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Bisectional curvature of complements of curves in $\mathbb{P}^2$

By

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Introduction

Classically it is well-known that $\mathbb{P}^1 \setminus \{3 \text{ points}\}$ is covered by the unit disc and so it admits a Hermitian metric of constant negative curvature. Very little is known in higher dimension. In the special case of complements of *hyperplanes* in $\mathbb{P}^n$, it is known (see [6], [7]) that $\mathbb{P}^n \setminus \{2^n + 1 \text{ hyperplanes in general position}\}$ admits a *Finsler* metric with *hsc* (= holomorphic sectional curvature) $\leq -c^2 < 0$ where c is a constant. We shall simply refer to this by saying that the *hsc* is strongly negative. Here the concept of holomorphic sectional curvature of a Finsler metric is defined as follows (see [1], [19], [20], [23]). Let h be a Finsler metric on a complex manifold M and X be a tangent vector at a point $x \in M$. Then

$$\text{hsc}(X) = \sup_C \{G_{h|_C}(X)\}$$

where the supremum is taken over all local smooth complex curve C through the point x and $G_{h|_C}(X)$ is the Gaussian curvature of the metric on C induced by h (for a Riemann surface, Finsler = Hermitian = Kähler). The definiton is equivalent to the usual concept if h is Hermitian. The result for complements of hyperplanes is improved in [25] and is further improved recently to the best possible form (see [26]), namely, $\mathbb{P}^n \setminus \{2n + 1 \text{ hyperplanes in general position}\}$ admits a strongly negatively curved Finsler metric. Unfortunately, the method is based on a very technical construction via the Nevanlinna-Ahlfors Theory on hyperplanes. This method does not seem to lend itself readily to further extensions, e.g., general hypersurfaces that are not hyperplanes.

In this article we initiate a more conceptual approach of constructing negatively curved metrics. This is based on the observation (see [3], also [4] and [5]) that, for a complex manifold M,

the cotangent bundle T^*M is ample if and only if there exists a Finsler metric on TM with strongly negative holomorphic bisectional curvature.

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A detailed account of the concepts of holomorphic bisectional (*hbsc*) and sectional curvature of a Finsler metric can be found in [1] (see also [3], [4] and [5]) with the implications:

$$\text{hbsc} \leq -c^2 \implies \text{hsc} \leq -c^2 \implies X \text{ is Kobayashi hyperbolic.}$$

This generalizes the usual concepts for Hermitian metrics which can be found in [17]. For examples of manifolds admitting Hermitian metric with strongly negatively curved hbsc see [24]. For the general theory of Cartan connection and Chern connection for Finsler metrics the readers are referred to [1] and [2]. We consider the problem of finding curves C in $\mathbb{P}^2$ such that $\mathbb{P}^2 \setminus C$ admit Finsler metrics with strongly negative holomorphic bisectional curvature. In view of the result of [3] stated above we look for curves C in $\mathbb{P}^2$ such that the logarithmic tangent bundle $T^*\mathbb{P}^2(\log C)$ (algebraic geometers usually use the notation $\Omega^1_{\mathbb{P}^2}(\log C)$) is ample. The result in [3] then implies the existence of a *complete* Finsler metrics on $\mathbb{P}^2 \setminus C$ with strongly negative holomorphic bisectional curvature. Completeness being a consequence of the fact that sections of $T^*\mathbb{P}^2(\log C)$ have poles along C. For the general theory of logarithmic bundles the readers are referred to the deep and important works [8], [15] and [21]. The main result of this article is the following theorem (see Theorem 2.2 in section 2):

Theorem. *Let E be a rank 2 vector bundle over a non-singular compact complex surface M. Assume that E is spanned, $c_1^2(E) - c_2(E) > 0$ and $\det E$ is ample. Then E is ample.*

The preceding result is obtained via the Riemann-Roch Theorem, the Bogomolov Theorem (see [9] and [10]) and a Lemma of Gieseker [12]. This Theorem is then applied to the case $X = \mathbb{P}^2$ and $E = T^*X(\log C)$ where C is a curve in $\mathbb{P}^2$ with simple normal crossings. Using the work of [21] (see also [10]) we obtain (see Theorem 3.6):

Theorem. *Let $C = C_1 + \cdots + C_q$ be a curve, of simple normal crossings, in $\mathbb{P}^2$ with smooth irreducible components C_i of degree d_i for $1 \leq i \leq q$. Assume that $K_{\mathbb{P}^2} + C$ is ample. Then $c_1^2(T^*\mathbb{P}^2(\log C)) - c_2(T^*\mathbb{P}^2(\log C)) > 0$ if and only if one of the following cases holds*:

$$\begin{cases} q \geq 5: & 1 \leq d_1 \leq d_2 \leq \cdots \leq d_q, \\ q = 4: & (i)\ d_1 = d_2 = 1, 2 \leq d_3 \leq d_4, \quad (ii)\ d_1 = 1, 2 \leq d_2 \leq d_3 \leq d_4, \\ & (iii)\ 2 \leq d_1 \leq d_2 \leq d_3 \leq d_4, \\ q = 3: & (i)\ d_1 = 1, d_2 = 3, 4 \leq d_3, \quad (ii)\ d_1 = 1, 4 \leq d_2 \leq d_3, \\ & (iii)\ d_1 = d_2 = 2, 3 \leq d_3, \quad (iv)\ d_1 = 2, 3 \leq d_2 \leq d_3, \\ & (v)\ 3 \leq d_1 \leq d_2 \leq d_3, \\ q = 2: & (i)\ d_1 = 4, 7 \leq d_2, \quad (ii)\ 5 \leq d_1 \leq d_2. \end{cases}$$

In fact we classify (see Theorem 3.12) all hypersurfaces X in $\mathbb{P}^3$ with $c_1^2(T^*X(\log C)) - c_2(T^*X(\log C)) > 0$.

It is much more difficult to verify the condition that $E = T^*X(\log C)$ is spanned (see sections 3 and 4) as required by Theorem 2.2. We obtain the following result (see Corollary 3.9 and, for a more general result, Theorem 4.4):

Corollary. *Let $C = C_1 + \cdots + C_q$ be any of the curves in the list below*:

$$\begin{cases} q \geq 5: & 1 \leq d_1 \leq d_2 \leq \cdots \leq d_q, \\ q = 4: & (i)\ d_1 = d_2 = 1, 2 \leq d_3 \leq d_4, \\ & (ii)\ d_1 = 1, 2 \leq d_2 \leq d_3 \leq d_4, \\ & (iii)\ 2 \leq d_1 \leq d_2 \leq d_3 \leq d_4. \end{cases}$$

Assume that C is of simple normal crossings and that $\cap[JF_I = 0] = \emptyset$ where I ranges over all subsets of $\{1, 2, \ldots, q\}$ consisting of 3 distinct elements. Then there exists a complete Finsler metric on $T^*(\mathbb{P}^2 \setminus C)$ with holomorphic bisectional curvature $\leq -c^2$ where c is a constant.

In the preceding Corollary JF_I is the Jacobian determinant of the map F_I where

$$F_I = (P_{i_0}, P_{i_1}, \ldots, P_{i_n})$$

with $C_i = [P_i = 0]$, and P_i is a homogeneous polynomial with $\deg P_i = d_i = \deg C_i$ (see section 3 for more details). The point is that a general configuration of $C = C_1 + \cdots + C_q$ in the list satisfies the condition $\cap[JF_I = 0] = \emptyset$. In other words, those configurations that do not satisfy this condition are Zariski closed and of strictly lower dimension. For example the result implies that the complement, in $\mathbb{P}^2$, of any 5 (or more) smooth curves in general position admits a Finsler metric with strongly negatively curved holomorphic *bisectional* curvature. This result is stronger, even in the case of 5 lines, than the results in [26] as the explicit construction there yields only strongly negatively curved holomorphic *sectional* curvature. Our Theorem also shows that the complement of a general configuration of 2 lines and 2 quadrics admits a Finsler metric with strongly negatively curved *hbsc* and a priori, Kobayashi hyperbolic. In [9] it was shown that actually the complement of a general configuration of 1 line and 2 quadrics is Kobayashi hyperbolic. However we do not know if this admits a strongly negatively curved metric; equivalently, we do not know if $T^*\mathbb{P}^2(\log C)$ is ample in this case.

The article is written with complex analysts and complex geometers in mind hence, instead of striving for efficiency, results from algebraic geometry are presented in a leisurely manner.

1. Projectivized vector bundles

In the literature the notations are not standardized. To avoid confusion we provide a brief account concerning projectivized vector bundles in this section. The monographs [11], [22] and the articles [13], [18] are excellent references for this section.

Given a vector space V (defined over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic) the dual, i.e., the vector space of all *hyperplanes* through the origin, will be denoted by $V^\vee$. For a vector bundle E over a scheme X, the dual $E^\vee$ is defined by dualizing each fiber. For a vector bundle E of rank $r \geq 2$, the projectivized bundle, $\mathbb{P}(E^\vee) = E^\vee_*/\mathbb{K}^*$ is defined by taking equivalence classes under the $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$ action. Here $E^\vee_* = E^\vee \setminus \{\text{zero} - \text{section}\}$. The projection is denoted by $p : \mathbb{P}(E^\vee) \to X$. There is a unique line bundle, henceforth referred to as the Serre bundle and denoted by $\mathcal{O}_{\mathbb{P}(E^\vee)}(1)$, which restricts to the hyperplane line bundle $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$ along the fibers of $p : \mathbb{P}(E^\vee) \to X$.

Definition 1.1. A vector bundle E of rank $r \geq 2$ over X is ample (resp. nef, spanned) if and only if the Serre line bundle $\mathcal{O}_{\mathbb{P}(E^\vee)}(1)$ over $\mathbb{P}(E^\vee)$ is ample (resp. nef, spanned). The dual is denoted by $\mathcal{O}_{\mathbb{P}(E^\vee)}(-1)$ and will be referred to as the tautological bundle. Tensor products of these bundles will be denoted by $\mathcal{O}_{\mathbb{P}(E^\vee)}(m), m \in \mathbb{Z}$.

