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Itô formula for the infinite-dimensional fractional Brownian motion

By

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Abstract

We introduce the stochastic integration with respect to the infinite-dimensional fractional Brownian motion. Using the techniques of the anticipating stochastic calculus, we derive an Itô formula for Hurst parameter bigger than $\frac{1}{2}$.

1. Introduction

The fractional Brownian motion (fBm) $B^h = (B^h_t)_{t\in[0,1]}$ is a centered Gaussian process, starting from zero, with covariance

$$R(t,s) = \frac{1}{2}(t^{2h} + s^{2h} - |t-s|^{2h})$$

for every $s,t \in [0,1]$. The parameter h belongs to $(0,1)$ and it is called Hurst parameter. If $h = \frac{1}{2}$ the associated process is the classical Brownian motion. The process $(B^h_t)_{t\in[0,1]}$ has stationary increments and it is self similar, that is, $B^h_{\alpha t}$ and $\alpha^h B^h_t$ have the same distribution for all $\alpha > 0$. These properties make the fBm a candidate as a model in different applications (like network traffic analysis or mathematical finance). Therefore a stochastic calculus with respect to the fractional Brownian motion was needed. Let us briefly recall the two principal directions considered in the fractional stochastic integration.

- a first approach is the anticipating (Skorohod) stochastic calculus and the white noise theory. This approach has been used in e.g. [2], [5] or [6] among others. An Itô's formula involving the Skorohod (divergence) integral has been proved. Note that for small Hurst parameters one need an extended divergence integral (see [4]).
- the pathwise integration theory has been considered in e.g. [8], [18] or [1]. An Itô formula has been proved also in this case and stochastic equations in the Stratonovich sense driven by the fBm has been considered. Again, a generalized integral was needed for small Hurst parameters.

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A natural extension of this problems is to develop a stochastic calculus with respect to a Hilbert space-valued fBm. Recently, the infinite dimensional fBm has been considered by several authors as a driving noise in the study of stochastic evolution equations. We refer, among others, to [7], [9] or [16]. Note that the stochastic integrals with respect to the infinite dimensional fBm appearing in these papers are only Wiener integrals. Our aim is to introduce the stochastic integration of *non-deterministic* integrands with respect to a Hilbert-valued fBm and to obtain an Itô formula in the Skorohod sense. Our work is motivated, on one side, by the recent development of the stochastic integration with respect to Gaussian (and even more general) processes and on the other side, by practical aspects. For example, we believe that a such stochastic integration could open the door to a further study of the connection between the ordinary and stochastic fractional calculus. Let us recall that, in the case of the infinite dimensional Wiener process, the last term appearing in the Itô formula (the "trace" term, see e.g. [15]) corresponds to the Kolmogorov equation associated to the infinite dimensional Ornstein-Uhlenbech process. See [15, Section 9] for details. This provides an intimated connection between the theory of the infinite dimensional diffusions and the theory of partial differential equations with concrete practical applications such that e.g. the wavefront propagation. In the case of the Hilbert space-valued fBm the last term is the one we could expect (see Section 4 below) and this seems to be related to the *fractional Kolmogorov equation* which is a reaction-diffusion equation with a fractional diffusion operator (see [3] and [12] for the definition and applications). We plan a more detailed study of the connection between (deterministic) fractional calculus and the stochastic calculus with respect to the infinite dimensional fBm.

Our approach extends, on one hand, the results of [10] in the case $h = \frac{1}{2}$ and, on the other hand, the results of [2] in the one-dimensional fBm case. We mention that the passage from the one-dimensional to the infinite dimensional case needs new techniques; for example the methods used in [2] to prove the Itô's formula are not directly applicable to the Hilbert-valued situation; the difficulty comes from the fact that the proof in [2] is based on a specific one-dimensional duality relation between the Skorohod integral and the Malliavin derivative. We prove here that, in the Wiener case, the approach of [15] to define stochastic integrals on Hilbert spaces and the method of [10] lead to the same integral and we take advantage from this fact. The method that we use to prove the Itô's formula for $h > \frac{1}{2}$ is the classical Taylor expansion.

