega. \end{aligned}$$

We take ψ sufficiently small so that q may satisfy $|q'(x)| < \varepsilon$. We take $a_1(t)$ suitably small, say $|a_1(t)| + |a_1'(t)| < \varepsilon/2$. By setting $t = (I - a_2)^{-1}(x)$, we obtain a functional equation with respect to $a_2(t)$

$$\begin{aligned} a_2(t) = \frac{1}{2}\Big(& \omega + q \circ (I - a_2)(t) \\ & + 2a_1 \circ (I + a_1) \circ (I + a_2)(t)\Big). \end{aligned} \tag{4.2}$$

We define a function $G(z, t)$ by

$$G(z, t) = z - \frac{1}{2}\Big(\omega + q(t - z) + 2a_1 \circ (I + a_1)(t + z)\Big).$$

Then since we have

$$G_z(z, t) = 1 - \frac{1}{2}\Big(- q'(t - z) + 2a_1' \circ (I + a_1)(t + z)(1 + a_1'(t + z))\Big),$$

it follows that

$$|G_z(z,t)| \geq 1 - \frac{1}{2}\left(\varepsilon + \varepsilon\left(1 + \frac{\varepsilon}{2}\right)\right) > \frac{1}{2}$$

for small ε. Hence it follows from the implicit function theorem that there exists a solution $a_2(t)$ of the equation (4.2). Taking ε suitably small as $6\varepsilon < \omega$, we have, from (4.2),

$$a_2(t) > \omega/2 - \frac{\varepsilon}{2} > \frac{\varepsilon}{2} > a_1(t)$$

for all $t \in R^1$. Since $a_1(t)$ and $q(t)$ are β-q.p., $a_2(t)$ also is β-q.p. Thus the proposition is proved. □

Remark 5. If $h(x,t)$ identically vanishes, then the condition in (C1) $0 < \inf_t a_2(t) - \sup_t a_1(t)$ is not necessary.

For simplicity we set

$$\beta = \frac{2\pi}{\eta} = \left(\frac{2\pi}{\eta_1}, \ldots, \frac{2\pi}{\eta_m}\right),$$
$$\lambda_i = \frac{2\pi}{\alpha_i} = \left(\frac{2\pi}{\alpha_i^1}, \ldots, \frac{2\pi}{\alpha_i^{m_i}}\right),$$
$$\gamma = \frac{2\pi}{\mu} = \left(\frac{2\pi}{\mu_1}, \ldots, \frac{2\pi}{\mu_p}\right),$$

where $\alpha_i = (\alpha_i^1, \ldots, \alpha_i^{m_i})$ and $\mu = (\mu_1, \ldots, \mu_p)$.

The following Diophantine condition from number theory is essential in order that each solution is q.p. in t.

(C4) β, λ_1, λ_2, γ and ω satisfy the following Diophantine condition: There exists a positive constant C depending on $\beta, \lambda_1, \lambda_2, \gamma$ and ω such that

$$|(k_1,\lambda_1) + (k_2,\lambda_2) + (k,\beta) + (j,\gamma) + \pi l/\omega| > \frac{C}{(|k_1| + |k_2| + |k| + |j|)^{m+p+m_1+m_2+1}}$$

holds for all $(k_1, k_2, k, j) \in Z^{m_1+m_2+m+p} \setminus \{0\}$ and all $l \in Z$.

Remark 6. It is well-known in number theory that almost all vector

$$(\lambda_1, \lambda_2, \beta, \gamma, \omega) \in R^{m_1+m_2+m+p+1}$$

satisfy (C4). "Almost all" means the Lebesgue measure sense. We can construct such vectors as solutions of algebraic equation of order $m_1+m_2+m+p+1$. For the construction, see Appendix in [Ya4].

The following Diophantine condition is a special case of (C4), but is important to apply the reduction theorem in Section 3 to the composed function A.

(**C5**) β and ω satisfy the following Diophantine condition: There exists a positive constant C_0 depending on β such that

$$|(k,\beta)+\pi l/\omega| > \frac{C_0}{|k|^{m+1}}$$

holds for all $k \in Z^m \setminus \{0\}$ and all $l \in Z$.

Remark 7. Applying the method in [Ya4] to IBVP (1.1)-(1.3), we can show that for suitable $a_i(t)$, $r_i(t)$ and $h(x,t)$ not satisfying (C4) or (C5), the solutions grow up as some time sequence $\{t_j\}$ tends to infinity.

The initial values are assumed to have the following differentiability and the usual compatibility condition.

(**C6**) The initial data ϕ and ψ are of C^∞-class in $(a_1(0), a_2(0))$, and of C^2-class and C^1-class (resp.) in $[a_1(0), a_2(0)]$. ϕ, ϕ'' and ψ vanish at $x = a_1(0)$ and $x = a_2(0)$.

Our theorem is as follows.

Theorem 4.1. *Assume that* (C1)–(C6) *hold. There exists a positive constant* ϵ *dependent on* C_0, β *and* r_0 *such that if* $\sup_{|\Im\theta|\le r_0} |\hat{q}(\theta)| < \epsilon$ *holds, IBVP* (1.1)–(1.3) *has a unique* C^2*-solution* u *that is* $(\alpha_1, \alpha_2, \eta, \omega)$*-q.p. in* t *and* x. u *is represented by the sum of functions* u_0, u_1, u_2 *and* u_3 *satisfying the following properties*:

1. *Solutions of BVP* (1.1)–(1.2)
   (a) u_0 *satisfies*

$$\partial_t^2 u - \partial_x^2 u = 0, \ \ (x,t) \in R^2,$$
$$u(a_1(t),t) = u(a_2(t),t) = 0, \ \ t \in R^1.$$

   (b) u_1 *satisfies*

$$\partial_t^2 u - \partial_x^2 u = 0, \ \ (x,t) \in R^2,$$
$$u(a_1(t),t) = r_1(t), \ \ u(a_2(t),t) = 0, \ \ t \in R^1.$$

   (c) u_2 *satisfies*

$$\partial_t^2 u - \partial_x^2 u = 0, \ \ (x,t) \in R^2,$$
$$u(a_1(t),t) = 0, \ \ u(a_2(t),t) = r_2(t), \ \ t \in R^1.$$

   (d) u_3 *satisfies*

$$\partial_t^2 u - \partial_x^2 u = \tilde{h}(x,t), \ \ (x,t) \in R^2,$$
$$u(a_1(t),t) = u(a_2(t),t) = 0, \ \ t \in R^1.$$

   Here $\tilde{h}(x,t)$ *is an extension of* $h(x,t)$ *defined in* D *to* R^2 *seen in Section* 6, *Remark* 9.

2. *Regularity*
   (a) u_0 *is of* C^2*-class in* $(x,t) \in R^2$ *and of* C^∞*-class in* $(x,t) \in R^2 \backslash \bar{S}$, *where* $\bar{S} = \{(x,t) \in R^2\,;\, x+t = A_1 \circ A^n(\mu),\ -x+t = A^n(\mu),\ \mu = 0, a_2(0),\ n \in Z\}$.
   (b) u_i, $i = 1,2,3$, *is of* C^∞*-class in* $(x,t) \in R^2$.
3. *Quasiperiodicity*
   (a) u_0 *is* (ω, η)*-q.p. in both* t *and* x.
   (b) u_i, $i = 1,2$, *is* (α_i, η)*-q.p.(resp.) in both* t *and* x.
   (c) u_3 *is* (μ, η)*-q.p.(resp.) in* t.

The proof of this theorem will be given in Sections 5 and 6.

5. IBVP for homogeneous wave equation

In this section we shall be concerned with a homogeneous IBVP with quasiperiodic boundary condition. We shall construct u_0, u_1 and u_2 in Theorem 4.1. Our strategy will be along the line of [Ya3].

Consider IBVP for a homogeneous wave equation with q.p. boundary conditions

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) = 0, \ \ (x,t) \in D, \tag{5.1}$$

$$u(a_1(t),t) = r_1(t), \quad u(a_2(t),t) = r_2(t), \ \ t \in R^1, \tag{5.2}$$

$$u(x,0) = \phi(x), \ \ \partial_t u(x,0) = \psi(x), \ \ x \in [a_1(0), a_2(0)]. \tag{5.3}$$

Noting that $a_1(t) < a_2(t)$ and $|a_i'(t)| < 1$, $i = 1,2$, in (C1) hold, we obtain the d'Alembert representation formula of solutions of the one-dimensional homogeneous wave equation (5.1)

$$u(x,t) = f(-x+t) + g(x+t). \tag{5.4}$$

We shall show that f and g are determined so that (5.4) may satisfy the boundary condition (5.2) and the initial condition (5.3). First, from (5.2) we have

$$f(-a_i(t)+t) + g(a_i(t)+t) = r_i(t), \ \ i = 1,2. \tag{5.5}$$

From (5.4) and (5.5) we obtain

$$u(x,t) = f(-x+t) - f \circ A_1^{-1}(x+t) + r_1 \circ (I + a_1)^{-1}(x+t)$$

and

$$f \circ A(\tau) - f(\tau) = r_1 \circ (I + a_1)^{-1} \circ A_2(\tau) - r_2 \circ (I - a_2)^{-1}(\tau). \tag{5.6}$$

Here we have set $\tau = (I - a_2)^{-1}(t)$ to obtain (5.6).

1. *Construction of* $u_1(x,t)$ *and* $u_2(x,t)$

The following lemmas shall be used to solve the functional equation (5.6) with respect to f for the given functions a_i, $i = 1,2$.

Lemma 5.1. *Let $R(x)$ be a q.p. function with basic frequencies $\Xi = (\Xi_1, \dots, \Xi_\nu)$ whose corresponding function $\hat{R}(\theta)$ is of C^∞ and satisfies $\int_{T^\nu} \hat{R}(\theta)d\theta = 0$. Let γ be a constant in $R^1 \backslash \{0\}$. Assume that $\Xi \in R^\nu$ and γ satisfy the following Diophantine condition: There exist positive constants $C = C(\Xi, \gamma)$ and $\tau \geq 1$ such that the following inequality*

$$|(k, \Xi) - (\pi/\gamma)l| > \frac{C}{|k|^{\nu+\tau}}$$

holds for all $k \in Z^\nu \setminus \{0\}$ and all $l \in Z$. Here $(\cdot, \cdot)$ is the usual inner product in R^ν and $|k| = |k_1| + \cdots + |k_\nu|$ for $k = (k_1, \dots, k_\nu) \in Z^\nu$. Then a functional equation

$$G(x + \gamma) - G(x) = R(x)$$

has a q.p. solution $G(x)$ with basic frequencies Ξ. The corresponding function $\hat{G}(\theta)$ is of C^∞. $G(x)$ is the only q.p. solution with basic frequencies Ξ which satisfies $\int_{T^\nu} \hat{G}(\theta)d\theta = 0$.