The following isomorphisms are well-known (see for example [14]) and shall be referred to as Grothendieck's isomorphisms ($R^i p_*$ denotes the i-th direct image of the projection):

Theorem 1.2. *Let E be a vector bundle of rank $r \geq 2$ over an algebraic scheme X. Then for $m \geq 0$, we have $R^0 p_* \mathcal{O}_{\mathbb{P}(E^\vee)}(m) = p_* \mathcal{O}_{\mathbb{P}(E^\vee)}(m) \cong \operatorname{sym}^m E$ and*

$$R^{r-1} p_* \mathcal{O}_{\mathbb{P}(E^\vee)}(-r-m) \cong \det E^\vee \otimes \operatorname{sym}^m E^\vee$$

and all other direct images vanish. In particular, the corresponding cohomological groups are isomorphic, more precisely, for any sheaf $\mathcal{S}$ on X

$$H^i(X, \operatorname{sym}^m E \otimes \mathcal{S}) \cong H^i(\mathbb{P}(E^\vee), \mathcal{O}_{\mathbb{P}(E^\vee)}(m) \otimes p^* \mathcal{S})$$

for all $i \geq 0$.

The Serre bundle is a quotient of the bundle p^*E, i.e., there is a surjection:

$$p^*E \to \mathcal{O}_{\mathbb{P}(E^\vee)}(1) \to 0$$

where $p : \mathbb{P}(E^\vee) \to X$ is the projection map. Equivalently we have an exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}(E^\vee)}(-1) \to p^* E^\vee \to Q \to 0. \tag{1.1}$$

The total Chern classes are related by Whitney's formula:

$$p^* c(E^\vee) = c(p^* E^\vee) = c(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1)) \cdot c(Q).$$

By eliminating the Chern classes of Q we arrive at:

Lemma 1.3. *Let E be a vector bundle of rank $= r \geq 2$ over a complex manifold X of dimension n. Let $p : P(E^\vee) \to X$ be the projectivized vector bundle. If $r \leq n = \dim X$ then*

$$\sum_{i=0}^{r} p^* c_i(E^\vee)\, c_1^{r-i}(\mathcal{O}_{\mathbf{P}(E^\vee)}(1)) = 0,$$

and if $r = n+k+1, k \geq 0$ then

$$\sum_{i=0}^{r-k-1} p^* c_i(E^\vee)\, c_1^{r-k-1-i}(\mathcal{O}_{\mathbf{P}(E^\vee)}(1)) = c_{r-k-1}(Q),$$

i.e.,

$$\sum_{i=0}^{n} p^* c_i(E^\vee)\, c_1^{n-i}(\mathcal{O}_{\mathbf{P}(E^\vee)}(1)) = c_n(Q)$$

where Q is the quotient bundle in (1.1).

Proof. Since rank $Q = r-1$, we have

$$\sum_{i=0}^{r} p^* c_i(E^\vee) = \big(1 + c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1)\big) \sum_{i=0}^{r-1} p^* c_i(Q)$$

hence

$$(*) \qquad p^* c_i(E^\vee) = c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\, c_{i-1}(Q) + c_i(Q)$$

for $1 \leq i \leq \min\{r, n\}$ with $c_0(Q) = 1$ and $c_i(Q) = 0$ for $i \geq \min\{r, n+1\}$. There are 3 cases to be considered: (1) $r \leq n$, (2) $r = n+1$ and (3) $r > n+1$.

Case 1. $r \leq n$

Then $\min\{r, n\} = \min\{r, n+1\} = r$. Starting from $i = r$ we get from $(*)$ the identities

$$\begin{aligned}
p^* c_r(E^\vee) &= c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\, c_{r-1}(Q),\\
p^* c_{r-1}(E^\vee) &= c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\, c_{r-2}(Q) + c_{r-1}(\mathcal{Q}),\\
&\dots\\
p^* c_2(E^\vee) &= c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\, c_1(Q) + c_2(Q),\\
p^* c_1(E^\vee) &= c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1)) + c_1(Q).
\end{aligned}$$

Multiplying the i-th identity above by $c_1^{i-1}(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-1))$ we get

$$\begin{aligned}
p^* c_r(E^\vee) &= c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\, c_{r-1}(Q),\\
p^* c_{r-1}(E^\vee)\, c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1)) &= c_1^2(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\, c_{r-2}(Q)\\
&\quad + c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\, c_{r-1}(Q),\\
&\dots\\
p^* c_1(E^\vee)\, c_1^{r-1}(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1)) &= c_1^r(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\\
&\quad + c_1^{r-1}(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\, c_1(Q).
\end{aligned}$$

Taking alternating sum eliminates the Chern classes of Q, resulting in:

$$\sum_{i=1}^{r}(-1)^{i-1}p^*c_i(E^\vee)\,c_1^{r-i}(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1)) = c_1^r(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1)).$$

This yields

$$\sum_{i=0}^{r}(-1)^i p^*c_i(E^\vee).c_1^{r-i}\mathcal{O}_{\mathbf{P}(E^\vee)}(-1) = 0$$

and since $c_1^{r-i}(\mathcal{O}_{\mathbf{P}(E^\vee)}(1)) = (-1)^{r-i}c_1^{r-i}(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))$, we get

$$\sum_{i=0}^{r} p^*c_i(E^\vee)\,c_1^{r-i}\mathcal{O}_{\mathbf{P}(E^\vee)}(1) = 0$$

as well.

Case 2. $r = n+1$

Then $\min\{r, n\} = n$ and $\min\{r, n+1\} = r = n+1$. Thus $p^*c_r(E^\vee) = 0$ and we get from $(*)$ the following identities:

$$\begin{aligned}
p^*c_{r-1}(E^\vee) &= c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\,c_{r-2}(\mathcal{Q}) + c_{r-1}(\mathcal{Q}),\\
p^*c_{r-2}(E^\vee) &= c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\,c_{r-3}(Q) + c_{r-2}(Q),\\
&\;\;\cdots\\
p^*c_2(E^\vee) &= c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\,c_1(Q) + c_2(Q),\\
p^*c_1(E^\vee) &= c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1)) + c_1(Q).
\end{aligned}$$

Multiplying the i-th identity above by $c_1^{i-1}(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))$ we get

$$\begin{aligned}
p^*c_{r-1}(E^\vee) &= c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\,c_{r-2}(Q) + c_{r-1}(Q),\\
p^*c_{r-2}(E^\vee)\,c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1)) &= c_1^2(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\,c_{r-3}(Q)\\
&\quad + c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\,c_{r-2}(Q),\\
&\;\;\cdots\\
p^*c_1(E^\vee)\,c_1^{r-2}(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1)) &= c_1^{r-1}(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1)) + c_1^{r-2}(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\,c_1(Q).
\end{aligned}$$

Taking alternating sum eliminates the Chern classes of Q, resulting in:

$$\begin{aligned}
\sum_{i=1}^{r-1}(-1)^{i-1}p^*c_i(E^\vee)\,c_1^{r-i-1}(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1)) &= c_1^{r-1}(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\\
&\quad + (-1)^{r-2}c_{r-1}(Q)
\end{aligned}$$

which is equivalent to

$$\sum_{i=0}^{r-1}(-1)^i p^*c_i(E^\vee)\,c_1^{r-i-1}(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1)) = (-1)^{r-1}c_{r-1}(Q)$$

and is also equivalent to

$$\sum_{i=0}^{r-1} p^* c_i(E^\vee)\, c_1^{r-i-1}(\mathcal{O}_{\mathbf{P}(E^\vee)}(1)) = c_{r-1}(Q).$$

Case 3. $r > n+1$

Let $r = n+1+k, k \geq 1$. Then $\min\{r,n\} = n$ and $\min\{r, n+1\} = n+1$. Thus

$$p^* c_r(E^\vee) = p^* c_{r-1}(E^\vee) = ... = p^* c_{r-k}(E^\vee) = 0$$

and we get from $(*)$ the following $r-k-1 = n$ identities:

$$\begin{aligned}
p^* c_{r-k-1}(E^\vee) &= c_1(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-1))\, c_{r-k-2}(Q) + c_{r-k-1}(Q),\\
p^* c_{r-k-2}(E^\vee) &= c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\, c_{r-k-3}(Q) + c_{r-k-2}(Q),\\
&\;\;...\\
p^* c_2(E^\vee) &= c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\, c_1(Q) + c_2(Q),\\
p^* c_1(E^\vee) &= c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1)) + c_1(Q).
\end{aligned}$$

Multiplying the i-th identity above by $c_1^{i-1}(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))$ we get

$$\begin{aligned}
p^* c_{r-k-1}(E^\vee) &= c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\, c_{r-k-2}(Q) + c_{r-k-1}(Q),\\
p^* c_{r-k-2}(E^\vee)\, c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1)) &= c_1^2(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-1))\, c_{r-k-3}(Q)\\
&\quad + c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\, c_{r-k-2}(Q),\\
&\;\;...\\
p^* c_1(E^\vee)\, c_1^{r-k-2}(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1)) &= c_1^{r-k-1}(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\\
&\quad + c_1^{r-k-2}(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\, c_1(Q).
\end{aligned}$$

Taking alternating sum eliminates the Chern classes of Q, resulting in:

$$\begin{aligned}
\sum_{i=1}^{r-k-1} (-1)^{i-1} p^* c_i(E^\vee)\, c_1^{r-k-i-1}(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1)) &= c_1^{r-k-1}(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1))\\
&\quad + (-1)^{r-k-2} c_{r-k-1}(Q)
\end{aligned}$$

which is equivalent to

$$\sum_{i=0}^{r-k-1} (-1)^i p^* c_i(E^\vee)\, c_1^{r-k-i-1}(\mathcal{O}_{\mathbf{P}(E^\vee)}(-1)) = (-1)^{r-k-1} c_{r-k-1}(Q)$$

and is also equivalent to

$$\sum_{i=0}^{r-k-1} p^* c_i(E^\vee)\, c_1^{r-k-i-1}(\mathcal{O}_{\mathbf{P}(E^\vee)}(1)) = c_{r-k-1}(Q).$$

□

We shall mainly be interested in the case $r = \operatorname{rank} E = \dim X = n = 2$. The Lemma in the general form here should be of great interest in higher dimension as well.