Our paper is organized as follows: Section 2 is devoted to recall the basic notions of the stochastic calculus of variations with respect to the Wiener process and fractional Brownian motion. In Section 3 we present the construction of the adapted and non-adapted infinite-dimensional stochastic analysis and we study the relation between different approaches. In Section 4 we introduce a fBm $\mathbf{B}^h$ in a Hilbert space and we obtain an Itô formula.

2. Preliminaries

2.1. Malliavin Calculus

Let $T = [0,1]$ the unit interval and $(W_t)_{t\in T}$ the standard Wiener process on the canonical Wiener space $(\Omega, \mathcal{F}, P)$. We will denote by $\mathcal{S}$ the class of Brownian functionals of the form

$$F = f(W_{t_1}, \dots, W_{t_n}) \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is an infinitely differentiable function such that f and all its derivatives are bounded and $t_1, \dots, t_n \in T$. The elements of $\mathcal{S}$ are called smooth random variables and form a dense subspace of $L^2(\Omega)$.

The Malliavin derivative of a smooth functional F of the form (1) is the stochastic process $\{D_t F; t \in T\}$ given by

$$D_t F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W_{t_1}, \dots, W_{t_n}) 1_{[0,t_i]}(t),\ t \in T$$

More generally, we can introduce the k-th derivative of $F \in \mathcal{S}$ by $D^{(k)}_{t_1,\dots,t_k} F = D_{t_1} D_{t_2} \cdots D_{t_k} F$.

Consider now V a real and separable Hilbert space and $\mathcal{S}_V$ the class of the smooth V-valued random variables that can be written as $F = \sum_{j=1}^{n} F_j v_j$ where $F_j \in \mathcal{S}$ and $v_j \in V$. We shall introduce the derivative of $F \in \mathcal{S}_V$ as

$$D_s F = \sum_{j=1}^{n} D_s F_j \otimes v_j\ ,\ \forall s \in T. \tag{2}$$

This operator is closeable from $L^p(\Omega; V)$ into $L^p(T \times \Omega; V)$ for any $p \geq 1$ and it can be extended to the completion of $\mathcal{S}_V$, denoted $\mathbb{D}^{k,p}(V)$, with respect to the norm

$$\|F\|^p_{k,p,V} = \mathbb{E}\|F\|^p_V + \sum_{j=1}^{k} \mathbb{E}\|D^{(j)} F\|^p_{L^2(T^j;V)}.$$

The adjoint of D, called the Skorohod integral and denoted by δ, is characterized by the duality relationship

$$\mathbb{E}\langle \delta(u), F\rangle_V = \mathbb{E}\langle DF, u\rangle_{L^2(T)\otimes V}$$

for all $F \in \mathcal{S}_V$, if u belongs to $Dom(\delta)$ (the domain of the operator δ), where

$$Dom(\delta) = \{u \in L^2(T \times \Omega; V) / |\mathbb{E}\langle DF, u\rangle_{L^2(T)\otimes V}| \leq C\|F\|_{L^2(\Omega;V)}\}.$$

We will need the following property of the Malliavin derivation in Hilbert spaces (see [13])

$$(D_\alpha F)(x) = D_\alpha(F(x)) \tag{3}$$

if U, V are two Hilbert spaces, F is a random variable taking values in $L(U, V)$, $x \in U$ and $\alpha \in T$.

2.2. Fractional Brownian motion

Let $B = (B_t)_{t\in T}$ be the fractional Brownian motion (fBm) with Hurst parameter $h \in (0, 1)$. We will omit in this section the superindex h. We know that B admits a representation as Wiener integral of the form $B_t = \int_0^t K(t, s)dW_s$, where $W = (W_t)_{t\in T}$ is a Wiener process, and $K(t, s)$ is the kernel (see [2], [5])

$$K(t,s) = c_h \, (t-s)^{h-\frac{1}{2}} + s^{h-\frac{1}{2}} F_1\left(\frac{t}{s}\right),$$

c_h being a constant and $F_1(z) = c_h \left(\frac{1}{2} - h\right) \int_0^{z-1} \theta^{h-\frac{3}{2}} \left(1 - (\theta+1)^{h-\frac{1}{2}}\right) d\theta$. This kernel satisfies the condition

$$\frac{\partial K}{\partial t}(t,s) = c_h \left(h - \frac{1}{2}\right) \left(\frac{s}{t}\right)^{\frac{1}{2}-h} (t-s)^{h-\frac{3}{2}}. \tag{4}$$

