J. Math. Kyoto Univ. (JMKYAZ)
45-3 (2005), 547–597

Characters of wreath products of finite groups with the infinite symmetric group

By

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Introduction

1. Let G be a countable discrete group, $K_1(G)$ the set of all positive definite class functions f on G normalized as $f(e) = 1$, and $E(G)$ the set of all extremal elements in the convex set $K_1(G)$, where e denotes the identity element of G. In [Tho1], a canonical bijective correspondence between $E(G)$ and the set of characters of all factor representations of finite type is established (cf. **1.2** below). In this sense every element $f \in E(G)$ is called a *character* of G.

The purpose of this paper is to give explicitly all the characters of the wreath product groups $G = \mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$ of any finite groups T with the infinite symmetric group $\mathfrak{S}_\infty$. This problem of determining all the characters of factor representations of finite type, or the problem of giving a general character formula for $f \in E(G)$, was worked out in [Tho2] for $G = \mathfrak{S}_\infty$. The result for $G = GL(\infty, \boldsymbol{F}_q)$ with a finite field $\boldsymbol{F}_q$ was given in [Sk].

The case of infinite symmetric group attracted interests of many mathematicians and we cite here, among others, works of Vershik-Kerov [VK], Kerov-Olshanski [KO] and Biane [Bi] in which they worked principally from the point of view of approximation from finite symmetric groups $\mathfrak{S}_n$ $(n \to \infty)$. Recently in [Hi3]–[Hi4], we reexamined the case of $\mathfrak{S}_\infty$ from the standpoint of taking limits of centralizations of positive definite functions obtained as matrix elements of simple unitary representations. Since this is one of our main ideas, let us explain it briefly here. For a subgroup G' of G a *centralization* of a function F on G with respect to G' is by definition

$$F^{G'}(g) = \frac{1}{|G'|} \sum_{g' \in G'} F(g' g\, {g'}^{-1}) \qquad (g \in G).$$

Taking an appropriate series of increasing subgroups $G_N \nearrow G$ as $N \to \infty$, we consider pointwise limit $f(g) = \lim_{N\to\infty} F^{G_N}(g)$. Here as a function F, we choose a positive definite matrix element of an induced representation $\rho = \mathrm{Ind}_H^G \pi$ of a (not necessary irreducible) unitary representation π of a subgroup

Received January 12, 2005

H of wreath product type. In [Ob1]–[Ob2] and [Hi1]–[Hi2], it is shown that appropriate choices of H and π give us a big family of irreducible unitary representations ρ of G. However here, to get characters of G as limits f of this kind, we found as a result that it is better to choose ρ rather far from to be irreducible.

Next to $\mathfrak{S}_\infty$, we proceed to the case of wreath products $\mathfrak{S}_\infty(T)$ with finite abelian groups T, and of their canonical subgroups. This case contains the cases of infinite Weyl groups $W_{\mathbf{B}_\infty} = \mathfrak{S}_\infty(\boldsymbol{Z}_2)$ of type $\mathbf{B}_\infty/\mathbf{C}_\infty$ and $W_{\mathbf{D}_\infty}$ of type $\mathbf{D}_\infty$, and limits $\mathfrak{S}_\infty(\boldsymbol{Z}_r)$ of complex reflexion groups $G(r,1,n)$ as $n \to \infty$ (for finite complex reflexion groups, cf. [Ka] and [Sh]). For this abelian case, a general explicit formula for characters $f \in E(G)$ is given in [HH1] with a sketch of proof, and so all the factor representations of finite type are classified for $G = \mathfrak{S}_\infty(T)$, T abelian.

Thus, we now come to the present case of $\mathfrak{S}_\infty(T)$ with any finite group T.

2. This paper is organized as follows. In §1, we define the wreath product group $G = \mathfrak{S}_\infty(T)$, and review the theory of characters of discrete groups. In §2, the structure of G is studied and conjugacy classes are given in Theorem 1, and then finite-dimensional irreducible representations are classified.

In §3, a general character formula for $f \in E(G)$ is given in Theorem 2 for $G = \mathfrak{S}_\infty(T)$ with T any finite group, whose proof occupies §§7–11 and §§13–16. Then, in §4, the case where T is abelian is treated. The character formula in this case has a much simpler form (Theorem 3). In §5, a canonical subgroup $G^e = \mathfrak{S}^e_\infty(T) = D^e_\infty(T) \rtimes \mathfrak{S}_\infty$ with T abelian is treated (Theorem 4). In §6, the infinite Weyl groups $W_{\mathbf{B}_\infty} = \mathfrak{S}_\infty(\boldsymbol{Z}_2)$ and its subgroup $W_{\mathbf{D}_\infty} = \mathfrak{S}^e_\infty(\boldsymbol{Z}_2)$ are treated (Theorem 6 and Theorem 7 respectively).

In §7, our method of proving Theorem 2 is explained, the first part of the proof in **7.1** and the second part in **7.2**. The first part occupies §§8–11 and §§13–14. In §8, the *centralization* of positive definite functions, is treated. In §9, an *inducing up* of a matrix element of π to that of $\rho = \mathrm{Ind}_H^G \pi$ is discussed. In §10, we discuss choices of H and π, refering the results in [Hi1]–[Hi2] and [Hi3]–[Hi4]. In §11, choices of series of increasing subgroups $G_N \nearrow G$ is discussed. Actually, our choices of the series give us as pointwise limits a big set $\mathcal{LIM}$ of positive definite class functions, which later turns out to be equal to $E(G)$.

In §12, we consider, for each $(\zeta, \varepsilon) \in \widehat{T} \times \{0,1\}$, a series of a special kind of irreducible representations ρ_n of $\mathfrak{S}_n(T)$ converging to an irreducible representation ρ of $G = \mathfrak{S}_\infty(T)$, and calculate limits of trace characters $F_{\zeta,\varepsilon}(g) := \lim_{n\to\infty} \mathrm{tr}(\rho_n(g))/\dim \rho_n$ $(g \in G)$ (Theorem 9). This result is applied in §16.

In §13, according to the choice in §11 of $G_N = \mathfrak{S}_{J_N}(T) = D_{J_N}(T) \rtimes \mathfrak{S}_{J_N}$ with $J_N \nearrow \boldsymbol{N}$, a partial centralization F^{D_N} of F with respect to the subgroup $D_N = D_{J_N}(T)$ of G_N is discussed (Proposition 10). In §14, the centralization with respect to another subgroup $S_N = \mathfrak{S}_{J_N}$ of G_N is performed on F^{D_N} to get finally $F^{G_N} = (F^{D_N})^{S_N}$. Thus we get a set $\mathcal{LIM}$ of positive definite class functions f, all of which are *factorizable* in the sense that $f(gg') = f(g)f(g')$ if the supports of g and g' are mutually disjoint. The set $\mathcal{LIM}$ coincides with the set $\mathcal{F}_\mathcal{A} = \{f_A\}$ of class functions f_A in Theorem 2 corresponding to a parameter $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ ;\ \mu\Big)$. Here every $\alpha_{\zeta,\varepsilon} = (\alpha_{\zeta,\varepsilon,i})_{i\in\boldsymbol{N}}$ is a decreasing sequence of non-negative real numbers, and $\mu = (\mu_\zeta)_{\zeta\in\widehat{T}}$ is a set of non-negative real numbers, which altogether satisfy the condition

$$\sum_{\zeta\in\widehat{T}}\ \sum_{\varepsilon\in\{0,1\}} \|\alpha_{\zeta,\varepsilon}\| \ +\ \|\mu\| \le 1\,,$$

with $\|\alpha_{\zeta,\varepsilon}\| = \sum_{i\in\boldsymbol{N}} \alpha_{\zeta,\varepsilon,i}$, $\|\mu\| = \sum_{\zeta\in\widehat{T}} \mu_\zeta$. Since we get $\mathcal{LIM} = \mathcal{F}_{\mathcal{A}}$, it is proved that every f_A is positive definite (Proposition 11). This completes the first part of the proof of Theorem 2, giving the assertion $\mathcal{F}_{\mathcal{A}} \subset K_1(G)$.

The second part of our proof of Theorem 2 is to prove that every f_A is extremal, and that $\mathcal{F}_{\mathcal{A}} = E(G)$. In §15, generalizing Satz 1 in [Tho2], we give a criterion for a positive definite class function $f \in K_1(G)$ to be extremal. Theorem 12 says that f *is extremal if and only if* f *is factorizable.* Then we see that every $f \in \mathcal{LIM}$ is extremal since f is factorizable, and so we have $\mathcal{LIM} \subset E(G)$.

In §16, the converse inclusion $\mathcal{LIM} \supset E(G)$ is proved (Proposition 13). To do so, we define a Fourier transform of a factorizable positive definite class function f with respect to $F_{\zeta,\varepsilon}$ and calculate it explicitly (Lemma 16.2). Thus the second part of the proof of Theorem 2 is now completed.

In §17, we deduce Theorem 4, character formula for the subgroup $\mathfrak{S}^e_\infty(T)$ in the case of T abelian, from Theorem 2. In §18, the wreath product $\mathfrak{A}_\infty(T) = D_\infty(T) \rtimes \mathfrak{A}_\infty$ of a finite group T with the infinite alternating group $\mathfrak{A}_\infty$ is discussed and its character formula is given in Theorem 14. In the case where T is abelian, we have also a canonical subgroup $\mathfrak{A}^e_\infty(T)$ and its character formula is given in Theorem 15.

In Appendix, we give several lemmas on integrals of matrix elements and characters for compact groups (containing finite groups).

1. Wreath product groups and characters

Let $G = \mathfrak{S}_\infty(T)$ be a wreath product of a finite group T with the infinite symmeric group $\mathfrak{S}_\infty$. The purpose of this paper is to give all the extremal (or indecomposable) positive definite class functions on G. The set $E(G)$ of such functions f, normalized as $f(e) = 1$, covers all the characters of factor representations of finite type, type II_1 or type I_n, $n < \infty$, of G. Here e denotes the identity element of G.

In this section, we first give a definition of wreath product groups, and then review briefly the relation between positive definite class functions and characters of factor representations of finite type, for countable discrete groups.

1.1. Wreath product groups with the infinite symmetric group

For a set I, denote by $\mathfrak{S}_I$ the group of all finite permutations on I. A permutation σ on I is called *finite* if its support $\mathrm{supp}(\sigma) := \{\, i \in I\ ;\ \sigma(i) \neq i\,\}$ is finite. The permutation group $\mathfrak{S}_{\boldsymbol{N}}$ on the set of natural numbers $\boldsymbol{N}$ is called the infinite symmetric group and the index $\boldsymbol{N}$ is frequently replaced by ∞. The symmetric group $\mathfrak{S}_n$ is naturally imbedded in $\mathfrak{S}_\infty$ as the permutation group of the set $\boldsymbol{I}_n := \{1, 2, \ldots, n\} \subset \boldsymbol{N}$.

Let T be a finite group. We consider a wreath product $\mathfrak{S}_I(T)$ of T with a permutation group $\mathfrak{S}_I$ as follows:

$$\mathfrak{S}_I(T) = D_I(T) \rtimes \mathfrak{S}_I\,, \quad D_I(T) = {\prod}'_{i\in I} T_i\,, \quad T_i = T \ (i \in I), \tag{1.1}$$

where the symbol $\prod'$ means the restricted direct product, and $\sigma \in \mathfrak{S}_I$ acts on $D_I(T)$ as

$$D_I(T) \ni d = (t_i)_{i\in I} \xrightarrow{\sigma} \sigma(d) = (t'_i)_{i\in I} \in D_I(T), \quad t'_i = t_{\sigma^{-1}(i)} \ (i \in I). \tag{1.2}$$

Identifying groups $D_I(T)$ and $\mathfrak{S}_I$ with their images in the semidirect product $\mathfrak{S}_I(T)$, we have $\sigma\, d\, \sigma^{-1} = \sigma(d)$. The group $\mathfrak{S}_{\boldsymbol{I}_n}(T)$ is denoted as $\mathfrak{S}_n(T)$, then $G := \mathfrak{S}_\infty(T)$ is an inductive limit of $G_n := \mathfrak{S}_n(T)$, and $G = \lim_{n\to\infty} G_n$ is countably infinite.

In the case where T is abelian, we put

$$P_I(d) = \prod_{i\in I} t_i \ \in T \quad \text{for} \quad d = (t_i)_{i\in I} \in D_I(T), \tag{1.3}$$

and define a subgroup of $\mathfrak{S}_I(T)$ as

$$\begin{aligned} \mathfrak{S}^e_I(T) \ &= \ D^e_I(T) \rtimes \mathfrak{S}_I \\ \text{with} \quad D^e_I(T) \ &:= \ \{\, d = (t_i)_{i\in I}\ ;\ P_I(d) = e_T \,\}, \end{aligned} \tag{1.4}$$

where e_T denotes the identity element of T.

This kind of wreath product groups contain the infinite Weyl groups of classical type, $W_{\mathbf{A}_\infty} = \mathfrak{S}_\infty, W_{\mathbf{B}_\infty} = \mathfrak{S}_\infty(\boldsymbol{Z}_2)$ and $W_{\mathbf{D}_\infty} = \mathfrak{S}^e_\infty(\boldsymbol{Z}_2)$, and moreover the inductive limits $\mathfrak{S}_\infty(\boldsymbol{Z}_r) = \lim_{n\to\infty} G(r, 1, n)$ of complex reflexion groups $G(r, 1, n) = \mathfrak{S}_n(\boldsymbol{Z}_r)$ (cf. [Ka], [Sh]).

1.2. Characters and positive definite functions on infinite discrete groups

Let G be an infinite discrete group. Denote by $C_c(G)$ the $*$-algebra of all compactly supported functions on G with the operations

$$(\psi_1 * \psi_2)(g) := \sum_{h\in G} \psi_1(gh^{-1})\psi_2(h), \quad \psi^{\,*}(g) = \overline{\psi(g^{-1})}, \tag{1.5}$$

for $\psi_1, \psi_2, \psi \in C_c(G)$, and $g \in G$. Then it has a basis $\{\, \delta_g\ ;\ g \in G \,\}$, where δ_g denotes a function on G having the value 1 at g, and zero elsewhere. The identity element of $C_c(G)$ is given by δ_e. The completion of $C_c(G)$ with respect to a certain special norm is the C^*-algebra $C^*(G)$ of G.

A unitary representation π of G corresponds bijectively to a non-degenerate representation of $C_c(G)$ and that of $C^*(G)$ through

$$\pi(\psi) := \textstyle\sum_{g\in G} \psi(g)\pi(g) \quad (\psi \in C_c(G)).$$

We refer [Di, §6] for the theory of traces and characters of representations of C^*-algebras, and for our special case of discrete groups, we refer also [Tho1].

For a C^*-algebra, a *character* $\boldsymbol{t}$ is, by definition, a trace which is semifinite, semicontinuous from below, and such that any such trace majorized by $\boldsymbol{t}$ is proportional to $\boldsymbol{t}$. This is translated on the level of $C_c(G)$ as follows. A *trace* $\boldsymbol{t}$ on $C_c(G)$ is a positive definite functional satisfying

$$\boldsymbol{t}(\psi_1 * \psi_2) = \boldsymbol{t}(\psi_2 * \psi_1) \quad (\psi_1, \psi_2 \in C_c(G)). \tag{1.6}$$

It is determined by a positive definite *invariant* (or *class*) function f as

$$\boldsymbol{t} : C_c(G) \ni \psi \longmapsto f(\psi) := \textstyle\sum_{g \in G} f(g)\psi(g) \in \boldsymbol{C}. \tag{1.7}$$

A trace $\boldsymbol{t}$ is a *character* in the sense of C^*-algebras if and only if the corresponding f is extremal or indecomposable ([Tho1, Korollar 2 to Lemma 2]).

Let us explain a little more. In general, for a non-zero positive definite function f on G, we can associate, by GNS construction, a cyclic representation π_f as follows. Introduce in $\mathfrak{A} := C_c(G)$ a positive semidefinite inner product as

$$\langle \psi_1, \psi_2 \rangle_f := \sum_{g,h \in G} f(h^{-1}g)\psi_1(g)\overline{\psi_2(h)} \;=\; f(\psi_2^* * \psi_1) \qquad (\psi_1, \psi_2 \in \mathfrak{A}).$$

Then, this is invariant under left G-action: $(L(g_0)\psi)(g) := \psi({g_0}^{-1}g)$ $(g_0, g \in G)$. Let J_f be the kernel of $\langle \cdot\,, \cdot \rangle_f$, and take a completion of the quotient space $\mathfrak{A}_f = \mathfrak{A}/J_f$, we get a Hilbert space $\mathfrak{H}_f$, on which a unitary representation π_f is induced from $L(g_0)$ $(g_0 \in G)$. Let $v_0 \in V(\pi_f) := \mathfrak{H}_f$ be the image of $\delta_e \in \mathfrak{A}$, then it is a cyclic vector of π_f, and moreover the original f is recovered as a matrix element as $f(g) = \langle \pi_f(g)v_0, v_0 \rangle$.

Let $K(G)$ be the set of all positive definite *class* functions on G, and $K_1(G)$ the set of all $f \in K(G)$ normalized as $f(e) = 1$, then $K_1(G)$ is a convex subset. Denote by $E(G)$ the set of all extremal points in $K_1(G)$. If we take $f \in K_1(G)$, since f is invariant, the kernel J_f is a two-sided $*$-ideal and $\mathfrak{A}_f = \mathfrak{A}/J_f$ is a $*$-algebra. The left and right multiplications of $\mathfrak{A}_f$ generate respectively representations $\pi_f(\psi)$, $\rho_f(\psi)$ of $\mathfrak{A}_f$ and accordingly of $\mathfrak{A}$. They generate von Neumann algebras $\mathfrak{U}_f := \pi_f(\mathfrak{A})''$, $\mathfrak{V}_f := \rho_f(\mathfrak{A})''$, which are mutually commutants of the other. The common center $\mathfrak{U}_f \cap \mathfrak{V}_f$ reduces to $\boldsymbol{C}{\cdot}I$ if and only if f is extremal or $f \in E(G)$, where I denotes the identity operator on $\mathfrak{H}_f$ ([Tho1], cited above).

When π_f is a factor representation, it is of finite type, and the character assosiated to it is given by $\boldsymbol{t} :\ C_c(G) \ni \psi \longmapsto f(\psi) \in \boldsymbol{C}$, or on the level of von Neumann algebras by

$$\mathfrak{U}_f \ni \pi_f(\psi) \longmapsto f(\psi) \in \boldsymbol{C} \qquad (\psi \in C_c(G)\ ).$$

This character has a finite value 1 at the identity operator $I \in \mathfrak{U}_f$, since $\pi_f(\delta_e) = I, f(\delta_e) = f(e) = 1$. Hence, when $\dim \pi_f = \infty$, the factor is of type II_1, and when $\dim \pi_f < \infty$, it is of type I_n with n such that $n \le \dim \pi_f \le n^2$ because π_f is cyclic.

Actually, for the present case of $G = \mathfrak{S}_\infty(T)$, finite-dimensional irreducible unitary representations (= IURs) are necessarily of dimension one (by Lemma 2.3 below). Therefore, if a factor representation π_f is finite-dimensional, it is necessarily a one-dimensional character of G.

1.3. Present problem

Our present problem is to determine explicitly all elements of $E(G)$ for the groups $G = \mathfrak{S}_\infty(T)$ for any finite groups T. As explained above, $E(G)$ corresponds bijectively to the set of all characters of factor representations of finite type of G. For the infinite symmetric group $\mathfrak{S}_\infty$, the problem was worked out in [Tho2], and it is reexamined in [VK], [KO], [Bi] etc. from the point of view of approximation from $\mathfrak{S}_n\,(n \to \infty)$, and recently in [Hi3]–[Hi4] from the standpoint of taking limits of centralizations of simple positive definite matrix elements of representations induced from subgroups of wreath product type of $\mathfrak{S}_\infty$. For the case where T is abelian, a general explicit formula for $f \in E(G)$ has been given in [HH1], and so all characters have been classified.

2. Structure of wreath product groups $\mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$

Fix a finite group T, and take the wreath product group $\mathfrak{S}_\infty(T)$ of T with $\mathfrak{S}_\infty$:

$$\mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty, \quad D_\infty(T) := \textstyle\prod'_{i\in \boldsymbol{N}} T_i\,, \quad T_i = T \quad (i \in \boldsymbol{N}). \tag{2.1}$$

Here $\sigma \in \mathfrak{S}_\infty$ acts on $d = (t_i)_{i\in\boldsymbol{N}} \in D_\infty(T)$ as $\sigma(d) = (t_{\sigma^{-1}(i)})_{i\in\boldsymbol{N}}$. We identify frequently d and σ with their images in $\mathfrak{S}_\infty(T)$ respectively, then $\sigma d \sigma^{-1} = \sigma(d)$ and

$$(d,\sigma)(d',\sigma') = \big(d(\sigma d' \sigma^{-1}), \sigma\sigma'\big) \qquad (d, d' \in D_\infty(T), \sigma, \sigma' \in \mathfrak{S}_\infty).$$

Notation. For $d = (t_i)_{i\in I} \in D_I(T)$, $I \subset \boldsymbol{N}$, put $\mathrm{supp}_I(d) := \{\, i \in I\ ;\ t_i \neq e_T \,\}$ and we omit the suffix I if $I = \boldsymbol{N}$ or I is specified from the context.

2.1. Standard decomposition of elements and conjugacy classes

An element $g = (d,\sigma) \in G = \mathfrak{S}_\infty(T)$ is called *basic* in the following two cases:

Case 1: σ is cyclic and $\mathrm{supp}(d) \subset \mathrm{supp}(\sigma)$;
Case 2: $\sigma = \mathbf{1}$ and for $d = (t_i)_{i\in\boldsymbol{N}}$, $t_q \neq e_T$ only for one $q \in \boldsymbol{N}$.

Here $\mathbf{1} \in \mathfrak{S}_\infty$ denotes the trivial permutation, and the element $(d,\mathbf{1})$ in Case 2 is denoted by ξ_q, and put $\mathrm{supp}(\xi_q) := \mathrm{supp}(d) = \{\, q \,\}$.

For a cyclic permutation $\sigma = (i_1\ i_2\ \ldots\ i_\ell)$ of ℓ integers, we define its *length* as $\ell(\sigma) = \ell$, and for the identity permutation $\mathbf{1}$, put $\ell(\mathbf{1}) = 1$ for convenience. In this connection, ξ_q is also denoted by $(t_q, (q))$ with a trivial cyclic permutation (q) of length 1. In Cases 1 and 2, put $\ell(g) = \ell(\sigma)$ for $g = (d,\sigma)$, and $\ell(\xi_q) = 1$.

An arbitrary element $g = (d, \sigma) \in G$, is expressed as a product of basic elements as

$$g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_m \tag{2.2}$$

with $g_j = (d_j, \sigma_j)$ in Case 1, in such a way that the supports of these components, $q_1, q_2, \ldots, q_r$, and $\mathrm{supp}(g_j) = \mathrm{supp}(\sigma_j)$ $(1 \le j \le m)$, are mutually disjoint. This expression of g is unique up to the orders of ξ_{q_k}'s and g_j's, and is called *standard decomposition* of g. Note that $\ell(\xi_{q_k}) = 1$ for $1 \le k \le r$ and $\ell(g_j) = \ell(\sigma_j) \ge 2$ for $1 \le j \le m$, and that, for $\mathfrak{S}_\infty$-components, $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ gives a cycle decomposition of σ.

To write down conjugacy class of $g = (d, \sigma)$, there appear products of components t_i of $d = (t_i)$, where the orders of taking products are crucial when T is not abelian. So we should fix notations well.

We denote by $[t]$ the conjugacy class of $t \in T$, and by $T/\!\sim$ the set of all conjugacy classes of T, and $t \sim t'$ denotes that $t, t' \in T$ are mutually conjugate in T. For a basic component $g_j = (d_j, \sigma_j)$ of g, let $\sigma_j = (i_{j,1}\ i_{j,2}\ \ldots\ i_{j,\ell_j})$ and put $K_j := \mathrm{supp}(\sigma_j) = \{\, i_{j,1},\, i_{j,2},\, \ldots,\, i_{j,\ell_j} \,\}$ with $\ell_j = \ell(\sigma_j)$. For $d_j = (t_i)_{i\in K_j}$, we put

$$P_{\sigma_j}(d_j) := \big[t'_{\ell_j}t'_{\ell_j-1}\cdots t'_2t'_1\big] \in T/\!\sim \quad \text{with} \quad t'_k = t_{i_{j,k}}\ \ (1 \le k \le \ell_j). \tag{2.3}$$

Note that the product $P_{\sigma_j}(d_j)$ is well-defined, because, for $t_1, t_2, \ldots, t_\ell \in T$, we have $t_1t_2\cdots t_\ell \sim t_kt_{k+1}\cdots t_\ell t_1\cdots t_{k-1}$ for any k, that is, the conjugacy class does not depend on any cyclic permutation of $(t_1, t_2, \ldots, t_\ell)$.