Let $\Delta_0 = 1, \Delta_1 = -c_1(E^\vee)$ and define inductively, for $j = 2, \dots, n+r-1 = \dim \mathbf{P}(E^\vee)$:

$$\Delta_j = \Delta_j(E^\vee) = -\sum_{i=1}^{j} \Delta_{j-i} \cdot c_i(E^\vee) \tag{1.2}$$

where it is understood that $c_i(E^\vee) = 0$ for $i > \min\{r = \text{rank } E^\vee, n = \dim X\}$. The Δ_i's are polynomials in the Chern classes of $E^\vee$. The first few polynomials are explicitly given by:

$$\begin{cases} \Delta_2 = -\Delta_1\, c_1(E^\vee) - \Delta_0\, c_2(E^\vee) = c_1^2(E^\vee) - c_2(E^\vee), \\ \Delta_3 = -\Delta_2\, c_1(E^\vee) - \Delta_1\, c_2(E^\vee) - \Delta_0\, c_3(E^\vee) \\ \quad = -c_1^3(E^\vee) + 2c_1(E^\vee)\, c_2(E^\vee) - c_3(E^\vee), \\ \Delta_4 = c_1^4(E^\vee) - 3c_1^2(E^\vee)\, c_2(E^\vee) + 2c_1(E^\vee)\, c_3(E^\vee) + c_2^2(E^\vee) - c_4(E^\vee), \end{cases}$$

etc. ...

Proposition 1.4. *Let $p : \mathbb{P}(E^\vee) \to X$ be the projectivized vector bundle where rank $E = r \geq 2$ over a complex manifold X of dimension n. If $r \leq n$ then the following intersection formulas hold*:

$$c_1^{r+j-1}(\mathcal{O}_{\mathbb{P}(E^\vee)}(1)) \cdot p^*D_1 \cdots p^*D_{n-j} = \Delta_{j+1} \cdot D_1 \cdots D_{n-j}$$

for $j = 0, 1, \dots, n$ and for any divisors $D_1, \dots, D_n$ in X. In particular, we have:

$$c_1^{r+n-1}(\mathcal{O}_{\mathbb{P}(E^\vee)}(1)) > 0 \Leftrightarrow \Delta_n > 0.$$

Proof. Note that the dimension of $\mathbb{P}(E^\vee)$ is $n + r - 1$ and the dimension of a fiber, $\mathbb{P}(E^\vee)_x$, is $n = r - 1$ and since the bundle $\mathcal{O}_{\mathbb{P}(E^\vee)}(1)|\mathbb{P}(E^\vee)_x$ is isomorphic to $\mathcal{O}_{\mathbf{P}^{r-1}}(1)$, we have:

$$\int_{\mathbb{P}(E^\vee)_x} c_1^{r-1}(\mathcal{O}_{\mathbb{P}(E^\vee)}(1))|\mathbb{P}(E^\vee)_x) = 1.$$

This implies that

$$c_1^{r-1}(\mathcal{O}_{\mathbb{P}(E^\vee)}(1)) \cdot p^*D_1 \cdots p^*D_n = D_1 \cdots D_n = \Delta_0 \cdot D_1 \cdots D_n. \tag{1.3}$$

For simplicity of notation write $\mathcal{H}$ for $\mathcal{O}_{\mathbb{P}(E^\vee)}(1)$. If $r \leq n$, we get by Lemma 1.3:

$$c_1^r(\mathcal{H}) + c_1^{r-1}(\mathcal{H}).p^*c_1(E^\vee) + c_1^{r-2}(\mathcal{H}).p^*c_2(E^\vee) + \cdots + p^*c_r(E^\vee) = 0. \tag{1.4}$$

Multiplying the above by $p^*D_1, \dots, p^*D_{n-1}$ we get

$$c_1^r(\mathcal{H}) \cdot p^*D_1 \cdots p^*D_{n-1} + c_1^{r-1}(\mathcal{H}) \cdot p^*c_1(E^\vee) \cdot p^*D_1 \cdots p^*D_{n-1} = 0$$

as the rest of the terms vanish for dimension reason. By (1.3) and the definition of Δ_1 we get from the preceding identity

$$\begin{aligned} c_1^r(\mathcal{H}) \cdot p^*D_1 \cdots p^*D_{n-1} &= -c_1^{r-1}(\mathcal{H}) \cdot p^*c_1(E^\vee) \cdot p^*D_1 \cdots p^*D_{n-1} \\ &= c_1^{r-1}(\mathcal{H}) \cdot p^*\Delta_1 \cdot p^*D_1 \cdots p^*D_{n-1} \\ &= \Delta_1 \cdot D_1 \cdots D_{n-1}. \end{aligned} \tag{1.5}$$

Multiplying (1.4) by $c_1(\mathcal{H})$ yields:

$$c_1^{r+1}(\mathcal{H}) = -c_1^r(\mathcal{H}) \cdot p^*c_1(E^\vee) - \cdots - c_1^{r+1-n}(\mathcal{H}) \cdot p^*c_n(E^\vee)$$

hence

$$\begin{aligned} &c_1^{r+1}(\mathcal{H}) \cdot p^*D_1 \cdots p^*D_{n-2} \\ &= -\big(c_1^r(\mathcal{H}) \cdot p^*c_1(E^\vee) + c_1^{r-1}(\mathcal{H}) \cdot p^*c_2(E^\vee)\big) \cdot p^*D_1 \cdots p^*D_{n-2} \end{aligned}$$

as the rest of the terms vanish, again, for dimension reason. By (1.3), (1.5) and the definition of Δ_2, we get

$$\begin{aligned} &c_1^{r+1}(\mathcal{H}) \cdot p^*D_1 \cdots p^*D_{n-2} \\ &= -\big(c_1^r(\mathcal{H}) \cdot p^*c_1(E^\vee) + c_1^{r-1}(\mathcal{H}) \cdot p^*c_2(E^\vee)\big) \cdot p^*D_1 \cdots p^*D_{n-2} \\ &= -\big(\Delta_1 \cdot c_1(E^\vee) + \Delta_0 \cdot c_2(E^\vee)\big) \cdot D_1 \cdots D_{n-2} \\ &= \Delta_2 \cdot D_1 \cdots D_{n-2}. \end{aligned}$$

Inductively, we have

$$\begin{aligned} &c_1^{r+j-1}(\mathcal{H}) \cdot p^*D_1 \cdots p^*D_{n-j} \\ &= -\{c_1^{r+j-1}(\mathcal{H}) \cdot p^*c_1(E^\vee) + \cdots + c_1^{r-1}(\mathcal{H}) \cdot p^*c_j(E^\vee)\} \cdot p^*D_1 \cdots p^*D_{n-2} \\ &= -\{\Delta_{j-1} \cdot c_1(E^\vee) + \Delta_{j-2} \cdot c_2(E^\vee) + \cdots + \Delta_0 \cdot c_3(E^\vee)\} \cdot D_1 \cdots D_{n-j} \\ &= \Delta_j \cdot D_1 \cdots D_{n-j} \end{aligned}$$

for $j \leq n$. In particular, we get $c_1^{r+n-1}(\mathcal{H}) = \Delta_n$ and the assertion of the Proposition follows. □

If rank $E = \dim X = 2$ we obtain from the preceding Proposition the following intersection formulas on $\mathbb{P}(E^\vee)$:

$$\begin{cases} c_1(\mathcal{O}_{\mathbb{P}(E^\vee)}(1)) \cdot p^*D \cdot p^*D' = D \cdot D', \\ c_1^2(\mathcal{O}_{\mathbb{P}(E^\vee)}(1)) \cdot p^*D = \Delta_1 \cdot D = -c_1(E^\vee) \cdot D = c_1(E) \cdot D, \\ c_1^3(\mathcal{O}_{\mathbb{P}(E^\vee)}(1)) = \Delta_2 = c_1^2(E^\vee) - c_2(E^\vee) = c_1^2(E) - c_2(E), \end{cases} \tag{1.6}$$

where D and D' are divisors in X. We have $c_1(E) = -c_1(E^\vee)$ and, for a rank 2 bundle, $E \cong (\det E) \otimes E^\vee$ (equivalently, $E^\vee \cong (\det E^\vee) \otimes E$) hence

$$c_2(E) = c_2((\det E) \otimes E^\vee) = c_1^2(E) + c_1(E)c_1(E^\vee) + c_2(E^\vee) = c_2(E^\vee).$$

Thus $c_1^2(E) - c_2(E) = c_1^2(E^\vee) - c_2(E^\vee)$.

Theorem 1.5 (Riemann-Roch)**.** *Let E be a holomorphic vector bundle of rank $r = 2$ over a compact surface X with $c_1^2(E) > c_2(E)$. Then $\chi(\mathrm{sym}^m E) = O(m^3)$.*

Proof. By Grothendieck's isomorphism $\chi(\mathrm{sym}^m E) = \chi(\mathcal{O}_{\mathbb{P}(E^\vee)}(m))$. Since rank of $E = r = 2$ and X is of dimension $n = 2$, it is clear that $\dim \mathbb{P}(E^\vee) = r + n - 1 = 3$ hence Riemann-Roch implies that

$$\chi(\mathcal{O}_{\mathbb{P}(E^\vee)}(m)) = \frac{c_1^3(\mathcal{O}_{\mathbb{P}(E^\vee)}(1))}{3!} m^3 + O(m^2).$$

By Proposition 1.4, $c_1^3(\mathcal{O}_{\mathbb{P}(E^\vee)}(1)) = \Delta_2 = c_1^2(E) - c_2(E) > 0$ and the Theorem follows. $\square$

Theorem 1.6 (Bogomolov)**.** *Let E be a holomorphic vector bundle of rank $r = 2$ over a compact surface X satisfying the conditions (i) $c_1^2(E) > c_2(E)$ and (ii) there exists a positive integer m_0 such that $K_X^{-1} \otimes (\det E)^{m_0}$ is effective. Then E is big, namely,*

$$h^0(X, \mathrm{sym}^m E) = \dim H^0(X, \mathrm{sym}^m E) = O(m^3).$$

Proof. By Riemann-Roch we have

$$h^0(\mathrm{sym}^m E) + h^2(\mathrm{sym}^m E) \geq \chi(\mathrm{sym}^m E) = O(m^3).$$