This lemma and its proof are seen in [Ya1], Lemma 2.9.

Lemma 5.2. *Let $a(t)$ and $b(t)$ be $2\pi/\alpha$-q.p. and $2\pi/\beta$-q.p. (resp.) whose corresponding functions are of C^∞, where $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n$ are rationally independent. Then a composed function $a \circ (I + b)(t)$ is $(2\pi/\alpha, 2\pi/\beta)$-q.p. Its corresponding function $F(\theta_1, \dots, \theta_m, \Theta_1, \dots, \Theta_n)$ is of the form*

$$\hat{a}(\theta_1 + \alpha_1 \hat{b}(\Theta), \dots, \theta_m + \alpha_m \hat{b}(\Theta))$$

of C^∞. Moreover it holds

$$\int_{T^m} F(\theta, \Theta)d\theta = \int_{T^m} \hat{a}(\theta)d\theta.$$

Proof. Proof is similarly done by the argument in [Ya1, Lemma 2.8]. Clearly $F(\theta, \Theta) = \hat{a}(\theta_1 + \alpha_1 \hat{b}(\Theta), \dots, \theta_m + \alpha_m \hat{b}(\Theta)))$ is 2π-periodic in each variables. We have

$$F(\alpha t, \beta t) = \hat{a}(\alpha(t + \hat{b}(\beta t))) = a \circ (I + b)(t).$$

So F is a corresponding function of $a \circ (I + b)(t)$. Denote the Fourier expansion of $\hat{a}(\theta)$ by $\sum_{k \in Z^m} r_k e^{i(k,\theta)}$. Then we calculate

$$\begin{aligned}
\int_{T^m} F(\theta, \Theta)d\theta &= \int_{T^m} \hat{a}(\theta_1 + \alpha_1 \hat{b}(\Theta), \dots, \theta_1 + \alpha_1 \hat{b}(\Theta))d\theta \\
&= \int_{T^m} \sum_{k \in Z^m} r_k e^{i(k, \theta + \alpha \hat{b}(\Theta))} d\theta \\
&= \sum_k r_k e^{i(k,\alpha)\hat{b}(\Theta)} \int_{T^m} e^{i(k,\theta)} d\theta \\
&= (2\pi)^m r_0 = \int_{T^m} \hat{a}(\theta)d\theta.
\end{aligned}$$

This proves the Lemma. □

Since by (C1) and (C4) A satisfies the assumptions of the Reduction Theorem, it follows that there exists a real analytic function $H(\xi) = \xi + h(\xi)$, where $h(\xi)$ is an η-q.p. function, such that

$$H^{-1} \circ A \circ H(\xi) = \xi + \omega. \tag{R}$$

Consider the following functional equations

$$g_i(\xi + \omega) - g_i(\xi) = \tilde{r}_i(\xi),\ i = 1, 2,$$

where

$$\tilde{r}_1(\xi) = r_1 \circ (I + a_1)^{-1} \circ A_2 \circ H(\xi),$$
$$\tilde{r}_2(\xi) = -r_2 \circ (I - a_2)^{-1} \circ H(\xi).$$

Since a_i and r_i are α_i-q.p. and η-q.p. (resp.), it follows from Lemma 5.2 that $\tilde{r}_i(\xi), i = 1, 2$, are (α_i, η)-q.p. whose corresponding functions are of C^∞ class and have the 0-mean value. By this fact and the Diophantine inequality in (C3), it follows from Lemma 5.1 that each equation has a unique (α_i, η)-q.p. solution g_i whose corresponding function is of C^∞. It is simple to see that composed functions $f_i(\tau) = g_i \circ H^{-1}(\tau), i = 1, 2$, are of C^∞ in R^1 and are the solutions of Eq.s (resp.)

$$f_i \circ A(\tau) - f_i(\tau) = \tilde{r}_i \circ H^{-1}(\tau), \quad i = 1, 2.$$

We set

$$u_1(x, t) = f_1(-x + t) - f_1 \circ A_1^{-1}(x + t) + r_1 \circ (I + a_1)^{-1}(x + t) \tag{5.7}$$

and

$$u_2(x, t) = f_2(-x + t) - f_2 \circ A_1^{-1}(x + t). \tag{5.8}$$

Then clearly u_1 and u_2 are of C^∞ in $(x, t) \in R^2$ and satisfy equation

$$\partial_t^2 u - \partial_x^2 u = 0,\ (x, t) \in R^2.$$

Also it follows that the boundary conditions

$$u_1(a_1(t), t) = r_1(t),\ u_1(a_2(t), t) = 0,$$
$$u_2(a_1(t), t) = 0,\ u_2(a_2(t), t) = r_2(t)$$

hold. Since $g_i(\xi)$ and $h(\xi)$ are (α_i, η)-q.p. and η-q.p. (resp.), it follows from Lemma 5.2 that f_i is (α_i, η)-q.p. Hence $u_i(x, t)$ is (α_i, η)-q.p. (resp.) both in x and t. Thus the assertions 1.(b), 1.(c), 2.(b) and 3.(b) of Theorem are proved.

2. *Construction of* $u_0(x, t)$

We shall construct an ω-periodic function $g_0(\xi)$ so that the function u_0 defined by

$$u_0(x,t) = f_0(-x+t) - f_0 \circ A_1^{-1}(x+t), \quad f_0 = g_0 \circ H^{-1}$$

may satisfy the assertions 1.(a), 2.(a) and 3.(a) of Theorem.

Let γ_1 and γ_2 be equal to $A_1^{-1}(a_2(0))$ and $-a_2(0)$ (resp.). For ϕ_0 and ψ_0 satisfying (C5) define $\hat{\phi}_0(x)$ and $\hat{\psi}_0(x)$ by

$$\hat{\phi}_0(x) = \begin{cases} \phi_0 \circ A_1(x) & (0 \le x \le \gamma_1), \\ -\phi_0(-x) & (\gamma_2 \le x < 0) \end{cases} \tag{5.9}$$

and

$$\hat{\psi}_0(x) = \begin{cases} \psi_0 \circ A_1(x) A_1'(x) & (0 \le x \le \gamma_1), \\ -\psi_0(-x) & (\gamma_2 \le x < 0). \end{cases} \tag{5.10}$$

Define $\hat{f}_0(x)$ by

$$\hat{f}_0(x) = -(1/2)\left(\hat{\phi}_0(x) + \int_0^x \hat{\psi}_0(\eta)d\eta\right). \tag{5.11}$$

Then we define $\hat{g}_0$ by

$$\hat{g}_0(\xi) = \hat{f}_0 \circ H(\xi).$$

$\hat{g}_0(\xi)$ is defined in $[\xi_2, \xi_1]$, where ξ_i, $i = 1, 2$, is the solution of equation $H(\xi_i) = \gamma_i$.

Lemma 5.3. *Let ϕ_0 and ψ_0 satisfy* (C6). *Let $\hat{g}_0$ be the function defined as above. Then there exists an ω-periodic function $g_0(\xi)$ such that*
(a) *$g_0(\xi) = \hat{g}_0(\xi)$ for $\xi \in [\xi_2, \xi_1]$,*
(b) *$g_0(\xi)$ is of C^2 class in R^1,*
(c) *$g_0(\xi)$ is of C^∞ class in $R^1 \setminus W$, where W is a set $\{\xi_i + n\omega; i = 0, 1, n \in Z\}$ and $\xi_0 = H^{-1}(0)$.*

To show this lemma we prepare some lemmas.

Lemma 5.3.1. *$\hat{\phi}_0$ and $\hat{\psi}_0$ are of C^2 and C^1 class (resp.) in $[\gamma_2, \gamma_1]$ and of C^∞ class in $(\gamma_2, 0) \cup (0, \gamma_1)$.*

Proof. It is clear from (C1) and (C5) that $\hat{\phi}_0$ and $\hat{\psi}_0$ are of C^∞ class in $(\gamma_2, 0) \cup (0, \gamma_1)$, and of C^2 and C^1 class (resp.) in $[\gamma_2, 0) \cup (0, \gamma_1]$. We show that $\hat{\phi}_0$ is of C^2 class in $[\gamma_2, \gamma_1]$. We have

$$\lim_{x \to -0} \hat{\phi}_0''(x) = \lim_{x \to -0} (-\phi_0''(-x)) = -\phi_0''(0) = 0.$$

We have, for $x > 0$,

$$\hat{\phi}_0''(x) = \phi_0'' \circ A_1(x)(A_1'(x))^2 + \phi_0' \circ A_1(x)A_1''(x).$$

Since one has

$$A_1'(x) = \frac{1 + a_1' \circ (I - a_1)^{-1}(x)}{1 - a_1' \circ (I - a_1)^{-1}(x)},$$
$$A_1''(x) = \frac{2a_1'' \circ (I - a_1)^{-1}(x)}{(1 - a_1' \circ (I - a_1)^{-1}(x))^3},$$

it follows that

$$A_1(0) = A_1''(0) = 0, \quad A_1'(0) = 1.$$

Hence we obtain

$$\lim_{x \to +0} \hat{\phi}_0''(x) = 0.$$

Thus we obtain

$$\lim_{x \to +0} \hat{\phi}_0''(x) = \lim_{x \to -0} \hat{\phi}_0''(x) = 0.$$

Similar calculations show

$$\lim_{x \to +0} \hat{\psi}_0'(x) = \lim_{x \to -0} \hat{\psi}_0'(x) = \psi_0'(0).$$

This means the C^2 and C^1-differentiability of $\hat{\phi}_0$ and $\hat{\psi}_0$ (resp.) in $[\gamma_2, \gamma_1]$. □

Lemma 5.3.2. *$\hat{f}_0(x)$ is of C^∞ class in $(\gamma_2, 0) \cup (0, \gamma_1)$, and of C^2 class in $[\gamma_2, \gamma_1]$. $\hat{f}_0(\gamma_2) = \hat{f}_0(\gamma_1)$ holds.*

Proof. The former part of this lemma is the direct conclusion of Lemma 5.3.1. One has

$$\hat{f}_0(\gamma_1) = -\frac{1}{2}\left(\phi_0(a_2(0)) + \int_0^{\gamma_1} \psi_0 \circ A_1(\eta)A_1'(\eta))d\eta\right)$$
$$= -\frac{1}{2}\int_0^{a_2(0)} \psi_0(\xi)d\xi \quad (\xi = A_1(\eta)).$$

The right hand side is equal to $\hat{f}_0(\gamma_2)$. This implies the assertion. □

Lemma 5.3.3. *$\hat{g}_0(x)$ is of C^∞ class in $(\xi_2, \xi_0) \cup (\xi_0, \xi_1)$, and of C^2 class in $[\xi_2, \xi_1]$. $\hat{g}_0(\xi_2) = \hat{g}_0(\xi_1)$ holds.*

This lemma is clear from the previous lemma.