Lemma 2.1. (i) *Let $\sigma \in \mathfrak{S}_\infty$ be a cycle, and put $K = \mathrm{supp}(\sigma)$. Then, an element $g = (d, \sigma) \in \mathfrak{S}_K(T) =: G_K$ (put) is conjugate in it to $g' = (d', \sigma) \in G_K$ with $d' = (t'_i)_{i\in K}, t'_i = e_T\ (i \ne i_0), [t'_{i_0}] = P_\sigma(d)$ for some $i_0 \in K$.*

(ii) *Identify $\tau \in \mathfrak{S}_\infty$ with its image in $G = \mathfrak{S}_\infty(T)$. Then we have, for $g = (d, \sigma)$,*

$$\tau\, g\tau^{-1} = (\tau(d), \tau\sigma\tau^{-1}) \ \ =: (d', \sigma'),$$

and $P_{\sigma'}(d') = P_\sigma(d)$.

Proof. (i) We may assume that $\sigma = (1\ \ 2\ \ \cdots\ \ \ell)$ and so $K = \boldsymbol{I}_\ell = \{1, 2, \ldots, \ell\}$. Then, for $s = (s_1, s_2, \ldots, s_\ell) \in D_K(T) \hookrightarrow \mathfrak{S}_K(T)$, we have $s\,g\,s^{-1} = (d'', \sigma)$ with $d'' = (t''_i)_{i\in K}$,

$$t''_i = s_it_i(s_{\sigma^{-1}(i)})^{-1} = s_it_i(s_{i-1})^{-1} \quad (1 \le i \le \ell,\ 0 \equiv \ell).$$

Therefore $t''_\ell t''_{\ell-1}\cdots t''_2t''_1 = s_\ell(t_\ell t_{\ell-1}\cdots t_2t_1)s_\ell^{-1}$, and so $P_\sigma(d'') = P_\sigma(d)$.

Take $s_\ell = e_T, s_1 = t_1^{-1}, s_2 = (t_2t_1)^{-1}, \ldots, s_{\ell-1} = (t_{\ell-1}\cdots t_2t_1)^{-1}$, then we get $t''_i = e_T$ $(1 \le i < \ell)$ and $t''_\ell = t_\ell t_{\ell-1}\cdots t_2t_1$.

(ii) With σ above, we have $\tau\sigma\tau^{-1} = (\tau(1)\ \ \tau(2)\ \ \cdots\ \ \tau(\ell)\,)$, and $d' = \tau(d) = (t'_j)_{j\in K'}, K' = \tau(K)$, with $t'_j = t_{\tau^{-1}(j)}$ and so $t'_{\tau(i)} = t_i$ $(i \in K)$. Hence $t'_{\tau(\ell)}t'_{\tau(\ell-1)}\cdots t'_{\tau(2)}\ t'_{\tau(1)} = t_\ell t_{\ell-1}\cdots t_2t_1$. This proves the assertion. □

Applying this lemma to each basic components $g_j = (d_j, \sigma_j)$ of $g \in G$ in (2.2), we get the following result.

Theorem 1. *Let T be a finite group. Take an element $g \in G = \mathfrak{S}_\infty(T)$ and let its standard decomposition into basic elements be*

$$g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m$$

in (2.2), with $\xi_{q_k} = (t_{q_k}, (q_k))$, and $g_j = (d_j, \sigma_j)$, σ_j cyclic, $\mathrm{supp}(d_j) \subset \mathrm{supp}(\sigma_j)$. *Then the conjugacy class of g is determined by*

$$\text{(2.4)} \qquad [t_{q_k}] \in T/\!\sim \quad (1 \le k \le r) \quad \text{and} \quad (P_{\sigma_j}(d_j), \ell(\sigma_j)) \;\; (1 \le j \le m),$$

where $P_{\sigma_j}(d_j) \in T/\!\sim$ and $\ell(\sigma_j) \ge 2$. (Note that we put $\ell(\xi_{q_k}) = 1$, $\ell(g_j) = \ell(\sigma_j) \ge 2$.)

2.2. The case where T is abelian

In the case where T is abelian, the set $T/\!\sim$ of conjugacy classes is equal to T itself. Take $g \in G$, and take its standard decompositon (2.2). For $g_j = (d_j, \sigma_j)$, put $g'_j := (d'_j, \sigma_j)$, where $d'_j = (t'_i)_{i \in \boldsymbol{N}}$ with $t'_{i_0} = P(d_j) = \prod_{i \in K_j} t_i$ for some $i_0 \in K_j := \mathrm{supp}(\sigma_j)$, and $t'_i = e_T$ elsewhere.

Lemma 2.2. *Let T be abelian. For a $g = (d, \sigma) \in \mathfrak{S}_\infty(T)$, let its standard decomposition be $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m$ in (2.2). Define g'_j ($1 \le j \le m$) as above and put $g' = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g'_1 g'_2 \cdots g'_m$. Then, g and g' are mutually conjugate in $\mathfrak{S}_\infty(T)$.*

The conjugacy class of g_j and g'_j is characterized by the pair of $P(d_j) = P(d'_j) \in T$ and $\ell_j = \ell(\sigma_j) \ge 2$. Thus we get the following corollary.

Corollary. *A complete set of parameters of the conjugacy classes of non-trivial elements $g \in \mathfrak{S}_\infty(T)$ is given by*

$$\text{(2.5)} \qquad \{\, t'_1, t'_2, \ldots, t'_r \,\} \quad \text{and} \quad \{\, (u_j, \ell_j)\;;\; 1 \le j \le m \,\},$$

where $t'_k = t_{q_k} \in T^* := T \setminus \{\, e_T \,\}$, $u_j = P(d_j) \in T$, $\ell_j \ge 2$, and $r + m > 0$.

2.3. Finite-dimensional irreducible representations

Let us study finite-dimensional irreducible representations (= IRs) of $G = \mathfrak{S}_\infty(T)$.

Let π be such an IR of G. Consider a series of subgroups $G_n := \mathfrak{S}_n(T)$. Then $G_n \nearrow G$ as $n \to \infty$. Since $\dim \pi < \infty$, there exists an n such that the restriction $\pi|_{G_n}$ of π on G_n is already irreducible. Then, $\pi(G_n)$ generates the full operator algebra $\mathfrak{B}$ of $V(\pi)$. Take the commutant $Z_G(G_n)$ of G_n in G. Then for any $h \in Z_G(G_n)$, the operator $\pi(h)$ commutes with every element in $\mathfrak{B}$, and so is a scalar operator.

On the other hand, any $g \in G$ is conjugate under G to an element $h \in Z_G(G_n)$. Therefore $\pi(g)$ is a scalar operator together with $\pi(h)$. This means that $\dim \pi = 1$. Thus we get the following result.

Lemma 2.3. *A finite-dimensional irreducible representation π of $\mathfrak{S}_\infty(T)$ is a one-dimensional character, and is given in the form $\pi = \pi_{\zeta,\varepsilon}$ with*

$$\pi_{\zeta,\varepsilon}(g) = \zeta(P(d))\,(\mathrm{sgn}_{\mathfrak{S}})^{\varepsilon}(\sigma) \quad \text{for } g = (d,\sigma) \in \mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty,$$

where ζ is a one-dimensional character of T, $P(d)$ is a product of components t_i of $d = (t_i)$, and $\mathrm{sgn}_{\mathfrak{S}}(\sigma)$ denotes the usual sign of σ and $\varepsilon = 0, 1$. (Since $\zeta(P(d)) = \prod_{i\in\boldsymbol{N}} \zeta(t_i)$, the order of taking product for $P(d)$ has no meaning even if T is not abelian.)

In the case where T is abelian, we can prove similarly the following fact for the subgroup $\mathfrak{S}^e_\infty(T)$ of $\mathfrak{S}_\infty(T)$.

Lemma 2.4. *Assume that T is abelian. Then, a finite-dimensional irreducible representation π of $\mathfrak{S}^e_\infty(T)$ is a one-dimensional character, and is given in the form*

$$\pi(g) = (\mathrm{sgn}_{\mathfrak{S}})^{\varepsilon}(\sigma) \quad \text{for } g = (d,\sigma) \in \mathfrak{S}^e_\infty(T) = D^e_\infty(T) \rtimes \mathfrak{S}_\infty.$$

3. Characters of wreath product group $\mathfrak{S}_\infty(T)$, T finite

In this section, we give our general results on characters of a wreath product group $G = \mathfrak{S}_\infty(T)$ for any finite group T.

First let us introduce several notations. Let $\widehat{T}$ be the dual of T consisting of all equivalence classes of irreducible representations. We identify every equivalence class with one of its representative. Thus $\zeta \in \widehat{T}$ is an IR and denote by χ_ζ its character:

$$\chi_\zeta(t) = \mathrm{tr}(\zeta(t)) \qquad (t \in T),$$

then $\dim\zeta = \chi_\zeta(e_T)$. Denote by $\mathbf{1}_T$ the identity representation of T, and put $\widehat{T}^* := \widehat{T} \setminus \{\mathbf{1}_T\}$, $T^* := T \setminus \{e_T\}$. Then

$$|T|\,\delta_{e_T} = \sum_{\zeta\in\widehat{T}} (\dim\zeta)\,\chi_\zeta\,, \quad \text{as functions on } T, \tag{3.1}$$

$$0 = \sum_{\zeta\in\widehat{T}} (\dim\zeta)\,\chi_\zeta \quad \text{and} \quad 1 = \chi_{\mathbf{1}_T} = -\sum_{\zeta\in\widehat{T}^*} (\dim\zeta)\,\chi_\zeta\,, \quad \text{on } T^*. \tag{3.2}$$

Take an element $g \in G = \mathfrak{S}_\infty(T)$ and let its standard decomposition into basic components be

$$g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m\,, \tag{3.3}$$

where the supports of components, $q_1, q_2, \ldots, q_r$, and $\mathrm{supp}(g_j) := \mathrm{supp}(\sigma_j)$ $(1 \le j \le m)$, are mutually disjoint. Furthermore, $\xi_{q_k} = (t_{q_k}, (q_k)), t_{q_k} \ne e_T$, with $\ell(\xi_{q_k}) = 1$ for $1 \le k \le r$, and σ_j is a cycle of length $\ell(\sigma_j) \ge 2$ and $\mathrm{supp}(d_j) \subset K_j = \mathrm{supp}(\sigma_j)$. For $\mathfrak{S}_\infty$-components, $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ gives the cycle decomposition of σ. For $d_j = (t_i)_{i\in K_j} \in D_{K_j}(T) \hookrightarrow D_\infty(T)$, put $P_{\sigma_j}(d_j)$ as in (2.3).

For one-dimensional characters of $\mathfrak{S}_\infty$, we introduce simple notation as

$$\chi_\varepsilon(\sigma) := \mathrm{sgn}_{\mathfrak{S}}(\sigma)^\varepsilon \quad (\sigma \in \mathfrak{S}_\infty\,;\ \varepsilon = 0,1). \tag{3.4}$$

As a parameter for characters of $G = \mathfrak{S}_\infty(T)$, we prepare a set

$$\alpha_{\zeta,\varepsilon}\ (\zeta \in \widehat{T}, \varepsilon \in \{\,0,1\,\}) \quad \text{and} \quad \mu = (\mu_\zeta)_{\zeta\in\widehat{T}}\,, \tag{3.5}$$

of decreasing sequences of non-negative real numbers

$$\alpha_{\zeta,\varepsilon} = (\alpha_{\zeta,\varepsilon,i})_{i\in\boldsymbol{N}}\,,\ \ \alpha_{\zeta,\varepsilon,1} \geq \alpha_{\zeta,\varepsilon,2} \geq \alpha_{\zeta,\varepsilon,3} \geq\ \cdots\ \geq 0\,;$$

and a set of non-negative real $\mu_\zeta \geq 0$ $(\zeta \in \widehat{T})$, which altogether satisfy the condition

$$\sum_{\zeta\in\widehat{T}}\ \sum_{\varepsilon\in\{\,0,1\,\}} \|\alpha_{\zeta,\varepsilon}\| \ +\ \|\mu\| \leq 1\,, \tag{3.6}$$

$$\text{with} \quad \|\alpha_{\zeta,\varepsilon}\| = \textstyle\sum_{i\in\boldsymbol{N}} \alpha_{\zeta,\varepsilon,i}\,, \quad \|\mu\| = \sum_{\zeta\in\widehat{T}} \mu_\zeta\,.$$

Then we have the following result.

Theorem 2. *Let $G = \mathfrak{S}_\infty(T)$ be a wreath product group of a finite group T with $\mathfrak{S}_\infty$. Then, for a parameter*

$$A := \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{\,0,1\,\}}\ ;\ \mu\Big)\,, \tag{3.7}$$

in (3.5)–(3.6)*, the following formula determines a character f_A of G: for an element $g \in G$, let* (3.3) *be its standard decomposition, then*

$$\begin{aligned} f_A(g) = \prod_{1\leq k\leq r} &\left\{\sum_{\zeta\in\widehat{T}}\left(\sum_{\varepsilon\in\{\,0,1\,\}}\ \sum_{i\in\boldsymbol{N}} \frac{\alpha_{\zeta,\varepsilon,i}}{\dim\zeta} \ +\ \frac{\mu_\zeta}{\dim\zeta}\right)\chi_\zeta(t_{q_k})\right\} \\ \times \prod_{1\leq j\leq m} &\left\{\sum_{\zeta\in\widehat{T}}\left(\sum_{\varepsilon\in\{\,0,1\,\}}\ \sum_{i\in\boldsymbol{N}}\left(\frac{\alpha_{\zeta,\varepsilon,i}}{\dim\zeta}\right)^{\ell(\sigma_j)} \chi_\varepsilon(\sigma_j)\right)\chi_\zeta\big(P_{\sigma_j}(d_j)\big)\right\}, \end{aligned} \tag{3.8}$$

where $\chi_\varepsilon(\sigma_j) = \mathrm{sgn}_{\mathfrak{S}}(\sigma_j)^\varepsilon = (-1)^{\varepsilon(\ell(\sigma_j)-1)}$.

Conversely any character of G is given in the form of f_A.

The parameter A of character is not necessarily unique because of the linear dependence (3.2) on T^* of functions χ_ζ , $\zeta \in \widehat{T}$. To establish uniqueness of parameter, we transfer from the parameter A, to another parameter $B = \phi(A)$ given by

$$B = \phi(A) := \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{\,0,1\,\}}\ ;\ \kappa\Big)\,, \tag{3.9}$$

$$\text{with} \quad \kappa = (\kappa_\zeta)_{\zeta\in\widehat{T}^*}\,,\ \ \kappa_\zeta = \mu_\zeta - (\dim\zeta)^2\,\mu_{\mathbf{1}_T} \ \ (\zeta \in \widehat{T}^*).$$

Then, we have from (3.1)–(3.2),

$$\sum_{\zeta\in\widehat{T}} \frac{\mu_\zeta}{\dim\zeta}\cdot\chi_\zeta(t_{q_k}) = \sum_{\zeta\in\widehat{T}^*} \frac{\kappa_\zeta}{\dim\zeta}\cdot\chi_\zeta(t_{q_k}),$$
$$\sum_{\zeta\in\widehat{T}^*} \kappa_\zeta \;=\; \sum_{\zeta\in\widehat{T}} \mu_\zeta - |T|\,\mu_{\mathbf{1}_T},$$

and the uniqueness of parameter is established. However the inequality (3.6) for the range of parameter A containing μ cannot be translated in a compact form in the parameter $\phi(A)$ containing κ in place of μ.

Note that the multiplicative factor for $\xi_{q_k} = (t_{q_k}, (q_k))$ in the formula is rewritten as

$$\sum_{\zeta\in\widehat{T}} (\|\alpha_{\zeta,0}\| + \|\alpha_{\zeta,1}\| + \mu_\zeta)\cdot\frac{\chi_\zeta(t_{q_k})}{\dim\zeta}$$
$$= \sum_{\zeta\in\widehat{T}} (\|\alpha_{\zeta,0}\| + \|\alpha_{\zeta,1}\|)\cdot\frac{\chi_\zeta(t_{q_k})}{\dim\zeta} + \sum_{\zeta\in\widehat{T}^*} \kappa_\zeta\cdot\frac{\chi_\zeta(t_{q_k})}{\dim\zeta}.$$

In this connection, we can propose two other choices of normalization of the parameter $\mu = (\mu_\zeta)_{\zeta\in\widehat{T}}$, $\mu_\zeta \geq 0$. The first one is given by taking into account of the relation (3.2) and

$$\sum_{\zeta\in\widehat{T}} \frac{\mu_\zeta}{\dim\zeta}\,\chi_\zeta = \sum_{\zeta\in\widehat{T}} \frac{\mu_\zeta}{(\dim\zeta)^2}\cdot(\dim\zeta)\chi_\zeta\,,$$

as the following minimum condition:

$$\text{(3.10)}\qquad \text{(MIN)}\qquad \min\left\{\frac{\mu_\zeta}{(\dim\zeta)^2}\;;\;\zeta\in\widehat{T}\right\} = 0\,.$$

The second one, in the case where T is non-trivial, is the following maximum condition on the parameter A, whose merit is that the character formula (3.8) is valid even for $t_{q_k} = e_T$ (not necessarily $t_{q_k}\in T^*$), whereas this is not the case in other normalizations:

$$\text{(3.11)}\qquad \text{(MAX)}\qquad \sum_{\zeta\in\widehat{T}}\sum_{\varepsilon\in\{0,1\}} \|\alpha_{\zeta,\varepsilon}\| + \|\mu\| = 1\,.$$

4. Characters of wreath product group $\mathfrak{S}_\infty(T)$, T abelian

When T is abelian, the general character formula (3.8) for $\mathfrak{S}_\infty(T) = D_\infty(T)\rtimes\mathfrak{S}_\infty$ with a finite group T has a simplified form.

First let us check simplification of notations. In this abelian case, $\widehat{T}$ is nothing but the dual group consisting of all one-dimensional characters of T, and for each $\zeta \in \widehat{T}$, its character χ_ζ is identified with ζ itself. The character identities (3.1)–(3.2) are written as

$$|T|\,\delta_{e_T} = \sum_{\zeta\in\widehat{T}} \zeta\,, \quad \text{as functions on } T, \tag{4.1}$$

$$0 = \sum_{\zeta\in\widehat{T}} \zeta \quad \text{and} \quad 1 = \mathbf{1}_T = -\sum_{\zeta\in\widehat{T}^*} \zeta\,, \qquad \text{on } T^*. \tag{4.2}$$

Take an element $g \in G = \mathfrak{S}_\infty(T)$. Let its standard decomposition be

$$g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m\,, \tag{4.3}$$

with $\xi_{q_k} = (t_{q_k}, (q_k)), t_{q_k} \neq e_T$, for $1 \le k \le r$, and $g_j = (d_j, \sigma_j)$ for $1 \le j \le m$. Put $K_j = \mathrm{supp}(\sigma_j)$, and for $d_j = (t_i)_{i\in K_j} \in D_{K_j}(T) \hookrightarrow D_\infty(T)$, put

$$P_{K_j}(d_j) = \textstyle\prod_{i\in K_j} t_i\,, \quad \zeta(d_j) := \zeta(P_{K_j}(d_j)) = \prod_{i\in K_j} \zeta(t_i)\,. \tag{4.4}$$

As a parameter for characters of $G = \mathfrak{S}_\infty(T)$, we prepare a set

$$\alpha_{\zeta,\varepsilon}\ (\zeta \in \widehat{T}, \varepsilon \in \{0,1\}), \quad \text{and} \quad \mu = (\mu_\zeta)_{\zeta\in\widehat{T}}\,, \tag{4.5}$$

of decreasing sequences of non-negative real numbers $\alpha_{\zeta,\varepsilon} = (\alpha_{\zeta,\varepsilon,i})_{i\in\boldsymbol{N}}$, and a set of non-negative real $\mu_\zeta \ge 0$ $(\zeta \in \widehat{T})$, which satisfy the condition

$$\sum_{\zeta\in\widehat{T}} \sum_{\varepsilon\in\{0,1\}} \|\alpha_{\zeta,\varepsilon}\| + \|\mu\| \le 1\,. \tag{4.6}$$

Theorem 3. *Let $G = \mathfrak{S}_\infty(T)$ be a wreath product group of a finite abelian group T with $\mathfrak{S}_\infty$. Then, for a parameter $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\Big)$, in* (4.5)–(4.6)*, the following formula determines a character f_A of G: for an element $g \in G$, let* (4.3) *be its standard decomposition, then*

$$\begin{aligned} f_A(g) = \prod_{1\le k\le r} & \left\{ \sum_{\zeta\in\widehat{T}} \left(\sum_{\varepsilon\in\{0,1\}} \sum_{i\in\boldsymbol{N}} \alpha_{\zeta,\varepsilon,i} + \mu_\zeta \right) \zeta(t_{q_k}) \right\} \\ \times \prod_{1\le j\le m} & \left\{ \sum_{\zeta\in\widehat{T}} \left(\sum_{\varepsilon\in\{0,1\}} \sum_{i\in\boldsymbol{N}} (\alpha_{\zeta,\varepsilon,i})^{\ell(\sigma_j)} \cdot \chi_\varepsilon(\sigma_j) \right) \zeta(d_j) \right\}, \end{aligned} \tag{4.7}$$

where $\chi_\varepsilon(\sigma_j) = \mathrm{sgn}_{\mathfrak{S}}(\sigma_j)^\varepsilon = (-1)^{\varepsilon(\ell(\sigma_j)-1)}$, and $\zeta(d_j)$ as in (4.4)*.*

Conversely any character of G is given in the form of f_A.

The parameter A of a character is not necessarily unique just as for Theorem 2. To establish uniqueness of parameter, we transfer from A to $B = \phi(A)$ given by

$$B = \phi(A) := \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\kappa\Big)\,, \tag{4.8}$$

$$\text{with} \quad \kappa = (\kappa_\zeta)_{\zeta\in\widehat{T}^*}\,, \quad \kappa_\zeta = \mu_\zeta - \mu_{\mathbf{1}_T}\ (\zeta \in \widehat{T}^*).$$

Then the uniqueness of parameter is established. Except the case of $T = \boldsymbol{Z}_2$ or the case of the infinite Weyl group of type $\mathbf{B}_\infty/\mathbf{C}_\infty$, the inequality (4.6) for the range of parameter A cannot be translated in a compact form in $B = \phi(A)$.

Note that the multiplicative factor for $\xi_{q_k} = (t_{q_k}, (q_k))$ in the formula is rewritten as

$$\sum_{\zeta\in\widehat{T}} (\|\alpha_{\zeta,0}\| + \|\alpha_{\zeta,1}\| + \mu_\zeta)\cdot\zeta(t_{q_k}) \\ = \sum_{\zeta\in\widehat{T}} (\|\alpha_{\zeta,0}\| + \|\alpha_{\zeta,1}\|)\cdot\zeta(t_{q_k}) + \sum_{\zeta\in\widehat{T}^*} \kappa_\zeta\cdot\zeta(t_{q_k}).$$

Another normalization condition (MIN) for $\mu = (\mu_\zeta), \mu_\zeta \geq 0$, is written as

$$\text{(MIN)} \qquad \min\left\{ \mu_\zeta \;;\; \zeta\in\widehat{T} \right\} \;=\; 0,$$

and one more normalization condition (MAX) is just as in (3.11).

Example 4.1. The case where $\alpha_{\zeta,\varepsilon,1} = 1$ for a fixed $(\zeta,\varepsilon) \in \widehat{T}\times\{0,1\}$ and all other parameters in A are zero, whence $\alpha_{\zeta,\varepsilon} = (1,0,0,\dots)$, corresponds to one-dimensional character $\pi_{\zeta,\varepsilon}$ of G in Lemma 2.3. Except these cases of one-dimensional representations of G, a character f_A given above corresponds to a factor representation of G of type II_1.

The case "$\alpha_{\zeta,\varepsilon} = (\alpha_{\zeta,\varepsilon,i})_{i\in\boldsymbol{N}} = \mathbf{0}$ for all $(\zeta,\varepsilon) \in \widehat{T}\times\{0,1\}$ and $\mu = (\mu_\zeta)_{\zeta\in\widehat{T}} = \mathbf{0}$" corresponds to the regular representation λ_G of G.