This implies that either

$$h^0(\mathrm{sym}^m E) = O(m^3) \quad \text{or} \quad h^2(\mathrm{sym}^m E) = O(m^3).$$

We are done if the first alternative holds. We use Serre's Duality to deal with the second alternative above:

$$h^2(\mathrm{sym}^m E) = h^0(K_X \otimes \mathrm{sym}^m E^\vee).$$

Recall that

$$E^\vee \cong (\det E^\vee) \otimes E$$

hence

$$\mathrm{sym}^m E^\vee \cong (\det E^\vee)^m \otimes \mathrm{sym}^m E$$

and so under the second alternative

$$h^0(K_X \otimes (\det E^\vee)^m \otimes \mathrm{sym}^m E) = h^2(\mathrm{sym}^m E) = O(m^3).$$

By the second assumption the sheaf $(\det E)^{m_0} \otimes K_X^{-1}$ admits a non-trivial section σ hence we have, for any positive integer λ, an injection:

$$0 \to H^0(K_X \otimes (\det E^\vee)^{\lambda m_0} \otimes \mathrm{sym}^{\lambda m_0} E) \xrightarrow{\otimes \sigma^\lambda} H^0(\mathrm{sym}^{\lambda m_0} E), \qquad \omega \mapsto \omega \otimes \sigma^\lambda.$$

Thus $h^0(\mathrm{sym}^{\lambda m_0} E) \geq h^0(K_X \otimes (\det E^\vee)^{\lambda m_0})$ which implies the first alternative holds, namely, $h^0(\mathrm{sym}^m E) = O(m^3)$. □

2. Big and spanned bundles

Let V be a vector space of dimension N and denote by $V_X = X \times V$ the trivial bundle over X. A vector bundle E of rank $r \geq 2$ is spanned if it is a quotient of V_X for some V, i.e., we have a surjection $V_X \to E \to 0$. This gives an injection

$$0 \to \mathbb{P}(E^\vee) \xrightarrow{\iota} \mathbb{P}(V_X^\vee) = X \times \mathbb{P}(V^\vee). \tag{2.1}$$

Note that the restriction of ι to a fiber $\mathbb{P}(E_x^\vee)$ over a point $x \in X$ is an injection of $\mathbb{P}(E_x^\vee)$ into $\{x\} \times \mathbb{P}(V^\vee)$. Define a map

$$\phi = p_2 \circ \iota : \mathbb{P}(E^\vee) \to \mathbb{P}(V^\vee) \tag{2.2}$$

by taking the composite of the projection $p_2 : X \times \mathbb{P}(V^\vee) \to \mathbb{P}(V^\vee) = \mathbb{P}^{N-1}$ with the injection map ι. Note that ϕ injects a fiber $\mathbb{P}(E_x^\vee)$ into $\mathbb{P}(V^\vee)$. By construction, we have

$$\mathcal{O}_{\mathbb{P}(E^\vee)}(1) = \phi^* \mathcal{O}_{\mathbb{P}(V^\vee)}(1). \tag{2.3}$$

For a point $x \in X$ the fiber E_x is a quotient of the vector space V or, equivalently, $\mathbb{P}(E_x^\vee)$ is a linear subspace of $\mathbb{P}(V^\vee)$. For a point $[v] \in \mathbb{P}(V^\vee)$ consider the set

$$Z_{[v]} = \{x \in X \mid [v] \in \mathbb{P}(E_x^\vee)\}. \tag{2.4}$$

It is well-known that:

Lemma 2.1. *Let E be a rank $r \geq 2$ vector bundle over a projective variety. Assume that E is spanned. Then the following conditions are equivalent*:
(a) *E is ample,*
(b) *the map ϕ is finite,*
(c) $\#Z_{[v]} < \infty$ *for all* $[v] \in \mathbb{P}(V^\vee)$.

Theorem 2.2. *Let E be a rank 2 vector bundle over a non-singular compact complex surface X. Assume that E is spanned, $c_1^2(E) - c_2(E) > 0$ and* $\det E$ *is ample. Then E is ample.*

Proof. Since $c_1^2(E) - c_2(E) > 0$ and $\det E$ is ample the assumptions of Bogomolov's Theorem are satisfied hence E is big. Gieseker's Lemma (see Gieseker [12]) implies that, *if E were non-ample, there exists an effective irreducible curve C in X such that $E|_C$ admits a trivial quotient.* Indeed the trivial quotient is constructed as follows. Since E is spanned we have a surjection:

$$X \times V \xrightarrow{ev} E \to 0, \qquad (x, \sigma) \mapsto \sigma(x)$$

via the evaluation map and $V = H^0(X, E)$. By the preceding proposition E is not ample if and only if the associate map ϕ in (2.2) is not finite. This is equivalent to saying that there is an effective irreducible curve $\tilde{C}$ in $\mathbb{P}(E^\vee)$ which is contracted, via ϕ, to a point $[v_0] \in \mathbb{P}(V^\vee)$. By (2.3) we have

$$\mathcal{O}_{\mathbb{P}(E^\vee)}(1)|_{\tilde{C}} = \phi^*(\mathcal{O}_{\mathbb{P}(V^\vee)}(1))_{[v_0]}$$

where the right hand side, being the pull-back of a single fiber of ϕ, is a trivial bundle over $\tilde{C}$. Since $\mathcal{O}_{\mathbb{P}(E^\vee)}(1)$ is a quotient of $p^*E, \mathcal{O}_{\mathbb{P}(E^\vee)}(1)|_{\tilde{C}}$ is a trivial quotient of $p^*E|_{\tilde{C}}$. Since ϕ is an injection of a fiber $\mathbb{P}(E_x)$ into $\mathbb{P}(V^\vee)$, the curve $\tilde{C}$ cannot be contained in a fiber of $p : \mathbb{P}(E^\vee) \to X$. Thus $p(\tilde{C}) = C$ is a curve in X. Moreover, the injectivity of ϕ implies that $p|_{\tilde{C}}$ is an isomorphism. This yields a trivial quotient $p_*\mathcal{O}_{\mathbb{P}(E^\vee)}(1)|_{\tilde{C}}$ of $p_*(p^*E|_C) = E|_C$.

Since $\mathcal{O}_{\mathbb{P}(E^\vee)}(1)|_{\tilde{C}}$ is a trivial quotient of $p^*E|_{\tilde{C}}$, we have $c_1(\mathcal{O}_{\mathbb{P}(E^\vee)}(1)|_{\tilde{C}}) = 0$. Consider the surface $p^{-1}(C)$ in $\mathbb{P}(E^\vee)$. This is a $\mathbb{P}^1$-bundle over C containing $\tilde{C}$. The Chern number $c_1^2(\mathcal{O}_{\mathbb{P}(E^\vee)}(1)|_{p^{-1}(C)})$ is computed via Proposition 1.4 (see formula (1.6)):

$$c_1^2(\mathcal{O}_{\mathbb{P}(E^\vee)}(1)|_{p^{-1}(C)}) = c_1^2(\mathcal{O}_{\mathbb{P}(E^\vee)}(1)) \cdot p^*C = \Delta_1 \cdot C = c_1(E) \cdot C.$$

The condition that $\det E$ is ample implies that $c_1(E) \cdot C > 0$ thus $E|_C$ is big (and spanned as E is spanned). By the vanishing theorem of Kawamata-Vieweg (see [11]) $H^1(C, E|_C) = 0$. Let Q be the trivial quotient of $E|_C$ (i.e., $Q \cong \mathcal{O}_C$) then $H^1(C, Q) = 0$ as there is a surjection $H^1(C, E|_C) \to H^1(C, Q) \to 0$. Since $\deg Q = 0$, Riemann-Roch implies that

$$H^0(C, \mathcal{O}_C) = H^0(C, Q) = H^0(C, Q) - H^1(C, Q) = 0$$

which is absurd. Thus E must be ample. ☐

By a (complex) Finsler metric (see [1] or [3] for more details) on a holomorphic vector bundle E we mean a non-negative function h on E with the following properties:

(FM1) *h is an upper semi-continuous function on E;*

(FM2) $h(z, \lambda v) = |\lambda| h(z, v)$ *for all* $\lambda \in \mathbb{C}$ *and* $(z, v) \in E_z$;

(FM3) $h(z, v) > 0$ *on* $E \setminus \{\text{zero} - \text{section}\}$;

(FM4) *for z and v fixed the function $h^2(z, \lambda v)$ is smooth even at* $\lambda = 0$.

For example the Kobayashi metric on a hyperbolic manifold is a Finsler metric on the tangent bundle. More generally, any intrinsic (i.e., depending only on the complex structure) (pseudo)-metric of a complex manifold is a Finsler (pseudo)-metric (i.e.,(FM3) is replaced by the weaker condition $h(z, v) \geq 0$ on E). Obviously the norm of a *Hermitian* metric on E is Finsler and satisfies, among others, the following additional conditions:

(FM5) *h is of class $\mathcal{C}^0$ on E and of class $\mathcal{C}^\infty$ on $E \setminus \{$zero-section$\}$;*

(FM6) *h is strictly pseudoconvex on $E_z \setminus \{0\}$ for all $z \in M$.*

This last two properties are, in general, not shared by the intrinsic metrics. There are many Finsler metrics with these additional properties which are not

Hermitian. With these conditions the mixed holomorphic bisectional curvature of E can be defined (see [3]). The term "mixed" refers to the fact that we shall be considering curvature in two directions: one in the space direction and the other in the fiber direction. If E is the tangent bundle and h a Hermitian metric this coincides with the usual notion of holomorphic bisectional curvature as introduced by Goldberg (see [17]). The significance of Finsler metric is the following characterization of ample vector bundles (see [3]):

Theorem 2.3. *Let E be a rank $r \geq 2$ holomorphic vector bundle over a compact complex manifold M. Then E is ample if and only if there exists a Finsler metric along the fibers of $E^\vee$ with negative mixed holomorphic bisectional curvature.*

Theorem 2.2 and Theorem 2.3 imply the following result:

Corollary 2.4. *Let E be a rank 2 vector bundle over a non-singular compact complex surface X. Assume that E is spanned, $c_1^2(E) - c_2(E) > 0$ and $\det E$ is ample, then E admits a Finsler metric along the fibers of $E^\vee$ with negative mixed holomorphic bisectional curvature. In particular, if $E = T^*X$ is the cotangent bundle then the preceding conditions imply that TX admits a Finsler metric along the fibers of TX with negative holomorphic bisectional curvature.*

If a Finsler metric on the tangent bundle is of negative holomorphic bisectional curvature then it is also of negative holomorphic sectional curvature (see [1]). It is well-known that the existence of a Finsler metric with holomorphic sectional curvature imples that the variety is Kobayashi hyperbolic (see [1], [19], [20], [23]). Thus we have

Corollary 2.5. *Let X be a non-singular compact complex surface. Assume that T^*X is spanned, $c_1^2(T^*X) - c_2(T^*X) > 0$ and that the canonical bundle $\det T^*X = K_X$ is ample. Then X is Kobayashi hyperbolic.*

3. Complements of curves

The main references of this sections are [8], [15], [16], [21]. For applications related to this section see [9], [10].