We consider the canonical Hilbert space of the fBm $\mathcal{H}$ as the closure of the linear space generated by the function $\{1_{[0,t]}, t \in T\}$ with respect to the scalar product $\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R(t, s)$. Then the mapping $1_{[0,t]} \to B_t$ gives an isometry between $\mathcal{H}$ and the first chaos generated by $\{B_t, t \in T\}$ and $B(\phi)$ denotes the image of a element $\phi \in \mathcal{H}$.

We recall that , if $\phi, \chi \in \mathcal{H}$ are such that $\int_T \int_T |\phi(s)||\chi(t)|t-s|^{2h-2} ds dt < \infty$, their scalar product in $\mathcal{H}$ is given , when $h > \frac{1}{2}$, by

$$\langle \phi, \chi \rangle_{\mathcal{H}} = h(2h-1) \int_0^1 \int_0^1 \phi(s)\chi(t)|t-s|^{2h-2} ds dt. \tag{5}$$

We can introduce a derivation and a Skorohod integration with respect to B. For a smooth $\mathcal{H}$-valued functionals $F = f(B(\varphi_1), \ldots, B(\varphi_n))$ with $n \geq 1$, $f \in C_b^\infty(\mathbb{R}^n)$ and $\varphi_1, \ldots, \varphi_n \in \mathcal{H}$ we put

$$D^B(F) = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(B(\varphi_1), \ldots, B(\varphi_n))\varphi_j.$$

and $D^B F$ will be closable from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$ for any $p \geq 1$. Therefore, we can extend D^B to the closure of smooth functionals $\mathbb{D}_B^{k,p}$ with respect to the norm

$$\|F\|_{B,k,p}^p = \mathbb{E}|F|^p + \sum_{j=1}^k \|(D^B)^{(j)} F\|_{L^p(\Omega;\mathcal{H}^{\otimes j})}^p.$$

Consider the adjoint δ^B of D^B. Its domain is the class of $u \in L^2(\Omega; \mathcal{H})$ such that

$$\mathbb{E}|\langle D^B F, u \rangle_{\mathcal{H}}| \leq C\|F\|_2 \text{ for all } F \in \mathbb{D}_B^{1,2}$$

and δ^B is the element of $L^2(\Omega)$ given by

$$\mathbb{E}(\delta^B(u)F) = \mathbb{E}\langle D^B F, u\rangle_{\mathcal{H}} \text{ for every } F \text{ smooth.}$$

The following relations will relate the derivation and the Skorohod integration with respect to W and B.

(i) $\mathbb{D}^{k,p} = \mathbb{D}_B^{k,p}$, and $K_1^* D^B F = DF$, for any $F \in \mathbb{D}^{k,p}$.

(ii) $Dom(\delta^B) = (K_1^*)^{-1}(Dom(\delta)$, and $\delta^B(u) = \delta(K_1^* u)$ for any $\mathcal{H}$-valued random variable u in $Dom(\delta^B)$, where

$$(K_1^*\varphi)(s) = \int_s^1 \varphi(r)\frac{\partial K}{\partial r}(r,s)dr. \tag{6}$$

We have the integration by parts formula, provided that all terms have sense,

$$F\delta^B(u) = \delta^B(Fu) + \langle D^B F, u\rangle_{\mathcal{H}}. \tag{7}$$

Throughout this paper, we denote by $\|\cdot\|_{HS}$ the Hilbert-Schmidt norm.