Consider the case where $\|\alpha_{\zeta,0}\| + \|\alpha_{\zeta,1}\| + \mu_\zeta = 1$ for a fixed $\zeta\in\widehat{T}$ and all other parameters in A are zero. Put $\alpha = \alpha_{\zeta,0}, \beta = \alpha_{\zeta,1}$, and let $f_{\alpha,\beta}$ be the character of $\mathfrak{S}_\infty$ given in [Tho2] (cf. (6.2) in **6.1**). Denote by Ψ the natural homomorphism from G onto $\mathfrak{S}_\infty \cong G/D$ with normal subgroup $D = D_\infty(T)$, and put $f^{\#}_{\alpha,\beta} := f_{\alpha,\beta}\circ\Psi$. Then the character $f_A(g)$ in this case is equal to $f^{\#}_{\alpha,\beta}(g)\cdot\pi_{\zeta,0}(g)$ with a one-dimensional character $\pi_{\zeta,0}$ of G in Lemma 2.3 (with $\varepsilon = 0$). In particular, the case where $\mu_\zeta = 1$ for a fixed $\zeta\in\widehat{T}$, corresponds to the induced representation $\mathrm{Ind}_D^G\zeta_D$, where $\zeta_D(d) := \zeta(P(d)), d\in D$, is a one-dimensional character of $D = D_\infty(T)$ (cf. **2.3**). The character f_A is equal to ζ_D on $D \hookrightarrow G$, and zero outside of D. In the case $\zeta = \mathbf{1}_T$, this induced representation is nothing but the regular representation of $G/D \cong \mathfrak{S}_\infty$.

5. Characters for the subgroup $\mathfrak{S}^e_\infty(T) \subset \mathfrak{S}_\infty(T)$, T abelian

Assume T be abelian. For the natural subgroup $G^e := \mathfrak{S}^e_\infty(T) = D^e_\infty(T) \rtimes \mathfrak{S}_\infty$ with

$$\text{(5.1)} \qquad D^e_\infty(T) := \{\, d = (t_i)_{i\in\boldsymbol{N}} \;;\; P(d) = e_T \,\}, \quad P(d) := \textstyle\prod_{i\in\boldsymbol{N}} t_i,$$

we deduce a general character formula from the one for $G := \mathfrak{S}_\infty(T)$.

Take an element $g\in G^e = \mathfrak{S}^e_\infty(T)$ and let its standard decomposition be

$$\text{(5.2)} \qquad g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2\cdots g_m$$

with $\xi_{q_k} = (t_{q_k}, (q_k))$ and $g_j = (d_j, \sigma_j), d_j = (t_i)_{i \in K_j}, K_j = \mathrm{supp}(\sigma_j)$. Note that each component ξ_{q_k} does not belong to G^e, and that the component $g_j = (d_j, \sigma_j)$ belongs to G^e if and only if $P(d_j) = \prod_{i \in K_j} t_i = e_T$. However, after careful discussions on the relation between G^e and G, we obtain the following result for the subgroup G^e from the result for G.

Theorem 4. *Let T be abelian, and let $G^e = \mathfrak{S}^e_\infty(T)$ be the subgroup of $G = \mathfrak{S}_\infty(T)$ given by* (5.1). *Then, for a parameter*

$$A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon) \in \widehat{T} \times \{0,1\}} \; ; \; \mu \Big), \tag{5.3}$$

in (4.5)–(4.6), *the following formula determines a character f^e_A of G^e: for an element $g \in G^e$, let* (5.2) *be its standard decomposition, then*

$$\begin{aligned} f^e_A(g) = & \prod_{1 \le k \le r} \left\{ \sum_{\zeta \in \widehat{T}} \left(\sum_{\varepsilon \in \{0,1\}} \sum_{i \in \boldsymbol{N}} \alpha_{\zeta,\varepsilon,i} \; + \; \mu_\zeta \right) \zeta(t_{q_k}) \right\} \\ & \times \prod_{1 \le j \le m} \left\{ \sum_{\zeta \in \widehat{T}} \left(\sum_{\varepsilon \in \{0,1\}} \sum_{i \in \boldsymbol{N}} (\alpha_{\zeta,\varepsilon,i})^{\ell(\sigma_j)} \cdot \chi_\varepsilon(\sigma_j) \right) \zeta(d_j) \right\}, \end{aligned} \tag{5.4}$$

where $\chi_\varepsilon(\sigma_j) = \mathrm{sgn}_{\mathfrak{S}}(\sigma_j)^\varepsilon = (-1)^{\varepsilon(\ell(\sigma_j)-1)}$, and $\zeta(d_j)$ as in (4.4).

Conversely any character of G^e is given in the form of f^e_A.

The parameter $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon) \in \widehat{T} \times \{0,1\}} \; ; \; \mu \Big)$ for f^e_A is not unique even under the normalization condition (MAX). To describe the correspondence of parameters, we introduce a translation $R(\zeta_0)$ on A by an element $\zeta_0 \in \widehat{T}$ as follows:

$$R(\zeta_0)A := \Big((\alpha'_{\zeta,\varepsilon})_{(\zeta,\varepsilon) \in \widehat{T} \times \{0,1\}} \; ; \; R(\zeta_0)\mu \Big) \tag{5.5}$$

with $\alpha'_{\zeta,\varepsilon} = \alpha_{\zeta\zeta_0^{-1},\varepsilon} \; \big((\zeta,\varepsilon) \in \widehat{T} \times \{0,1\} \big)$; $\quad R(\zeta_0)\mu = (\mu'_\zeta)_{\zeta \in \widehat{T}}, \;\; \mu'_\zeta = \mu_{\zeta\zeta_0^{-1}}$.

Proposition 5. *Let T be abelian. Assume that two parameters for characters*

$$A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon) \in \widehat{T} \times \{0,1\}} \; ; \; \mu \Big) \quad \text{and} \quad A' = \Big((\alpha'_{\zeta,\varepsilon})_{(\zeta,\varepsilon) \in \widehat{T} \times \{0,1\}} \; ; \; \mu' \Big)$$

satisfy the normalization condition (MAX) *for μ and μ' respectively. Then, they determine the same character, that is, $f^e_A = f^e_{A'}$, if and only if $A' = R(\zeta_0)A$ for some $\zeta_0 \in \widehat{T}$.*

By Lemma 2.3, we know all one-dimensional characters of $G = \mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$. Among them take those which depend only on $D_\infty(T)$-component $d = (t_i)_{i \in \boldsymbol{N}}, t_i \in T$, of $g = (d, \sigma) \in G$. Then they are given for $g = (d, \sigma)$ in (5.2) as

$$\pi_{\zeta,0}(g) := \zeta(P(d)) = \zeta\left(\prod_{i \in \boldsymbol{N}} t_i \right) = \prod_{i \in \boldsymbol{N}} \zeta(t_i) = \prod_{1 \le k \le r} \zeta(t_{q_k}) \prod_{1 \le j \le m} \zeta(d_j),$$

for some $\zeta \in \widehat{T}$. Therefore we see that, as characters on $G \supset G^e$,

$$f_{A'}(g) = \pi_{\zeta_0,0}(g) \cdot f_A(g) \quad (g \in G) \qquad \text{for} \quad A' = R(\zeta_0)A.$$

6. Characters for the infinite Weyl groups

The infinite symmetric group $\mathfrak{S}_\infty$ is the Weyl group of type $\mathbf{A}_\infty$, and the symmetric group $\mathfrak{S}_n$, which is the Weyl group of type $\mathbf{A}_n$, is imbedded in $\mathfrak{S}_\infty$ as $\mathfrak{S}_n = \mathfrak{S}_{\boldsymbol{I}_n}$ with $\boldsymbol{I}_n := \{1, 2, \ldots, n\} \subset \boldsymbol{N}$. Take $\sigma \neq \mathbf{1}$ from $\mathfrak{S}_\infty$, and decompose it into a product of mutually disjoint cycles (= cyclic permutations) as

$$\sigma = \sigma_1\sigma_2\cdots\sigma_m\,, \quad \sigma_j = (i_{j,1}\ i_{j,2}\ \ldots\ i_{j,\ell_j}). \tag{6.1}$$

By definition, $\ell_j = \ell(\sigma_j)$ is the length of the cycle σ_j, and put $n_\ell(\sigma) = \big|\{ j\,;\, \ell_j = \ell\}\big|$ the number of cycles σ_j with length ℓ. The set of multiplicities $\{ n_\ell(\sigma)\ ;\ \ell \geq 2\}$ determines the conjugacy class of σ.

6.1. Review of the case of infinite symmetric group $\mathfrak{S}_\infty$

The formula of characters of $W_{\mathbf{A}_\infty} \cong \mathfrak{S}_\infty$ in [Tho2] is written as follows. As a parameter for such a character, take (α, β) with $\alpha = (\alpha_i)_{i\geq 1}, \beta = (\beta_i)_{i\geq 1}$, decreasing sequences of non-negative real numbers satisfying $\|\alpha\| + \|\beta\| \leq 1$. Then

$$f_{\alpha,\beta}(\sigma) = \prod_{\ell\geq 2}\left(\sum_{1\leq i<\infty}\alpha_i^{\,\ell} + (-1)^{\ell-1}\sum_{1\leq i<\infty}\beta_i^{\,\ell}\right)^{n_\ell(\sigma)}. \tag{6.2}$$

We rewrite this in the form of our formula for $\mathfrak{S}_\infty(T)$. Put

$$\chi_\varepsilon(\sigma) := \big(\mathrm{sgn}_{\mathfrak{S}}(\sigma)\big)^\varepsilon \ \ (\sigma \in \mathfrak{S}_\infty); \quad \alpha_{0,i} = \alpha_i\,,\ \ \alpha_{1,i} = \beta_i\,,$$

for $\varepsilon = 0, 1$, and $i = 1, 2, \ldots$. For a cycle σ_j in the decomposition $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ of $\sigma \in \mathfrak{S}_\infty$, we have $\mathrm{sgn}_{\mathfrak{S}}(\sigma_j) = (-1)^{\ell(\sigma_j)-1}$, and the formula above is rewritten as

$$f_{\alpha,\beta}(\sigma) = \prod_{1\leq j\leq m}\left(\sum_{\varepsilon=0,1}\chi_\varepsilon(\sigma_j)\sum_{1\leq i<\infty}(\alpha_{\varepsilon,i})^{\ell(\sigma_j)}\right). \tag{6.3}$$

In [Hi3]–[Hi4], it is shown that all these characters $f_{\alpha,\beta}$ are obtained as various limits of centralizations of one matrix element $F = \mathrm{Ind}_H^G f_\pi$ of a unitary representation $\rho = \mathrm{Ind}_H^G \pi$, induced from one-dimensional character π of a certain subgroup H of wreath product type (cf. [Hi4, §15], in particular).

6.2. Character formula for infinite Weyl group of type $\mathbf{B}_\infty/\mathbf{C}_\infty$

For the infinite Weyl group $G = W_{\mathbf{B}_\infty}$ of type $\mathbf{B}_\infty/\mathbf{C}_\infty$, all the characters or extremal positive definite class functions are given as follows [HH1]. Recall that G is naturally realized as a semidirect product group as

$$G = W_{\mathbf{B}_\infty} = \mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty \quad \text{with} \quad T = \boldsymbol{Z}_2\,. \tag{6.4}$$

A one-dimensional character of G is given as a tensor product of such ones $(\mathrm{sgn}_D)^a$ of $D = D_\infty(T)$ and $(\mathrm{sgn}_{\mathfrak{S}})^b$ of $\mathfrak{S}_\infty$: for $g = (d, \sigma) \in G = D \rtimes \mathfrak{S}_\infty$,

$$\chi_{a,b}(g) := (\mathrm{sgn}_D)^a(d) \cdot (\mathrm{sgn}_{\mathfrak{S}})^b(\sigma) \quad (\, a, b \in \{\, 0, 1 \,\}), \tag{6.5}$$

with $\mathrm{sgn}_D(d) = \prod_{i \in \boldsymbol{N}} t_i$ for $d = (t_i)_{i \in \boldsymbol{N}} \in D$.

We prepare a set of parameters $(\alpha, \beta, \gamma, \delta, \kappa)$ as

$$\begin{array}{ll} \alpha = (\alpha_i)_{i \in \boldsymbol{N}}, & \alpha_1 \geq \alpha_2 \geq \cdots \geq 0, \\ \beta = (\beta_i)_{i \in \boldsymbol{N}}, & \beta_1 \geq \beta_2 \geq \cdots \geq 0, \\ \gamma = (\gamma_i)_{i \in \boldsymbol{N}}, & \gamma_1 \geq \gamma_2 \geq \cdots \geq 0, \\ \delta = (\delta_i)_{i \in \boldsymbol{N}}, & \delta_1 \geq \delta_2 \geq \cdots \geq 0, \\ \text{and} \quad \kappa\,, & \text{a real number.} \end{array} \tag{6.6}$$

Here $\alpha, \beta, \gamma, \delta$ and κ satisfy the condition

$$\|\alpha\| + \|\beta\| + \|\gamma\| + \|\delta\| + |\kappa| \leq 1\,, \quad \text{or} \tag{6.7}$$

$$-1 + (\|\alpha\| + \|\beta\| + \|\gamma\| + \|\delta\|) \leq \kappa \leq 1 - (\|\alpha\| + \|\beta\| + \|\gamma\| + \|\delta\|),$$

with

$$\begin{array}{ll} \|\alpha\| = \sum_{1 \leq i < \infty} \alpha_i\,, & \|\beta\| = \sum_{1 \leq i < \infty} \beta_i\,, \\ \|\gamma\| = \sum_{1 \leq i < \infty} \gamma_i\,, & \|\delta\| = \sum_{1 \leq i < \infty} \delta_i\,. \end{array} \tag{6.8}$$

Take a $g \in G$ and let

$$g = \xi_{q_1} \xi_{q_2} \cdots \xi_{q_r} g_1 g_2 \cdots g_m \tag{6.9}$$

be a standard decomposition with $\xi_{q_k} = (t_{q_k}, (q_k))$ and $g_j = (d_j, \sigma_j)$. Here, for $d_j = (t_i)_{i \in \boldsymbol{N}} \in D_{\boldsymbol{N}}(T)$, we have $\mathrm{supp}(d_j) \subset K_j := \mathrm{supp}(\sigma_j)$, and $\mathrm{sgn}_D(d_j) = P_{K_j}(d_j) = \prod_{i \in K_j} t_i$. Put $\ell(g_j) = \ell(\sigma_j)$, and by definition, $\ell(\xi_{q_k}) = 1$. For $(a, b) \in \{\, 0, 1 \,\} \times \{\, 0, 1 \,\}$, we put

$$\chi_{a,b}(g_j) = \mathrm{sgn}_D(d_j)^a \cdot \mathrm{sgn}_{\mathfrak{S}}(\sigma_j)^b = \Big(\prod_{i \in K_j} t_i\Big)^a \cdot (-1)^{b(\ell(g_j) - 1)}.$$

Theorem 6 ([HH1]). *Let $G = W_{\mathbf{B}_\infty} = \mathfrak{S}_\infty(T)$ with $T = \boldsymbol{Z}_2$ be the infinite Weyl group of type $\mathbf{B}_\infty$. To a character of G, there coreseponds uniquely a parameter $(\alpha, \beta, \gamma, \delta, \kappa)$ given in* (6.6)–(6.7)*, and it is expressed as $f_{\alpha,\beta,\gamma,\delta,\kappa}$ in the following formula: for a $g \in G$, express it as in* (6.9)*, then*

$$\begin{aligned} f_{\alpha,\beta,\gamma,\delta,\kappa}(g) = s_1^{\,r} \times \prod_{1 \leq j \leq m} \Bigg(& \sum_{1 \leq i < \infty} {\alpha_i}^{\ell(g_j)} + \chi_{0,1}(g_j) \sum_{1 \leq i < \infty} {\beta_i}^{\ell(g_j)} + \\ & + \chi_{1,0}(g_j) \sum_{1 \leq i < \infty} {\gamma_i}^{\ell(g_j)} + \chi_{1,1}(g_j) \sum_{1 \leq i < \infty} {\delta_i}^{\ell(g_j)} \Bigg) \end{aligned} \tag{6.10}$$

with

$$s_1 = s_{1;0} + \kappa\,, \quad s_{1;0} := \|\alpha\| + \|\beta\| - (\|\gamma\| + \|\delta\|)\,. \tag{6.11}$$

Another expression of characters. We can rewrite the formula (6.10) in more compact form. For $a, b \in \{0,1\}$, put $\alpha_{a,b;i} \geq 0$ as

$$\alpha_{0,0;i} = \alpha_i\,, \quad \alpha_{0,1;i} = \beta_i\,, \quad \alpha_{1,0;i} = \gamma_i\,, \quad \alpha_{1,1;i} = \delta_i \quad (k \geq 1). \tag{6.12}$$

For a basic element $h = (d, \sigma) \in G$ with $\ell = \ell(h) := \ell(\sigma), \mathrm{sgn}_D(d) = \epsilon 1$ ($\epsilon = \pm$), we have $\chi_{a,b}(h) = (\epsilon 1)^a \cdot (-1)^{b(\ell-1)}$, and put

$$\begin{aligned}\phi(h) &= \sum_{a,b\in\{0,1\}} \chi_{a,b}(h) \sum_{1\leq i<\infty} (\alpha_{a,b;i})^{\ell(h)} \\ &= \sum_{1\leq i<\infty} \left\{ \alpha_i^{\ \ell} + (-1)^{\ell-1}\beta_i^{\ \ell} + (\epsilon 1)\gamma_i^{\ \ell} + (-1)^{\ell-1}(\epsilon 1)\delta_i^{\ \ell} \right\} =: s_{\epsilon;\ell}\,.\end{aligned} \tag{6.13}$$

In case $\ell = 1$, we have $\ell(\xi_q) = 1, \mathrm{sgn}_D(\xi_q) := \mathrm{sgn}_D(t_q) = -1$ for $\xi_q = (t_q, (q))$, and

$$\begin{aligned}\phi(\xi_q) &= \textstyle\sum_{a,b\in\{0,1\}} \chi_{a,b}(\xi_q) \left(\sum_{1\leq i<\infty} \alpha_{a,b;i} \right) \\ &= \|\alpha\| + \|\beta\| - \|\gamma\| - \|\delta\|\,.\end{aligned} \tag{6.14}$$

We define $s_{-;1}$ adding some deviation κ to $\phi(\xi_q)$ as

$$\phi(\xi_q) + \kappa =: s_{-;1}\,. \tag{6.15}$$

The formula (6.10) of positive definite function $f_{\alpha,\beta,\gamma,\delta,\kappa}$ is written as

$$f_{\alpha,\beta,\gamma,\delta,\kappa}(g) = \prod_{1\leq k\leq r} \big(\phi(\xi_{q_k}) + \kappa\big) \times \prod_{1\leq j\leq m} \phi(g_j). \tag{6.16}$$

For $\ell \geq 2$ and $\epsilon = \pm$, let $n_{\epsilon;\ell}(g)$ be the multiplicity of $g_j = (d_j, \sigma_j)$ with $\ell(\sigma_j) = \ell$ and $\mathrm{sgn}_D(d_j) = \epsilon 1$:

$$n_{\epsilon;\ell}(g) = \big|\{ j\,;\ \ell(g_j) = \ell,\ \xi_{1,0}(g_j) = \mathrm{sgn}_D(d_j) = \epsilon 1 \}\big|\,,$$

and let $n_{-;1}(g) = r$ be the multiplicity of ξ_q in (6.9). Then, the formula (6.16) is written as

$$f_{\alpha,\beta,\gamma,\delta,\kappa}(g) = (s_{-;1})^{\,n_{-;1}(g)} \times \prod_{\epsilon=\pm,\ell\geq 2} (s_{\epsilon;\ell})^{n_{\epsilon;\ell}(g)}. \tag{6.17}$$

6.3. Characters for the infinite Weyl group $W_{\mathbf{D}_\infty}$ of type $\mathbf{D}_\infty$

For the infinite Weyl group $G^D := W_{\mathbf{D}_\infty}$ of type $\mathbf{D}_\infty$, all the characters of G are given explicitly as follows. Recall that G^D is realized as a semidirect product group as

$$\begin{aligned}G^D &= W_{\mathbf{D}_\infty} = \mathfrak{S}^e_\infty(T) = D^e_\infty(T) \rtimes \mathfrak{S}_\infty \ \text{ with } \ T = \boldsymbol{Z}_2\,, \\ D^e_\infty(\boldsymbol{Z}_2) &= \{ d = (t_i)_{i\in\boldsymbol{N}} \in D_\infty(\boldsymbol{Z}_2),\ \mathrm{sgn}_D(d) = 1 \},\end{aligned} \tag{6.18}$$

where $\mathrm{sgn}_D(d) = \prod_{i\in\boldsymbol{N}} t_i$. For a subset $I \subset \boldsymbol{N}$, we put also $\mathrm{sgn}_D(d) = P_I(d) = \prod_{i\in I} t_i$ for $d = (t_i)_{i\in I} \in D_I(\boldsymbol{Z}_2) \hookrightarrow D_\infty(\boldsymbol{Z}_2)$.

A one-dimensional character of the group G^D is given as $(\mathrm{sgn}_{\mathfrak{S}})^b, b = 0, 1$. However we need one-dimensional characters of so-called wreath product type subgroups H of G^D, and so we keep notations in the case of the Weyl group $G^B := W_{\mathbf{B}_\infty}$.

Similarly as in the case of $W_{\mathbf{B}_\infty}$, we prepare a parameter $(\alpha, \beta, \gamma, \delta, \kappa)$ just as in (6.6) which satisfies the inequality (6.7).

Take a $g \in G^D$ and let $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_m$ be a standard decomposition with basic components $\xi_{q_k} = (t_{q_k}, (q_k))$ and $g_j = (d_j, \sigma_j)$. We have $\mathrm{supp}(d_j) \subset K_j = \mathrm{supp}(\sigma_j)$, and $d_j = (t_i)_{i\in K_j} \in D_{K_j}(\boldsymbol{Z}_2) \hookrightarrow D_\infty(\boldsymbol{Z}_2)$, $\mathrm{sgn}_D(d_j) = P_{K_j}(d_j)$.

Note that each component ξ_{q_k} does not belong to G^D but to G^B, and that $g_j = (d_j, \sigma_j)$ belongs to G^D if and only if $\mathrm{sgn}_D(d_j) = 1$.

Theorem 7 ([HH1]). *Let $G^D = W_{\mathbf{D}_\infty} = \mathfrak{S}^e_\infty(\boldsymbol{Z}_2)$ be the infinite Weyl group of type $\mathbf{D}_\infty$. Let $f_{\alpha,\beta,\gamma,\delta,\kappa}$ be a character of $G^B = W_{\mathbf{B}_\infty} = \mathfrak{S}_\infty(\boldsymbol{Z}_2)$ with a parameter $(\alpha, \beta, \gamma, \delta, \kappa)$ in* (6.6)–(6.7), *and $f^e_{\alpha,\beta,\gamma,\delta,\kappa}$ be its restriction onto $G^D \subset G^B$. Then, $f^e_{\alpha,\beta,\gamma,\delta,\kappa}$ is a character of G^D and is expressed by the same formulas as* (6.10)–(6.11).

Conversely any character of G^D is equal to $f^e_{\alpha,\beta,\gamma,\delta,\kappa}$ for some parameter $(\alpha, \beta, \gamma, \delta, \kappa)$ in (6.6)–(6.7).

Two parameters $(\alpha, \beta, \gamma, \delta, \kappa)$ and $(\alpha', \beta', \gamma', \delta', \kappa')$ determine the same character of G^D, or $f^e_{\alpha,\beta,\gamma,\delta,\kappa} = f^e_{\alpha',\beta',\gamma',\delta',\kappa'}$, if and only if they coincide with each other or

$$(\alpha', \beta') = (\gamma, \delta), \ \ (\gamma', \delta') = (\alpha, \beta), \ \ \kappa' = -\kappa. \tag{6.19}$$

7. Method of proving Theorem 2

Let us explain our method of proving Theorem 2 (for this, see also the part **2** of Introduction). Our proof consists of two parts. The first part is to prepare seemingly sufficiently big family of factorizable (hence extremal by the criterion in the second part) positive definite class functions on $G = \mathfrak{S}_\infty(T)$. The second part is to guarantee that actually all extremal positive definite class functions or characters have been already obtained in the first part.

7.1. The first part of the proof

The first part of our proof has two important ingredients.

One is a method of *taking limits of centralizations* of positive definite functions. This method, which will be explained in the next section, §8, has been applied in [Hi3]–[Hi4] to the case of $\mathfrak{S}_\infty$ and reestablished the results in [Tho2], and also applied in [HH1] to the case of the wreath product groups $\mathfrak{S}_\infty(T)$ with T abelian to get the character formula, which is given here as Theorem 3 in §4.

The other is *inducing up positive definite functions* from appropriate subgroups. After choosing subgroups H and their representations π appropriately, we use their matrix elements f_π as positive definite functions on H to be induced up to G, and then to be centralized.

We have constructed in [Hi1] a huge family of irreducible unitary representations (= IURs) of a wreath product group $G = \mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$ with any finite group T, by taking so-called wreath product type subgroups H in a 'saturated fashion', and their IURs π of a certain form to get IURs of G as induced representations $\rho = \mathrm{Ind}_H^G \pi$.

For our present purpose of getting (possibly) all extremal positive definite class functions on G as pointwise limits of centralizations of their matrix elements, we choose simpler subgroups of degenerate wreath product type and their IURs. In this case, we get unitary representations $\rho = \mathrm{Ind}_H^G \pi$ which are very far from being irreducible, but enough for our purpose to get a sufficiently big set $\mathcal{LIM}$ of positive definite class functions, as such limits of centralizations. This ingredient will be discussed in §§9–10.