The logarithmic cotangent bundle $T^*X(\log D)$, with log-pole along a divisor D, is defined for any divisor of simple normal crossings in a smooth variety X. By simple normal crossings we mean

(i) each irreducible component D_i of D is smooth;

(ii) for any point $x \in D$ there exists a coordinate neighborhood $(U, t_1, \ldots, t_n)$ of x such that $U \cap D = \{z \in U \mid t_1(z) \cdots t_k(z) = 0\}, 1 \leq k \leq n$.

The second condition means that the components $\{D_i\}$ intersect transversally. The number k in (ii) is the intersection number at x, i.e., $\#\{i \mid x \in D_i\} = k$. This number shall be referred to as the *crossing number* at x. A local section

$\omega \in H^0(U, T^*X(\log D))$ is (just as in the case of hyperplanes) of the form

$$\sum_{i=1}^{k} a_i(z)\frac{dt_i}{t_i} + \sum_{j \geq k+1} b_j(z)dt_j$$

where a_i and b_j are regular functions on U.

In the preceding section we obtained results concerning the bisectional curvature of certain projective surfaces. Differential geometers are also interested in non-compact manifolds. Our results in the preceding sections are valid for general vector bundles E not merely for the cotangent bundle. This is not just for the sake of generality. In fact, if we take $E = T^*X(\log C)$ (the logarithmic cotangent bundle) where C is a curve in X with simple normal crossings then what we obtained in section 2 are results concerning the existence of Finsler metric on the quasi-projective surfaces $X \setminus C$ with negative bisectional curvature. The fact that we are using the cotangent sheaf with log-poles along C means that the metric blows up along C which means that the metric is complete on $X \setminus C$. Under this situation Theorem 2.2 maybe restated as follows:

Theorem 3.1. *Let X be a non-singular compact complex surface and let C be a curve in X with simple normal crossings. Assume that the logarithmic cotangent bundle $E = T^*X(\log C)$ is spanned, $c_1^2(E) - c_2(E) > 0$, and the logarithmic canonical bundle* $\det E = K_X + C$ *is ample. Then $X \setminus C$ admits a complete Finsler metric with negative holomorphic bisectional curvature. In particular, $X \setminus C$ is Kobayashi hyperbolic.*

We proceed to construct examples satisfying the conditions of Theorem 3.1. First we deal with the "spanned" condition.

Let $H = \sum_{i=0}^{n} H_i = [z_0 \cdots z_n = 0]$ be the union of the coordinate hyperplanes in $\mathbb{P}^n$. Denote by $E_H = T^*\mathbb{P}^n(\log H)$ the logarithmic cotangent bundle with poles along the coordinate hyperplanes. The logarithmic cotangent bundle can be described in terms of its presheaf of local sections as follows. Let U be an open set. Then $U \cap H = \{z_{i_1} \cdots z_{i_k} = 0\}$ for some $1 \leq k \leq n$. The local sections of E_H over U $(H^0(U, E_H))$ are of the form:

$$\sum_{i \in I} a_i(z)\frac{dt_i}{t_i} + \sum_{j \in J} b_j(z)dt_j \tag{3.1}$$

where $I \cap J = \emptyset, I \cup J = \{1, \ldots, n\}, a_i$ and b_j are regular functions on U. A section is said to vanish at a point z if $a_i(z) = b_j(z) = 0$ for all i, j.

Lemma 3.2. *Let $H = \sum_{i=0}^{n} H_i = [z_0 \cdots z_n = 0]$. Then the bundle $E_H = T^*\mathbb{P}^n(\log H)$ is spanned.*

Proof. There is no loss of generality in assuming that $i_j = j$ for $j = 1, \ldots, k$ in the remark preceding the Lemma. We may take $t_i = z_i/z_0, i =$

$1, \dots, n$ to be a local coordinate system on U. The logarithmic 1-forms

$$\left\{\omega_i = d\log t_i = \frac{dt_i}{t_i} = d\log\frac{z_i}{z_0}, i = 1, \dots, n\right\}$$

are globally defined regular sections of E_H because $t_i = z_i/z_0, 1 \le i \le n$, are globally defined rational functions and each dt_i/t_i is of logarithmic type with simple poles along $[z_i = 0]$ and $[z_0 = 0]$. Since $t_1, \dots, t_n$ is a local coordinate system on $\mathbb{P}^n \setminus [z_0 = 0]$ it is clear that ω_i are linearly independent. Note that for any $i \neq j$,

$$d\log(z_i/z_j) = d\log(z_i z_0/z_0 z_j) = d\log(z_i/z_0) - d\log(z_j/z_0) = dt_i/t_i - dt_j/t_j.$$

Since rank $E_H = n$ this shows that the bundle E_H is spanned. $\square$

The following simple general result is often useful.

Lemma 3.3. *Let $f : X \to Y$ be a holomorphic surjective map between complex varieties. Let $\pi : E \to Y$ be a holomorphic vector bundle over Y. If E is spanned then the pull-back f^*E is also spanned.*

Proof. By definition

$$f^*E = \{(x, v) \in X \times E \mid \pi(v) = f(x)\},$$

i.e., the fiber $(f^*E)_x$ over $x \in X$ is identified with the fiber E_y over $y = f(x)$. If E is spanned then, for any $v \in E_y$ there exists $\sigma \in H^0(Y, E)$ such that $\sigma(y) = v$. The pull-back $f^*(\sigma)$ is a global section of f^*E. By definition, $f^*\sigma(x) = (x, v) \in f^*E$ if $y = f(x)$. Thus f^*E is spanned. $\square$

Let $D = D_0 + D_1 + \cdots + D_n$ be a divisor in $\mathbb{P}^n$ with smooth irreducible components D_i in simple normal crossings. Assume that $\deg D_i = d \ge 1$ for all i. Let $D_i = [P_i = 0]$ where $P_i(z_0, \dots, z_n)$ is a homogeneous polynomial of degree $d = \deg D_i$. Then the map

$$f = [P_0, P_1, \dots, P_n] : \mathbb{P}^n \to \mathbb{P}^n$$

is a well-defined holomorphic map. It is well-defined because each component is homogeneous of the same degree. It is holomorphic because D is of simple normal crossings hence, for any $x \in D$ there exists i such that $x \notin D_i$, i.e., $P_i(x) \neq 0$. This shows that $P_0, P_1, \dots, P_n$ have no common zeros hence f is holomorphic. Indeed,

$$f^*([z_i = 0]) = D_i$$

where $z_i = 0$ is the i-th coordinate hyperplane. Thus the map

$$F = (P_0, P_1, \dots, P_n) : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}^{n+1} \setminus \{0\}$$

is a well-defined holomorphic map with

$$f \circ \pi = \pi \circ F,$$

where $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is the Hopf fibration. The Jacobian determinant

$$JF = \det \begin{pmatrix} \partial P_0/\partial z_0 & \partial P_0/\partial z_1 & . & . & . & \partial P_0/\partial z_n \\ \partial P_1/\partial z_0 & \partial P_1/\partial z_1 & . & . & . & \partial P_1/\partial z_n \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \partial P_n/\partial z_0 & \partial P_n/\partial z_1 & . & . & . & \partial P_n/\partial z_n \end{pmatrix} \tag{3.2}$$

is a homogeneous polynomial of degree $(d-1)(n+1)$. If $d = 1$ then $[JF = 0]$ is empty, otherwise it is a well-defined divisor of degree $(d-1)(n+1)$ in $\mathbb{P}^n$. By Lemma 3.3 the pull-back bundle $f^{-1}(T^*\mathbb{P}^n(\log H))$ is spanned. Since $f^*(H) = D$ where $H = H_0 + H_1 + \cdots + H_n$ is the union of the coordinate hyperplanes we verify readily that, for any $\omega \in H^0(\mathbb{P}^n, T^*\mathbb{P}^n(\log H))$, the pull-back of ω via f is a regular 1-form with logarithmic singularity along D, i.e., a global section of $T^*\mathbb{P}^n(\log D))$. Indeed, $f^{-1}(T^*\mathbb{P}^n(\log H))$ is isomorphic to $T^*\mathbb{P}^n(\log D)$ on the complement of the divisor $[JF = 0]$. Thus we have,

Corollary 3.4. *Let $D = D_0 + D_1 + \cdots + D_n$ be a divisor in $\mathbb{P}^n$ with irreducible components D_i in simple normal crossings. If $\deg D_i = d$ for all i then the restriction of $T^*\mathbb{P}^n(\log D)$ to $\mathbb{P}^n \setminus [JF = 0]$ is spanned, where JF is given by* (3.2).