Lemma 5.3.4. *$\xi_1 - \xi_2 = \omega$ holds.*

Proof. We have

$$H(\xi_2+\omega) = A \circ H(\xi_2) = A_1^{-1}(a_2(0)) = H(\xi_1).$$

This implies $\xi_1 = \xi_2 + \omega$. □

Proof of Lemma 5.3. From Lemma 5.3.3 and Lemma 5.3.4 it follows that $\hat{g}_0$ is extended to an ω-periodic continuous function g_0 in R^1 which is of C^∞ in $R^1 \setminus W$. We show that g_0 is of C^2 in R^1. To this end we show

$$\lim_{\xi\to\xi_1-0} \hat{g_0}'(\xi) = \lim_{\xi\to\xi_2+0} \hat{g_0}'(\xi), \tag{5.12}$$

$$\lim_{\xi\to\xi_1-0} \hat{g_0}''(\xi) = \lim_{\xi\to\xi_2+0} \hat{g_0}''(\xi). \tag{5.13}$$

First we have

$$\hat{g_0}'(\xi_i) = \hat{f_0}'(\gamma_i)H'(\xi_i),\ i=1,2.$$

Differentiating $H(\xi+\omega) = A\circ H(\xi)$ and setting $\xi=\xi_2$, we have $H'(\xi_1) = A'(\gamma_2)H'(\xi_2)$. Using $A(\gamma_2) = \gamma_1, A_2'(\gamma_2) = 1$, we have $A'(\gamma_2) = 1/A_1'(\gamma_1)$, whence

$$H'(\xi_1) = \frac{H'(\xi_2)}{A_1'(\gamma_1)}. \tag{5.14}$$

From these and

$$\hat{f}_0'(\gamma_1) = -\frac{1}{2}\,\phi_0'(a_2(0))A_1'(\gamma_1), \quad \hat{f}_0'(\gamma_2) = -\frac{1}{2}\,\phi_0'(a_2(0))$$

we obtain (5.12).

Next we show (5.13). We have

$$\hat{g}_0''(\xi_i) = \hat{f}_0''(\gamma_i)H'(\xi_i)^2 + \hat{f}_0'(\gamma_i)H''(\xi_i),\ i=1,2. \tag{5.15}$$

we have

$$\begin{aligned}\hat{\phi}_0''(\gamma_1) &= \phi_0''(a_2(0))(A_1'(\gamma_1))^2 + \phi_0'(a_2(0))A_1''(\gamma_1)\\ &= \phi_0'(a_2(0))A_1''(\gamma_1),\\ \hat{\psi}_0'(\gamma_1) &= \psi_0'(a_2(0))A_1'(\gamma_1)^2 + \psi_0(a_2(0))A_1''(\gamma_1)\\ &= \psi_0'(a_2(0))A_1'(\gamma_1)^2.\end{aligned}$$

Here we have used (C6). Thus we obtain

$$\hat{f}_0''(\gamma_1) = -\frac{1}{2}\left(\phi_0'(a_2(0))A_1''(\gamma_1) + \psi_0'(a_2(0))A_1'(\gamma_1)^2\right).$$

We also obtain

$$\hat{f}_0''(\gamma_2) = -\frac{1}{2}\,\psi_0'(a_2(0)).$$

Hence we have

$$\hat{f}_0''(\gamma_1) = \hat{f}_0'(\gamma_2)A_1''(\gamma_1) + \hat{f}_0''(\gamma_2)A_1'(\gamma_1)^2. \tag{5.16}$$

Differentiating $H(\xi + \omega) = A \circ H(\xi)$ twice and setting $\xi = \xi_2$, we have

$$H''(\xi_1) = A''(\gamma_2)H'(\xi_2)^2 + A'(\gamma_2)H''(\xi_2).$$

Simple computations give

$$A''(\gamma_2) = \frac{A_2''(\gamma_2)}{A_1'(\gamma_1)} - \frac{A_1''(\gamma_1)}{A_1'(\gamma_1)^3}.$$

From (C1) we have $a_2'' \circ (I - a_2)^{-1}(\gamma_2) = 0$, which leads to

$$A_2''(\gamma_2) = 2a_2'' \circ (I - a_2)^{-1}(\gamma_2)/(1 - a_2' \circ (I - a_2)^{-1}(\gamma_2))^3 = 0.$$

This implies

$$A''(\gamma_2) = -\frac{A_1''(\gamma_1)}{A_1'(\gamma_1)^3}.$$

Hence we have

$$H''(\xi_1) = -\frac{A_1''(\gamma_1)}{A_1'(\gamma_1)^3}\, H'(\xi_2)^2 + \frac{1}{A_1'(\gamma_1)}\, H''(\xi_2). \tag{5.17}$$

From (5.15) and (5.16) we have

$$\begin{aligned}\hat{g}_0''(\xi_1) &= \left(\hat{f}_0'(\gamma_2)A_1''(\gamma_1) + \hat{f}_0''(\gamma_2)A_1'(\gamma_1)^2\right) H'(\xi_1)^2 + \hat{f}_0'(\gamma_2)A_1'(\gamma_1)H''(\xi_1)\\ &= \hat{f}_0''(\gamma_2)A_1'(\gamma_1)^2 H'(\xi_1)^2 + \hat{f}_0'(\gamma_2)\left(A_1''(\gamma_1)H'(\xi_1)^2 + A_1'(\gamma_1)H''(\xi_1)\right).\end{aligned}$$

(5.14) and (5.17) give

$$\begin{aligned}A_1''(\gamma_1)H'(\xi_1)^2 + A_1'(\gamma_1)H''(\xi_1) &= \frac{A_1''(\gamma_1)H'(\xi_2)^2}{A_1'(\gamma_1)^2}\\ &\quad + A_1'(\gamma_1)\left[-\frac{A_1''(\gamma_1)}{A_1'(\gamma_1)^3}\, H'(\xi_2)^2 + \frac{1}{A_1'(\gamma_1)}\, H''(\xi_2)\right].\end{aligned}$$

This is equal to $H''(\xi_2)$. Therefore we obtain

$$\hat{g}_0''(\xi_1) = \hat{g}_0''(\xi_2).$$

Hence Lemma 5.3 is proved. □

Lemma 5.4. *Let ϕ_0 and ψ_0 satisfy* (C5). *Let $g_0(\xi)$ be the ω-periodic function in Lemma* 5.3. *Let $f_0(x)$ be defined by $f_0(x) = g_0 \circ H^{-1}(x)$. Then*

$$u_0(x,t) = f_0(-x+t) - f_0 \circ A_1^{-1}(x+t) \tag{5.18}$$

is the solution of IBVP

$$\partial_t^2 u - \partial_x^2 u = 0, \ \ (x,t) \in (a_1(t), a_2(t)) \times R^1, \tag{5.19}$$

$$u(a_1(t), t) = u(a_2(t), t) = 0, \ \ t \in R^1, \tag{5.20}$$

$$u(x,0) = \phi_0(x), \ \ \partial_t u(x,0) = \psi_0(x), \ \ x \in [a_1(0), a_2(0)], \tag{5.21}$$

and of C^2 in R^2 and of C^∞ in $R^2 \setminus \bar{S}$, where $\bar{S} = \{(x,t) \in R^2; x+t = A_1 \circ A^n(\mu), -x+t = A^n(\mu), \mu = -a_2(0), a_1(0), n \in Z\}$. u_0 is (ω,η)-q.p. both in t and x.

Proof. Since by Lemma 5.3 g_0 is of C^2 in R^1 and of C^∞ in $R^1 \setminus W$, $f_0(x)$ also is of C^2 and of C^∞ in $R^1 \setminus V$, where $V = HW = \{A^n(-a_2(0)), A^n(a_1(0)); n \in Z\}$. Thus $u_0(x,t)$ is of C^2 in R^2 and of C^∞ in $R^2 \setminus \bar{S}$. Clearly u_0 is the solution of $\partial_t^2 u - \partial_x^2 u = 0$ in R^2. We simply have

$$u_0(a_1(t), t) = f_0(-a_1(t)+t) - f_0 \circ A_1^{-1}(a_1(t)+t) = 0.$$

By the change of variables $\tau = H^{-1} \circ (I - a_2)(t)$ we also have

$$\begin{aligned} u_0(a_2(t), t) &= f_0(-a_2(t)+t) - f_0 \circ A_1^{-1}(a_2(t)+t) \\ &= g_0(\tau) - g_0 \circ H^{-1} \circ A \circ H(\tau) \\ &= g_0(\tau) - g_0(\tau+\omega) = 0. \end{aligned}$$

Therefore u_0 satisfies the boundary conditions. Next noting that

$$f_0(x) = \hat{f}_0(x), \quad x \in [-a_2(0), A_1^{-1}(a_2(0))],$$

we have, in $[a_1(0), a_2(0)] = [0, a_2(0)]$

$$u_0(x,0) = \hat{f}_0(-x) - \hat{f}_0 \circ A_1^{-1}(x).$$

Since $-x \in [\gamma_2, 0]$ and $A_1^{-1}(x) \in [0, \gamma_1]$, from (5.9), (5.10) and (5.11) we have

$$\begin{aligned} \hat{f}_0(-x) &= -\frac{1}{2}\left(-\phi_0(x) + \int_0^{-x} -\psi_0(-\eta)d\eta\right) \\ &= \frac{1}{2}\left(\phi_0(x) - \int_0^x \psi_0(\xi)d\xi\right), \\ \hat{f}_0 \circ A_1^{-1}(x) &= -\frac{1}{2}\left(\hat{\phi}_0 \circ A_1^{-1}(x) + \int_0^{A_1^{-1}(x)} \hat{\psi}_0(\eta)d\eta\right) \\ &= -\frac{1}{2}\left(\phi_0(x) + \int_0^{A_1^{-1}(x)} \psi_0 \circ A_1(\eta)A_1'(\eta)d\eta\right) \\ &= -\frac{1}{2}\left(\phi_0(x) + \int_0^x \psi_0(\xi)d\xi\right). \end{aligned}$$

This gives $u_0(x,0) = \phi_0(x)$. Differentiating (5.18) with respect to t and setting $t=0$, we have

$$\partial_t u_0(x,0) = \hat{f}_0'(-x) - \hat{f}_0' \circ A_1^{-1}(x)(A_1^{-1})'(x).$$

Since one has

$$\hat{f}_0'(-x) = -\frac{1}{2}\,(\phi_0'(x) - \psi_0(x)),$$
$$\hat{f}_0' \circ A_1^{-1}(x) = -\frac{1}{2}\,(\phi_0'(x) + \psi_0(x))A_1' \circ A_1^{-1}(x).$$

Therefore we obtain $\partial_t u_0(x, 0) = \psi_0(x)$. □

Lemma 5.5. *Let $u_i, i = 1, 2$, be the functions defined by* (5.7) *and* (5.8). *Let $u_i(x, 0)$ and $\partial_t u_i(x, 0), i = 1, 2$, be denoted by $\phi_i(x)$ and $\psi_i(x)$, $i = 1, 2$ (resp.). Then the restrictions of ϕ_i and $\psi_i, i = 1, 2$ to $[a_1(0), a_2(0)]$ satisfy* (C6).