Altogether the first part occupies §§8–11 and §§13–14.

7.2. The second part of the proof

The second part contains also two important ingredients.

The first one is a *criterion* for a positive definite class function f to be extremal or indecomposable. Our criterion is given in §15 as Theorem 12 which says that f *is extremal if and only if it is factorizable.*

The second one is a kind of *partial Fourier transform* of class functions on $G = D_\infty(T) \rtimes \mathfrak{S}_\infty$ with respect to the subgroup $D_\infty(T)$. We utilize it in §16 to reduce the problem "when is a factorizable class function f on G positive definite ?" to the level of the infinite symmetric group $\mathfrak{S}_\infty$, and then appeal to [Tho2, Korollar 1 to Satz 2].

As the results, such a factorizable class function f is positive definite if and only if it has the same form as f_A in Theorem 2 with a parameter $A = \big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ ;\ \mu\big)$ satisfying the same condition as in §3.

Altogether the second part occupies §§15–16, and the result in §12 is applied in §16.

8. Centralizations of positive definite functions

Let us explain our method of taking limits of centralizations of positive definite functions. For a function f on a countable discrete group G and a finite subgroup $G' \subset G$, we define a *centralization* of f with respect to G' as

$$f^{G'}(g) := \frac{1}{|G'|} \sum_{g' \in G'} f(g' g {g'}^{-1}). \tag{8.1}$$

Taking an increasing sequence of finite subgroups $G_N \nearrow G$, we consider a series f^{G_N} of centralizations of f with respect to G_N and study its pointwise convergence limit, $\lim_{N\to\infty} f^{G_N}$, which depends heavily on the choice of the series $G_N \nearrow G$.

In our previous papers [Hi3]–[Hi4], we studied positive definite functions

$f(\sigma)$ on $G = \mathfrak{S}_\infty$ of three different types given in [Bo], [BS]: for $\sigma \in G$,

$$r^{|\sigma|} \quad (-1 \le r \le 1); \qquad q^{\|\sigma\|} \quad (0 \le q \le 1); \qquad \mathrm{sgn}(\sigma) q^{\|\sigma\|} \quad (0 \le q \le 1),$$

where r and q are constants. Here $|\sigma|$ denotes the usual length of a permutation σ coming from its reduced expressions by simple transpositions, and $\|\sigma\|$ denotes the block length of σ, which is by definition the number of different simple transpositions appearing in a reduced expression of σ. Then we have proved the following.

Theorem 8. *Let f be one of the above positive definite functions, and $G_N = \mathfrak{S}_N$ $(N \ge 1)$. Assume $|r| < 1$ or $0 < q < 1$ correspondingly. Then the series of centralizations f^{G_N} of f converges pointwise to the delta function δ_e on $G = \mathfrak{S}_\infty$ as N tends to ∞.*

In other words, in the topology of weak containment of unitary representations, this means that each of the representations π_f, associated to f by GNS construction, contains weakly the regular representation λ_G of $G = \mathfrak{S}_\infty$.

We have also calculated various limits of centralizations of positive definite matrix elements of irreducible or non-irreducible representations which are induced from subgroups of wreath product type.

In the recent paper [HH1], we have treated the case of $\mathfrak{S}_\infty(T)$ with T any finite abelian group, which contains the case of infinite Weyl groups $W_{\mathbf{B}_\infty}$ and $W_{\mathbf{D}_\infty}$.

Especially we observed in [Hi3]–[Hi4] and in [HH1] the following fact, for the infinite symmetric group $\mathfrak{S}_\infty$ and wreath product groups $\mathfrak{S}_\infty(T)$ with T abelian.

Observation. *For a certain choice of a subgroup H and one of its unitary representation π, the family of limits of centralizations of matrix elements of the induced representation $\rho = \mathrm{Ind}_H^G \pi$ covers all the characters of the group G.*

9. Inducing up of positive definite functions

9.1. Matrix elements of induced representations

In a general setting, let G be a discrete group, and H its subgroup. Take a unitary representation π of H on a Hilbert space $V(\pi)$, and consider an induced representation $\rho = \mathrm{Ind}_H^G \pi$.

The representation space $V(\rho)$ of ρ is given as follows. For a vector $v \in V(\pi)$, and a representative g_0 of a right coset $Hg_0 \in H\backslash G$, put

$$E_{v,g_0}(g) = \begin{cases} \pi(h)v & (g = hg_0, h \in H), \\ 0 & (\, g \notin Hg_0). \end{cases} \tag{9.1}$$

Let $\mathcal{V}$ be a linear span of these $V(\pi)$-valued functions on G, and define an inner product on it as

$$\langle E_{v,g_0}, E_{v',g_0'} \rangle = \begin{cases} \langle \pi(h)v, v' \rangle & \text{if } \ hg_0 = g_0' \ (\exists h \in H), \\ 0 & \text{if } \ Hg_0 \ne Hg_0'. \end{cases} \tag{9.2}$$

The space $V(\rho)$ is nothing but the completion of $\mathcal{V}$.

The representation ρ is given as $\rho(g_1)E(g) = E(gg_1)$ $(g_1, g \in G, E \in V(\rho))$.

Now take a non-zero vector $v \in V(\pi)$ and put $E = E_{v,e} \in V(\rho)$. Consider a positive definite function on H associated to π as

$$f_\pi(h) = \langle \pi(h)v, v \rangle \qquad (h \in H), \tag{9.3}$$

and also such a one on G associated to ρ as

$$F(g) = \langle \rho(g)E, E \rangle \qquad (g \in G). \tag{9.4}$$

Then, we can easily prove the following lemma.

Lemma 9.1. *The positive definite function F on G in (9.4) associated to $\rho = \mathrm{Ind}_H^G \pi$ is equal to the inducing up of the positive definite function f_π on H associated to π: $F = \mathrm{Ind}_H^G f_\pi$, which is, by definition, equal to f_π on H and to zero outside of H.*

9.2. Centralizations of $F = \mathrm{Ind}_H^G f_\pi$

Let $G_N \nearrow G$ be an increasing sequence of finite subgroups going up to G, and consider a series of centralizations F^{G_N} of F.

Since F is zero outside of H, the value of centralization $F^{G_N}(g)$ is $\neq 0$ only for elements g which are conjugate under G_N to some $h \in H$. Moreover, for $h \in H$, we get

$$F^{G_N}(h) = \frac{1}{|G_N|} \sum_{g' \in G_N \,:\, g'hg'^{-1} \in H} f_\pi(g'hg'^{-1}). \tag{9.5}$$

The condition $g'hg'^{-1} \in H$ for $g' \in G_N$, is translated into certain combinatorial conditions, and to get the limit as $N \to \infty$, we have to calculate asymptotic behavior of several ratios of combinatorial numbers.

The details in the case of $G = \mathfrak{S}_\infty$ are given in [Hi3]–[Hi4]. For the infinite Weyl groups, $G = W_{\mathbf{B}_\infty}$ and $W_{\mathbf{D}_\infty}$, and moreover for wreath product groups $G = \mathfrak{S}_\infty(T)$ with T any finite abelian groups, essential parts of these calculations are sketched in [HH1].

10. Subgroups and their representations for $\mathfrak{S}_\infty(T)$

10.1. IURs of $\mathfrak{S}_\infty(T)$, T a finite group, as induced representations

In the previous paper [Hi1], we have constructed a big family of IURs by the method of inducing up from wreath product type subgroups. Let us review it briefly.

Take a subgroup H of $G = \mathfrak{S}_\infty(T)$ of the form

$$\begin{aligned} H = H_0 \times {\textstyle\prod'_{p \in P}} H_p\,, \quad H_0 = \mathfrak{S}_{I_0}(T), \\ H_p = \mathfrak{S}_{I_p}(T_p) = D_{I_p}(T_p) \rtimes \mathfrak{S}_{I_p}\,, \end{aligned} \tag{10.1}$$

where I_0 is a finite subset (we admits empty set), and I_p's are infinite subsets of $\boldsymbol{N}$ all mutually disjoint, and T_p's are subgroups of T. Thus H is determined by the datum

$$\mathfrak{c} := \Big(I_0, (I_p, T_p)_{p\in P}\Big)$$

and is denoted also by $H^{\mathfrak{c}}$. To get IURs as induced representations from $H = H^{\mathfrak{c}}$, we assume that H is "*saturated*" in G in the sense that $\boldsymbol{N} = I_0 \sqcup (\sqcup_{p\in P} I_p)$ is a partition of $\boldsymbol{N}$.

As an IUR of H, we take so-called factorizable one:

$$\pi = \pi_0 \otimes \big(\otimes^b_{p\in P}\pi_p\big) \quad \text{for} \;\; H_0 \times {\textstyle\prod'_{p\in P}} H_p\,. \tag{10.2}$$

Here $b = (b_p)_{p\in P}, b_p \in V(\pi_p), \|b_p\| = 1$, is a reference vector to take tensor product of π_p's, when P is infinite, and IURs π_0 and π_p are given as follows. First choose an IUR $\zeta_p \in \widehat{T_p}$ (resp. $\zeta_0 \in \widehat{T}$). Then, for the subgroup $D_{I_p}(T_p)$ (resp. $D_{I_0}(T)$), we take an IUR given as a tensor product $\pi^D_p := \otimes^{a_p}_{i\in I_p}\zeta_{p,i}\,,\ \zeta_{p,i} = \zeta_p$ (resp. $\pi^D_0 := \otimes_{i\in I_0}\zeta_{0,i}\,,\ \zeta_{0,i} = \zeta_0$), where $a_p = (a_{p,i})_{i\in I_p}\,,\ a_{p,i} \in V(\zeta_{p,i}),\ \|a_{p,i}\| = 1$, is a reference vector with respect to which the infinite tensor product of $\zeta_{p,i} = \zeta_p$ $(i \in I_p)$ is taken. Then, a $\sigma \in \mathfrak{S}_{I_p}$ acts on the space $V := \otimes^{a_p}_{i\in I_p} V(\zeta_{p,i})$ as a permutation of components as

$$I(\sigma) : V \ni v = \otimes_{i\in I_p} v_i \;\longmapsto \otimes_{i\in I_p} v'_i \in V,\ v'_i = v_{\sigma^{-1}(i)}\,, \tag{10.3}$$

where $v_i \in V(\zeta_{p,i}), i \in I_p$. Take a one-dimensional character $\chi^{\mathfrak{S}}_p$ of $\mathfrak{S}_{I_p}$, then we get an IUR π_p of $H_p = \mathfrak{S}_{I_p}(T_p)$ by the formula:

$$\pi_p\big((d,\sigma)\big) := \pi^D_p(d)\, I(\sigma)\, \chi^{\mathfrak{S}}_p(\sigma) \quad \big((d,\sigma) \in D_{I_p}(T_p) \rtimes \mathfrak{S}_{I_p}\big),$$

and similarly for $H_0 = \mathfrak{S}_{I_0}(T)$. In case ζ_p is one-dimensional or P is finite, the reference vector a_p or b is not necessary.

Thus the IUR π of $H = H^{\mathfrak{c}}$ is determined by the datum $(\mathfrak{c}, \mathfrak{d})$ with

$$\mathfrak{d} := \big(\,(\zeta_0, \chi^{\mathfrak{S}}_0)\,,\, (\zeta_p, a_p, \chi^{\mathfrak{S}}_p)_{p\in P}\,;\ b\,\big),$$

and is denoted also by $\pi(\mathfrak{c}, \mathfrak{d})$. We know in [Hi1] that, under the saturation condition: $\boldsymbol{N} = I_0 \sqcup (\sqcup_{p\in P} I_p)$, the induced representation

$$\rho(\mathfrak{c}, \mathfrak{d}) = \mathrm{Ind}^G_H \pi(\mathfrak{c}, \mathfrak{d})$$

is irreducible, and equivalence relations among these IURs are also clarified there.

In the previous paper [HH1], we gave Conjecture 2002-5 to generalize this method of constructing IURs. One point is that, to have the irreducibility for induced representation $\rho = \mathrm{Ind}^G_H\pi$, we may start with π coming from H_p and π_p in (10.2) such that the full group T is taken as T_p, and a cyclic representation of T as ζ_p, and $a_p = (a_{p,i})_{i\in I_p}$ with cyclic vectors $a_{p,i} \in V(\zeta_{p,i}) = V(\zeta_p)$.

10.2. Subgroups and their representations for matrix elements f_π

In place of the purpose in [Hi1] of getting IURs, our present purpose is to get all the characters of $G = \mathfrak{S}_\infty(T)$ as limits of centralizations of matrix elements $F = \mathrm{Ind}_H^G f_\pi$ of $\rho = \mathrm{Ind}_H^G \pi$, where f_π is a positive definite matrix element of a UR π of H. To this purpose, we look for the best choice of a pair of H and π, following principally the case of [Hi1], but simplifying the situation without paying attention on *irreducibility* of the induced representation.

To give such subgroups H, we take first a partition of $\boldsymbol{N}$ as

$$\boldsymbol{N} = \Big(\bigsqcup_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \Big(\bigsqcup_{p\in P_{\zeta,\varepsilon}} I_p\Big)\Big) \bigsqcup \Big(\bigsqcup_{\zeta\in\widehat{T}} I_\zeta\Big) \bigsqcup I_e\,, \tag{10.4}$$

where each $P_{\zeta,\varepsilon}$ is an infinite index set, and the subsets I_* are all infinite. Corresponding to this partition, we define a subgroup

$$H = \Big(\prod_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \Big(\prod'_{p\in P_{\zeta,\varepsilon}} H_p\Big)\Big) \times \Big(\prod_{\zeta\in\widehat{T}} H_\zeta\Big) \times H_e\,, \tag{10.5}$$

$$\text{with} \qquad H_p = \mathfrak{S}_{I_p}(T), \quad H_\zeta = D_{I_\zeta}(T), \quad H_e = \{\, e\,\}.$$

Here e is the identity element of G, and we consider H_e as a trivial subgroup of $\mathfrak{S}_{I_e}(T)$. We call this kind of subgroups *of degenerate wreath product type.*

For a representation π of H to be induced up to G, we take

$$\pi = \Big(\otimes_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \Big(\otimes^{b_{\zeta,\varepsilon}}_{p\in P_{\zeta,\varepsilon}} \pi_p\Big)\Big) \otimes \Big(\otimes_{\zeta\in\widehat{T}} \pi_\zeta\Big) \otimes \mathbf{1}_{H_e}\,. \tag{10.6}$$

Here $b_{\zeta,\varepsilon} = (b_p)_{p\in P_{\zeta,\varepsilon}}$ is a reference vector with $b_p \in V(\pi_p)$, $\|b_p\| = 1$ $(p \in P_{\zeta,\varepsilon})$, and for $p \in P_{\zeta,\varepsilon}$, π_p for $H_p = \mathfrak{S}_{I_p}(T)$ is given as

$$\begin{aligned} \pi_p\big((d,\sigma)\big) \;&=\; \Big(\otimes^{a_p}_{i\in I_p} \zeta_i(t_i)\Big)\, I(\sigma)\, \mathrm{sgn}_{\mathfrak{S}}(\sigma)^\varepsilon \\ &\text{for} \quad d = (t_i)_{i\in I_p} \in D_{I_p}(T),\ \sigma \in \mathfrak{S}_{I_p}, \end{aligned} \tag{10.7}$$

where $a_p = (a_i)_{i\in I_p}$ is a reference vector with $a_i \in V(\zeta_i)$, $\|a_i\| = 1$, and $\zeta_i = \zeta$ as a representation of $T_i = T$ $(i \in I_p)$, and $I(\sigma)$ as in (10.3); and for $\zeta \in \widehat{T}$, π_ζ for $H_\zeta = D_{I_\zeta}(T)$ is given as

$$\pi_\zeta(d) = \otimes^{a_\zeta}_{i\in I_\zeta} \zeta_i(t_i) \quad \text{for} \;\; d = (t_i)_{i\in I_\zeta} \in H_\zeta, \tag{10.8}$$

where $a_\zeta = (a_i)_{i\in I_\zeta}$ is a reference vector with $a_i \in V(\zeta_i)$, $\|a_i\| = 1$, and $\zeta_i = \zeta$ as a representation of $T_i = T$ $(i \in I_\zeta)$.

11. Increasing sequences of subgroups $G_N \nearrow G = \mathfrak{S}_\infty(T)$.

Depending on the choice of increasing series $G_N \nearrow G$ of subgroups, we get various positive definite class functions of G as limits of centralizations F^{G_N} for $F = \mathrm{Ind}_H^G f_\pi$, which turn out to be characters. We choose a series G_N as $G_N = \mathfrak{S}_{J_N}(T)$, $J_N \nearrow \boldsymbol{N}$, and demand an asymptotic condition as

$$\frac{|I_p \cap J_N|}{|J_N|} \to \lambda_p \;\; (p \in P), \qquad \frac{|I_\zeta \cap J_N|}{|J_N|} \to \mu_\zeta \;\; (\zeta \in \widehat{T}), \tag{11.1}$$

where $P := \sqcup_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} P_{\zeta,\varepsilon}$ is the union of index sets. Note that even in this case, $\lim_{N\to\infty} |I_e \cap J_N|/|J_N|$ may not exist. Anyhow we have

$$\sum_{p\in P} \lambda_p + \sum_{\zeta\in\widehat{T}} \mu_\zeta \le 1 . \tag{11.2}$$

For each $(\zeta,\varepsilon) \in \widehat{T}\times\{0,1\}$, let reorder the numbers $\{\lambda_p\,;\, p \in P_{\zeta,\varepsilon}\}$ in the decreasing order and put it as $\alpha_{\zeta,\varepsilon} := (\alpha_{\zeta,\varepsilon,i})_{i\in\boldsymbol{N}}$, and also put $\mu := (\mu_\zeta)_{\zeta\in\widehat{T}}$. Then,

$$\sum_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \|\alpha_{\zeta,\varepsilon}\| + \|\mu\| \le 1 ,$$

which is nothing but the condition (3.6). As a pointwise limit of the series of centralizations F^{G_N}, we obtain the character f_A with

$$A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ ;\ \mu\Big)$$

in Theorem 2. The calculation will be given later in §§13–14, and it is also explained in detail in [Hi3]–[Hi4] in the case of $\mathfrak{S}_\infty$.

Finally we remark that, to obtain all the characters of G, it is actually sufficient for us to use only one set of H and π above, and this means that the induced representation $\rho = \mathrm{Ind}_H^G \pi$ contains weakly all the factor representations of finite type of G.

Example 11.1. Non-existence of $\lim_{N\to\infty} |J_N \cap I_e|/|J_N|$ happens due to $|P| = \infty$. Let us give an example. Let the index set P be equal to $\boldsymbol{N}$. We define J_N as a disjoint union of $J'_N \subset \cup_{n\in\boldsymbol{N}} I_n$ and $J''_N \subset I_e$ (we put I_ζ's aside, for simplicity). Choose J'_N in such a way that $\big|J'_N \cap I_n\big| = N - n$ $(n < N)$; $\big|J'_N \cap I_n\big| = 0$ $(n \ge N)$. Then, $|J'_N| = N(N-1)/2$ and so

$$0 \le \lambda_n = \lim_{N\to\infty} \frac{|J_N \cap I_n|}{|J_N|} \le \lim_{N\to\infty} \frac{|J'_N \cap I_n|}{|J'_N|} = 0 \quad (\forall n).$$

To define J''_N, we determine $N_1 < N_2 < \cdots$ inductively as follows. Put $N_1 = 1$ and $J''_1 = \emptyset$, and for $N_k < N < N_{k+1}$, put $J''_N = J''_{N_k}$. Here N_{k+1} is the first integer $N > N_k$ for which $e_N := |J_N \cap I_e|/|J_N| = |J''_{N_k}|/(|J'_N| + |J''_{N_k}|)$ becomes smaller than $1/10^{k+1}$. Then put $J''_{N_{k+1}} = J''_{N_k} \sqcup J'''_k$, where $J'''_k \subset I_e$ is so taken as

$$a_{k+1} := \frac{|J_{N_{k+1}} \cap I_e|}{|J_{N_{k+1}}|} = \frac{|J''_{N_k}| + |J'''_k|}{|J'_{N_{k+1}}| + |J''_{N_k}| + |J'''_k|} \ \ge \frac{1}{2}.$$

Since e_N for $N_k < N < N_{k+1}$ decreases from a_k to $\le 1/10^{k+1}$, we have $\overline{\lim}_{N\to\infty} e_N = \overline{\lim}_{k\to\infty} a_k \ge 1/2$, and $\underline{\lim}_{N\to\infty} e_N = 0$.

12. Limits of trace characters of representations of $\mathfrak{S}_n(T)$

In certain cases, we can calculate characters of $G = \mathfrak{S}_\infty(T)$ as limits of trace characters of representations of $G_n = \mathfrak{S}_n(T)$ as $n \to \infty$. This result will be applied later.

First we take IURs of a degenerate form. Take a $\zeta \in \widehat{T}$, and put $I = \boldsymbol{N}$. Define tensor product representation of $D_I(T)$ as

$$\otimes^a_{i\in I}\zeta_i \quad \text{with} \quad \zeta_i = \zeta \quad \text{for} \quad T_i = T \ (i \in I = \boldsymbol{N})$$

with respect to a reference vector $a = (a_i)_{i\in I}, a_i \in V(\zeta_i), \|a_i\| = 1$, for which the representation space is $V = \otimes^a_{i\in I} V(\zeta_i)$. For $\sigma \in \mathfrak{S}_\infty$, put

$$I(\sigma)\big(\otimes^a_{i\in I}\, v_i\big) := \otimes^a_{i\in I} v_{\sigma^{-1}(i)} \quad \text{with} \quad v_i \in V(\zeta_i),\ v_i = a_i \ (i \gg 1).$$

Then, for $(\zeta,\varepsilon) \in \widehat{T} \times \{0,1\}$, we get an IUR ρ of G: for $g = (d,\sigma) \in G, d = (t_i)_{i\in I}$,

$$\rho(g) = \rho\big((d,\sigma)\big) := \big(\otimes^a_{i\in I}\, \zeta_i(t_i)\big) I(\sigma)\,\mathrm{sgn}(\sigma)^\varepsilon. \tag{12.1}$$

Now for $\boldsymbol{I}_n = \{1,2,\ldots,n\}$, we take a similar representation ρ_n of G_n. This is given on the space $V_n := \otimes_{i\in \boldsymbol{I}_n} V(\zeta_i)$ and, for $g = (d,\sigma) \in G_n = \mathfrak{S}_n(T) = D_{\boldsymbol{I}_n}(T) \rtimes \mathfrak{S}_n$,

$$\rho_n(g) = \rho_n\big((d,\sigma)\big) = \big(\otimes_{i\in \boldsymbol{I}_n}\, \zeta_i(t_i)\big) I(\sigma)\,\mathrm{sgn}(\sigma)^\varepsilon. \tag{12.2}$$

Then, we may consider as $V_n \nearrow V$, and then $\rho_n \nearrow \rho$ acccording to $G_n \nearrow G$.

Take a $g \in G$. Then, starting from a certain n, g belongs to G_n, and so we can consider the limit of trace characters as $\lim_{n\to\infty} \mathrm{trace}\big(\rho_n(g)\big)$. As a result, it is better to consider the normalized one as $\mathrm{trace}\big(\rho_n(g)\big)/\dim \rho_n$.