Suppose now that $D = D_0 + \cdots + D_n + D_{n+1} + \cdots + D_{n+k}$ is a divisor with $n + 1 + k$ $(k \geq 1)$ irreducible components in simple normal crossings. Assume that each component $D_i = [P_i = 0]$ where $P_i(z_0, \ldots, z_n)$ is a homogeneous polynomial of degree $d = \deg D_i$. Let $I = (i_0, \ldots, i_n)$ be any subset of $\{0, 1, \ldots, n + k\}$ consisting of $n + 1$ distinct elements. Then the map

$$f_I = [P_{i_0}, P_{i_1}, \ldots, P_{i_n}] : \mathbb{P}^n \to \mathbb{P}^n$$

is a well-defined holomorphic map. The preceding Corollary implies that the restriction of $T^*\mathbb{P}^n(\log D)$ to $\mathbb{P}^n \setminus [JF_I = 0]$ is spanned where JF_I is the Jacobian of the map $F_I = (P_{i_0}, P_{i_1}, \ldots, P_{i_n})$.

Corollary 3.5. *Let $D = D_0 + D_1 + \cdots + D_{n+k}, k \geq 1,$ be a divisor in $\mathbb{P}^n$ with irreducible components D_i in simple normal crossings and $\deg D_i = d$ for all i. Assume that*

$$\cap_I [JF_I = 0] = \emptyset$$

where I *ranges over all subsets of $\{0, 1, \ldots, n + k\}$ consisting of $n + 1$ distinct elements. Then the bundle $T^*\mathbb{P}^n(\log D)$ is spanned.*

Proof. This is clear as the condition $\cap_I [JF_I = 0] = \emptyset$ implies that every point $x \in \mathbb{P}^n$ is contained in $\mathbb{P}^n \setminus JF_I$ for some I and the restriction of $T^*\mathbb{P}^n(\log D)$ to $\mathbb{P}^n \setminus [JF_I = 0]$ is spanned. □

Now we consider the case of a surface X and $C = C_1 + \cdots + C_q$ a divisor in X with irreducible components $C_i, i = 1, \ldots, q$. Assume that C is of simple normal crossings. In particular this implies that each component is smooth and $C_i \neq C_j$ if $i \neq j$. Denote by $E = T^*X(\log C)$ the logarithmic cotangent bundle. The logarithmic Chern numbers $\bar{c}_1 = c_1(E) = c_1(K_X + C)$ (as $\det E = K_X + C$ is the logarithmic canonical bundle) and $\bar{c}_2 = c_2(E)$ are defined and are given by the following formulas (for more details see [21] and [10] in which more general situations are considered):

$$\begin{cases} \bar{c}_1^2 = (c_1(K_X + C))^2, \\ \bar{c}_2 = 3 - \sum_{i=1}^q (2 - 2g(C_i)) + \sum_{1 \leq i < j \leq q} C_i C_j, \end{cases} \tag{3.3}$$

where $g(C_i)$ is the genus of C_i.

Take $X = \mathbb{P}^2$ and assume, without loss of generality, that $d_1 \leq d_2 \leq \ldots \leq d_q$. Then

$$\bar{c}_1 = \sum_{i=1}^q d_i - 3$$

hence

$$\det E \text{ is ample if and only if } \deg C = \sum_{i=1}^q d_i \geq 4. \tag{3.4}$$

Using the genus formula, we get from (3.3)

$$\begin{aligned} \bar{c}_2 &= 3 - 2q + \sum_{i=1}^q (d_i - 1)(d_i - 2) + \sum_{1 \leq i < j \leq q} d_i d_j \\ &= 3 + \sum_{i=1}^q d_i^2 + \sum_{1 \leq i < j \leq q} d_i d_j - 3 \sum_{i=1}^q d_i \end{aligned}$$

so that

$$\bar{c}_1^2 - \bar{c}_2 = \sum_{1 \leq i < j \leq q} d_i d_j - 3 \sum_{i=1}^q d_i + 6 \tag{3.5}$$

By a direct verification using the preceding formula we have:

Theorem 3.6. *Let $C = C_1 + \cdots + C_q$ be a curve, of simple normal crossings, in $\mathbb{P}^2$ with irreducible components C_i of degree d_i for $1 \leq i \leq q$. Assume that $K_{\mathbb{P}^2} + C$ is ample. Then $\bar{c}_1^2 - \bar{c}_2 > 0$ if and only if one of the following cases hold*:

$$\begin{cases} q \geq 5: & 1 \leq d_1 \leq d_2 \leq \cdots \leq d_q, \\ q = 4: & (i)\ d_1 = d_2 = 1, 2 \leq d_3 \leq d_4, \ \ (ii)\ d_1 = 1, 2 \leq d_2 \leq d_3 \leq d_4, \\ & (iii)\ 2 \leq d_1 \leq d_2 \leq d_3 \leq d_4, \\ q = 3: & (i)\ d_1 = 1, d_2 = 3, 4 \leq d_3, \ \ (ii)\ d_1 = 1, 4 \leq d_2 \leq d_3, \\ & (iii)\ d_1 = d_2 = 2, 3 \leq d_3, \ \ (iv)\ d_1 = 2, 3 \leq d_2 \leq d_3, \\ & (v)\ 3 \leq d_1 \leq d_2 \leq d_3, \\ q = 2: & (i)\ d_1 = 4, 7 \leq d_2, \ \ (ii)\ 5 \leq d_1 \leq d_2. \end{cases}$$

Proof. If $q = 1$ then $\Delta_1 = -3(d_1 - 2)$ hence $\Delta_1 > 0$ if and only if $d_1 = 1$. This case is eliminated by the condition that $K_{\mathbb{P}^2} + C$ is ample.

If $q = 2$ then $\Delta_2 = \bar{c}_1^2 - \bar{c}_2 = d_1 d_2 - 3(d_1 + d_2) + 6$. If $d_1 = 1$ then $\Delta_2 = -3d_1 - 2d_2 + 6$. The ampleness of $K_{\mathbb{P}^2} + C$ is equivalent to $1 + d_2 = d_1 + d_2 \geq 4$ or equivalently, $d_2 \geq 4$ but then $\Delta_2 = -3 - 2d_2 + 6 = -2d_2 + 3 < 0$. Thus the case $d_1 = 1$ is eliminated. If $d_1 = 2$ then $d_2 \geq d_1 \geq 2$ hence $\Delta_2 = -3d_1 - d_2 + 6 = -d_2$ is strictly negative. Thus the case $d_1 = 2$ is also eliminated. Analogously, if $d_2 \geq d_1 = 3$ then $\Delta_2 = -3d_1 + 6 = -3 < 0$ hence this case is also eliminated. If $d_2 \geq d_1 = 4$ then $\Delta_2 = -3d_1 + d_2 + 6 = d_2 - 6$ which is strictly positive if and only if $d_2 \geq 7$. If $d_2 \geq d_1 = 5 + k \geq 5$ then $\Delta_2 = -3d_1 + (2 + k)d_2 + 6 = (2 + k)d_2 - 9$ which is strictly positive if and only if $d_2 \geq d_1$.

If $q = 3$ then $\Delta_3 = \bar{c}_1^2 - \bar{c}_2 = d_1 d_2 + d_2 d_3 + d_3 d_1 - 3(d_1 + d_2 + d_3) + 6 = \Delta_2 + d_2 d_3 + d_3 d_1 - 3d_3 = \Delta_2 + (d_1 + d_2 - 3)d_3$ where $\Delta_2 = d_1 d_2 - 3(d_1 + d_2) + 6$. Thus, if $d_3 \geq d_2 \geq d_1 = 1$ then $\Delta_3 = -2d_2 + 3 + (d_1 + d_2 - 3)d_3 = -2d_2 + 3 + (d_2 - 2)d_3$ which is $\leq \Delta_2$ if $d_2 \geq 2$. Thus the cases $(d_1 = 1, d_2 \leq 2, d_2 \leq d_3)$ are eliminated. If $d_3 \geq d_2 \geq d_1 = 2$ then $\Delta_3 = -d_2 + (d_1 + d_2 - 3)d_3 = -d_2 + (d_2 - 1)d_3$ which is ≤ 0 if and only if $(d_3 - 1)d_2 \leq d_3$ if and only if $d_2 = d_3 = 2$. Thus only the case $d_1 = d_2 = d_3 = 2$ is eliminated. If $d_3 \geq d_2 \geq d_1 = 3 + k \geq 3$ then $\Delta_3 = -3(3 + k) + kd_2 + 6 + d_2 d_3 = (d_2 - 3)k - 3 + d_2 d_3 \geq -3 + d_2 d_3 > 0$.

If $q = 4$ then $\Delta_4 = \bar{c}_1^2 - \bar{c}_2 = \Delta_3 + (d_1 + d_2 + d_3 - 3)d_4$ where $\Delta_3 = d_1 d_2 + d_2 d_3 + d_3 d_1 - 3(d_1 + d_2 + d_3) + 6$. Thus $\Delta_4 \geq \Delta_3$ unless $d_1 = d_2 = d_3 = 1$. Thus we must eliminate the cases $d_4 \geq d_3 = d_2 = d_1 = 1$. If $d_3 = 2 + k \geq 2$ then $(d_1 + d_2 + d_3 - 3)d_4 \geq (d_1 + d_2 - 1 + k)d_4 \geq 2(d_1 + d_2 - 1 + k)$ and $\Delta_3 = d_1 d_2 + (2 + k)d_2 + (2 + k)d_1 - 3(d_1 + d_2 + 2 + k) + 6 = d_1 d_2 - (1 - k)(d_1 + d_2)$. Thus $\Delta_4 \geq d_1 d_2 + (1 + k)(d_1 + d_2) - 2 - k > 0$ for all $k \geq 0$.

If $q = 5$ then $\Delta_5 = \bar{c}_1^2 - \bar{c}_2 = \Delta_4 + (d_1 + d_2 + d_3 + d_4 - 3)d_5$ where $\Delta_4 = d_1 d_2 + d_1 d_3 + d_1 d_4 + d_2 d_3 + d_2 d_4 + d_3 d_4 - 3(d_1 + d_2 + d_3 + d_4) + 6$. For $d_5 \geq d_4 \geq d_3 \geq d_2 \geq d_1 \geq 1$, $\Delta_5 > \Delta_4$. Thus $\Delta_5 > 0$ if $\Delta_4 > 0$ and so all the cases that work for $q = 4$ certainly work for $q = 5$. The only remaining cases to be checked are $d_3 = d_2 = d_1 = 1$. In these cases $\Delta_4 = 3 + 3d_4 - 3(3 + d_4) + 6 = 0$ hence $\Delta_5 > 0$ even in these cases.