Proof. Since $u_i(x, t), i = 1, 2$ are of C^∞ in R^2, $\phi_i(x)$ and $\psi_i(x), i = 1, 2$ are of C^∞ in R^1. So we have only to show that all $\phi_i(x), \phi_i''(x)$ and $\psi_i(x), i = 1, 2$ vanish at $x = a_1(0), a_2(0)$. We simply have

$$\phi_1(a_i(0)) = u_1(a_i(t), t)|_{t=0} = \delta_{i1} r_1(0) = 0,\ \ i = 1, 2,$$

where δ_{ij} is the Kronecker delta. We also have

$$\psi_1(a_i(0)) = \partial_t u_1(a_i(0), 0) = \partial_t u_1(a_i(t), t)|_{t=0}.$$

By differentiation of $u_1(a_i(t), t) = \delta_{i1} r_1(t)$ with respect to t, we have

$$\partial_x u_1(a_i(t), t)a_i'(t) + \partial_t u_1(a_i(t), t) = \delta_{i1} r_1'(t).$$

Setting $t = 0$ and using $a_i'(0) = r_1'(0) = 0$ from (C1) and (C2), we obtain $\psi_1(a_i(0)) = 0, i = 1, 2$. Next we differentiate $u_1(a_i(t), t) = \delta_{i1} r_1(t)$ twice. Then we have

$$\partial_x^2 u_1(a_i(t), t)a_i'(t)^2 + 2\partial_t\partial_x u_1(a_i(t), t)a_i'(t) + \partial_x u_1(a_i(t), t)a_i''(t) + \partial_t^2 u_1(a_i(t), t) = \delta_{i1} r_1''(t).$$

Setting $t = 0$ and using $a_i'(0) = a_i''(0) = r_1''(0) = 0$ and $\partial_t^2 u_1(x, t) = \partial_x^2 u_1(x, t)$, we have $\partial_x^2 u_1(a_i(0), 0) = 0$, which implies $\phi_1''(a_i(0)) = 0$. Similarly we can prove $\phi_2(a_i(0)) = \phi_2''(a_i(0)) = 0$. □

6. BVP for nonhomogeneous wave equation

We shall consider BVP (1.1)–(1.2) with $r_i(t) \equiv 0$ for the nonhomogeneous equation. We shall show the existence of q.p. solutions of BVP. In order to deal with this problem, we shall introduce a fine domain transformation of the noncylindrical domain $\bar{D}$ onto the cylindrical domain $E = [0, \omega/2] \times R^1$. This transformation preserves the d'Alembert operator and does not produce any lower order differential operators.

We consider BVP for a nonhomogeneous wave equation with the zero Dirichlet boundary condition

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) = h(x,t), \ \ (x,t) \in D, \tag{1.1}$$

$$u(a_1(t),t) = 0, \quad u(a_2(t),t) = 0, \ \ t \in R^1. \tag{1.2$'$}$$

Now let H be the function in (R) in section 5. Let X be a mapping of R^2 to R^2 defined by

$$y = \frac{1}{2}\left(H^{-1} \circ A_1^{-1}(-x+t) - H^{-1}(x+t)\right), \tag{6.1}$$

$$s = \frac{1}{2}\left(H^{-1} \circ A_1^{-1}(-x+t) + H^{-1}(x+t)\right). \tag{6.2}$$

Such transformations were considered in [Ya5], [Ya-Yo] in case where $A(t)$ is a periodic DS. Without any difficulty, we are able to extend the transformations for periodic DS to one for quasiperiodic DS $A(t)$ due to the Reduction Theorem in Section 3. Similarly to Propositions 4.1 and 4.2 in [Ya5], we are able to show the following Proposition 6.1. We assume

(**C1′**) $a_i(t)$, $i = 1,2$, are η-q.p. functions, where η belongs to R^m. $\hat{a}_i(\theta)$ are of C^2 in T^m. $a_i(t)$ satisfy $0 < \inf_{t\in R^1} a_2(t) - \sup_{t \in R^1} a_1(t))$ and $|a_i'(t)| < 1$ for $t \in R^1$.

(**R1**) A is reducible in the following sense: there exists a conjugate function $H \in D_\beta$, where $\hat{H}$ and $(H^{-1})\hat{}$ are of C^2 in T^m, such that

$$H^{-1} \circ A \circ H(x) = x + \omega. \tag{R}$$

Remark 8. Assume (C1) and (C5). Then it follows from the Reduction Theorem that there exists a positive constant ε such that for q with $|q|_r < \varepsilon$ A is reducible by a real analytic conjugate function H. Hence (C1′) and (R1) are satisfied.

Proposition 6.1. *Assume* (C1′) *and* (R1). *The mapping X defined by* (6.1)–(6.2) *is the bijection of $\bar{D}$ to $\bar{E}$, and maps the boundaries $x = a_1(t)$ and $x = a_2(t)$ onto the boundaries $y = 0$ and $y = \omega/2$ (resp.) bijectively. Moreover the d'Alembert operator is preserved by X as follows. Let $u(x,t)$ be of C^2 in $(x,t) \in R^2$ and $v(y,s)$ be defined by $u(X^{-1}(y,s))$. Then the following identity holds*

$$(\partial_t^2 - \partial_x^2)u(x,t) = K(y,s)(\partial_s^2 - \partial_y^2)v(y,s), \tag{6.3}$$

where $K(y,s)$ is defined by

$$(H^{-1})' \circ H(y+s)(H^{-1})' \circ H(-y+s)(A_1^{-1})' \circ A_1 \circ H(y+s).$$

$K(y,s)$ is η-q.p. in s and its corresponding function $\hat{K}(y,\theta)$ is real analytic.

We apply X to BVP (1.1)–(1.2′). Then we obtain BVP in the cylindrical domain E:

$$\partial_s^2 v(y,s) - \partial_y^2 v(y,s) = g(y,s), \quad (y,s) \in E, \tag{6.4}$$

$$v(0,s) = v(\omega/2,s) = 0, \quad s \in R^1. \tag{6.5}$$

Here $v(y,s) = u \circ X^{-1}(y,s)$ and $g(y,s) = (1/K(y,s))h \circ X^{-1}(y,s)$. Clearly $g(y,s)$ is (η,μ)-q.p. and its corresponding function is analytic in Π_r. We have the following proposition.

(C4′) β, γ and ω satisfy the following Diophantine condition: There exists a positive constant C depending on β, γ and ω such that

$$|(k,\beta) + (j,\gamma) + \pi l/\omega| > \frac{C}{(|k| + |j|)^{m+p+1}}$$

holds for all $(k,j) \in Z^{m+p} \setminus \{0\}$ and all $l \in Z$.

Proposition 6.2. *Assume* (C1), (C3) *and* (C4′). *Then BVP* (6.4)–(6.5) *has a* (η,μ)*-q.p. solution* $v(y,s)$*. The corresponding function* $\hat{v}(y,\theta)$ *of* v *is of* C^∞ *in* $(0,\omega/2) \times T^{m+p}$.

Proof. The proof is done in the similar way to [Ya5, Proposition 5.1]. Since $\hat{g}(y,\theta)$ is of C^∞ with respect to $(y,\theta) \in W = (0,\omega/2) \times T^{m+p}$, $\hat{g}(y,\theta)$ is expanded into the Fourier series:

$$\hat{g}(y,\theta) = \sum_{j\in Z^+,\, k\in Z^{m+p}} g_{jk}\, e_{jk}, \tag{6.6}$$

where $e_{jk} = \exp i(k,\theta)\, \sin(2\pi j/\omega)y$. We shall expand v into the Fourier series formally:

$$\hat{v}(y,\theta) = \sum_{j\in Z^+,\, k\in Z^{m+p}} v_{jk}\, e_{jk}. \tag{6.7}$$

Substitute (6.6) and (6.7) into (6.4) and compare the Fourier coefficients. Then we have

$$v_{ik} = \frac{g_{jk}}{(2\pi j/\omega)^2 - (k,\zeta)^2}, \tag{6.8}$$

where $\zeta = (\beta,\gamma) \in R^{m+p}$. By (C4′) we have

$$|v_{jk}| \le \text{Const.}\, |g_{jk}|\, |k|^{m+p}. \tag{6.9}$$

Since $\hat{g}(y,\theta)$ is of C^∞ with respect to (y,θ), the Fourier coefficients g_{ik} satisfy

$$|g_{jk}| \le \text{Const.}\, \frac{C_d}{(1+|k|+j)^d}, \tag{6.10}$$

where

$$C_d = \sup_{(y,\theta)\in W,\, |\alpha|+i\le d} |\partial_y^i\, \partial_\theta^\alpha \hat{g}(y,\theta)|$$

and Const. is a positive constant independent of g and d. From (6.9) and (6.10) we obtain

$$\begin{aligned} |v_{jk}| &\le \text{Const.}\, C_d \frac{|k|^{m+p}}{(1+|k|+j)^d} \\ &\le \text{Const.} \frac{C_d}{(1+|k|+j)^{d-(m+p)}}. \end{aligned}$$

Since d is taken arbitrarily large, this means that the Fourier series (6.7) converges and $\hat{v}$ is of C^∞ in W. □

It follows from Proposition 6.2 that $u_3(x,t) = v \circ X(x,t)$ is a (η,μ)-q.p. solution of BVP (1.1)–(1.2′) that is of C^∞ in D.