Theorem 9. *Let ρ_n be an IUR of $G_n = \mathfrak{S}_n(T)$ constructed from $(\zeta,\varepsilon) \in \widehat{T} \times \{0,1\}$ as above. Then, there exists a pointwise limit $F_{\zeta,\varepsilon}$ on $G = \mathfrak{S}_\infty(T)$ given as follows. For $g = (d,\sigma) \in G$, let*

$$g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m, \quad \xi_q = \big(t_q,(q)\big), \ \ g_j = (d_j,\sigma_j),$$

be a standard decomposition, i.e., a decomposition into mutually disjoint basic elements. Then,

$$\begin{aligned} F_{\zeta,\varepsilon}(g) &:= \lim_{n\to\infty} \frac{\mathrm{trace}\big(\rho_n(g)\big)}{\dim \rho_n} \\ &= \prod_{1\le k\le r} \frac{\chi_\zeta(t_{q_k})}{\dim \zeta} \times \prod_{1\le j\le m} \frac{\chi_\zeta\big(P_{\sigma_j}(d_j)\big)}{(\dim \zeta)^{\ell(\sigma_j)}}\,\mathrm{sgn}(\sigma_j)^\varepsilon, \end{aligned} \tag{12.3}$$

where, for $\sigma_j = (i_1\ \ i_2\ \ \ldots\ \ i_{\ell_j})$ with $\ell_j = \ell(\sigma_j)$, and $d_j = (t_i)_{i\in K_j}$ with $K_j := \mathrm{supp}(\sigma_j)$,

$$P_{\sigma_j}(d_j) := \big[t'_{\ell_j} t'_{\ell_j - 1} \cdots t'_2 t'_1\big] \in T/{\sim} \qquad \text{with} \quad t'_k = t_{i_k}.$$

Proof. Since ζ is unitary, we have an orthonormal basis $\{ e_1, e_2, \ldots, e_m \}$ with $m = \dim \zeta$ of the representation space $V(\zeta)$. Then, a basis of V_n is given by

$$\big\{ e_M \;;\; M = (j_1, j_2, \ldots, j_n),\, j_1, j_2, \ldots, j_n \in \boldsymbol{I}_m \big\}$$
$$\text{with} \quad e_M = e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_n}.$$

Take a basic element $g' = (d', \sigma') \in \mathfrak{S}_n(T) = D_{\boldsymbol{I}_n}(T) \rtimes \mathfrak{S}_n$ with a cycle $\sigma' = (1\ 2\ \ldots\ n)$ and $d' = (t_i)_{i \in \boldsymbol{I}_n}$. On the space V_n, it operates as

$$\rho_n(g') = \rho_n((d', \sigma')) = \big(\otimes_{i \in \boldsymbol{I}_n} \zeta_i(t_i) \big)\, I(\sigma')\, \mathrm{sgn}(\sigma')^{\varepsilon}.$$

Let us calculate the trace of $\rho_n(g')$. Recall that $I(\tau) e_M = e_{\tau M}$ with $\tau M := (j_{\tau^{-1}(1)}, j_{\tau^{-1}(2)}, \ldots, j_{\tau^{-1}(n)})$. Let the matrix elements of $\zeta(t)$ with respect to the basis $\{ e_j \;;\; 1 \le j \le m \}$ be $\zeta_{jk}(t)$, that is, $\zeta(t) e_k = \sum_{1 \le j \le m} \zeta_{jk}(t) e_j$. Then, taking into account of $\sigma^{-1}(i) = i - 1$ ($1 \le i \le n$, $0 \equiv n$), we get for $g' = (d', \sigma')$,

$$\begin{aligned}
&\mathrm{trace}\big(\rho_n(g')\big) = \sum_M \langle \rho_n(g') e_M, e_M \rangle \\
&= \mathrm{sgn}(\sigma')^{\varepsilon} \sum_{1 \le j_1 \le m} \sum_{1 \le j_2 \le m} \cdots \sum_{1 \le j_n \le m} \prod_{i \in \boldsymbol{I}_n} \langle \zeta(t_i) e_{j_{\sigma^{-1}(i)}}, e_{j_i} \rangle \\
&= \mathrm{sgn}(\sigma')^{\varepsilon} \sum_{1 \le j_1 \le m} \sum_{1 \le j_2 \le m} \cdots \sum_{1 \le j_n \le m} \zeta_{j_1, j_n}(t_1)\, \zeta_{j_2, j_1}(t_2) \cdots \zeta_{j_n, j_{n-1}}(t_n) \\
&= \mathrm{sgn}(\sigma')^{\varepsilon}\ \mathrm{trace}\big(\zeta(t_n t_{n-1} \cdots t_2 t_1)\big) = \mathrm{sgn}(\sigma')^{\varepsilon}\ \chi_{\zeta}\big(P_{\sigma'}(d')\big).
\end{aligned}$$

The calculation is similar for other choice of cycle σ', and the proof is now complete. □

Note 12.1. (i) The positive definite class function $F_{\zeta,\varepsilon}$ is a special case of f_A in (3.8) in Theorem 2, for which $\alpha_{\zeta,\varepsilon} = (1, 0, 0, \ldots)$ and other parameters $\alpha_{\zeta',\varepsilon'}$ and $\mu_{\zeta'}$ are all zero.

(ii) For $\zeta = \boldsymbol{1}_T \in \widehat{T}$, the trivial representation of T, we have $F_{\boldsymbol{1}_T,\varepsilon}(g) = \mathrm{sgn}(\sigma)^{\varepsilon}$ for $g = (d, \sigma) \in G$, and for any $\zeta \in \widehat{T}$, $F_{\zeta,\varepsilon}(g) = F_{\zeta,0}(g)\, F_{\boldsymbol{1}_T,\varepsilon}(g)$ ($g \in G$).

13. Partial centralization with respect to $D_{J_N}(T)$

As an increasing sequence $G_N \nearrow G = \mathfrak{S}_{\infty}(T)$ of subgroups, we have chosen $G_N = \mathfrak{S}_{J_N}(T) = D_{J_N}(T) \rtimes \mathfrak{S}_{J_N}$ with $J_N \nearrow \boldsymbol{N}$. Put $D_N = D_{J_N}(T)$ and $S_N = \mathfrak{S}_{J_N}$ for simplicity, then $G_N = D_N \rtimes S_N$, and we identify $d' \in D_N$ and $\sigma' \in S_N$ with their images in G_N respectively. Our task is to calculate centralizations F^{G_N} of a positive definite matrix element $F = \mathrm{Ind}_H^G f_{\pi}$ of $\rho = \mathrm{Ind}_H^G \pi$, and to determine their limits. From the formula (9.5) for F^{G_N} and the explicit form of the subgroup H in (10.5), we see that for $h \in H$,

$$F^{G_N}(h) = \frac{1}{|G_N|} \sum_{g' \in G_N \,:\, g'hg'^{-1} \in H} f_\pi(g'hg'^{-1}) = \frac{1}{|S_N|} \sum_{\sigma' \in S_N \,:\, \sigma' h \sigma'^{-1} \in H} \widetilde{f_\pi}(\sigma' h \sigma'^{-1}), \tag{13.1}$$

where $\widetilde{f_\pi}$ is a *partial centralization* of f_π with respect to $D_N \cong T^{J_N}$ defined as

$$\widetilde{f_\pi}(h') = \int_{D_N} f_\pi(d'h'd'^{-1})\, d\mu_{D_N}(d') \qquad (h' \in H), \tag{13.2}$$

with the normalized Haar measure $d\mu_{D_N}$ on D_N. (Hereafter we apply the notations in the case of compact groups by using the integration instead of the summation.)

Note that for a finite number of $h' \in H$, the partial centralization $\widetilde{f_\pi}(h')$ is stable as N is sufficiently large. To calculate it, we apply the explicit form of representation π of H given in (10.6)–(10.7). Then we see that it is essentially enough to treat two cases of basic elements:

(i) $h' = \xi_q = (t_q, (q))$ with $t_q \in T^*$, and

(ii) $h' = (d', \sigma')$ with σ' a cycle and $\mathrm{supp}(d') \subset \mathrm{supp}(\sigma')$.

For this, we prepare two lemmas, one for a compact group T, and the other for a wreath product group $\mathfrak{S}_n(T)$ of a compact group T with the symmetric group $\mathfrak{S}_n$.

Lemma 13.1. *Let T be a compact group and $\zeta \in \widehat{T}$. Take $v, w \in V(\zeta)$, then*

$$\int_T \langle \zeta(sts^{-1})v, w\rangle\, d\mu_T(s) = \frac{\chi_\zeta(t)}{\dim \zeta}\, \langle v, w\rangle, \tag{13.3}$$

where $d\mu_T$ denotes the normalized Haar measure on T.

Put $K = \{1, 2, \ldots, \ell\}$, and let $\sigma = (1\ 2\ \ldots\ \ell)$ be a cycle with $\mathrm{supp}(\sigma) = K$ and $g = (d, \sigma)$ a basic element in $\mathfrak{S}_K(T)$ with $d = (t_i)_{i\in K}$. Then, for $d' = (s_i)_{i \in K} \in D_K(T)$, we have

$$d'gd'^{-1} = (d'', \sigma) \quad \text{with} \quad d'' = d'd \cdot \sigma(d'^{-1}) = (s_i t_i s_{i-1}^{-1})_{i\in K} \quad (0 \equiv \ell). \tag{13.4}$$

On the other hand, for a decomposable vector $v = \otimes_{i\in K} v_i \in V\big(\otimes_{i\in K} \zeta_i\big)$ with $v_i \in V(\zeta_i), \zeta_i = \zeta$, the subrepresentation Π of π_p for $\mathfrak{S}_K(T) \subset \mathfrak{S}_{I_p}(T)$ is given as

$$\Pi(g)v = \bigotimes\nolimits_{i\in K} \big(\zeta(t_i) v_{\sigma^{-1}(i)}\big) = \bigotimes\nolimits_{i \in K} \big(\zeta(t_i) v_{i-1}\big).$$

Therefore the partial centralization with respect to $D_K(T)$ is given as follows.

Lemma 13.2. *Let $\otimes_{i\in K}\zeta_i$ be a tensor product representation of $D_K(T) \cong T^K$ of $\zeta_i = \zeta$ of $T_i = T$ $(i \in K)$, and take decomposable vectors $v = \otimes_{i\in K}v_i$ and $w = \otimes_{i\in K}w_i$ from $V\big(\otimes_{i\in K}\zeta_i\big)$ with $v_i, w_i \in V(\zeta_i)$. Then, as an integration with respect to the normalized Haar measure $d\mu_{D_K(T)}(s) = \prod_{i\in K} d\mu_T(s_i)$, $s = (s_i)_{i\in K} \in T^K \cong D_K(T)$, we have*

$$\begin{aligned}
&\int_{D_N(T)} \langle \Pi(sgs^{-1})v, w\rangle \, d\mu_{D_K(T)}(s) \\
&= \int_{D_K(T)} \big\langle \otimes_{i\in K} \big(\zeta(s_i t_i s_{i-1}^{-1})v_{i-1}\big), \otimes_{i\in K} w_i \big\rangle \, d\mu_{D_K(T)}(s) \\
&= \int_T \cdots \int_T \langle \zeta(s_1 t_1 s_\ell^{-1})v_\ell, w_1\rangle \langle \zeta(s_2 t_2 s_1^{-1})v_1, w_2\rangle \cdots \langle \zeta(s_\ell t_\ell s_{\ell-1}^{-1})v_{\ell-1}, w_\ell\rangle \\
&\qquad\qquad d\mu_T(s_1)\, d\mu_T(s_2) \cdots d\mu_T(s_\ell) \\
&= \frac{\chi_\zeta(t_\ell t_{\ell-1}\cdots t_2 t_1)}{(\dim\zeta)^\ell} \prod_{i\in K} \langle v_i, w_i\rangle \;=\; \frac{\chi_\zeta\big(P_\sigma(d)\big)}{(\dim\zeta)^\ell} \prod_{i\in K}\langle v_i, w_i\rangle.
\end{aligned}$$

For a proof, see Lemma A.5 in Appendix.

Take $v = \otimes_{i\in K}v_i$ with unit vectors $v_i \in V(\zeta_i)$, and put $w = v$ in Lemma 13.2, then we get $\chi_\zeta\big(P_\sigma(d)\big)/(\dim\zeta)^\ell$ as the result of partial centralization above.

Let H be a subgroup of G given by (10.4)–(10.5), and π its unitary representation given in (10.6)–(10.8). For a unit vector $v \in V(\pi)$, we put $f_\pi(h) = \langle \pi(h)v, v\rangle$ $(h \in H)$. Since we are now concerned with centralizations with respect to finite subgroups G_N, the role of reference vectors is not important, and we may take v as a tensor product of unit vectors from $V(\zeta_i)$ for $i \in I_p, p \in P_{\zeta,\varepsilon}$, for every (ζ,ε), and similarly for I_ζ's. Then, by Lemmas 13.1 and 13.2, we get the following result.

Proposition 10. *Take a $g = (d,\sigma)$ from H and let*

$$g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m, \quad \xi_q = \big(t_q, (q)\big), \quad g_j = (d_j, \sigma_j),$$

be a standard decomposition. Then, the partial centralization $\widetilde{f_\pi}(g)$ of matrix element f_π is given as follows. Let $K(\zeta)$ be the set of k, $1 \le k \le r$, such that $\xi_{q_k} \in H_p$ with $p \in \sqcup_{\varepsilon\in\{0,1\}} P_{\zeta,\varepsilon}$ or $\xi_{q_k} \in H_\zeta$, and $J(\zeta,\varepsilon)$ be the set of j, $1 \le j \le m$, such that $g_j = (d_j, \sigma_j) \in H_p$ with $p \in P_{\zeta,\varepsilon}$. Then,

(13.5)

$$\widetilde{f_\pi}(g) = \\ = \left(\prod_{\zeta\in\widehat{T}} \prod_{k\in K(\zeta)} \frac{\chi_\zeta(t_{q_k})}{\dim\zeta}\right) \left(\prod_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \prod_{j\in J(\zeta,\varepsilon)} \frac{\chi_\zeta\big(P_{\sigma_j}(d_j)\big)}{(\dim\zeta)^{\ell(\sigma_j)}} \operatorname{sgn}(\sigma_j)^\varepsilon\right),$$

where, for $\sigma_j = (i_1\ i_2\ \ldots\ i_{\ell_j})$ with $\ell_j = \ell(\sigma_j)$ and $d_j = (t_i)_{i\in K_j}$ with $K_j :=$ $\mathrm{supp}(\sigma_j)$,

$$P_{\sigma_j}(d_j) := \big[t'_{\ell_j} t'_{\ell_j-1}\cdots t'_2 t'_1\big] \in T/\!\sim \qquad \text{with} \quad t'_k = t_{i_k}.$$

14. Limits of centralizations of positive definite functions

We are now on the way of calculating centralizations of F^{G_N} of a positive definite matrix element $F = \mathrm{Ind}_H^G f_\pi$ of $\rho = \mathrm{Ind}_H^G \pi$ with respect to $G_N = D_{J_N}(T) \rtimes \mathfrak{S}_{J_N}$, and to determine their limits. Recall the formula (13.1) as

$$F^{G_N}(g) = \frac{1}{|S_N|} \sum_{\tau \in S_N \,:\, \tau g \tau^{-1} \in H} \widetilde{f_\pi}(\tau g \tau^{-1}) \qquad (g \in H), \tag{14.1}$$

where $S_N = \mathfrak{S}_{J_N}$, and the partial centralization $\widetilde{f_\pi}$ with respect to $D_N = D_{J_N}(T)$ is defined by (13.2) and is calculated in Proposition 10.

14.1. Limit of centralizations for a ' monomials ' term

For any element in G, there exists an element in H conjugate to it. Therefore it is enough for us to determine the value F^{G_N} on H. Take $g = (d, \sigma) \in H$ and let $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m$, $\xi_q = \big(t_q, (q)\big)$, $g_j = (d_j, \sigma_j)$, be its standard decomposition. Put $P = \bigsqcup_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} P_{\zeta,\varepsilon}$, then,

$$H = \Big(\textstyle\prod'_{p\in P} H_p\Big) \times \Big(\prod_{\zeta\in\widehat{T}} H_\zeta\Big) \times H_e,$$

and the condition $g \in H$ means that each ξ_{q_k} belongs to one of H_p and H_ζ, and that each g_j belongs to one of H_p. Furthermore, the latter condition can be expressed by means of supports as

$$\begin{aligned} &\mathrm{supp}(\xi_{q_k}) = \{\, q_k \,\} \subset I_p \ \text{ or } \ \subset I_\zeta \\ \text{and} \quad &K_j = \mathrm{supp}(g_j) = \mathrm{supp}(\sigma_j) \subset I_p. \end{aligned} \tag{14.2}$$

For $p \in P$, choose (ζ, ε) such that $p \in P_{\zeta,\varepsilon}$, and put for basic elements $\xi_q = (t_q, (q))$ and $g_j = (d_j, \sigma_j)$ in H_p,

$$\begin{aligned} \chi_p(\xi_q) &= \frac{\chi_\zeta(t_q)}{\dim\zeta} \quad \text{for} \quad t_q \in T^* \,; \\ \chi_p(g_j) &= \frac{\chi_\zeta\big(P_{\sigma_j}(d_j)\big)}{(\dim\zeta)^{\ell(\sigma_j)}}\, \mathrm{sgn}(\sigma_j)^\varepsilon. \end{aligned} \tag{14.3}$$

Then the formula (13.5) for $\widetilde{f_\pi}(g)$ is rewritten as

$$\begin{aligned} &\widetilde{f_\pi}(g) = \\ &= \prod_{\zeta\in\widehat{T}} \left(\prod_{k\,:\,q_k\in I_\zeta} \frac{\chi_\zeta(t_{q_k})}{\dim\zeta} \right) \times \prod_{p\in P} \left(\prod_{k\,:\,q_k\in I_p} \chi_p(\xi_{q_k}) \times \prod_{j\,:\,K_j\subset I_p} \chi_p(g_j) \right), \end{aligned} \tag{14.4}$$

where $1 \le k \le r, 1 \le j \le m$. The term corresponding to ζ in the first product comes from $\xi_{q_k} \in H_\zeta$, and the term corresponding to $p \in P$ in the second product comes from $\xi_{q_k} \in H_p$ and $g_j \in H_p$.

Let $Q(g, I_\zeta)$ be the union of supports $\{ q_k \} = \mathrm{supp}(\xi_{q_k}) \subset I_\zeta$, and $QK(g, I_p)$ be the union of supports $\{ q_k \} \subset I_p$ and $K_j = \mathrm{supp}(g_j) \subset I_p$. Since $g \in H$, they give a partition of $\mathrm{supp}(g)$. Let their orders be $n(\zeta)$ and $n(p)$ respectively, then

$$\begin{gathered}\Big(\sum\nolimits_{\zeta \in \widehat{T}} Q(g, I_\zeta)\Big) \bigsqcup \Big(\sum\nolimits_{p \in P} QK(g, I_p)\Big) \;=\; \mathrm{supp}(g), \\ \sum\nolimits_{\zeta \in \widehat{T}} n(\zeta) \;+\; \sum\nolimits_{p \in P} n(p) \;=\; |\mathrm{supp}(g)|. \end{gathered} \tag{14.5}$$

Now, for $\tau \in \mathfrak{S}_\infty$, put ${}^\tau g = \tau g \tau^{-1}, {}^\tau \xi_q = \tau \xi_q \tau^{-1}$, and ${}^\tau g_j = \tau g_j \tau^{-1}$. Then, the standard decomposition of ${}^\tau g$ into mutually disjoint basic elements is given as

$$\begin{gathered}{}^\tau g = {}^\tau \xi_{q_1} {}^\tau \xi_{q_2} \cdots {}^\tau \xi_{q_r} {}^\tau g_1 {}^\tau g_2 \cdots {}^\tau g_m, \\ {}^\tau \xi_q = \big(t_q, (\tau(q))\big), \quad {}^\tau g_j = (\tau(d_j), \tau \sigma_j \tau^{-1}).\end{gathered}$$

For ξ_q, we have $\chi_p({}^\tau \xi_q) = \chi_p(\xi_q)$ if ${}^\tau \xi_q$ is still in H_p, or equivalently if $\tau(q) \in I_p$. For $d_j = (t_i)_{i \in K_j}$, recall that $\tau(d_j) = (t_{\tau^{-1}(i')})_{i' \in \tau(K_j)}$, and $P_{\tau \sigma_j \tau^{-1}}(\tau(d_j)) = P_{\sigma_j}(d_j)$ and so $\chi_p({}^\tau g_j) = \chi_p(g_j)$ if ${}^\tau g_j$ is still in H_p, or equivalently if $\tau(K_j) \subset I_p$.

Let us now consider a partial sum of (14.1), where $\tau \in S_N = \mathfrak{S}_{J_N}$ runs over all such elements that it preserves $Q_\zeta := Q(g, I_\zeta)$ and $QK_p := QK(g, I_p)$ inside of I_ζ and I_p respectively.

Suppose that N is sufficiently large so that g is contained in $H \cap G_N$, then this condition on $\tau \in S_N$ is written as

$$\tau(Q_\zeta) \subset I_\zeta \cap J_N, \quad \tau(QK_p) \subset I_p \cap J_N. \tag{14.6}$$

Put $\mathcal{Q} := \{ Q_\zeta \ (\zeta \in \widehat{T}),\ QK_p \ (p \in P) \}$, and denote by $\mathcal{T}(\mathcal{Q}, N)$ the set of $\tau \in S_N = \mathfrak{S}_{J_N}$ satisfying the condition (14.6). Then, for $\tau \in \mathcal{T}(\mathcal{Q}, N)$, we see from the above consideration that $\widetilde{f_\pi}({}^\tau g) = \widetilde{f_\pi}(g)$. Therefore the partial sum over $\tau \in \mathcal{T}(\mathcal{Q}, N)$ is calculated as

$$\frac{1}{|S_N|} \sum_{\tau \in \mathcal{T}(\mathcal{Q}, N)} \widetilde{f_\pi}({}^\tau g) = \frac{|\mathcal{T}(\mathcal{Q}, N)|}{|J_N|!} \widetilde{f_\pi}(g). \tag{14.7}$$

Let us calculate the order $|\mathcal{T}(\mathcal{Q}, N)|$. For $n(\zeta)$ numbers of $i \in Q_\zeta$, $\tau(i)$'s can be freely chosen inside of $I_\zeta \cap J_N$. Therefore the number of possible choices is

$$N(\zeta) := |I_\zeta \cap J_N| \big(|I_\zeta \cap J_N| - 1\big)\big(|I_\zeta \cap J_N| - 2\big) \cdots \big(|I_\zeta \cap J_N| - n(\zeta) + 1\big).$$

Similarly, the number of possible choices of $\tau(i), i \in QK_p$, inside of $I_p \cap J_N$ is equal to

$$N(p) := |I_p \cap J_N| \big(|I_p \cap J_N| - 1\big)\big(|I_p \cap J_N| - 2\big) \cdots \big(|I_p \cap J_N| - n(p) + 1\big).$$

Recall that the union of Q_ζ's and QK_p's is $\mathrm{supp}(g)$. After choosing $\tau(i), i \in \mathrm{supp}(g)$, we can choose $\tau(i)$ for $i \in J_N \setminus \mathrm{supp}(g)$ freely from $J_N \setminus \tau(\mathrm{supp}(g))$. Hence the number of possible choices is $|J_N \setminus \mathrm{supp}(g)|!$.

Thus we can evaluate, under the asymptotic condition (11.1),

(14.8)
$$\frac{|\mathcal{T}(\mathcal{Q},N)|}{|J_N|!} = \frac{1}{|J_N|!}\prod_{\zeta\in\widehat{T}} N(\zeta) \ \times \prod_{p\in P} N(p) \ \times |J_N\setminus \mathrm{supp}(g)|!$$
$$= \prod_{\zeta\in\widehat{T}} \frac{N(\zeta)}{|J_N|^{n(\zeta)}} \times \prod_{p\in P}\frac{N(p)}{|J_N|^{n(p)}} \ \times \frac{|J_N|^{|\mathrm{supp}(g)|}}{|J_N|\big(|J_N|-1\big)\cdots\big(|J_N|-|\mathrm{supp}(g)|+1\big)}$$
$$\longrightarrow \prod_{\zeta\in\widehat{T}} \mu_\zeta^{\,n(\zeta)} \times \prod_{p\in P}\lambda_p^{\,n(p)}.$$

Applying the formulas (14.7) and (14.4), we obtain

$$\lim_{N\to\infty}\frac{1}{|S_N|}\sum_{\tau\in\mathcal{T}(\mathcal{Q},N)} \widetilde{f_\pi}({}^\tau g) = \lim_{N\to\infty}\frac{|\mathcal{T}(\mathcal{Q},N)|}{|J_N|!}\,\widetilde{f_\pi}(g)$$
$$= \prod_{\zeta\in\widehat{T}}\left(\prod_{k\,:\,q_k\in I_\zeta}\frac{\mu_\zeta}{\dim\zeta}\chi_\zeta(t_{q_k})\right)$$
$$\times\prod_{p\in P}\left(\prod_{k\,:\,q_k\in I_p}\lambda_p\chi_p(\xi_{q_k})\prod_{j\,:\,K_j\subset I_p}\lambda_p^{\ell(\sigma_j)}\chi_p(g_j)\right),$$

where for $p\in P_{\zeta,\varepsilon}$, $(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}$,

$$\lambda_p\chi_p(\xi_{q_k}) = \frac{\lambda_p}{\dim\zeta}\chi_\zeta(t_{q_k}),$$
$$\lambda_p^{\ell(\sigma_j)}\chi_p(g_j) = \left(\frac{\lambda_p}{\dim\zeta}\right)^{\ell(\sigma_j)}\chi_\zeta\big(P_{\sigma_j}(d_j)\big)\,\mathrm{sgn}(\sigma_j)^\varepsilon.$$

The above calculation for a partial sum over $\tau\in\mathcal{T}(\mathcal{Q},N)\subset\mathfrak{S}_{J_N}$ can be applied to other partial sums. These partial sums come from possible cases of ${}^\tau g$ corresponding to which of I_ζ or I_p contains $\mathrm{supp}({}^\tau\xi_{q_k})=\tau(q_k)$, and which of I_p contains $\mathrm{supp}({}^\tau g_j)=\tau(K_j)$. All these cases give us limits of partial centralizations similarly as above, and they correspond altogether exactly to all the ‘monomial’ terms of the expansion of the right hand side of (3.8) in Theorem 2 into ‘monomials’ as explained below.