It is clear that $\Delta_q > 0$ for $q \geq 6$. $\square$

Remark 3.7. If we drop the ampleness condition on $K_{\mathbb{P}^2} + C$ in the preceding theorem then there are two more cases: (a) $q = 1, d_1 = 1$ and (b) $q = 2, d_1 = d_2 = 1$.

Remark 3.8. It is convenient to list all the cases violating Theorem

3.6:

$q=2$	$q=3$	$q=4$	$q\geq 5$
$(1,\geq 2)$	$(1,1,\geq 1)$	$(1,1,1,\geq 1)$	none
$(2,\geq 2)$	$(1,2,\geq 2)$		
$(3,\geq 3)$	$(1,3,3)$		
$(4,4)$	$(2,2,2)$		
$(4,5)$			
$(4,6)$			

Corollary 3.9. *Let $C = C_1 + \cdots + C_q$ be any of the curves in the list below*:

$$\begin{cases} q \geq 5: & 1 \leq d_1 = d_2 = \cdots = d_q, \\ q = 4: & 2 \leq d_1 = d_2 = d_3 = d_4. \end{cases}$$

Assume that $\cap[JF_I = 0] = \emptyset$ where I ranges over all subsets of $\{1, 2, \ldots, q\}$ consisting of 3 distinct elements. Then there exists a complete Finsler metric on $T^*(\mathbb{P}^2 \setminus C)$ with holomorphic bisectional curvature $\leq -c^2$ where c is a constant.

Proof. Any curve C in the list of Theorem 3.6 has the property that the bundle $E = T^*\mathbb{P}^2 \setminus C$ satisfies the conditions of Theorem 2.2 except for "spannedness". This last condition is satisfied, via Corollary 3.5, under the assumption that $\cap[JF_I = 0] = \emptyset$. The existence of a Finsler metric on $T^*(\mathbb{P}^2 \setminus C)$ with holomorphic bisectional curvature $\leq -c^2$, where c is a constant, is guaranteed by Theorem 2.2. By [3] this metric is constructed as follows. Since E is ample there exists a positive integer m such that $\mathrm{sym}^m E$ is very ample. Let $\omega_0, ..., \omega_N$ be a basis of $H^0(X, \mathrm{sym}^m E)$. Then

$$\sum_{i=0}^{N} |\omega_i|^{2/m}$$

is the desired Finsler metric. Since $E = T^*X(\log C)$ the metric blows up along C and completeness follows. $\square$

Example 3.10. Let $Q_0 = z_0^2, Q_1 = z_1^2, Q_2 = z_2^2$ and

$$Q_3 = \sum_{i,j=0}^{2} a_{ij} z_i z_j$$

with $a_{ij} = a_{ji}$. The coefficients of Q_3 are chosen such that the linear forms:

$$L_0 = \sum_{j=0}^{2} a_{0j} z_j, \quad L_1 = \sum_{j=0}^{2} a_{1j} z_j, \quad L_2 = \sum_{j=0}^{2} a_{2j} z_j$$

are linearly independent and $a_{ij} \neq 0$ for all i, j. The linearly independent condition is equivalent to the condition that the hyperplanes $[L_0 = 0], [L_1 = 0], [L_2 = 0]$ have no common zero. It is clear that a general quadrics satisfies these conditions. Let $F_{i_0 i_1 i_2} = (Q_{i_0}, Q_{i_1}, Q_{i_2})$. Then the Jacobian determinant

$$JF_{012} = 8 \det \begin{pmatrix} z_0 & 0 & 0 \\ 0 & z_1 & 0 \\ 0 & 0 & z_2 \end{pmatrix} = 8 z_0 z_1 z_2,$$

$$JF_{013} = 8 \det \begin{pmatrix} z_0 & 0 & 0 \\ 0 & z_1 & 0 \\ \sum_{j=0}^2 a_{0j} z_j & \sum_{j=0}^2 a_{1j} z_j & \sum_{j=0}^2 a_{2j} z_j \end{pmatrix} = 8 z_0 z_1 \sum_{j=0}^2 a_{2j} z_j,$$

$$JF_{032} = 8 \det \begin{pmatrix} z_0 & 0 & 0 \\ \sum_{j=0}^2 a_{0j} z_j & \sum_{j=0}^2 a_{1j} z_j & \sum_{j=0}^2 a_{2j} z_j \\ 0 & 0 & z_2 \end{pmatrix} = 8 z_0 z_2 \sum_{j=0}^2 a_{1j} z_j,$$

$$JF_{312} = 8 \det \begin{pmatrix} \sum_{j=0}^2 a_{0j} z_j & \sum_{j=0}^2 a_{1j} z_j & \sum_{j=0}^2 a_{2j} z_j \\ 0 & z_1 & 0 \\ 0 & 0 & z_2 \end{pmatrix} = 8 z_1 z_2 \sum_{j=0}^2 a_{0j} z_j.$$

By the choice of Q_3 we have

$$(*) \qquad [JF_{012} = 0] \cap [JF_{013} = 0] \cap [JF_{032} = 0] \cap [JF_{312} = 0] = \emptyset.$$

The configurations satisfying condition $(*)$ is Zariski open and, as was demonstrated above, is non-empty. In particular, we may deform, obtaining: $Q_{0,t}, Q_{1,t}, Q_{2,t}, Q_{3,t}$ so that $Q_{i,0} = Q_i$ and $C_{i,t} = [Q_{i,t} = 0]$ is a smooth quadrics for $t \neq 0$ ($|t|$ small), $C_t = C_{0,t} + C_{1,t} + C_{2,t} + C_{3,t}$ is of simple normal crossings and condition $(*)$ is satisfied for C_t.

Remark 3.11. The example can obviously be carried out for any degree. Thus the condition "$\cap [JF_I = 0] = \emptyset$ where I ranges over all subsets of $\{1, 2, \ldots, q\}$ consisting of 3 distinct elements" in Corollary 3.9 may be replaced by requiring that the curve be general (meaning that the exceptional set is Zariski closed and of strictly lower dimension) :

Let $C = C_1 + \cdots + C_q$ be a general member of any of the curves in the list below:

$$\begin{cases} q \geq 5: & 1 \leq d_1 = d_2 = \cdots = d_q, \\ q = 4: & 2 \leq d_1 = d_2 = d_3 = d_4. \end{cases}$$

Then there exists a complete Finsler metric on $T^*(\mathbb{P}^2 \setminus C)$ with holomorphic bisectional curvature $\leq -c^2$ where c is a constant.

Consider now the case of a *smooth* hypersurface X in $\mathbb{P}^3$ of degree a and let $C_i = X \cap Y_i, i = 1, \ldots, q$, be an irreducible smooth curve in X, where Y_i is a hypersurface of degree b_i in $\mathbb{P}^3$. Let $C = C_1 + C_2 + \cdots + C_q$. The Euler numbers of X, C_i and C are given by the following formulas (see [10]):

$$\begin{cases} e(X) = a(2 + (a-2)^2), \\ e(C_i) = a(4 - a - b_i)b_i, \\ e(C) = \sum_{i=1}^{q} e(C_i) - \sum_{1 \le i < j \le q} C_i \cdot C_j \end{cases} \tag{3.6}$$

where $\sum_{1 \le i < j \le q} C_i \cdot C_j = ab_ib_j$. Hence

(3.7)

$$e(C) = a\left(\sum_{i=1}^{q}(4 - a - b_i)b_i - \sum_{1 \le i < j \le q} b_ib_j\right) = a\left((4 - a - b)b + \sum_{1 \le i < j \le q} b_ib_j\right)$$

where $b = \sum_{i=1}^{q} b_i$. Let $E = T^*X(\log C)$ and $\bar{c}_i = c_i(E)$. Then (by Sakai [21]):

$$\bar{c}_1^2 - \bar{c}_2 = c_1^2(K_X + C) - e(X) + e(C).$$

By the adjunction formula we have $K_X = \mathcal{O}_{\mathbb{P}^3}(a-4)|_X$, hence $K_X + C = \mathcal{O}_{\mathbb{P}^3}(a + b - 4)|_X$, which is ample if and only if $a + b \ge 5$ and

$$\bar{c}_1^2 = c_1^2(K_X + C) = (\det E)^2 = a(a + b - 4)^2, \;\; b = \sum_{i=1}^{q} b_i. \tag{3.8}$$

This together with (3.6) and (3.7) yield

$$\bar{c}_1^2 - \bar{c}_2 = a(a + b - 4)^2 - a(2 + (a-2)^2) + a\left((4 - a - b)b + \sum_{1 \le i < j \le q} b_ib_j\right).$$

This shows that $\bar{c}_1^2 - \bar{c}_2 > 0$ if and only if

$$(a + b - 4)^2 - (2 + (a-2)^2) + \left((4 - a - b)b + \sum_{1 \le i < j \le q} b_ib_j\right) > 0.$$

This last inequality is equivalent to the condition that

$$(a-4)(b-4) - 6 + \sum_{1 \le i < j \le q} b_ib_j > 0.$$

If $a = 1$ then $X = \mathbb{P}^2$, a case which was dealt with earlier. In the following a q-tuple $(b_1, \ldots, b_q)$ represents the degrees of $Y_i, i = 1, \ldots, q$ with the convention that $b_1 \le b_2 \le \cdots \le b_q$.