Lemma 6.1. *Let $u_3(x,0)$ and $\partial_t u_3(x,0)$ be denoted by $\phi_3(x)$ and $\psi_3(x)$ (resp.). Then ϕ_3 and ψ_3 satisfy* (C6)*.*

Proof. Clearly ϕ_3 and ψ_3 are of C^∞ in R^1. It follows from the proof of Proposition 6.2 that $v(y,s)$, $\partial_y^2 v(y,s)$ and $\partial_s v(y,s)$, $\partial_s^2 v(y,s)$ vanish at the boundaries $y=0$ and $y=\omega/2$. Simple calculation shows

$$\begin{aligned} \partial_t u_3(x,t) &= \partial_y v(y,s)\partial_t y(x,t) + \partial_s v(y,s)\partial_t s(x,t), \\ \partial_x^2 u_3(x,t) &= \partial_y^2 v(y,s)(\partial_x y(x,t))^2 + 2\partial_y\partial_s v(y,s)\partial_x y(x,t)\partial_x s(x,t) \\ &\quad + \partial_s^2 v(y,s)(\partial_x s(x,t))^2 \\ &\quad + \partial_y v(y,s)\partial_x^2 y(x,t) + \partial_s v(y,s)\partial_x^2 s(x,t). \end{aligned}$$

By $y(x,t)|_{x=a_1(t)} = 0$ and $a_1'(0)=0$ we have $\partial_t y(x,0)|_{x=a_1(0)} = 0$, whence

$$\psi_3(x)|_{x=a_1(0)} = \partial_t u_3(x,0)|_{x=a_1(0)} = 0.$$

Also since $\partial_x s(x,t) = -\partial_t y(x,t)$ and $\partial_t^2 y(x,t) = \partial_x^2 y(x,t)$ hold, we have

$$\begin{aligned} \partial_x s(x,0)|_{x=a_1(0)} &= -\partial_t y(x,0)|_{x=a_1(0)} = 0, \\ \partial_x^2 y(x,0)|_{x=a_1(0)} &= \partial_t^2 y(x,0)|_{x=a_1(0)} = 0. \end{aligned}$$

Hence we obtain

$$\partial_x^2 \phi_3(x)|_{x=a_1(0)} = \partial_x^2 u_3(x,0)|_{x=a_1(0)} = 0.$$

In case where $x = a_2(0)$ we are able to show (C6) in the same way. Thus the lemma is proved. □

Thus we have obtained the following proposition.

Proposition 6.3. *Assume* (C1), (C3) *and* (C4′). *Then BVP* (1.1)-(1.2′) *has a* (η, μ)*-q.p. solution* $u_3(x,t)$ *of* C^∞ *in* D. *Moreover* $\phi_3(x) = u_3(x,0)$ *and* $\psi_3(x) = \partial_t u_3(x,0)$ *satisfy* (C6).

Remark 9. From the proof of the above proposition the solution is extended to $R^1_x \times R^1_t$ if we extend $h(x,t)$ to $R^1_x \times R^1_t$ as $\tilde{h}(x,t) = g \circ \Phi(x,t)$.

We shall complete the proof of Theorem. Let ϕ_0 and ψ_0 be defined by

$$\phi_0 = \phi - (\phi_1 + \phi_2 + \phi_3), \quad \psi_0 = \psi - (\psi_1 + \psi_2 + \psi_3),$$

where ϕ, ψ are the initial values in (1.3) satisfying (C6), and $\phi_i, \psi_i, i = 1,2$ are defined in Lemma 5.5: $\phi_i(x) = u_i(x,0)$, $\psi_i(x) = \partial_t u_i(x,0)$. Then since ϕ_i and $\psi_i, i = 1,2$ and ϕ_3 and ψ_3 satisfy (C6) from Lemma 5.5 and Proposition 6.3 (resp.). Then ϕ_0 and ψ_0 also satisfy (C6). Therefore it follows from Lemma 5.4 that u_0 defined by (5.18) is the (ω, η)-q.p. solution of IBVP (5.19)–(5.21) and satisfies the regularity conditions 2.(a) of Theorem. Clearly $u = u_0 + u_1 + u_2 + u_3$ is the unique solution of IBVP (1.1)–(1.3). Thus we have proved the theorem.

7. Quasiperiodic solutions by the superposition of unbounded waves

As we have seen in the previous sections, the solutions of IBVP (1.1)–(1.3) are represented as the superposition of the forward waves and the backward waves that are almost periodic in t. The almost-periodicity of solutions is shown, provided that the Diopantine condition and the differentiability of a_i, r_i, h are supposed. In this section we shall construct r_i so that every almost periodic solution of IBVP (1.1)–(1.3) is the superposition of the forward wave and the backward wave that are *sequentially time-unbounded*. The order of the growth rate of the waves depends on the differentiability of r_i and the order of the Diophantine index. As we stated in Section 1, in [Ya4] for the fixed end case *i.e.,* the case where $a_1(t) = r_i(t) = 0$, $i = 1,2$, and $a_2(t)$ is equal to a constant, we have already constructed $h(x,t)$ so that every solution of IBVP (1.1)–(1.3) may be time-unbounded. Hence in this section we shall treat the case where $h(x,t)$ identically vanishes. By the similar number-theoretic arguments to [Ya4], we shall take appropriate basic frequencies, and then by use of the basic frequencies we shall construct $r_i(t)$ as lacunary Fourier series for which every solution of BVP (1.1)–(1.3) is the superposition of sequentially time-unbounded waves. In this section we shall assume that $a_i(t)$, $i = 1,2$, are periodic functions with the same period 1.

Consider IBVP for a linear homogeneous wave equation:

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) = 0, \quad (x,t) \in D, \tag{7.1}$$

$$u(a_1(t),t) = r_1(t), \quad u(a_2(t),t) = r_2(t), \quad t \in R^1, \tag{7.2}$$

$$u(x,0) = \phi(x), \;\; \partial_t u(x,0) = \psi(x), \quad x \in [a_1(0), a_2(0)]. \tag{7.3}$$

Here $a_i(t)$, $i = 1, 2$, are C^∞ periodic functions satisfying (1.4), and $r_i(t)$, $i = 1, 2$, are q.p. with basic frequencies $\lambda = (\lambda_1, \dots, \lambda_m)$ given later. For simplicity we take the periods of $a_i(t)$ as 1. The initial values ϕ, ψ satisfy (C6) in Section 4.

First we shall change IBVP (7.1)–(7.3) to IBVP that gives boudary values at fixed x in the same way in Section 6.

Consider the composed function $A(t)$ in (1.6). Since $a_i(t)$ are periodic, it follows from (1.4) that A is a periodic dynamical system in $D(T^1)$, where $D(T^1)$ is the set of all one-dimensional continuous and 1-periodic dynamical systems. Poincaré showed (see [Yoc]) that the rotation number of any one-dimensional continuous periodic dynamical system always and uniquely exist and is independent of t . Let the rotation number of A be ω. It is shown ([Ya1]) that ω is positive. Assume the following condition on ω:

(**AH**) ω satisfies the Diophantine condition: There exist positive constants c and δ such that the following Diopahantine inequality holds

$$|q\omega - p| \geq \frac{c}{q^\delta}$$

for any $p \in N$, $q \in N$.

It is well-known that the set of ω that satisfy (AH) for $\delta > 1$ is of full measure in R^1_+.

By (AH) we are able to apply the Herman-Yoccoz Theorem to $A(t)$ (see [Ya3]). Then we obtain

$$H^{-1} \circ A \circ H(\xi) = \xi + \omega$$

for some $H \in D(T^1)$. Then by the same argument as that of Section 6, using the domain transformation X of R^2 onto R^2 defined by

$$y = \frac{1}{2}\left(H^{-1} \circ A_1^{-1}(-x+t) - H^{-1}(x+t)\right), \tag{6.1}$$

$$s = \frac{1}{2}\left(H^{-1} \circ A_1^{-1}(-x+t) + H^{-1}(x+t)\right), \tag{6.2}$$

we transform D onto $K = (0, \omega/2) \times R^1$. Also by this transformation IBVP (7.1)–(7.3) becomes

$$\partial_t^2 v(y,s) - \partial_x^2 v(y,s) = 0, \quad (y,s) \in K, \tag{7.4}$$

$$v(0,s) = \rho_1(s), \quad v(\omega/2, s) = \rho_2(s), \quad s \in R^1, \tag{7.5}$$

$$v(y,0) = \phi_1(y), \quad \partial_t v(y,0) = \psi_1(y), \quad y \in (0, \omega/2), \tag{7.6}$$

where $v(y,s) = u \circ X^{-1}(y,s)$, $\rho_i(s)$ are q.p. functions with basic frequencies $(1, \lambda)$ whose corresponding function $\hat{\rho}_i(\theta)$ have the same smoothness as $\hat{r}_i$ in T^{m+1}, and ϕ_1, ψ_1 are C^∞ in $(0, \omega/2)$ and C^2 in $[0, \omega/2]$ and satisfy suitable compatibility conditions.

In order to show our assertion we shall need some number-theoretic arguments. We shall prepare some lemmas.

Lemma 7.1. *Let a be any natural number. Then there exist countably many a-dimensional real vectors $\zeta = (\zeta_1, \dots, \zeta_a)$; $\zeta_i > 0$, $i = 1, \dots, a$, with the following property: There exist positive constants C_0 and C_1, and a sequence $\{k_j\} \subset Z^a \setminus \{0\}$ with $|k_j| \to \infty \, (j \to \infty)$ such that*

$$\frac{C_0}{|k_j|^a} \le |(k_j, \zeta)| \le \frac{C_1}{|k_j|^a} \tag{7.7}$$

holds.

This lemma is simply proved by using [Ca, Theorem VI and VIII in Chapter I].

First we shall give the basic frequencies of $r_i(t)$, $i = 1, 2$, by Lemma 7.1. Let a be equal to $m + 1$ and $(\zeta_1, \dots, \zeta_{m+1})$ be the vector given in Lemma 7.1. We take $(\lambda_1, \dots, \lambda_m) = (\zeta_1/\zeta_{m+1}, \dots, \zeta_m/\zeta_{m+1})$ as the bacic frequencies of $r_i(t)$. Set $\Lambda = (\lambda, 1) \in R^{m+1}$. Then it follows from Lemma 7.1 that there exists a sequence $\{k_j\} \subset Z^{m+1} \setminus \{0\}$ such that

$$\frac{C_2}{|k_j|^{m+1}} \le |(k_j, \Lambda)| \le \frac{C_3}{|k_j|^{m+1}} \tag{7.8}$$

holds for any j. Here C_2 and C_3 are positive constants equal to C_0/ζ_{m+1} and C_1/ζ_{m+1} (resp.).

Remark 10. The real vectors $(\zeta_1, \dots, \zeta_{m+1})$ satisfying (7.7) are constructed as algebraic solutions of some algebraic equations of degree $m + 1$. There are infinitely many vectors in R^{m+1} that satisfy (7.7). See [Ca] and [Ya4].

The following lemma is shown similarly to [Ya4, Proposition 2.4].