14.2. Summing up all ‘monomial’ terms to the whole formula

Put newly $P_{\zeta,\varepsilon}=\{(\zeta,\varepsilon,i); i\in\boldsymbol{N}\}$ for $(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}$, and $P=\sqcup_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}P_{\zeta,\varepsilon}$. For $p\in P$, put $X_p(\xi_q)$ for $\xi_q=(t_q,(q))$ and $X_p(g_j)$ for $g_j=(d_j,\sigma_j)$ as in (14.3), then

$$|X_\zeta(\xi_q)|\le 1\,,\quad |X_p(\xi_q)|\le 1\,,\quad |X_p(g_j)|\le\frac{1}{(\dim\zeta)^{\ell(\sigma_j)-1}}\le 1. \tag{14.9}$$

Let further $\{\lambda_p ; p \in P_{\zeta,\varepsilon}\}$ be a reordering of $\{\alpha_{\zeta,\varepsilon,i} ; i \in \boldsymbol{N}\}$. Then by (3.6) we have

$$\sum_{\zeta\in\widehat{T}} \mu_\zeta + \sum_{p\in P} \lambda_p \leq 1, \tag{14.10}$$

and the formula (3.8) of $f_A(g)$ is rewritten as

(14.11)

$$f_A(g) =$$
$$= \prod_{1\leq k\leq r} \left(\sum_{p\in P} \lambda_p X_p(\xi_{q_k}) + \sum_{\zeta\in\widehat{T}} \mu_\zeta X_\zeta(\xi_{q_k}) \right) \cdot \prod_{1\leq j\leq m} \left(\sum_{p\in P} \lambda_p^{\ell(\sigma_j)} X_p(g_j) \right).$$

Note that by (14.9) each multiplicative factor in (14.11) is evaluated in its absolute value as ≤ 1.

Let $\mathcal{P}_m$ be the set of all partitions $\delta = \{J_p \ (p \in P)\}$ indexed by P of the set of indices $j \in \boldsymbol{I}_m = \{1, 2, \ldots, m\}$ of g_j's, and $\mathcal{Q}_r$ be the set of all partitions $\gamma = \{K_\zeta \ (\zeta \in \widehat{T}), K_p \ (p \in P)\}$ be the set of partitions indexed by $\widehat{T} \sqcup P$ of the set of indices $k \in \boldsymbol{I}_r$ of ξ_{q_k}'s. Put $\gamma \cdot \delta := \{ K_\zeta \ (\zeta \in \widehat{T}),\ K_p,\ J_p \ (p \in P) \}$, and let $\mathcal{KJ}$ be the set of all these $\gamma \cdot \delta$. Then the expansion of $f_A(g)$ of the right hand side of (14.11) into monomial terms are parametrized by the set $\gamma \cdot \delta \in \mathcal{KJ}$ as

$$\begin{aligned}
f_A(g) &= \sum_{\gamma\cdot\delta\in\mathcal{KJ}} \Xi_{\gamma\cdot\delta}(g), \\
\Xi_{\gamma\cdot\delta}(g) &= \prod_{\zeta\in\widehat{T}} \prod_{k\in K_\zeta} \mu_\zeta X_\zeta(\xi_{q_k}) \\
&\quad \times \prod_{p\in P} \left(\prod_{k\in K_p} \lambda_p X_p(\xi_{q_k}) \cdot \prod_{j\in J_p} \lambda_p X_p(g_j) \right) \\
&= \prod_{\zeta\in\widehat{T}} \prod_{k\in K_\zeta} X_\zeta(\xi_{q_k}) \cdot \prod_{p\in P} \left(\prod_{k\in K_p} X_p(\xi_{q_k}) \prod_{j\in J_p} X_p(g_j) \right) \\
&\quad \times \prod_{k\in K_\zeta} \mu_\zeta^{n(\zeta)} \cdot \prod_{p\in P} \lambda_p^{n(p)},
\end{aligned} \tag{14.12}$$

where the product over $p \in P$ are actually finite, and

$$n(\zeta) = |K_\zeta|, \quad n(p) = |K_p| + \sum_{j\in J_p} \ell(\sigma_j) = |K_p| + \sum_{j\in J_p} |\mathrm{supp}(g_j)|.$$

Now we come back to the centralization F^{G_N} in **14.1**. Take $\gamma \cdot \delta = \{ K_\zeta \ (\zeta \in \widehat{T}),\ K_p,\ J_p \ (p \in P) \}$, and put

$$Y^N_{\gamma\cdot\delta}(g) := \frac{1}{|S_N|} \sum_{\tau\in\mathcal{T}(\gamma\cdot\delta)} \widetilde{f_\pi}({}^\tau g) \qquad \text{with}$$

$$\mathcal{T}(\gamma\cdot\delta) := \{\tau \in S_N ; {}^\tau\xi_{q_k} \in H_\zeta \ (k \in K_\zeta), {}^\tau\xi_{q_k} \in H_p \ (k \in K_p), {}^\tau g_j \in H_p \ (j \in J_p)\}.$$

Then, by a similar calculation as in **14.1**, we have

$$F^{G_N}(g) = \sum_{\gamma\cdot\delta\in\mathcal{K}\mathcal{J}} Y^N_{\gamma\cdot\delta}(g), \tag{14.13}$$

$$Y^N_{\gamma\cdot\delta}(g) = \prod_{\zeta\in\widehat{T}} \prod_{k\in K_\zeta} X_\zeta(\xi_{q_k}) \cdot \prod_{p\in P} \left(\prod_{k\in K_p} X_p(\xi_{q_k}) \prod_{j\in J_p} X_p(g_j) \right) \times \frac{C^N_{\gamma\cdot\delta}}{|J_N|!},$$

where $C^N_{\gamma\cdot\delta} := |\mathcal{T}(\gamma\cdot\delta)|$. Note that $\mathcal{T}(\gamma\cdot\delta)$ is defined by the following condition on $\tau \in S_N$:

$$^\tau q_k \in I_\zeta \cap J_N \ (k \in K_\zeta), \quad ^\tau q_k \in I_p \cap J_N \ \ (k \in K_p),$$
$$^\tau(\mathrm{supp}(g_j)) \subset I_p \cap J_N \ \ (j \in J_p).$$

Then, similarly as in **14.1**, the order $C^N_{\gamma\cdot\delta} = |\mathcal{T}(\gamma\cdot\delta)|$ can be calculated as in (14.8) and so

$$\frac{C^N_{\gamma\cdot\delta}}{|J_N|!} \longrightarrow \prod_{\zeta\in\widehat{T}} \lambda_\zeta^{n(\zeta)} \times \prod_{p\in P} \lambda_p^{n(p)}. \tag{14.14}$$

We note that, for $\mathcal{Q} = \{Q_\zeta\,(\zeta\in\widehat{T}),\, QK_p\,(p\in P)\}$ in **14.1**, there corresponds a $\gamma\cdot\delta = \{\,K_\zeta\ (\zeta\in\widehat{T}),\ K_p,\ J_p\ (p\in P)\,\}$ given by

$$K_\zeta = \{k \in \boldsymbol{I}_r\ ;\ \xi_{q_k} \in H_\zeta\},$$
$$K_p = \{k \in \boldsymbol{I}_r\ ;\ \xi_{q_k} \in H_p\}, \quad J_p = \{j \in \boldsymbol{I}_m\ ;\ g_j \in H_p\}.$$

Now we can prove the following proposition, a half of Theorem 2.

Proposition 11. *Let T be a finite group. Let f_A be the class function on $G = \mathfrak{S}_\infty(T)$ given by the formula* (3.8) *in Theorem* 2*, with parameter in* (3.7):

$$A := \left((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{\,0,1\,\}}\ ;\ \mu \right).$$

(i) *If the parameter A satisfies the conditions* (3.5)–(3.6)*, then f_A is obtained as a pointwise limit of centralizations of a positive definite function $F = \mathrm{Ind}_H^G f_\pi$ with (H,π) given above. The limit is taken according to an appropriate increasing sequence of subgroups $G_N = \mathfrak{S}_{J_N}(T)$ with $J_N \nearrow \boldsymbol{N}$ obeying the asymptotic condition* (11.1)*. Moreover if the equality holds in* (3.6) *or $\sum_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \|\alpha_{\zeta,\varepsilon}\| + \|\mu\| = 1$, then the limit is obtained according to any such increasing sequence.*

(ii) *All the class functions f_A thus obtained are positive definite under the conditions* (3.5)–(3.6)*.*

Proof. Note that the equality in (3.6) is nothing but the equality in (14.10). Under this equality condition we evaluate $|f_A(g) - F^{G_N}(g)|$ as follows. The case where the inequality holds in (3.6) or in (14.10) can be treated after these discussions in the equality case.

(1) Let $\varepsilon_n \searrow 0$ $(n \to \infty)$ be a decreasing sequence of positive numbers. Let $P_n \subset P$ be a finite subset such that

$$\sum_{\zeta\in\widehat{T}} \mu_\zeta + \sum_{p\in P_n} \lambda_p \;>\; 1-\varepsilon_n. \tag{14.15}$$

Put $\lambda_{\zeta,N} = \dfrac{|I_\zeta \cap J_N|}{|J_N|}, \lambda_{p,N} = \dfrac{|I_p \cap J_N|}{|J_N|}$, then $\sum_{\zeta\in\widehat{T}} \mu_{\zeta,N} + \sum_{p\in P} \lambda_{p,N} = 1$. Since $\mu_{\zeta,N} \to \lambda_\zeta, \lambda_{p,N} \to \lambda_p$ $(N \to \infty)$ by assumption, we can take N_n sufficiently large so that for any $N \geq N_n$

$$\sum_{\zeta\in\widehat{T}} |\mu_\zeta - \mu_{\zeta,N}| + \sum_{p\in P_n} |\lambda_p - \lambda_{p,N}| \;<\; \varepsilon_n. \tag{14.16}$$

Then we have

$$\sum_{p\notin P_n} \lambda_p \;<\; \varepsilon_n, \qquad \sum_{p\notin P_n} \lambda_{p,N} \;<\; 2\varepsilon_n. \tag{14.17}$$

(2) Let $\mathcal{KJ}_n$ be the set of $\gamma\cdot\delta = \{\, K_\zeta\ (\zeta\in\widehat{T}),\ K_p,\ J_p\ (p\in P)\,\}$ such that $K_p = J_p = \emptyset$ for $p \notin P_n$. Then $\mathcal{KJ}_n$ is finite. In the formula (14.12) of $f_A(g)$, we divide the sum over $\gamma\cdot\delta \in \mathcal{KJ}$ of $\Xi_{\gamma\cdot\delta}(g)$ into two cases depending on $\gamma\cdot\delta$ belongs to $\mathcal{KJ}_n$ or not as

$$f_A(g) = f_A^n(g) + f_A^{\sharp n}(g), \tag{14.18}$$

$$f_A^n(g) := \sum_{\gamma\cdot\delta\in\mathcal{KJ}_n} \Xi_{\gamma\cdot\delta}(g), \quad f_A^{\sharp n}(g) := \sum_{\gamma\cdot\delta\notin\mathcal{KJ}_n} \Xi_{\gamma\cdot\delta}(g).$$

Similarly, in the formula (14.13) of $F^{G_N}(g)$, we divide the sum over $\gamma\cdot\delta\in\mathcal{KJ}$ of $Y^N_{\gamma\cdot\delta}(g)$ into two cases according as $\gamma\cdot\delta\in\mathcal{KJ}_n$ or $\gamma\cdot\delta\notin\mathcal{KJ}_n$ as above, and express F^{G_N} as

$$F^{G_N}(g) = F^{G_N,n}(g) + F^{G_N,\sharp n}(g), \tag{14.19}$$

Then we have

$$|f_A(g) - F^{G_N}(g)| \;\leq\; |f_A^n(g) - F^{G_N,n}(g)| \;+\; |f_A^{\sharp n}(g)| \;+\; |F^{G_N,\sharp n}(g)|.$$

(3) We denote by $R_{1,N}, R_{2,N}$ and $R_{3,N}$ the first, the second and the third term in the right hand side respectively. Then $R_{1,N}$ is a finite sum of the terms $\Xi_{\gamma\cdot\delta}(g) - Y^N_{\gamma\cdot\delta}(g)$ each of which tends to 0 as $N\to\infty$. So we can choose $N'_n \geq N_n$ such that, for any $N \geq N'_n$, we have $R_{1,N} < \varepsilon_n$.

For the second term $R_{2,N}$, using the evaluation (14.9) and the note just after (14.11), we get

$$R_{2,N} \;\leq\; \sum_{1\leq k\leq r} \Big(\sum_{p\notin P_n} \lambda_p\Big) \;+\; \sum_{1\leq j\leq m} \Big(\sum_{p\notin P_n} \lambda_p^{\ell(\sigma_j)}\Big) \;<\; (r+m)\varepsilon_n.$$

For the third term $R_{3,N}$, first evaluate each $|Y^N_{\gamma\cdot\delta}(g)|$ as

$$|Y^N_{\gamma\cdot\delta}(g)| \;\leq\; C^N_{\gamma\cdot\delta}/|J_N|! \leq C\cdot \prod_{\zeta\in\widehat{T}} {\lambda_{\zeta,N}}^{n(\zeta)} \cdot \prod_{p\in P} {\lambda_{p,N}}^{n(p)},$$

where C is a constant, for example, we can take $C = 2^{|\mathrm{supp}(g)|}$ if $N \geq 2|\mathrm{supp}(g)|$. Then, a similar evaluation as that for $R_{2,N}$ (using $\lambda_{\zeta,N}, \lambda_{p,N}$ instead of λ_ζ, λ_p respectively) gives us $R_{3,N} < C\,(r+m)\cdot 2\varepsilon_n$.

Thus altogether we get for any $N \geq N'_n$,

$$|f_A(g) - F^{G_N}(g)| \;<\; \{1 + (r+m) + 2C(r+m)\}\,\varepsilon_n .$$

□

15. Criterion for extremal positive definite class functions

In this section, we give a criterion for extremality as follows (cf. [Tho2, Satz 1]). This is the first ingredient of the second part of the proof of Theorem 2.

Theorem 12. *Let T be a finite group, and f a positive definite class function on $G = \mathfrak{S}_\infty(T)$ normalized as $f(e) = 1$. Then f is extremal if and only if it has one of the following properties which are mutually equivalent*:

(FTP) [Factorizability Property] *For any $g = (d,\sigma) \in G$, let*

$$g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m, \quad \xi_q = \big(t_q,(q)\big), \;\; g_j = (d_j,\sigma_j),$$

be a standard decomposition. Then,

$$f(g) = \textstyle\prod_{1\leq k\leq r} f(\xi_{q_k}) \;\times\; \prod_{1\leq j\leq m} f(g_j). \tag{15.1}$$

(FTP′) *For any two elements $g, g' \in G$ with disjoint supports, $f(gg') = f(g)f(g')$.*

Let us rewrite these conditions in another form. As is proved in Theorem 1, conjugacy classes of basic elements in G is given by the set Ω of the following objects ω:

$$\begin{aligned} &\omega = ([t], \ell = 1) \;\text{ with }\; [t] \in T^*/\!\sim, \\ \text{and}\quad &\omega = ([t], \ell\,) \in (T/\!\sim) \times \{\,\ell \geq 2\,\}, \end{aligned} \tag{15.2}$$

and the conjugacy class of $g \in G \setminus \{\,e\,\}$ with the above standard decomposition is determined by the collection of

$$\begin{aligned} &([t_{q_k}], \ell = 1) \quad (1 \leq k \leq r) \\ \text{and}\quad &(P_{\sigma_j}(d_j), \ell(\sigma_j)\,) \quad (1 \leq j \leq m). \end{aligned} \tag{15.3}$$

Denote by $n_\omega(g)$ the multiplicity of $\omega \in \Omega$ for $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m$.

Put $\boldsymbol{Z}_{\geq 0} := \{\,n \in \boldsymbol{Z}\;;\; n \geq 0\,\}$ and denote by $(\boldsymbol{Z}_{\geq 0})^{(\Omega)}$ the set of all $\boldsymbol{n} = (n_\omega)_{\omega\in\Omega}$, $n_\omega \in \boldsymbol{Z}_{\geq 0}$, with $n_\omega = 0$ for almost all ω. Then, $\boldsymbol{n}(g) := (n_\omega(g))_{\omega\in\Omega}$ is an element of $(\boldsymbol{Z}_{\geq 0})^{(\Omega)}$, and the correspondence

$$\Phi \;:\; G/\!\sim\, \ni [g] \longmapsto \boldsymbol{n}(g) \,\in\, (\boldsymbol{Z}_{\geq 0})^{(\Omega)}$$

gives a bijective map from the set of all conjugacy classes $[g]$ of $g \in G, g \neq e$, onto $(\boldsymbol{Z}_{\geq 0})^{(\Omega)}$.

For $\omega = ([t], \ell) \in \Omega$, put $\omega^{-1} := ([t^{-1}], \ell)$. Then, if ω is the conjugacy class of $\xi_q = (t_q, (q))$ or of $g_j = (d_j, \sigma_j)$, then ω^{-1} is that of ξ_q^{-1} or of g_j^{-1} respectively. Hence, $n_\omega(g^{-1}) = n_{\omega^{-1}}(g)$, and the transformation $[g] \mapsto [g^{-1}]$ in the set $G/\!\sim$ of conjugacy classes of elements in G induces an involutive transformation ι on $(\boldsymbol{Z}_{\geq 0})^{(\Omega)}$ given as

$$\iota\ :\ (\boldsymbol{Z}_{\geq 0})^{(\Omega)} \ni \boldsymbol{n} = (n_\omega)_{\omega\in\Omega} \longmapsto \boldsymbol{n'} = (n'_\omega)_{\omega\in\Omega} \quad \text{with } n'_\omega = n_{\omega^{-1}} \ \ (\omega \in \Omega).$$

For a positive definite class function f on G, put $s(f) = (s_\omega)_{\omega\in\Omega}$ with $s_\omega = f(g_\omega)$, where g_ω denotes a basic element in the class ω. Then, since ω^{-1} is represented by g_ω^{-1}, and since $f(g^{-1}) := \overline{f(g)}, g \in G$, we have

$$s_{\omega^{-1}} = \overline{s_\omega} \quad \text{(complex conjugate).} \tag{15.4}$$

Define a positive definite class function $\overline{f}$ by $\overline{f}(g) = \overline{f(g)}\quad (g \in G)$, then $s(\overline{f}) = \overline{s(f)}$. Here, for $s = (s_\omega)_{\omega\in\Omega}$, we put $\overline{s} := (\overline{s_\omega})_{\omega\in\Omega}$ with $\overline{s_\omega} = s_\omega$ for $\omega \in \Omega_{re} := \{\,\omega \in \Omega\ ;\ \omega^{-1} = \omega\,\}$.

Put $\Omega_c := \{\,\omega \in \Omega\ ;\ \omega^{-1} \neq \omega\,\}$, then $\Omega = \Omega_{re} \sqcup \Omega_c$. Let $I_\omega := [-1, 1] \subset \boldsymbol{R}$ for $\omega \in \Omega_{re}$, and $D_\omega := \{\, z \in \boldsymbol{C}\ ;\ |z| \leq 1\,\} \subset \boldsymbol{C}$ for $\omega \in \Omega_c$, and put $S := S_{re} \times S_c$ with

$$S_{re} := \prod_{\omega\in\Omega_{re}} I_\omega\,, \quad S_c := \left\{ (z_\omega)_{\omega\in\Omega_c} \in \prod_{\omega\in\Omega_c} D_\omega\ ;\ z_{\omega^{-1}} = \overline{z_\omega}\ (\forall\omega) \right\}. \tag{15.5}$$

Every $s = (s_\omega)_{\omega\in\Omega} \in S$ defines a function Ψ_s on $(\boldsymbol{Z}_{\geq 0})^{(\Omega)} \cong G/\!\sim$ by

$$\Psi_s\ :\ (\boldsymbol{Z}_{\geq 0})^{(\Omega)} \ni \boldsymbol{n} = (n_\omega)_{\omega\in\Omega} \longmapsto \prod_{\omega\in\Omega} {s_\omega}^{n_\omega} \ \in\ \boldsymbol{K},$$

where $\boldsymbol{K} = \boldsymbol{R}$ or $\boldsymbol{C}$ according as $\Omega_c = \emptyset$ or $\neq \emptyset$. Then we get a class function $f_s := \Psi_s \circ \Phi$ on G satisfying $f_s(g^{-1}) = \overline{f_s(g)}$.

Now the condition (FTP) above is rewritten in these notations as follows:

(FTP$''$) *There exists an* $s = (s_\omega)_{\omega\in\Omega}$ *in* $S = S_{re} \times S_c$ *such that* $f = f_s$*, that is, that for a* $g \in G$ *with standard decomposition* $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_m$*, let* $n_\omega(g)$ *be the multiplicity of* $\omega \in \Omega$ *in these basic components, then*

$$f(g) = \textstyle\prod_{\omega\in\Omega} {s_\omega}^{n_\omega(g)}, \quad \textit{where} \quad {s_\omega}^0 := 1. \tag{15.6}$$

Proof of Theorem 12.

1. The proof of "only if" part is carried out similarly as for [Tho2, Satz 1] with several appropriate changes.

2. For the proof of "if" part, we utilize a kind of Stone-Weierstrass theorem on the uniform convergence of continuous functions on a compact set. For any pair $\{\omega, \omega^{-1}\}$ in Ω_c, choose once for all a representative, say ω, from the pair, and denote by Ω'_c the set of all such representatives. Put

$$S' := S_{re} \times S'_c \quad \text{with} \quad S'_c := \prod_{\omega \in \Omega'_c} D_\omega, \quad \text{and} \quad \Omega' = \Omega_{re} \sqcup \Omega'_c\,,$$

then S' is isomorphic to $S = S_{re} \times S_c$ as a compact space through an isomorphism $\Gamma : S \ni s = (s_\omega)_{\omega\in\Omega} \mapsto s' = (s_\omega)_{\omega\in\Omega'} \in S'$. Let $C(S')$ be the space of all $\boldsymbol{K}$-valued continuous functions F on the compact set $S' = S_{re} \times S'_c$.

For an element $\boldsymbol{n} = (n_\omega)_{\omega\in\Omega} \in (\boldsymbol{Z}_{\geq 0})^{(\Omega)}$, consider $\Psi_s(\boldsymbol{n}) = \prod_{\omega\in\Omega} {s_\omega}^{n_\omega}$ as a function of $s' = (s_\omega)_{\omega\in\Omega'} \in S'$, we have a monomial function

$$P_{\boldsymbol{n}}(s') = \prod_{\omega\in\Omega_{re}} {s_\omega}^{n_\omega} \times \prod_{\omega\in\Omega'_c} \left({s_\omega}^{n_\omega}\, \overline{s_\omega}^{\,n_{\omega^{-1}}}\right).$$

Lemma 15.1. *The set $\mathcal{P}$ of the following monomial functions in $s' = (s_\omega)_{\omega\in\Omega'} \in S'$*

$$P_{\boldsymbol{n}}(s'), \qquad \boldsymbol{n} = (n_\omega)_{\omega\in\Omega} \in (\boldsymbol{Z}_{\geq 0})^{(\Omega)},$$

is total in $C(S')$ with uniform convergence norm.

Let $K_1(G)$ be the set of all positive definite class functions f on G normalized as $f(e) = 1$, and $E(G)$ the set of all extremal points of the convex set $K_1(G)$. With the pointwise convergence topology $K_1(G)$ is compact. Then, we can apply here the Choquet-Bishop-de Leeuw representation theorem ([BL, Theorem 5.6]) for the compact convex subset $X := K_1(G)$ in a real locally convex linear space (spanned by $K(G)$) and the set $X_e := E(G)$ of its extremal points. Denote by $\mathfrak{B}$ the σ-ring generated by X_e and the Baire subset of X. Each $f^0 \in X$ has a representation of the form

$$f^0 \;=\; \int_X f\, d\mu(f)$$

with respect to a non-negative measure μ on $\mathfrak{B}$ such that $\mu(X) = \mu(X_e) = 1$.

Let $M \subset K_1(G)$ be the subset consisting of factorizable f or of the form $f_s = \Psi_s \circ \Phi$ with $s \in S$, then it is closed and so compact. By the "only if" part mentioned above, we have $E(G) \subset M$. The above measure μ can be considered as a measure on M such that $\mu(M) = \mu(E(G)) = 1$. We map $M \subset K_1(G)$ into S by $f \mapsto s$ through $f = f_s$ (put $s = s(f)$), then M is homeomorphic to its image $\widetilde{M}$ in S. Let $\widetilde{E}$ be the image of $E(G)$.