3. Infinite-dimensional stochastic analysis

Consider T the unit interval and let U be a real separable Hilbert space. We consider Q a nuclear, self-adjoint and positive operator on U ($Q \in L_1(U), Q = Q^\star > 0$). It is well-known that Q admits a sequence $(\lambda_j)_{j\geq 1}$ of eigenvalues with $0 < \lambda_j \searrow 0$ and $\sum_{j\geq 1}\lambda_j < \infty$. Moreover, the corresponding eigenvectors $(e_j)_{j\geq 1}$ form an orthonormal basis in U.

A process $(X_t)_{t\in T}$ with values in U is called Gaussian if for every $t_1, \dots, t_n \in T$ and $u_1, \dots, u_n \in U$ the real random variable $\sum_{j=1}^n \langle u_j, X_{t_j}\rangle_U$ has a normal distribution.

We define the expectation m_X of X as $m_X : T \to U, m_X(t) = \mathbb{E}[X_t]$ and the covariance $C_X : T^2 \to L_1(U)$ by, for every $s,t \in T$, $\langle C_X(t,s)u, v\rangle_U = \mathbb{E}[\langle X_t - m_X(t), v\rangle_U \langle X_s - m_X(s), u\rangle_U]$.

Definition 1. We call $(\mathbf{W}_t)_{t\in T}$ a Q-Wiener process for the filtration $\mathcal{F}_s$ if $\mathbf{W}$ is a U -valued process and the following properties hold:

i) $\mathbf{W}_0 = 0$ and $\mathbf{W}$ has continuous trajectories.

ii) For every $s < t$, $\mathbf{W}_t - \mathbf{W}_s$ is independent of $\mathcal{F}_s$ and $\mathbf{W}_t - \mathbf{W}_s \in N(0,(t-s)Q)$.

We have equivalent definitions for a Q-Wiener process (see [17]).

Theorem 1. *Let* $(\boldsymbol{W}_t)_{t\in T}$ *a* U*-valued process such that* $\boldsymbol{W}_0 = 0$ *and* $\boldsymbol{W}$ *has continuous trajectories. The following are equivalent*:

i) $\boldsymbol{W}_t$ *is a* Q*-Wiener process.*

ii) $\boldsymbol{W}_t$ *is a centered Gaussian process with covariance*

$$C_{\boldsymbol{W}}(t,s) = min(t,s)Q.$$

iii) *There exist real and independent Brownian motions* $\{(\beta_j(t))_{t\in T})\}_{j\geq 1}$ *such that*

$$\boldsymbol{W}_t = \sum_{j\geq 1} \sqrt{\lambda_j}\beta_j(t)e_j. \tag{8}$$

Let H be another real separable Hilbert space with $(h_k)_k$ an orthonormal system in H and put $U_0 = Q^{\frac{1}{2}}(U) \subset U$. We endow U_0 with the norm

$$\|u\|_0 = \|Q^{-\frac{1}{2}}(u)\|_U.$$

Then $(U_0, \|\cdot\|_0)$ is a separable Hilbert space and $(\sqrt{\lambda_j}e_j)_{j\geq 1}$ is a orthonormal basis in U_0.

For a $L_2(U_0; H)$-valued, adapted process with $\mathbb{E}\left(\int_0^1 \|\Phi_s\|^2_{HS} ds\right) < \infty$, the Itô stochastic integral of Φ with respect to $\mathbf{W}$ is defined as being the H-valued process $(I^\Phi(t))_{t\in T}$ given by

$$I^\Phi(t) = \sum_{j\geq 1} \sqrt{\lambda_j} I_j^\Phi(t) \tag{9}$$

with

$$I_j^\Phi(t) = \sum_{k\geq 1} \left(\int_0^t \langle \Phi_s e_j, h_k\rangle d\beta_j(s) \right) h_k$$

and the two series above are convergent in $L^2(\Omega; H)$, uniformly on T. Note that I^Φ is a martingale on H (see [15] or [17] for the definition).