Lemma 7.2. *There exists a subsequence of the above $\{k_j\}$ with the following properties. We again write the subsequence by $\{k_j\}$.*

1. *$0 < (k_j, \Lambda) \le 2\pi/3\omega$ for any $j \in N$.*
2. *$\{(k_j, \Lambda)\}$ is monotone decreasing: $(k_{j+1}, \Lambda) \le (k_j, \Lambda)$.*
3. *There exists a positive constant $M < 1$ such that $|k_j| \le M|k_{j+1}|$ holds for any $j \in N$.*

Let $\{k_j\}$ be the sequence given in Lemma 7.2. We define $\hat{f}(\theta)$ by a lacunary Fourier series

$$\hat{f}(\theta) = \sum_{j=1}^{\infty} f_j \cos{(k_j, \theta)}. \tag{7.9}$$

Let the Fourier coefficients f_j satisfy the following: There exists positive constasnts $c_i = c_i(f)$, $i = 1, 2$, such that

$$\frac{c_1}{|k_j|^N} \le f_j \le \frac{c_2}{|k_j|^N} \tag{7.10}$$

for a given $N \in Z_+$. We denote the set of such functions $\hat{f}(\theta)$ by $\hat{Q}^N$ and the set of q.p. functions $f(t)$ defined by $f(t) = \hat{f}(\Lambda t)$ by Q_Λ^N. Clearly Q_Λ^N is a subset of D_Λ.

Lemma 7.3. *Consider a functional equation*

$$F(t+\omega) - F(t) = G(t), \quad t \in R^1, \tag{7.11}$$

where $G(t)$ *is an element of* Q_Λ^N *with* $\int_{T^{m+1}} \hat{G}(\theta)d\theta = 0$. *Let* $m+1 > N$. *Then there exists a solution* $F(t)$ *of* (7.11) *of the form*

$$F(t) = S(t) + G(t)/2 + U(t) \tag{7.12}$$

that is unique except ω*-periodic functions* $U(t)$. $S(t)$ *satisfies the following: There exist positive constants* C, $\tilde{C}$ *and a sequence* $\{t_\nu\}$ *with* $t_\nu \to \infty$ $(\nu \to \infty)$ *such that*

$$C\, t_\nu^{1-N/(m+1)} \le S(t_\nu) \le \tilde{C}\, t_\nu^{1-N/(m+1)}, \quad \nu \in N. \tag{7.13}$$

The proof of this lemma is done in the similar way to that of [Ya4, Theorem 3.1].

Proof. Without loss of generality we assume $U(t) \equiv 0$. $\hat{G}(\theta)$ is expanded into the lacunary Fourier series

$$\hat{G}(\theta) = \sum_{j=1}^{\infty} G_j \cos(k_j, \theta). \tag{7.14}$$

We look for a solution of (7.11) in the Fourier series form. We formally expand $\bar{F}$ into the Fourier series

$$\bar{F}(\theta) = \sum_{j=1}^{\infty} \left(P_j \sin(k_j, \theta) + Q_j \cos(k_j, \theta)\right). \tag{7.15}$$

Substituting (7.14) and (7.15) into (7.11) and comparing the Fourier coefficients, we have

$$\begin{aligned} &P_j(\cos(k_j, \Lambda)\omega - 1) - Q_j \sin(k_j, \Lambda)\omega = 0, \\ &P_j \sin(k_j, \Lambda)\omega + Q_j(\cos(k_j, \Lambda)\omega - 1) = G_j. \end{aligned}$$

Solving this system, we obtain

$$\begin{aligned} P_j &= \frac{G_j}{2}\, \frac{\cos(k_j, \Lambda)\omega/2}{\sin(k_j, \Lambda)\omega/2}, \\ Q_j &= \frac{G_j}{2}, \quad j = 1, 2, \ldots. \end{aligned}$$

Take

$$\bar{S}(\theta) = \sum_{j=1}^{\infty} P_j \sin(k_j, \theta).$$

We shall show (7.13). Setting $\Lambda_1 = (\omega/2)\,\Lambda$, we have

$$S(t) = \sum_{j=1}^{\infty} \frac{G_j}{2} \frac{\cos{(k_j, \Lambda_1)}}{\sin{(k_j, \Lambda_1)}} \sin{(k_j, \Lambda)t}.$$

We decompose $S(t)$ into two sums:

$$\begin{aligned} S(t) &= \left(\sum_{j=1}^{n_0-1} + \sum_{j=n_0}^{\infty} \right) \frac{G_j}{2} \frac{\cos{(k_j, \Lambda_1)}}{\sin{(k_j, \Lambda_1)}} \sin{(k_j, \Lambda)t} \\ &\equiv I^1_{n_0}(t) + I^2_{n_0}(t). \end{aligned} \tag{7.16}$$

We take

$$t_\nu = \frac{\pi}{2}\,(k_\nu, \Lambda)^{-1} \tag{7.17}$$

for any $\nu \in N$. Then it follows from (7.8) that

$$c_4\,|k_\nu|^{m+1} < t_\nu < c_5\,|k_\nu|^{m+1}. \tag{7.18}$$

First we shall estimate $I^1_\nu(t_\nu)$. Set $L = \min_{0<x\leq\pi/2}(\sin x/x)$. Since by use of Lemma 7.2, no. 1 and (7.8) we have

$$\sin(k_j, \Lambda_1) \geq \frac{\omega\,L}{2}(k_j, \Lambda) \geq \frac{C_2\,\omega\,L}{2}\,\frac{1}{|k_j|^{m+1}},$$

it follows from Lemma 7.2, no. 3 and (7.10) that

$$\begin{aligned} |I^1_\nu(t_\nu)| &\leq \sum_{j=1}^{\nu-1} \left| \frac{G_j}{2} \right| \frac{1}{|\sin(k_j, \Lambda_1)|} \\ &\leq \frac{c_2}{C_2\omega L} \sum_{j=1}^{\nu-1} |k_j|^{m+1-N} \\ &\leq \frac{c_2}{C_2\omega L} |k_\nu|^{m+1-N} \sum_{j=1}^{\nu-1} M^{(m+1-N)j} \\ &\leq \frac{c_2}{C_2\omega L} |k_\nu|^{m+1-N} \frac{M^{m+1-N}}{1 - M^{m+1-N}}. \end{aligned} \tag{7.19}$$

Next we shall estimate $I^2_\nu(t_\nu)$:

$$I^2_\nu(t_\nu) = \sum_{j=\nu}^{\infty} \frac{G_j}{2} \frac{\cos(k_j, \Lambda_1)}{\sin(k_j, \Lambda_1)} \sin\left(\frac{\pi}{2}\,\frac{(k_j, \Lambda)}{(k_\nu, \Lambda)} \right).$$

Since $\{(k_j, \Lambda)\}$ satisfies $(k_j, \Lambda) \leq (k_\nu, \Lambda)$, $j \geq \nu$ from Lemma 7.2, we have

$$\sin\left(\frac{\pi}{2}\,\frac{(k_j, \Lambda)}{(k_\nu, \Lambda)} \right) \geq L \frac{\pi}{2}\,\frac{(k_j, \Lambda)}{(k_\nu, \Lambda)}. \tag{7.20}$$

Also we have

$$\cos(k_j, \Lambda_1) \geq \frac{1}{2}.$$

Therefore it follows from (7.20) and (7.10) that

$$I_\nu^2(t_\nu) \geq \frac{\pi L}{8} \sum_{j=\nu}^{\infty} G_j \frac{1}{\sin(k_j, \Lambda_1)} \frac{(k_j, \Lambda)}{(k_\nu, \Lambda)} \geq \frac{\pi c_1 L}{4\omega} \frac{1}{(k_\nu, \Lambda)} \sum_{j=\nu}^{\infty} |k_j|^{-N} \geq \frac{\pi c_1 L}{4\omega C_3} |k_\nu|^{m+1-N}. \tag{7.21}$$

It follows from (7.16), (7.19) and (7.21) that

$$S(t_\nu) \geq \left(\frac{\pi c_1 L}{4\omega C_3} - \frac{c_2}{C_2 \omega L} \frac{M^{m+1-N}}{1 - M^{m+1-N}} \right) |k_\nu|^{m+1-N}.$$

Thus by taking M sufficiently small and using (7.17) and (7.18), we obtain

$$S(t_\nu) \geq C\, t_\nu^{1-N/(m+1)},$$

where C is a positive constant independent of ν. This proves the former part of (7.13).

We shall show the latter part of (7.13). We have, from Lemma 7.2 and (7.8)

$$|I_\nu^2(t_\nu)| \leq \sum_{j=\nu}^{\infty} \left| \frac{G_j}{2} \right| \frac{1}{\sin(k_j, \Lambda_1)} \sin\left(\frac{\pi}{2} \frac{(k_j, \Lambda)}{(k_\nu, \Lambda)} \right) \leq \frac{\pi c_2}{2\omega L} \frac{1}{(k_\nu, \Lambda)} \sum_{j=\nu}^{\infty} \frac{1}{|k_j|^N} \leq \frac{\pi c_2}{2\omega L} \frac{1}{(k_\nu, \Lambda)} \frac{1}{|k_\nu|^N} \sum_{j=1}^{\infty} M^{jN} \leq \frac{\pi c_2}{2\omega L C_2} |k_\nu|^{m+1-N} \frac{M^N}{1 - M^N} \leq \left(\frac{\pi c_2}{2\omega L C_2} \frac{M^N}{1 - M^N} \right) t_\nu^{1-N/(m+1)}. \tag{7.22}$$

Therefore it follows from (7.19) and (7.22) that

$$S(t_\nu) \leq C_4\, t_\nu^{1-N/(m+1)}.$$

Thus Lemma 7.3 is proved. □

Now we show the quasiperiodicity of the solution $v(y,s)$ of IBVP (7.4)–(7.6). v is represented by the superposition of the forward wave and the backward wave by the d'Alembert formula (See Section 5)

$$v(y,s) = f(-y+s) - f(y+s) + \rho_1(y+s), \tag{7.23}$$

where $f(s)$ satisfies

$$f(s+\omega) - f(s) = \rho_1(s+\omega) - \rho_2(s+\omega/2) \equiv \rho(s). \tag{7.24}$$

The right hand side $\rho(s)$ is q.p. with basic frequencies Λ. We take ρ as an element of Q_Λ^N. We apply Lemma 7.3 to (7.24). Then there exists a solution $f(s)$ of the form

$$f(s) = S(s) + \rho(s)/2 + U(t),$$

where $S(s)$ satisfies the following: There exist a sequence $\{s_\nu\}$, $s_\nu \to \infty$, and positive constants C, $\tilde{C}$ such that

$$C\, s_\nu^{1-N/(m+1)} \le S(s_\nu) \le \tilde{C}\, s_\nu^{1-N/(m+1)} \tag{7.25}$$

holds. $U(t)$ is an ω-periodic function. $U(t)$ is determined by the initial data ϕ_1, ψ_1 in the similar but simpler way as g_0 in Section 5.