Now take $f^0 = f_{s^0} \in M$ and prove that f^0 is extremal or $f^0 \in E(G)$. Through the correspondence $f \mapsto s(f)$, we discuss this in the space S. So, it should be proved that $s^0 = s(f^0) \in \widetilde{M}$ actually belongs to $\widetilde{E}$. Corresponding to the integral expression of f^0 given above, we have, as functions on $(\boldsymbol{Z}_{\geq 0})^{(\Omega)}$, s^0 is expressed as an integral on $\widetilde{M}$ with respect to a measure $\widetilde{\mu}$ for which $\widetilde{\mu}(\widetilde{M}) = \widetilde{\mu}(\widetilde{E}) = 1$:

$$s^0 = \int_{\widetilde{M}} s\, d\widetilde{\mu}(s) \qquad \text{or} \qquad \Psi_{s^0} = \int_{\widetilde{M}} \Psi_s\, d\widetilde{\mu}(s).$$

Evaluate the latter integral at each point $\boldsymbol{n} = (n_\omega)_{\omega\in\Omega} \in (\boldsymbol{Z}_{\geq 0})^{(\Omega)}$, and rewrite it on S' through the isomorphism $\Gamma : S \ni s \mapsto s' \in S'$, and with a probability measure $\widetilde{\mu}' := \widetilde{\mu} \circ \Gamma^{-1}$ on a compact subset $\Gamma(\widetilde{M})$ of S', supported by $\Gamma(\widetilde{E})$, we have

$$P_{\boldsymbol{n}}((s^0)') = \int_{\Gamma(\widetilde{M})} P_{\boldsymbol{n}}(s')\, d\widetilde{\mu}'(s') \qquad \text{for} \qquad \boldsymbol{n} = (n_\omega)_{\omega\in\Omega} \in (\boldsymbol{Z}_{\geq 0})^{(\Omega)}.$$

By Lemma 15.1, it follows from this that

$$F((s^0)') = \int_{\Gamma(\widetilde{M})} F(s')\, d\widetilde{\mu}'(s') \qquad \text{for any} \quad F \in C(S').$$

From this, we see that $\widetilde{\mu}'$ is supported by one point set $\{ (s^0)' \}$. This means that $(s^0)' \in \Gamma(\widetilde{E})$ or $s^0 \in \widetilde{E}$ and so $f^0 = f_{s^0} \in E(G)$.

Thus the proof of the "if" part of Theorem 12 is now complete.

16. Final step of the proof of Theorem 2

By the "only if" part of the proof of Theorem 12, for each $f \in E(G)$, there corresponds an element $s \in S$ such that $f = f_s = \Psi_s \circ \Phi$. As the final step of the proof of Theorem 2, we specify the parameter $s = (s_\omega)_{\omega\in\Omega}$ and prove the following.

Proposition 13. *An extremal positive definite class function* (*or a character*) f *on* $G = \mathfrak{S}_\infty(T)$, *normalized as* $f(e) = 1$, *is given in the form of* f_A *in the formula* (3.8) *in Theorem* 2, *with parameter* $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ ;\ \mu \Big)$ *in* (3.7) *satisfying the condition* (3.6).

Proof. By the "only if" part already proved, we should examine a positive definite class function f of the form (15.6). We define a class function on T by putting

$$X(t) = \begin{cases} 1 & \text{for} \quad t = e_T, \\ s_{([t],1)} & \text{for} \quad t \in T^*, \end{cases}$$

where $s_{([t],1)} = s_\omega$ for $\omega = ([t], 1) \in \Omega$. Then, since X is a class function on T, it is expressed as a linear combination of $\chi_\zeta, \zeta \in \widehat{T}$, as

$$X(t) = \sum_{\zeta\in\widehat{T}} b_\zeta\, \chi_\zeta(t) \quad (t \in T) \tag{16.1}$$

$$\text{with} \qquad b_\zeta = \int_T X(t)\, \overline{\chi_\zeta(t)}\, d\mu_T(t), \qquad \sum_{\zeta\in\widehat{T}} (\dim\zeta)\, b_\zeta \ = \ 1\,. \tag{16.2}$$

For $\ell \geq 2$, we define also a class function $Y_\ell(t)$ on T by putting

$$Y_\ell(t) = s_{([t],\ell)} \quad (t \in T),$$

where $s_{([t],\ell)} = s_\omega$ for $\omega = ([t], \ell) \in \Omega$. Then, similarly as for X, it is expressed as

$$Y_\ell(t) = \sum_{\zeta \in \widehat{T}} a_{\zeta,\ell}\, \chi_\zeta(t) \quad (t \in T) \tag{16.3}$$

$$\text{with} \qquad a_{\zeta,\ell} = \int_T Y_\ell(t)\, \overline{\chi_\zeta(t)}\, d\mu_T(t). \tag{16.4}$$

Then, for $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m$, we have from (15.1) and (15.6)

$$f(g) = \prod_{i \le k \le r} \left(\sum_{\zeta \in \widehat{T}} b_\zeta\, \chi_\zeta(t_{q_k}) \right) \times \prod_{1 \le j \le m} \left(\sum_{\zeta \in \widehat{T}} a_{\zeta,\ell(\sigma_j)}\, \chi_\zeta(P_{\sigma_j}(d_j)) \right). \tag{16.5}$$

Now we apply the following lemma.

Lemma 16.1. (i) *Let f_1 and f_2 be positive definite functions on a group G, then the product $(f_1 f_2)(g) := f_1(g) f_2(g)$ $(g \in G)$ is again positive definite.*

(ii) *Let D be a compact normal subgroup of a locally compact group G. For a continuous positive definite function f on G, put*

$$f^o(g) := \int_D f(gd)\, d\mu_D(d),$$

where $d\mu_D$ denotes the normalized Haar measure on D. Suppose that for each $g \in G$, the automorphism $D \ni d \mapsto gdg^{-1} \in D$ is measure-preserving. Then, f^o gives a continuous positive definite function on the quotient group G/D, and it is also expressed as $f^o(g) := \int_D f(d'g)\, d\mu_D(d')$.

Proof. Let us prove (ii). Take $g_1, g_2, \ldots, g_n \in G$, $d_1, d_2, \ldots, d_n \in D$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \boldsymbol{C}$. Then,

$$\sum_{1 \le i,j \le n} \lambda_i\, \overline{\lambda_j}\, f(g_i d_i d_j^{-1} g_j^{-1}) \ \ge\ 0.$$

Integrate this inequality with respect to $d\mu_D(d_1)\, d\mu_D(d_2)\ \cdots\ d\mu_D(d_n)$, then we get

$$\begin{aligned} 0 &\le \sum_{1 \le i,j \le n} \lambda_i\, \overline{\lambda_j} \int_D f(g_i d g_j^{-1})\, d\mu_D(d) \\ &= \sum_{1 \le i,j \le n} \lambda_i\, \overline{\lambda_j} \int_D f(g_i g_j^{-1} d)\, d\mu_D(d). \end{aligned}$$

This proves that the function f^o is positive definite. □

Fix a $(\zeta_0, \varepsilon) \in \widehat{T} \times \{0, 1\}$, and take a positive definite class function $F_{\zeta_0,\varepsilon}$ in (12.3) in Theorem 9: for $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m$, $\xi_q = \big(t_q, (q)\big)$, $g_j = (d_j, \sigma_j)$,

$$F_{\zeta_0,\varepsilon}(g) = \prod_{1 \le k \le r} \frac{\chi_{\zeta_0}(t_{q_k})}{\dim \zeta_0} \times \prod_{1 \le j \le m} \frac{\chi_{\zeta_0}\big(P_{\sigma_j}(d_j)\big)}{(\dim \zeta_0)^{\ell(\sigma_j)}}\, \mathrm{sgn}(\sigma_j)^\varepsilon.$$

Then the product $f'(g) := \big(f\,\overline{F_{\zeta_0,\varepsilon}}\big)(g) = f(g)\,\overline{F_{\zeta_0,\varepsilon}}(g)$ is positive definite. Take a subgroup $D_n := D_{\boldsymbol{I}_n}(T)$ with n sufficiently large so that $\mathrm{supp}(g) \subset \boldsymbol{I}_n$. *Fourier transform* $\mathcal{F}_{\zeta_0,\varepsilon;n}(f)$ *of* f *with respect to* $F_{\zeta_0,\varepsilon}$ is by definition the integral of f' with respect to D_n:

$$\mathcal{F}_{\zeta_0,\varepsilon;n}(f)(g) := \int_{D_n} f(d'g)\,\overline{F_{\zeta_0,\varepsilon}}(d'g)\,d\mu_{D_n}(d').$$

Let us calculate $\mathcal{F}_{\zeta_0,0;n}(f)(g)$. Taking multiplicative factors of $F_{\zeta_0,0}$, we put

$$X_{\zeta_0}(t) \;=\; \frac{\chi_{\zeta_0}(t)}{\dim\zeta_0}\,, \quad Y_{\ell,\zeta_0}(t) \;=\; \frac{\chi_{\zeta_0}(t)}{(\dim\zeta_0)^{\ell}} \quad (t \in T).$$

Then, by (16.1)–(16.4), we need the following formulas. Firstly,

$$\int_T X(t)\,\overline{X_{\zeta_0}(t)}\,d\mu_T(t) \;=\; \frac{b_{\zeta_0}}{\dim\zeta_0}\,.$$

Secondly, for a basic element (d',σ') with $d' = (t_1,t_2,\ldots,t_\ell)$, $\sigma' = (1\ 2\ \cdots\ \ell)$, we have $P_{\sigma'}(d') = t_\ell t_{\ell-1}\cdots t_2 t_1$, and therefore

$$\int_{T^\ell} \big(Y_\ell\,\overline{Y_{\ell,\zeta_0}}\big)\big(t_\ell t_{\ell-1}\cdots t_2 t_1\big)\,d\mu_T(t_1)d\mu_T(t_2)\,\cdots\,d\mu_T(t_\ell) = \frac{a_{\zeta_0,\ell}}{(\dim\zeta_0)^{\ell}}\,.$$

Lemma 16.2. *Let* f *be a factorizable positive definite class function* f *in* (15.6) *given as* $f(g) = \prod_{\omega\in\Omega} {s_\omega}^{n_\omega(g)}$. *Then, through* (16.1)–(16.3), *it is expressed as in* (16.5), *and the Fourier transform* $\mathcal{F}_{\zeta_0,0;n}(f)$ *of* f *with respect to* $F_{\zeta_0,0}$ *is given as follows: for* $\sigma \in \mathfrak{S}_n$, *let* $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ *be its decomposition into mutually disjoint cycles, then*

$$\mathcal{F}_{\zeta_0,0;n}(f)(\sigma) = \left(\frac{b_{\zeta_0}}{\dim\zeta_0}\right)^{n-|\mathrm{supp}(\sigma)|} \times \prod_{1\le j\le m} \frac{a_{\zeta_0,\ell(\sigma_j)}}{(\dim\zeta_0)^{\ell(\sigma_j)}}\,.$$

By Lemma 16.1(ii), the Fourier transform $\mathcal{F}_{\zeta_0,0;n}(f)$ is a positive definite class function on the symmetric group $\mathfrak{S}_n$ for any n.

We continue the proof of Proposition 13. For $\sigma \in \mathfrak{S}_\infty$, let $n_\ell(\sigma)$ be as in §6, the multiplicity in σ of disjoint cycles of length ℓ. For a series of complex numbers $s = (s_1, s_2, \ldots)$, consider a class function α_s^n on each subgroup $\mathfrak{S}_n$ given by

$$\alpha_s^n(\sigma) := {s_1}^{n-|\mathrm{supp}(\sigma)|}{s_2}^{n_2(\sigma)}\cdots{s_\ell}^{n_\ell(\sigma)} \quad (\sigma \in \mathfrak{S}_n),$$

where $2\,n_2(\sigma) + 3\,n_3(\sigma) + \cdots + \ell\,n_\ell(\sigma) = |\mathrm{supp}(\sigma)| \le n$. Then, [Tho2, Korollar 1 of Satz 2] says that

($\star$) *The class function* α_s^n *is positive definite on* $\mathfrak{S}_n$ *for all* $n \ge 1$ *if and only if there exist series of non-negative real numbers* $\alpha = (\alpha_i)_{i\in\boldsymbol{N}}$, $\beta = (\beta_i)_{i\in\boldsymbol{N}}$ *with* $\|\alpha\| < +\infty$, $\|\beta\| < +\infty$, *such that*

$$\|\alpha\| \;+\; \|\beta\| \;\le s_1, \quad s_\ell = \sum_{i\in\boldsymbol{N}} {\alpha_i}^{\ell} \;+\; (-1)^{\ell-1}\sum_{i\in\boldsymbol{N}} {\beta_i}^{\ell} \quad (\ell \ge 2).$$

In our case, by $(\star)$, we have $\alpha = (\alpha_i)_{i\in\boldsymbol{N}},\ \beta = (\beta_i)_{i\in\boldsymbol{N}}$ (naturally depending on ζ_0) such that

$$\|\alpha\| + \|\beta\| \le \frac{b_{\zeta_0}}{\dim\zeta_0},$$
$$\sum_{i\in\boldsymbol{N}} \alpha_i^{\ell} + (-1)^{\ell-1}\sum_{i\in\boldsymbol{N}} \beta_i^{\ell} = \frac{a_{\zeta_0,\ell}}{(\dim\zeta_0)^{\ell}} \quad (\ell\ge 2).$$

Rearrange α_i's and β_i's in decreasing order and put

$$\alpha_{\zeta_0,0,i} = (\dim\zeta_0)^2\,\alpha_i, \qquad \alpha_{\zeta_0,1,i} = (\dim\zeta_0)^2\,\beta_i,$$
$$\alpha_{\zeta_0,0} = (\alpha_{\zeta_0,0,i})_{i\in\boldsymbol{N}}, \qquad \alpha_{\zeta_0,1} = (\alpha_{\zeta_0,1,i})_{i\in\boldsymbol{N}},$$
$$\mu_{\zeta_0} = (\dim_{\zeta_0})\,b_{\zeta_0} - \|\alpha_{\zeta_0,0}\| - \|\alpha_{\zeta_0,1}\|.$$

Then we have

$$\frac{\|\alpha_{\zeta_0,0}\|}{\dim_{\zeta_0}} + \frac{\|\alpha_{\zeta_0,1}\|}{\dim_{\zeta_0}} + \frac{\mu_{\zeta_0}}{\dim_{\zeta_0}} = b_{\zeta_0},$$
$$\sum_{i\in\boldsymbol{N}}\left(\frac{\alpha_{\zeta_0,0,i}}{\dim\zeta_0}\right)^{\ell} + (-1)^{\ell-1}\sum_{i\in\boldsymbol{N}}\left(\frac{\alpha_{\zeta_0,1,i}}{\dim\zeta_0}\right)^{\ell} = a_{\zeta_0,\ell} \quad (\ell\ge 2).$$

Now put $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ ;\ \mu\Big)$ with $\mu = (\mu_\zeta)_{\zeta\in\widehat{T}}$. Then we have from (16.2)

$$\sum_{\zeta\in\widehat{T}}\ \sum_{\varepsilon\in\{0,1\}} \|\alpha_{\zeta,\varepsilon}\| + \|\mu\| = 1,$$

which is nothing but the maximum condition (MAX) in (3.11) on the parameter A.

Finally we get the following. For $\omega = ([t],1)$ with $t\in T^*$, the value $s_\omega = f(\xi_q)$ for $\xi_q = (t,(q))$ is given by

$$s_\omega = f(\xi_q) = \sum_{\zeta\in\widehat{T}}\left(\frac{\|\alpha_{\zeta,0}\|}{\dim\zeta} + \frac{\|\alpha_{\zeta,1}\|}{\dim\zeta} + \frac{\mu_\zeta}{\dim\zeta}\right)\chi_\zeta(t),$$

and for $\omega = ([t],\ell), \ell\ge 2$, the value $s_\omega = f((d,\sigma))$ for a basic (d,σ), with $P_\sigma(d) = [t]$ and $\ell(\sigma) = \ell$, is given by

$$s_\omega = f((d,\sigma)) = \sum_{\zeta\in\widehat{T}}\left\{\sum_{i\in\boldsymbol{N}}\left(\frac{\alpha_{\zeta,0,i}}{\dim\zeta}\right)^{\ell} + (-1)^{\ell-1}\sum_{i\in\boldsymbol{N}}\left(\frac{\alpha_{\zeta,1,i}}{\dim\zeta}\right)^{\ell}\right\}\chi_\zeta(t).$$

This completes the proof of Proposition 13. $\square$

Hence the proof of Theorem 2 is now complete.

(For a historical reason, we add here the reference [ASW] in addition to [Tho2].)

17. Deduction from $\mathfrak{S}_\infty(T)$ to $\mathfrak{S}^e_\infty(T)$, T abelian

17.1. Proof of Theorem 4

To prove Theorem 4, first we quote some results from [Tho1]. As a general setting, let G be a countable discrete group and N its normal subgroup. Let $K_1(G), E(G), F(G)$ be as in §15, and further let $K_1(N,G)$ be the set of positive definite class functions f on N which is normalized as $f(e)=1$, and invariant under G, and let $E(N,G)$ be the set of all extremal points in the convex compact set $K_1(N,G)$. Then, [Tho1, Lemma 14 and Lemma 16] assert respectively the following.

(1) *For an $F \in E(G)$, its restriction $f = F|_N$ on N belongs to $E(N,G)$.*

(2) *Let $f \in E(N,G)$ and $F = \mathrm{Ind}_N^G f \in K_1(G)$ the trivial inducing up of f. Express F as an integral on the closure $F(G)$ of $E(G)$ as $F = \int_{F(G)} F'\, d\mu(F')$, where μ is a measure on the compact set $F(G)$ such that $\mu(F(G)\setminus E(G)) = 0$. Denote by* $\mathrm{supp}(F)$ *the support of the measure μ. Then, each $F' \in \mathrm{supp}(F)$ is an extension of f onto G. In particular, $F' \in \mathrm{supp}(F) \cap E(G) \neq \emptyset$ is an extremal extension of f.*

Now we apply these results to the case of $G = \mathfrak{S}_\infty(T)$ and $N = G^e$. Then, we see that every element in $E(N,G)$ is a restriction of some element in $E(G)$.

On the other hand, remark that a conjugacy class in G is either disjoint with N or equals to a conjugacy class in $N = G^e$. This means that $K_1(N,G) = K_1(N)$ and so $E(N,G) = E(N)$.

Therefore, each element of $E(N) = E(G^e)$ is equal to a restriction of some element in $E(G)$. This proves Theorem 4. □

17.2. Proof of Proposition 5

To prove Proposition 5, we study the surjective correspondence

$$E(G) \ni f_A \longmapsto f_A^e := (f_A)|_{G^e} \in E(G^e),$$

in detail and prove that $f_A|_{G^e} = f_{A'}|_{G^e}$ if and only if $A' = R(\zeta_0)A$, or if and only if $f_{A'}(g) = \pi_{\zeta_0,0}(g)\, f_A(g)$ $(g \in G)$, when the condition (MAX) in (3.11) is assumed both for A and A'. For $g \in G$, let

$$g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m\,, \quad \xi_{q_k} = (t_{q_k}, (q_k)), \quad g_j = (d_j, \sigma_j), \tag{17.1}$$

be its standard decomposition. Denote the number m of disjoint cycles in σ by $m(g) = m(\sigma)$, then it is a class function on G and also on $\mathfrak{S}_\infty$. Here the supports of components, $q_1, q_2, \ldots, q_r$, and $\mathrm{supp}(g_j) := \mathrm{supp}(\sigma_j)$ $(1 \le j \le m)$, are mutually disjoint, and σ_j is a cycle of length $\ell(\sigma_j) \ge 2$ and $\mathrm{supp}(d_j) \subset \mathrm{supp}(\sigma_j) =: K_j$. For $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\, \mu\Big)$ in (3.5), we assume the condition (MAX), instead of (3.6), that is,

$$\text{(MAX)} \qquad \sum_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \|\alpha_{\zeta,\varepsilon}\| \;+\; \|\mu\| \;=\; 1\,. \tag{17.2}$$

The formula of a character f_A of G in Theorem 2 is rewritten as

$$f_A(g) = \prod_{1\le k\le r} Y_1(t_{q_k}) \times \prod_{1\le j\le m} Y_{\ell_j}(P_{K_j}(d_j)), \tag{17.3}$$

where $\ell_j = \ell(\sigma_j)$, $P_{K_j}(d_j) = \prod_{i\in K_j} t_i$ for $d_j = (t_i)_{i\in K_j}$, and $Y_\ell(t)$ $(\ell \ge 1, t \in T)$, are the multiplicative factors of f_A given as

$$\begin{aligned} Y_1(t) &= \sum_{\zeta\in\widehat{T}} \left(\sum_{\varepsilon\in\{0,1\}} \sum_{i\in\boldsymbol{N}} \alpha_{\zeta,\varepsilon,i} \ + \ \mu_\zeta \right) \zeta(t), \\ Y_\ell(t) &= \sum_{\zeta\in\widehat{T}} \left(\sum_{\varepsilon\in\{0,1\}} \sum_{i\in\boldsymbol{N}} (\alpha_{\zeta,\varepsilon,i})^\ell \, (-1)^{\varepsilon(\ell-1)} \right) \zeta(t) \quad (\ell \ge 2). \end{aligned}$$

Since the condition (MAX) is assumed for A, the above formula is valid even in the case where $t_{q_k} = e_T$ because $Y_1(e_T) = 1$.

For another $A' := \left((\alpha'_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \ ; \ \mu' \right)$ a character $f_{A'}$ of G is given similarly as

$$f_{A'}(g) = \prod_{1\le k\le r} Y'_1(t_{q_k}) \times \prod_{1\le j\le m} Y'_{\ell_j}(P_{K_j}(d_j)), \tag{17.4}$$

$Y'_\ell(t)$ $(\ell \ge 1, t \in T)$, are similarly given corresponding to the parameter A'.

Now assume that $f_A|_{G^e} = f_{A'}|_{G^e}$. Put $n = r + m$, and denote newly by $\{ (\ell_s, t_s) \ ; \ 1 \le s \le n \}$, $\ell_s \ge 1, t_s \in T$, the set of the pairs $(1, t_{q_k})$, $1 \le k \le r$ (here $\ell = 1$), and (ℓ_j, t'_j), $1 \le j \le m$, with $\ell_j \ge 2, t'_j = P_{K_j}(d_j) \in T$, we see that the above condition is equivalent to

$$\prod_{1\le s\le n} Y_{\ell_s}(t_s) \ = \ \prod_{1\le s\le n} Y'_{\ell_s}(t_s) \qquad \text{under} \ \prod_{1\le s\le n} t_s = e_T, \tag{17.5}$$

for any choice of pairs $(\ell_s, t_s) \in \boldsymbol{N} \times T$ $(1 \le s \le n)$ satisfying the condition $\prod_{1\le s\le n} t_s = e_T$.

Put $T(\ell) = \{ t \in T \ ; \ Y_\ell(t) \ne 0 \}$. Then, $T(\ell)$ is stable under the map $t \mapsto t^{-1}$ because $Y_\ell(t^{-1}) = \overline{Y_\ell(t)}$. Moreover, $T(\ell)$ is just the set of such $t \in T$ that $Y'_\ell(t) \ne 0$, because $Y_\ell(t)Y_\ell(t^{-1}) = Y'_\ell(t)Y'_\ell(t^{-1})$. Put $\chi_\ell(t) = Y'_\ell(t)/Y_\ell(t)$ on $T(\ell)$, then $|\chi_\ell(t)| = 1$, $\chi_\ell(t^{-1}) = \chi_\ell(t)^{-1}$. For a set of elements $t_s \in T(\ell_s), 1 \le s \le n$, the equality (17.5) gives us

$$\prod_{1\le s\le n} \chi_{\ell_s}(t_s) = 1 \qquad \text{if} \quad \prod_{1\le s\le n} t_s = e_T.$$

From this we obtain in particular $\chi_\ell(t) = \chi_{\ell'}(t)$ for $t \in T(\ell) \cap T(\ell')$. Putting $\chi(t) = \chi_\ell(t)$ for $t \in T(\ell)$, and taking into account of the above equality, we see that χ on $T' := \cup_{\ell\ge1} T(\ell)$ can be extended to a one-dimensional chatacter on the group $\langle T' \rangle \hookrightarrow T$ generated by T'.

Since $\langle T' \rangle$ is determined by the parameter A, we denote it by T_A. Put $D_\infty(T_A) := \prod'_{i\in \boldsymbol{N}} T_{A,i}$, $T_{A,i} = T_A$ $(i \in \boldsymbol{N})$, then the character f_A of G vanishes outside of $\mathfrak{S}_\infty(T_A) = D_\infty(T_A) \rtimes \mathfrak{S}_\infty \subset G$. Since $T_{A'} = T_A$, we have similar fact for $f_{A'}$. The formulas (17.3) and (17.4) gives us

$$f_{A'}(g) = \chi(g) \cdot f_A(g) \;\; (g \in \mathfrak{S}_\infty(T_A)\,) \quad \text{with} \quad \chi(g) := \chi(P(d)) \;\text{ for }\; g = (d, \sigma).$$

Take a one-dimensional character $\zeta_0 \in \widehat{T}$ extending χ on T_A to T, then we get $f_{A'}(g) = \zeta_0(P(d)) \cdot f_A(g)$ for $g = (d, \sigma) \in G$. Since $\pi_{\zeta_0,0}(g) = \zeta_0(P(d))$ by definition, this is written as $f_{A'} = \pi_{\zeta_0,0} \cdot f_A$, as desired.