Theorem 3.12. *Under the assumptions above the condition $\bar{c}_1^2 - \bar{c}_2 > 0$ is satisfied if and only if C is in the list below:*

I. a = 2

$$\begin{cases} q \geq 5: & \text{all}, \\ q = 4: & \text{all except } (1,1,1,1), \\ q = 3: & (i)\ \ b_1 = 1, 2 \leq b_2 \leq b_3,\ \ (ii)\ \ 2 \leq b_1 \leq b_2 \leq b_3, \\ q = 2: & (i)\ \ b_1 = 3, b_2 \geq 5,\ \ (ii)\ \ 4 \leq b_1 \leq b_2. \end{cases}$$

II. a = 3

$$\begin{cases} q \geq 5: & \text{all}, \\ q = 4: & \text{all except } (1,1,1,1), \\ q = 3: & (i)\ \ b_1 = b_2 = 1, b_3 \geq 4, \\ & (ii)\ \ b_1 = 1, 2 \leq b_2 \leq b_3, \\ & (iii)\ \ 2 \leq b_1 \leq b_2 \leq b_3, \\ q = 2: & (i)\ \ b_1 = 2, b_2 \geq 5,\ \ (ii)\ \ 3 \leq b_1 \leq b_2. \end{cases}$$

III. a = 4

$$\begin{cases} q \geq 5: & \text{all}, \\ q = 4: & \text{all except } (1,1,1,1), \\ q = 3: & (i)\ \ b_1 = b_2 = 1, b_3 \geq 3, \\ & (ii)\ \ b_1 = 1, 2 \leq b_2 \leq b_3, \\ & (iii)\ \ 2 \leq b_1 \leq b_2 \leq b_3, \\ q = 2: & (i)\ \ b_1 = 1, b_2 \geq 7,\ \ (ii)\ \ b_1 = 2, b_2 \geq 4,\ \ (iii)\ \ 3 \leq b_1 \leq b_2. \end{cases}$$

IV. a $\geq$ 5

$$\begin{cases} q \geq 5: & \text{all}, \\ q = 4: & \text{all except } (1,1,1,1), \\ q = 3: & (i)\ \ b_1 = b_2 = 1, b_3 \geq 3, \\ & (ii)\ \ b_1 = 1, 2 \leq b_2 \leq b_3, \\ & (iii)\ \ 2 \leq b_1 \leq b_2 \leq b_3, \\ q = 2: & (i)\ \ b_1 = 1, b_2 > \dfrac{3a-6}{a-3},\ \ (ii)\ \ b_1 = 2, b_2 \geq 3,\ \ (iii)\ \ 3 \leq b_1 \leq b_2, \\ q = 1: & b_1 > \dfrac{4a-10}{a-4}. \end{cases}$$

We omit the straight-forward verification. Note that all cases in the list of the preceding theorem satisfies the condition that $K_X + C$ is ample because $a + b \geq 5$. We also have the analogue of Corollary 3.9 (Remark 3.11):

Corollary 3.13. *Let X be a smooth hypersurface of degree a in $\mathbb{P}^3$ and $C_i = X \cap Y_i, i = 1, \ldots, q$, be an irreducible smooth curve in X where Y_i is a hypersurface of degree b_i in $\mathbb{P}^3$. Let $C = C_1 + \cdots + C_q$ be a general member of* *any of the families of curves in the list below*:

$$\begin{cases} q \geq 5: & 1 \leq b_1 = b_2 = \cdots = b_q, \\ q = 4: & 2 \leq b_1 = b_2 = b_3 = b_4. \end{cases}$$

Then there exists a complete Finsler metric on $T^*(X \setminus C)$ with holomorphic bisectional curvature $\leq -c^2$ where c is a constant.

4. The case where the components may have different degrees

Denote by $\mathcal{M}_D(X)^*$ (or simply $\mathcal{M}_D^*$) the sheaf of germs of not identically zero meromorphic functions on X with zeros and poles contained in a divisor D in X. Given a local meromorphic function f ($\not\equiv 0$) on an open neighborhood $U \subset M$ with zeros and poles contained in $U \cap D$, then $d\log f$ is a meromorphic 1-form on U with logarithmic poles in $U \cap D$. Thus $d\log(\mathcal{M}_D(M)^*)$ (or simply $d\log\mathcal{M}_D^*$) is a subsheaf of $T^*X^1(D)(= T^*X \otimes [D])$ = the sheaf of germs of meromorphic 1-forms with poles along D. It is immediate from the definition that the following sequence is exact

$$0 \to \mathbb{C}^* \to \mathcal{M}_D^* \overset{d\log}{\longrightarrow} d\log(\mathcal{M}_D^*) \to 0$$

and we have the induced exact sequence of cohomology groups

(4.1)
$$0 \to H^0(X, \mathbb{C}^*) \to H^0(X, \mathcal{M}_D^*) \overset{d\log}{\longrightarrow} H^0(X, d\log\mathcal{M}_D^*) \overset{\Delta}{\longrightarrow} H^1(X, \mathbb{C}^*) \to \cdots$$

Definition 4.1. The sheaf of germs of logarithmic 1-forms with poles along an arbitrary divisor D, denoted by $T^*X^1(\log D)$, is by definition the sheaf generated by $d\log\mathcal{M}_D(X)^*$ over $\mathcal{O}_X$, i.e., it is the sheaf of germs of rational 1-forms of the type

$$\sum f_i \omega_i$$

where $f_i \in (\mathcal{O}_X)_x$ and $\omega_i \in (d\log\mathcal{M}_D(X)^*)_x$.

The definition coincides with our earlier definition in the case where D is of simple normal crossings. In fact the following result is known ([8], [15]):

Proposition 4.2. *Let $\beta : (\hat{X}, \hat{D}) \to (X, D)$ be a succession of monodial transformations with the properties that*

(i) $\beta^* : \hat{X} \setminus \hat{D} \to X \setminus D$ is biholomorphic;

(ii) $\hat{D}$ is a divisor of simple normal crossings.

Then the induced maps

$$\beta^* : H^0(X, d\log\mathcal{M}_D(X)^*)) \to H^0(\hat{X}, d\log\mathcal{M}_{\hat{D}}(\hat{X})^*))$$

and

$$\beta^* : H^0(X, T^*X(\log D)) \to H^0(\hat{X}, T^*X(\log \hat{D}))$$

are isomorphisms.

To extend the results in the preceding to the case where the components of the divisor D (assumed to be simple normal crossings) may have different degrees d_i, we need only the fact that for any rational function $f \in H^0(X, \mathcal{M}_D^*)$ the singularity of $d \log f$ is of log-type along D (otherwise regular). Hence we have an injection

$$0 \to H^0(d \log \mathcal{M}_D^*) \to H^0(T^* X^1(\log D)).$$

Let $D_i = [P_i = 0]$ where $P_i(z_0, \ldots, z_n)$ is a homogeneous polynomial of degree $d_i = \deg D_i$. Let m be the least common multiple of $d_0, \ldots, d_n$ and let $m = d_i m_i$. Then $\deg P_i^{m_i} = m$ and the map

$$f = [P_0^{m_0}, P_1^{m_1}, \ldots, P_n^{m_n}] : \mathbb{P}^n \to \mathbb{P}^n$$

is a well-defined holomorphic map. This is well-defined because each component $P_i^{m_i}$ is homogeneous of the same degree m. It is holomorphic because D is of simple normal crossings hence, for any $x \in \mathbb{P}^n$ there exists i such that $x \notin D_i$, i.e., $P_i(x) \neq 0$. This is equivalent to the condition that $P_i^{m_i}(x) \neq 0$. This shows that $P_0^{m_0}, P_1^{m_1}, \ldots, P_n^{m_n}$ have no common zeros hence f is holomorphic. By Lemma 3.2, $E = \mathbb{P}^n(\log H)$ (where $H = [z_0 = 0] + [z_1 = 0] + \cdots + [z_n = 0]$) is spanned. Thus f^*E is also spanned. In fact $\omega_i = dt_i/t_i, t_i = z_i/z_0, i = 1, \ldots, n$ span E. The pull back $f^* t_i = P_i^{m_i}/P_0^{m_0}$ is an element of $H^0(\mathbb{P}^n, \mathcal{M}_D^*)$ hence by the observation above,

$$f^*\omega_i = f^* d \log t_i \in H^0(\mathbb{P}^n, d \log \mathcal{M}_D^*) \subset H^0(\mathbb{P}^n, T^*\mathbb{P}^n(\log D)).$$

Thus we obtain as in section 2:

Corollary 4.3. *Let $D = D_0 + D_1 + \cdots + D_n$ be a divisor in $\mathbb{P}^n$ with irreducible components D_i, of degree d_i, in simple normal crossings. Then the restriction of $T^*\mathbb{P}^n(\log D)$ to $\mathbb{P}^n \setminus [JF = 0]$ is spanned where JF is given by*

$$JF = \det \begin{pmatrix} \partial P_0^{m_0}/\partial z_0 & \partial P_0^{m_0}/\partial z_1 & . & . & . & \partial P_0^{m_0}/\partial z_n \\ \partial P_1^{m_1}/\partial z_0 & \partial P_1^{m_1}/\partial z_1 & . & . & . & \partial P_1^{m_1}/\partial z_n \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \partial P_n^{m_n}/\partial z_0 & \partial P_n^{m_n}/\partial z_1 & . & . & . & \partial P_n^{m_n}/\partial z_n \end{pmatrix},$$

a homogeneous polynomial of degree $(m-1)(n+1)$ where m is the least common multiple of $d_0, \ldots, d_n$ and $m = d_i m_i$.

Theorem 4.4. *Let X be a smooth hypersurface of degree a in $\mathbb{P}^3$ and $C_i = X \cap Y_i, i = 1, \ldots, q$, be an irreducible smooth curve in X where Y_i is a hypersurface of degree b_i in $\mathbb{P}^3$. Let $C = C_1 + \cdots + C_q$ be a general member of any of the families of curves in the list below:*

(i) a $= 1$

$$\begin{cases} q \geq 5: & 1 \leq b_1 \leq b_2 \leq ... \leq b_q, \\ q = 4: & (i)\ b_1 = b_2 = 1, 2 \leq b_3 \leq b_4, \\ & (ii)\ b_1 = 1, 2 \leq b_2 \leq b_3 \leq b_4, \\ & (iii)\ 2 \leq b_1 \leq b_2 \leq b_3 \leq b_4. \end{cases}$$

(ii) a $= 2$

$$\begin{cases} q \geq 5: & 1 \leq b_1 \leq b_2 \leq ... \leq b_q, \\ q = 4: & 1 \leq b_1 \leq b_2 \leq b_3 \leq b_4 \ \text{ except the case } \ b_1 = b_2 = b_3 = b_4 = 1. \end{cases}$$

Then there exists a complete Finsler metric on $T^*(X \setminus C)$ *with holomorphic bisectional curvature* $\leq -c^2$ *where c is a constant.*

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