Next we shall show the quasiperiodicity of the solution v. Since we have

$$\begin{aligned} f(-y+s) - f(y+s) &= S(-y+s) - S(y+s) \\ &\quad + \big(\rho(-y+s) - \rho(y+s)\big)/2 + U(-y+s) - U(y+s), \end{aligned}$$

and the latter part is q.p. in s, we have only to estimate the former part:

$$\begin{aligned} |S(-y+s) - S(y+s)| &\le \sum_{j=1}^{\infty} \frac{G_j}{2} \frac{\cos(k_j,\Lambda_1)}{\sin(k_j,\Lambda_1)} \\ &\quad \times \Big| \sin(k_j,\Lambda)(-y+s) - \sin(k_j,\Lambda)(y+s) \Big| \\ &\le \sum_{j=1}^{\infty} G_j \frac{1}{\sin(k_j,\Lambda_1)} \Big| \sin(k_j,\Lambda) y \cos(k_j,\Lambda) s \Big| \\ &\le C_3 \sum_{j=1}^{\infty} \frac{1}{|k_j|^N} \\ &< +\infty. \end{aligned}$$

This means the quasiperiodicy of $S(-y+s) - S(y+s)$. Therefore our assertion is proved.

Since from (6.1) t is written of the form $s + w(y,s)$ with $w(y,s)$ periodic in both (y,s), t tends to ∞ as s tends to ∞. Since we have, from (6.1)–(6.2),

$$u(x,t) = v \circ X(x,t),$$

it follows that the solution $u(x,t)$ also is the superposition of the sequentially time-unbounded waves. Hence we have seen the conclusion.

8. Proof of the Reduction Theorem

In this section we shall prove the Reduction Theorem in Section 3. We shall deal with

$$x_1 = Q(x) = x + \omega + q(x), \tag{3.1}$$

and by a suitable transformation

$$x = H(\xi) = \xi + h(\xi) \tag{3.2}$$

reduce the mapping (3.1) to the affine mapping

$$\xi_1 = H^{-1} \circ Q \circ H(\xi) = R(\xi) = \xi + \omega. \tag{3.3}$$

Let κ and α be constants in $(0,1)$ and $(1,\infty)$ (resp.). We set $r_0 = r$ and

$$a_\alpha = \sum_{s=1}^{\infty} \frac{1}{s^\alpha},$$
$$d_0 = \min\left(\frac{r_0}{4a_\alpha}, \frac{1}{2a_\alpha}\right),$$
$$d_s = \frac{d_0}{s^\alpha}, \ s = 1, 2, \ldots .$$

We take α so as to satisfy $r_0 > 2d_0$. We define sequences $\{r_s\}, \{\rho_s\}, \{\zeta_s\}$ and $\{M_s\}$ by

$$r_{s+1} = r_s - 2d_s, \qquad \rho_s = r_s - d_s, \qquad \zeta_s = r_s - \frac{d_s}{2},$$
$$M_s = M^{(1+\kappa)^s}$$

for every $s = 0, 1, \ldots$ and $M = |\hat{q}|_r$. For later use we note that

$$r_{s+1} - \rho_s = -d_s,$$
$$\zeta_{s+1} = \rho_{s+1} + \frac{d_{s+1}}{2},$$
$$\rho_s - \zeta_{s+1} = d_s + \frac{d_{s+1}}{2}.$$

In this section all C are positive constants dependent on some or all of $m, \beta, \omega, \alpha, r$.

Consider a sequence of mappings $\{Q_s\}$ of the form

$$(T)_s \qquad x_{1s} = Q_s(x_s) = x_s + \omega + q_s(x_s), \ s = 0, 1, \ldots, \tag{8.1}$$

where q_s is a $2\pi/\beta$-q.p. function and ω is the upper rotation number $\bar{\rho}(Q)(a_0)$, the same constant in (3.1), and for $s = 0$ we set $x_0 = x, x_{10} = x_1, Q_0 = Q$ and $q_0 = q$, where x_1, x, Q, q are seen in (3.1). We shall successively construct a sequence $\{H_s\}$ of transformations in D_β of the variables x_s

$$x_s = H_s(x_{s+1}) = (I + h_s)(x_{s+1}), \; s = 0, 1, \dots,$$

where h_s is a $2\pi/\beta$-q.p. function and I is an identity, so that the mappings $(T)_s$ may become closer and closer to the affine mapping (3.3) by the successive transformations. It should be noted that in this process we shall keep the upper rotation number of Q_s fixed i.e., for every $s = 0, 1, \dots$ there exists $a_s \in R^1$ such that $\bar{\rho}(Q_s)(a_s) = \bar{\rho}(Q)(a_0) = \omega$. This is assured by Proposition 2.1.

We assume

(A_s) $\hat{q_s}(\theta)$ is real analytic in $\hat{\Pi}_{\rho_s}$, continuous in Π_{ρ_s} and satisfies

$$|\hat{q}_s|_{\rho_s} \leq M_s. \tag{8.2}$$

The following proposition is fundamental to prove Reduction Theorem.

Proposition 8.1. *Consider a mapping* (8.1). *Assume that* (C) *holds. Then there exists a positive constant* $M^0 = M^0(\kappa, \alpha, \|\beta\|, r, m, C_0)$ *independent of* s *such that for any* $M \in [0, M^0)$, *under the assumption* (A_s) *the mapping* $(T)_s$ *is transformed to*

$$(T)_{s+1} \qquad x_{1s+1} = Q_{s+1}(x_{s+1}) = x_{s+1} + \omega + q_{s+1}(x_{s+1}) \tag{8.3}$$

by a transformation with a $2\pi/\beta$*-q.p.term* h_s

$$x_s = H_s(x_{s+1}) = (I + h_s)(x_{s+1}) \tag{8.4}$$

with the following properties:

1. q_{s+1} *is a* $2\pi/\beta$*-q.p. function, and* $\hat{q}_{s+1}$ *is real analytic in* $\hat{\Pi}_{\rho_{s+1}}$, *continuous in* $\Pi_{\rho_{s+1}}$ *and satisfies*

$$|\hat{q}_{s+1}|_{\rho_{s+1}} \leq M_{s+1} = M_s^{1+\kappa}. \tag{8.5}$$

2. h_s *is a solution of the following functional equation*

$$h_s(x + \omega) - h_s(x) = q_s(x) - \nu_s, \tag{8.6}$$

where q_s *is expanded into the Fouriuer series*

$$q_s(x) = \sum_{n \in Z^m} q_s^n e^{i(n,\beta)x}, \quad q_s^n = \left(\frac{1}{2\pi}\right)^m \int_{T^m} \hat{q_s}(\theta) e^{-i(n,\theta)} d\theta, \tag{8.7}$$

and ν_s *is given by*

$$\nu_s = q_s^0 = \left(\frac{1}{2\pi}\right)^m \int_{T^m} \hat{q_s}(\theta) d\theta. \tag{8.8}$$

$\hat{h}_s$ is real analytic in $\hat{\Pi}_{r_{s+1}}$, continuous in $\Pi_{r_{s+1}}$ and satisfies

$$|\hat{h}_s|_{r_{s+1}} \leq \frac{M_s^{(5\kappa+3)/8}}{10} \leq \frac{M_s^\kappa}{10}, \tag{8.9}$$

$$|D_\theta \hat{h}_s|_{r_{s+1}} \leq \frac{M_s^{(5\kappa+3)/8}}{10} \leq \frac{M_s^\kappa}{10}. \tag{8.10}$$

In the following lemmas in this section M is taken suitably small.

Proof of Proposition 8.1. Proof of Proposition 8.1 shall be done by several steps.

1. *Estimates of $\hat{h}_s$ and $\partial_\theta \hat{h}_s$*

We show (8.9) and (8.10). Consider the functional equation (8.6) with (8.7) and (8.8). Setting

$$h_s(x) = \sum_{n \in Z^m} h_s^n e^{i(n,\beta)x}, \tag{8.11}$$

we obtain from (8.6), (8.7) and (8.11)

$$h_s^n = \frac{q_s^n}{e^{i(n,\beta)\omega} - 1} \tag{8.12}$$

for every $n \neq 0$. h_s^0 is arbitrary. So we take $h_s^0 = 0$. We estimate h_s as follows. For $\theta \in \hat{\Pi}_{r_{s+1}}$ we have

$$\begin{aligned} |\hat{h}_s(\theta)| &\leq \sum |h_s^n| e^{|n||\Im\theta|} \\ &\leq \sum |h_s^n| e^{|n| r_{s+1}}, \end{aligned}$$

where $|n| = |n_1| + \cdots + |n_m|$ for $n = (n_1, \ldots, n_m)$. Using the assumption (C), we have, for a suitable $l \in Z$

$$\begin{aligned} |e^{i(n,\beta)\omega} - 1| &\geq \frac{2}{\pi} |(n,\beta)\omega - \pi l| \\ &\geq \left(\frac{2|\omega| C_0}{\pi} \right) \frac{1}{|n|^{m+1}}. \end{aligned}$$

Noting that $|q_s^n| \leq M_s \exp(-|n|\rho_s)$ holds for each $n \in Z^m$, we have

$$\begin{aligned} |\hat{h}_s(\theta)| &\leq \left(\frac{\pi}{2|\omega| C_0} \right) \sum |n|^{m+1} M_s\, e^{-|n|\rho_s}\, e^{|n| r_{s+1}} \\ &= \left(\frac{\pi}{2|\omega| C_0} \right) M_s \sum |n|^{m+1} e^{-|n| d_s}. \end{aligned} \tag{8.13}$$

Since the inequality

$$\sum_{n \in Z^m} |n|^k e^{-|n|d} \leq C_m d^{-(k+m+1)}$$

holds for any positive d, it follows from (8.13) that

$$|\hat{h}_s(\theta)| \leq C \frac{M_s}{d_s^{2m+2}}.$$

If we take M sufficiently small, then the right-hand side is estimated

$$|\hat{h}_s(\theta)| \leq \frac{M_s^{(5\kappa+3)/8}}{10} \leq \frac{M_s^{\kappa}}{10}$$

in $\Pi_{r_{s+1}}$. Hence (8.9) is proved. Similarly we can obtain (8.10). In fact, since $\partial_{\theta_j} \hat{h}_s(\theta) = \sum i\, n_j h_s^n \exp i(n, \theta)$, the same procedure as the above implies (8.10). It is clear to show that $h_s(x)$ is real-valued for each real x. In fact, from (8.12) $h_s^{-n} = \overline{h_s^n}$ holds.