The proof of Proposition 5 is now complete. □

17.3. The case where the parameter A is unique for f_A^e

As seen above, the parameter $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\;;\;\mu\Big)$ for the character $f_A^e = f_A|_{G^e}$ is not unique in general, even under the condition (MAX) on $\mu = (\mu_\zeta)_{\zeta\in\widehat{T}}$. However, in a very special case of A, the parameter becomes unique. This case is characterized by

$$R(\zeta_0)A \;=\; A \quad (\forall \zeta_0 \in \widehat{T}).$$

Let us study the explicit form of the character f_A in this special case. From the above condition on A, we have $\alpha_{\zeta,\varepsilon} = \alpha_{\mathbf{1}_T,\varepsilon}$, $\mu_\zeta = \mu_{\mathbf{1}_T}$ for any $\zeta \in \widehat{T}$. Then put

$$\alpha = (\alpha_i)_{i\in\boldsymbol{N}} := \alpha_{\mathbf{1}_T,0}, \quad \beta = (\beta_i)_{i\in\boldsymbol{N}} := \alpha_{\mathbf{1}_T,1}, \quad \nu = \mu_{\mathbf{1}_T}.$$

Then, by the condition (MAX) on A, we have

$$(17.6) \qquad |T|\,(\|\alpha\| \,+\, \|\beta\| \,+\, \nu) \;=\; 1 \quad \text{or} \quad |T|\,(\|\alpha\| \,+\, \|\beta\|) \;\le\; 1.$$

Hence the multiplicative factors $Y_\ell(t)$ in this case are

$$\begin{aligned} Y_1(t) &= (\|\alpha\| \,+\, \|\beta\| \,+\, \nu) \textstyle\sum_{\zeta\in\widehat{T}} \zeta(t) = \delta_{e_T}(t)\,, \\ Y_\ell(t) &= \big(\textstyle\sum_{i\in\boldsymbol{N}} \alpha_i^{\,\ell} \,+\, (-1)^{\ell-1} \sum_{i\in\boldsymbol{N}} \beta_i^{\,\ell}\big) \sum_{\zeta\in\widehat{T}} \zeta(t) \\ &= |T| \big(\textstyle\sum_{i\in\boldsymbol{N}} \alpha_i^{\,\ell} \,+\, (-1)^{\ell-1} \sum_{i\in\boldsymbol{N}} \beta_i^{\,\ell}\big)\; \delta_{e_T}(t) \qquad (\ell \ge 2), \end{aligned}$$

where δ_{e_T} denotes the delta function on T supported by the unit element e_T .

It follows from these formulas for multiplicative factors Y_1 and Y_ℓ that, for $g \in G$ in (17.1), the value of the character $f_A(g)$ is not zero only if $r = 0$, and for each j, $1 \le j \le m$, the $D_{K_j}(T)$-component d_j of $g_j = (d_j, \sigma_j)$ has product $P_{K_j}(d_j) = e_T$. This condition on g means exactly that $g = (d, \sigma)$ is conjugate to $\sigma \in \mathfrak{S}_\infty \hookrightarrow G$, so that f_A is supported by the set of conjugacy classes having representatives from $\mathfrak{S}_\infty$. Thus we have

$$f_A(g) = \begin{cases} |T|^{m(\sigma)} f_{\alpha,\beta}(\sigma) & \text{if } \; g = (d, \sigma) \; \text{ is conjugate to } \; \sigma \in \mathfrak{S}_\infty, \\ \qquad 0 & \text{if } \; g \; \text{ is not conjugate to any } \; \tau \in \mathfrak{S}_\infty, \end{cases}$$

where $m(\sigma)$ denotes the number of disjoint cycles in σ, and $f_{\alpha,\beta}(\sigma)$ denotes the character of $\mathfrak{S}_\infty$ in **6.1** with parameter (α, β).

18. Wreath product of a finite group with $\mathfrak{A}_\infty$

18.1. The case of the group $\mathfrak{A}_\infty(T)$

Let us consider a normal subgroup $\mathfrak{A}_\infty(T) := D_\infty(T) \rtimes \mathfrak{A}_\infty$ of $G = \mathfrak{S}_\infty(T)$ for the infinite alternating group $\mathfrak{A}_\infty$. Here we prove the following result.

Theorem 14. *All the characters of the group* $G' := \mathfrak{A}_\infty(T)$ *are given as restrictions of those of the group* $G = \mathfrak{S}_\infty(T)$.

For two characters f_A *and* $f_{A'}$ *on* G *with parameters*

$$A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ ;\ \mu\Big) \quad \text{and} \quad A' = \Big((\alpha'_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ ;\ \mu'\Big)$$

as in (3.5) *respectively, their restrictions on* G' *coincide with each other if and only if* $f_{A'} = (\mathrm{sgn}_{\mathfrak{S}})^a f_A$ $(a = 0$ *or* $1)$, *or, under the condition* (MAX) *for both of* A *and* A', *if and only if* $A' = A$ *or* $A' = {}^tA$, *where*

$${}^tA \;=\; \Big((\alpha''_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ \ \mu''\Big)$$

is defined as

$$\alpha''_{\zeta,0} \;=\; \alpha_{\zeta,1}\ ,\ \ \alpha''_{\zeta,1} \;=\; \alpha_{\zeta,0} \quad (\zeta \in \widehat{T}), \quad \text{and} \quad \mu'' = \mu.$$

Proof. The first assertion can be proved just as in **17.1**.

For the second assertion, if the above condition holds between A and A', then we see easily that the equality $f_A|_{G'} = f_{A'}|_{G'}$ holds. Therefore it rests only to prove the converse.

Suppose $f_A|_{G'} = f_{A'}|_{G'}$. For the parameter A, put for $\zeta \in \widehat{T}$,

$$\begin{aligned} Z^A_{\zeta,1} &= \|\alpha_{\zeta,0}\| \;+\; \|\alpha_{\zeta,1}\| \;+\; \mu_\zeta, \\ Z^A_{\zeta,\ell} &= \sum_{\varepsilon\in\{0,1\}}\ \sum_{i\in\boldsymbol{N}} (-1)^{\varepsilon(\ell-1)}\,(\alpha_{\zeta,\varepsilon,i})^\ell \qquad (\ell \geq 2), \end{aligned}$$

and similarly put $Z^{A'}_{\zeta,\ell}$ $(\ell \geq 1)$ for A'.

1. Firstly take a basic element $g = \xi_q = (t, (q)) \in D_\infty(T) \subset G'$ and write down the relation $f_A(g) = f_{A'}(g)$, then we have

$$\sum_{\zeta\in\widehat{T}} Z^A_{\zeta,1}\,\frac{\chi_\zeta(t)}{\dim\zeta} \;=\; \sum_{\zeta\in\widehat{T}} Z^{A'}_{\zeta,1}\,\frac{\chi_\zeta(t)}{\dim\zeta} \qquad (t \in T).$$

2. Secondly take a basic element $g = (d, \sigma)$ with σ a cycle and $\mathrm{supp}(d) \subset \mathrm{supp}(\sigma)$. Suppose $\ell = \ell(\sigma)$ is odd, then $g \in G'$. Put $t = P_\sigma(d)$, then the relation $f_A(g) = f_{A'}(g)$ gives us

$$\sum_{\zeta\in\widehat{T}} Z^A_{\zeta,\ell}\,\frac{\chi_\zeta(t)}{(\dim\zeta)^\ell} \;=\; \sum_{\zeta\in\widehat{T}} Z^{A'}_{\zeta,\ell}\,\frac{\chi_\zeta(t)}{(\dim\zeta)^\ell} \qquad (t \in T).$$

3. Thirdly take $g=(d_1,\sigma_1)(d_2,\sigma_2)$ a product of two basic elements with $\ell_k=\ell(\sigma_k)$ both even and $P_{\sigma_k}(d_k)=t_k$, then $g\in G'$. We get from $f_A(g)=f_{A'}(g)$ the following:

$$\prod_{k=1,2}\left(\sum_{\zeta\in\widehat{T}} Z^A_{\zeta,\ell_k}\frac{\chi_\zeta(t_k)}{(\dim\zeta)^{\ell_k}}\right) = \prod_{k=1,2}\left(\sum_{\zeta\in\widehat{T}} Z^{A'}_{\zeta,\ell_k}\frac{\chi_\zeta(t_k)}{(\dim\zeta)^{\ell_k}}\right) \qquad (t_1,t_2\in T).$$

From these equations, we get

$$\|\alpha_{\zeta,0}\| + \|\alpha_{\zeta,1}\| + \mu_\zeta = \|\alpha'_{\zeta,0}\| + \|\alpha'_{\zeta,1}\| + \mu'_\zeta\,, \tag{18.1}$$

$$Z^A_{\zeta,\ell} = Z^{A'}_{\zeta,\ell} \qquad (\ell\ge 2\,,\ \text{odd}) \tag{18.2}$$

$$Z^A_{\zeta_1,\ell_1}\,Z^A_{\zeta_2,\ell_2} = Z^{A'}_{\zeta_1,\ell_1}\,Z^{A'}_{\zeta_2,\ell_2} \quad (\zeta_1,\zeta_2\in\widehat{T}\ ;\ \ell_1,\ell_2\ \text{ even}).$$

From the third equation, there exists $a=0,1$ such that

$$Z^A_{\zeta,\ell} = (-1)^a Z^{A'}_{\zeta,\ell} \qquad (\ell\ \text{ even}). \tag{18.3}$$

The equations (18.1)–(18.3) prove that $f_{A'}=(\mathrm{sgn}_{\mathfrak{S}})^a f_A$.

Furthermore, fix ζ and put $\alpha=\alpha_{\zeta,0},\beta=\alpha_{\zeta,1}$ and $\alpha'=\alpha'_{\zeta,0},\beta'=\alpha'_{\zeta,1}$. Then, $\|\alpha\|+\|\beta\|\le 1$, $\|\alpha'\|+\|\beta'\|\le 1$, and the equations (18.2)–(18.3) give us for $a=0$ or 1 respectively, as meromorphic functions on $\boldsymbol{C}$,

$$\sum_{i\ge1}\frac{\alpha_i^{\,2}}{1-\alpha_i z} - \sum_{i\ge1}\frac{\beta_i^{\,2}}{1+\beta_i z} = \sum_{i\ge1}\frac{(\alpha'_i)^2}{1-\alpha'_i z} - \sum_{i\ge1}\frac{(\beta'_i)^2}{1+\beta'_i z}\ ;$$

or

$$\sum_{i\ge1}\frac{\alpha_i^{\,2}}{1-\alpha_i z} - \sum_{i\ge1}\frac{\beta_i^{\,2}}{1+\beta_i z} = -\sum_{i\ge1}\frac{(\alpha'_i)^2}{1+\alpha'_i z} + \sum_{i\ge1}\frac{(\beta'_i)^2}{1-\beta'_i z}\,.$$

Comparing poles in both sides, we can conclude for $i\ge1$, $\alpha_i=\alpha'_i,\beta_i=\beta'_i$, or $\alpha_i=\beta'_i,\beta_i=\alpha'_i$, according as $a=0$ or 1. Hence, $\alpha'_{\zeta,\varepsilon}=\alpha_{\zeta,\varepsilon}$ for any (ζ,ε), or $\alpha'_{\zeta,0}=\alpha_{\zeta,1}$, $\alpha'_{\zeta,1}=\alpha_{\zeta,0}$, according as $a=0$ or 1. In any case we get $\mu'_\zeta=\mu_\zeta$ $(\zeta\in\widehat{T})$.

This completes the proof of Theorem 14. □

Example 18.1. Let us study the case where the restriction $f_A|_{G'}$ has its unique parameter A. By Theorem 14, this corresponds to the case of A such that ${}^tA=A$. Then, $\alpha_{\zeta,1}=\alpha_{\zeta,0}$ $(\zeta\in\widehat{T})$. From the formula (3.8), we have in this case the following expression of f_A. For $g=(d,\sigma)\in G$ with standard decomposition $g=\xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_m$, $\xi_q=\big(t_q,(q)\big)$, $g_j=(d_j,\sigma_j)$ with $\ell(\sigma_j)\ge2$,

$$f_A(g)=0, \qquad \text{if some of } \ell(\sigma_j) \text{ is even;} \quad \text{and otherwise,}$$

$$f_A(g)=\prod_{k=1}^{r}\left(\sum_{\zeta\in\widehat{T}}\big(2\|\alpha_{\zeta,0}\|+\mu_\zeta\big)\frac{\chi_\zeta(t_{q_k})}{\dim\zeta}\right)\times$$
$$\times\prod_{j=1}^{m}\left(\sum_{\zeta\in\widehat{T}}\sum_{i\in\boldsymbol{N}}2\left(\frac{\alpha_{\zeta,0,i}}{\dim\zeta}\right)^{\ell(\sigma_j)}\chi_\zeta\big(P_{\sigma_j}(d_j)\big)\right).$$

18.2. The case of the group $\mathfrak{A}^e_\infty(T)$, T abelian

Assume T be abelian, then we have another normal subgroup

$$\mathfrak{A}^e_\infty(T) \;=\; \{\, g=(d,\sigma)\in\mathfrak{S}_\infty(T)\;;\;\mathrm{sgn}_{\mathfrak{S}}(\sigma)=1,\; P(d)=e_T\,\}$$

of $G=\mathfrak{S}_\infty(T)$. For this group, we can prove the following result, analogously as for $G^e=\mathfrak{S}^e_\infty(T)$ and $G'=\mathfrak{A}_\infty(T)$.

Theorem 15. *Let T be abelian. Then for the normal subgroup ${G'}^e = \mathfrak{A}^e_\infty(T)$ of $G=\mathfrak{S}_\infty(T)$, every character of ${G'}^e$ is given as the restriction of some characters of G.*

Two characters f_A and $f_{A'}$ of G with parameters

$$A=\Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\;;\;\mu\Big)\quad\text{and}\quad A'=\Big((\alpha'_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\;;\;\mu'\Big)$$

have the same restriction on ${G'}^e$ if and only if $f_{A'}=\pi_{\zeta_0,a}f_A$ with a one-dimensional character

$$\pi_{\zeta_0,a}(g)=\zeta_0\big(P(d)\big)\,(\mathrm{sgn}_{\mathfrak{S}})^a(\sigma)\quad\text{for}\;\; g=(d,\sigma)\in D_\infty(T)\rtimes\mathfrak{S}_\infty,$$

in Lemma 3, *where $\zeta_0\in\widehat{T}$, $a=0,1$. This corresponds to the following relation between parameters A and A' both satisfying the condition* (MAX)*,*

$$\begin{aligned} A' &= R(\zeta_0)A \quad \text{in (5.5)} && \text{in case}\;\; a=0,\\ A' &= R(\zeta_0)({}^tA) && \text{in case}\;\; a=1.\end{aligned}$$

19. Appendix: Lemmas for compact groups

A finite group T is a kind of compact group, and we see in our discussions above that it is sometimes simpler to use notations and notions for the case of compact groups. For example, the notations

$$\frac{1}{|T|}\sum_{t\in T}F(t)\qquad\text{and}\qquad\int_T F(t)\,d\mu(t),$$

when T is finite or compact respectively, can be unified with the latter one. Here $|T|$ denotes the number of elements in T, and $d\mu$ denotes the nomalized Haar measure on a compact group T. The Haar measure μ on a finite group T is given by $\mu(\{\,t\,\})=1/|T|$ $(t\in T)$.

In the present paper, when induced representations $\rho=\mathrm{Ind}_H^G\pi$ from subgroups H of wreath product type are taken as ingredients, we have chosen representations π of H in (10.5) constructed as in (10.6)–(10.7) from irreducible representations $\zeta\in\widehat{T}$ of components $\cong T$. However, as we remarked in **10.1**, some IURs ρ of G can be constructed starting from π given by means of cyclic representations of T. Therefore it is worthwhile to check what happens when we use, in our discussions, cyclic representations ϖ of T instead of irreducible ones $\zeta\in\widehat{T}$. Actually we have done the calculations for this cyclic case on the way of preparing this paper, and on doing this, the following lemmas on representations of compact groups were utilized. So, we expose them here in a little more general form than directly necessary in the present paper.

Now let T be a compact group and any representation of T treated here is assumed to be unitary. An equivalence class in $\widehat{T}$ is identified with a representation in that class.

Take a finite dimensional representation $(\varpi, V(\varpi))$ of T with representation space $V(\varpi)$. Its character, denoted by χ_ϖ, is a positive definite class function on T. For a $\zeta \in \widehat{T}$, denote by $m(\zeta) = [\varpi : \zeta]$ the multiplicity of ζ in ϖ, and by V_ζ the subspace of $V(\varpi)$ consisting of vectors on which T acts according to a multiple of ζ, then an irreducible decomposition of ϖ is given as

$$V(\varpi) = \bigoplus_{\zeta \in \widehat{T}} V_\zeta \ , \qquad V_\zeta = \bigoplus_{1 \le k \le m(\zeta)} V_\zeta^k, \tag{19.1}$$

where V_ζ^k denote mutually orthogonal subspaces isomorphic to $V(\zeta)$ as T-modules.

Lemma A.1. *Let $v, w \in V(\varpi)$ and $v_\zeta, w_\zeta \in V_\zeta$ $(\zeta \in \widehat{T})$ be their components in the direct sum decomposition* (19.1)*. Then*

$$\int_T \langle \varpi(sts^{-1})v, w\rangle \, d\mu(s) = \sum_{\zeta \in \widehat{T}} \frac{\chi_\zeta(t)}{\dim \zeta} \, \langle v_\zeta, w_\zeta \rangle. \tag{19.2}$$

Lemma A.2. *Let $\zeta \in \widehat{T}$ and $v_1, v_2, w_1, w_2 \in V(\zeta)$. Then,*

$$\int_T \langle \zeta(s)v_1, w_1\rangle \, \langle \zeta(s^{-1})v_2, w_2\rangle \, d\mu(s) = \frac{1}{\dim \zeta} \, \langle v_1, w_2\rangle \langle v_2, w_1\rangle. \tag{19.3}$$

Lemma A.3. *Let $v_i, w_i \in V(\varpi)$ $(i = 1, 2)$ and $v_{i,\zeta}^k, w_{i,\zeta}^k \in V_\zeta^k$ $(\zeta \in \widehat{T}, 1 \le k \le m(\zeta))$ be their components in the finer direct sum decomposition* (19.1)*: $v_i = \sum_{\zeta \in \widehat{T}} \sum_{1 \le k \le m(\zeta)} v_{i,\zeta}^k$ etc. Denote by $\hat{v}_{i,\zeta}^k$ the image of $v_{i,\zeta}^k$ under a fixed equivalence map from V_ζ^k onto $V(\zeta)$. Then*

$$\begin{aligned} &\int_T \langle \varpi(st)v_1, w_1\rangle \langle \varpi(s^{-1})v_2, w_2\rangle \, d\mu(s) = \\ &\qquad = \sum_{\zeta \in \widehat{T}} \frac{1}{\dim \zeta} \sum_{1 \le k_1, k_2 \le m(\zeta)} \langle \zeta(t)\hat{v}_{1,\zeta}^{k_1}, \hat{w}_{2,\zeta}^{k_2}\rangle \langle \hat{v}_{2,\zeta}^{k_2}, \hat{w}_{1,\zeta}^{k_1}\rangle. \end{aligned} \tag{19.4}$$

Proof. It is enough to prove the equality for $t = e_T$, the identity element of T. Then,

$$\begin{aligned} &\int_T \langle \varpi(s)v_1, w_1\rangle \langle \varpi(s^{-1})v_2, w_2\rangle \, d\mu(s) \ = \\ &\quad = \sum_{\zeta \in \widehat{T}} \int_T \sum_{1 \le k_1, k_2 \le m(\zeta)} \langle \zeta(s)\hat{v}_{1,\zeta}^{k_1}, \hat{w}_{1,\zeta}^{k_1}\rangle \langle \zeta(s^{-1})\hat{v}_{2,\zeta}^{k_2}, \hat{w}_{2,\zeta}^{k_2}\rangle \, d\mu(s). \end{aligned}$$

Here we apply Lemma A2. □

Lemma A.4. *Let $v_i, w_i \in V(\varpi)$ $(1 \le i \le \ell)$ and $v^k_{i,\zeta}, w^k_{i,\zeta} \in V^k_\zeta$ $(\zeta \in \widehat{T}, 1 \le k \le m(\zeta))$ be respectively their components in the finer direct sum decomposition* (19.1). *Then*

$$\int_T \cdots \int_T \langle \varpi(s_1 t_1 s_\ell^{-1}) v_1, w_1 \rangle \langle \varpi(s_2 t_2 s_1^{-1}) v_2, w_2 \rangle \cdots \langle \varpi(s_\ell t_\ell s_{\ell-1}^{-1}) v_\ell, w_\ell \rangle \, d\mu(s_1)\, d\mu(s_2) \cdots d\mu(s_\ell)$$
$$= \sum_{\zeta \in \widehat{T}} \frac{\chi_\zeta(t_\ell t_{\ell-1} \cdots t_2 t_1)}{(\dim \zeta)^\ell} \sum_{1 \le k_1, k_2, \ldots, k_\ell \le m(\zeta)} \langle \hat{v}^{k_\ell}_{\ell,\zeta}, \hat{w}^{k_{\ell-1}}_{\ell-1,\zeta} \rangle \langle \hat{v}^{k_{\ell-1}}_{\ell-1,\zeta}, \hat{w}^{k_{\ell-2}}_{\ell-2,\zeta} \rangle \cdots \cdots \langle \hat{v}^{k_2}_{2,\zeta}, \hat{w}^{k_1}_{1,\zeta} \rangle \langle \hat{v}^{k_1}_{1,\zeta}, \hat{w}^{k_\ell}_{\ell,\zeta} \rangle.$$

For the proof, we apply Lemmas A2 and A3.

Lemma A.5. *Let $v_i \in V(\varpi)$ $(1 \le i \le \ell)$ and $v_{i,\zeta} \in V_\zeta$ $(\zeta \in \widehat{T})$ be their components in the direct sum decomposition* (19.1). *Then*

$$\int_T \cdots \int_T \langle \varpi(s_1 t_1 s_\ell^{-1}) v_\ell, v_1 \rangle \langle \varpi(s_2 t_2 s_1^{-1}) v_1, v_2 \rangle \cdots \langle \varpi(s_\ell t_\ell s_{\ell-1}^{-1}) v_{\ell-1}, v_\ell \rangle \, d\mu(s_1)\, d\mu(s_2) \cdots d\mu(s_\ell)$$
$$= \sum_{\zeta \in \widehat{T}} \frac{\chi_\zeta(t_\ell t_{\ell-1} \cdots t_2 t_1)}{(\dim \zeta)^\ell} \prod_{1 \le i \le \ell} \|v_{i,\zeta}\|^2.$$

Proof. Let $v^k_{i,\zeta} \in V^k_\zeta$ $(\zeta \in \widehat{T}, 1 \le k \le m(\zeta))$ be the components in the finer direct sum decomposition (19.1). By Lemma A4, the integral is equal to

$$\sum_{\zeta \in \widehat{T}} \frac{\chi_\zeta(t_\ell t_{\ell-1} \cdots t_2 t_1)}{(\dim \zeta)^\ell} \sum_{1 \le k_1, k_2, \ldots, k_\ell \le m(\zeta)} \|\hat{v}^{k_{\ell-1}}_{\ell-1,\zeta}\|^2 \, \|\hat{v}^{k_{\ell-2}}_{\ell-2,\zeta}\|^2 \cdots \|\hat{v}^{k_1}_{1,\zeta}\|^2 \, \|\hat{v}^{k_\ell}_{\ell,\zeta}\|^2.$$

□

Acknowledgements. We express our thanks to Professor M. Bożejko for interesting discussions on several subjects for Coxeter groups, especially on positive definite functions. We also thank Professor N. Obata for everlasting mutual stimulations about studies on harmonic analysis on the infinite symmetric group etc. Our thanks go also to Professors N. Kawanaka, K. Nishiyama and H. Yamashita for inviting us to the study of characters on the inifinite Weyl groups, and to Professor K. Shinoda for advising us to extend our results for infinite Weyl groups to the case of infinite complex reflexion groups $G(r, 1, \infty)$. Thus, altogether, they provide us a natural motivation to the present study on characters of wreath products $\mathfrak{S}_\infty(T)$ of *any* finite groups T with the infinite symmetric group $\mathfrak{S}_\infty$.

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