2. *Estimate of* $\hat{q}_{s+1}$

We denote H_s^{-1} by $I + g_s$. Clearly $g_s = -h_s \circ H_s^{-1}$.

Lemma 8.1. *g_s is $2\pi/\beta$-q.p., and $\hat{g}_s$ is real analytic in $\Pi_{\zeta_{s+1}}$.*

This lemma proved in [S-M, p. 261–263].

Lemma 8.2. *For any $\Theta \in \Pi_{\zeta_{s+1}}$ there exists $\theta \in \Pi_{r_{s+1}}$ such that*

$$\Theta = \theta + \beta \hat{h}_s(\theta). \tag{8.14}$$

Proof. Define a sequence $\{\theta^j\} \subset C$ by

$$\theta^{j+1} + \beta \hat{h}_s(\theta^j) = \Theta, \quad \theta^0 = 0.$$

We show that $\{\theta^j\}$ is a contracting sequence in $\Pi_{r_{s+1}}$. Let θ^j satisfy $|\Im \theta^j| \leq r_{s+1}$. Then we have

$$\begin{aligned} |\Im \theta^{j+1}| &\leq |\Im \Theta| + |\Im \hat{h}_s(\theta^j)| \, \|\beta\| \\ &\leq \zeta_{s+1} + |\hat{h}_s(\theta^j)| \, \|\beta\|. \end{aligned}$$

Since $|\Im \theta^j| \leq r_{s+1}$ holds, by (8.9) we have $|\hat{h}_s(\theta^j)| \leq |\hat{h}_s|_{r_{s+1}} \leq M_s^{\kappa}/10$. Taking M small such that $M_s^{\kappa}/10 < d_{s+1}/2 \,||\, \beta \,||$, we have

$$|\Im \theta^{j+1}| \leq \zeta_{s+1} + \frac{d_{s+1}}{2} = r_{s+1}.$$

Next $\{\theta^j\}$ satisfies, by the mean value theorem

$$\begin{aligned}\theta^{j+1}-\theta^j &= \beta(\hat{h}_s(\theta^{j-1})-\hat{h}_s(\theta^j))\\ &= \beta\sum_{k=1}^{m}\partial_{\theta_k}\hat{h}_s(\theta^{j-1}+t(\theta^j-\theta^{j-1}))(\theta_k^{j-1}-\theta_k^j),\end{aligned}$$

whence by (8.10)

$$|\theta^{j+1}-\theta^j| \le \frac{M_s^\kappa}{10}\,|\theta^{j-1}-\theta^j|\,m\|\beta\|.$$

Therefore if we take M small, the sequence is the contracting sequence. Thus there exists $\theta \in \Pi_{r_{s+1}}$ such that (8.14) holds. □

It follows from (8.1), (8.3), (8.4) and (8.6) that q_{s+1} is represented as

$$\begin{aligned}q_{s+1}(x_{s+1}) = (I+h_s)^{-1}\Big(&(I+h_s)(x_{s+1}+\omega)+\nu_s\\ &+[q_s\circ(I+h_s)(x_{s+1})-q_s(x_{s+1})]\Big)-(x_{s+1}+\omega).\end{aligned} \tag{8.15}$$

Note that $q_{s+1}(x)$ is real-valued for real x. For brevity we set $x = x_{s+1}$ and

$$\begin{aligned}J_0 &= (I+h_s)(x+\omega),\\ J_1 &= q_s\circ(I+h_s)(x)-q_s(x),\\ J &= J_1+\nu_s.\end{aligned}$$

Then using the mean-value theorem, we simply obtain

$$q_{s+1}(x) = J\int_0^1((I+h_s)^{-1})'(J_0+tJ)dt. \tag{8.16}$$

First we estimate J.

Lemma 8.3. *J satisfies*

$$|\hat{J}|_{\rho_{s+1}} \le \frac{1}{5}\,M_s^{\kappa+1}.$$

Proof. By the mean-value theorem we have

$$J_1 = h_s(x)\int_0^1 q_s'(x+th_s(x))dt.$$

Using (8.9), we have, for $\theta \in \Pi_{\rho_{s+1}}$

$$\begin{aligned}|\Im(\theta_i+t\beta_i\hat{h}_s(\theta))| &\le |\Im\theta_i| + |\Im\hat{h}_s(\theta)|\,|\beta_i|\\ &\le \rho_{s+1}+\frac{M_s^\kappa\,|\beta_i|}{10} \le \zeta_{s+1}.\end{aligned}$$

Here we have taken M small so as to satisfy $M_s^\kappa/10 < d_{s+1}/(2\|\beta\|)$. Since $|\hat{q}_s(\theta)| \le M_s$ for $|\Im\theta| \le \rho_s$, the Cauchy integral formula gives, for $\theta \in \Pi_{\rho_{s+1}}$

$$|\hat{q'}_s(\theta + t\beta\hat{h}_s(\theta))| \le ||\beta|| m \frac{M_s}{(\rho_s - \zeta_{s+1})^m} \le ||\beta|| m \frac{M_s}{d_s^m}.$$

Hence we obtain

$$|\hat{J}_1|_{\rho_{s+1}} \le ||\beta|| m \frac{M_s^{(5\kappa+3)/8+1}}{10 d_s^m} \le \frac{M_s^{1+\kappa}}{20}.$$

Next we estimate ν_s as follows. By Proposition 2.1 the upper rotation number can be kept invariant under the homeomorphism so that the upper rotation number of Q_{s+1} may be equal to ω. Therefore Proposition 2.2 shows that by choosing x suitably we can take $q_{s+1}(x)$ as small as possible. By this fact there exists a point $z \in R^1$ such that

$$|q_{s+1}(z)| \le \frac{M_s^{1+\kappa}}{40}. \tag{8.17}$$

From (8.15) it follows that

$$\nu_s = \Big((I + h_s)(z + \omega + q_{s+1}(z)) - (I + h_s)(z + \omega)\Big) + \Big(q_s(z) - q_s \circ (I + h_s)(z)\Big). \tag{8.18}$$

Then by the similar way to the estimate of J_1 we have

$$|q_s(z) - q_s \circ (I + h_s)(z)| \le \frac{M_s^{1+\kappa}}{20}.$$

For the first term of the right hand side of (8.18), using the mean-value theorem, (8.10) and (8.17), we have

$$\begin{aligned} &|(I + h_s)(z + \omega + q_{s+1}(z)) - (I + h_s)(z + \omega)| \\ &\quad \le (1 + |h_s'(\eta)|\,)\,|q_{s+1}(z)| \\ &\quad \le \left(1 + \frac{M_s^\kappa m\|\beta\|}{10}\right) \frac{M_s^{1+\kappa}}{40} \\ &\quad \le \frac{M_s^{1+\kappa}}{20}. \end{aligned}$$

Therefore we obtain

$$|\nu_s| \le \frac{M_s^{1+\kappa}}{10}.$$

Thus we have the conclusion. □

Lemma 8.4. *The inverse of $I + h_s$ satisfies*

$$|((I + h_s)^{-1})'(\theta)|_{\zeta_{s+1}} \le \frac{4}{3}.$$

Proof. It follows from Lemma 8.2 that for $\Theta, |\Im\Theta| \le \zeta_{s+1}$ we have $|\Im(I + \beta\hat{h}_s)^{-1}(\Theta)| \le r_{s+1}$. Therefore, (8.10) gives

$$|((I+h_s)^{-1})'\hat{}(\theta)|_{\zeta_{s+1}} \le \frac{1}{1 - \|\beta\| m\, |\partial_\theta \hat{h}_s|_{\zeta_{s+1}}} \le \frac{1}{1 - M_s^\kappa \|\beta\| m/10} < \frac{4}{3},$$

where we have taken M small so as to satisfy $M_s^\kappa \|\beta\| m < 5/2$. □

From (8.15) and Lemma 8.2 clearly $\hat{q}_{s+1}$ is real analytic in $\Pi_{\rho_{s+1}}$. We estimate $\hat{q}_{s+1}$. Using (8.16), Lemma 8.3 and Lemma 8.4, we have

$$|\hat{q}_{s+1}|_{\rho_{s+1}} \le \frac{M_s^{1+\kappa}}{5}\,\frac{4}{3} \le M_s^{1+\kappa} = M_{s+1}.$$

This proves (8.5). Thus Proposition 8.1 is proved.

Now using Proposition 8.1, we shall prove the Reduction Theorem. Set

$$G_s = H_1 \circ H_2 \circ \cdots \circ H_s, \quad F_s = G_s - I.$$

Then we have

Lemma 8.5. *The sequence $\{\hat{F}_s\}$ converges uniformly to a function $\hat{F}$ in $\{|\Im\theta| \le r/2\}$.*

Proof. We show the following estimate

$$|\hat{F}_s - \hat{F}_{s-1}|_{r/2} \le C M_s^\kappa. \tag{8.19}$$

The differentiation of the composed functions and (8.9)–(8.10) gives

$$|(G_s')\hat{}|_{r_{s+1}} \le \prod_{j=1}^{\infty}(1 + m\|\beta\|\, M_j^\kappa) \le \frac{3}{2}.$$

We have

$$F_s(x) - F_{s-1}(x) = G_s(x) - G_{s-1}(x) = G_{s-1}(H_s(x)) - G_{s-1}(x).$$

Hence using the mean-value theorem, we have,

$$F_s(x) - F_{s-1}(x) = h_s(x)\, G_{s-1}'(x + t h_s(x)) = \hat{h}_s(\beta x)\,(G_{s-1}')\hat{}(\beta(x + t h_s(x))).$$

Then we have, for $\theta \in \{|\Im\theta| \le r/2\}$

$$|\hat{F}_s(\theta) - \hat{F}_{s-1}(\theta)| = |\hat{h}_s(\theta)\,(G_{s-1}')\hat{}(\theta + t\beta\hat{h}_s(\theta))| \le \frac{M_s^\kappa}{10}\,\frac{3}{2} = \frac{3}{20}\,M_s^\kappa.$$

Thus we have (8.19). (8.19) implies that $\{\hat{F}_s\}$ is a Cauchy sequence with respect to the norm $|\cdot|_{r/2}$. Therefore the lemma is proved. □

Let H be the limit of the sequence $\{G_s\}$. Since

$$G_s^{-1} \circ Q \circ G_s(x) = x + \omega + q_s(x)$$

and $q_s(x)$ converges to 0 as $s \to \infty$, we obtain

$$H^{-1} \circ Q \circ H(x) = x + \omega.$$

Therefore the Reduction Theorem is proved.

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