We will also recall some notions of the non-adapted stochastic calculus with respect to $\mathbf{W}$ following [10]. A different construction, based on chaos expansion, was given in [11]. A Malliavin type derivative operator and a Skorohod integral with respect to the infinite-dimensional Brownian motion $(\mathbf{W}_t)_{t\in T}$ has been introduced in [10] as follows. For $h \in L^2(T; U_0)$ and if $\mathbf{W}$ is given by (8) we denote by $\mathbf{W}(h) = \int_0^1 h(s) d\mathbf{W}_s$ the random variable

$$\mathbf{W}(h) := \sum_{j\geq 1} \sqrt{\lambda_j} \int_0^1 \langle h(s), e_j\rangle_{U_0} d\beta_j(s). \tag{10}$$

Note that the above quantity (10) is a well-defined random variable in $L^2(\Omega)$ since for every $h \in L^2(T; U_0)$

$$\begin{aligned}\mathbb{E}\left(\mathbf{W}(h)\right)^2 &= \sum_{j\geq 1} \int_0^1 \langle h(s), \sqrt{\lambda_j} e_j\rangle^2_{U_0} ds \\ &= \int_0^1 \|h(s)\|^2_{U_0} ds = \|h\|^2_{L^2(T;U_0)}.\end{aligned}$$

Consider $\mathbf{S}_H$ the subspace of $L^2(\Omega; H)$ of smooth functionals

$$F = \sum_{p=1}^m f_p(\mathbf{W}(h_{p1}), \ldots, \mathbf{W}(h_{pn_p}))k_p$$

where $m, n_1, \ldots, n_m \in \mathbb{N}$, $k_1, \ldots, k_m \in H$, $f_p \in C_b^\infty(\mathbb{R}^{n_p})$ for all $p = 1, \ldots, m$ and $h_{pi} \in L^2(T; U_0)$ for all $i = 1, \ldots, n_p$. Then the derivative of F is the $L^2(U_0; H)$-valued process $\{\mathbf{D}_t F; t \in T\}$ with

$$\mathbf{D}_t F = \sum_{p=1}^{m} \sum_{i=1}^{n_p} \frac{\partial f_p}{\partial x_i}(\mathbf{W}(h_{p1}), \ldots, \mathbf{W}(h_{pn_p}))k_p \otimes h_{pi}(t).$$

The operator $\mathbf{D}$ can be extended to the closure of $\mathbf{S}_H$ with respect to the norm

$$\|F\|_{1,2}^2 = \mathbb{E}\|F\|_H^2 + \mathbb{E}\int_0^1 \|\mathbf{D}_s F\|_{L_2(U_0;H)}^2 ds.$$

We define the Skorohod integral $\boldsymbol{\delta}(\Phi)$ of a process $\Phi \in L^2(T \times \Omega; L_2(U_0; H))$ as being the H-valued random variable $\boldsymbol{\delta}(\Phi)$ characterized by the following duality

$$\mathbb{E}\langle F, \boldsymbol{\delta}(\Phi)\rangle_H = \mathbb{E}\int_0^1 \langle \mathbf{D}_s F, \Phi_s\rangle_{HS} ds \tag{11}$$

and this operator is well-defined on the set of processes $\Phi \in L^2(T\times\Omega; L_2(U_0; H))$ such that

$$\left|\mathbb{E}\int_0^1 \langle \mathbf{D}_s F, \Phi_s\rangle_{HS} ds\right| \leq C\|F\|_{L^2(\Omega;H)}.$$

When $\mathbf{W}$ is a real one-dimensional Brownian motion, the operators $\mathbf{D}$ and $\boldsymbol{\delta}$ coincide with the ones defined in Section 2.1.

We denote by $L^{1,2,Q}(H)$ the Hilbert space of processes Φ belonging to $L^2(T \times \Omega; L_2(U_0; H))$ with the norm

$$\|\Phi\|_{1,2,Q}^2 = \mathbb{E}\int_0^1 \|\Phi_s\|_{HS}^2 ds + \mathbb{E}\int_0^1\int_0^1 \|D_\alpha \Phi_s\|_{HS}^2 d\alpha ds.$$

By [10, Prop. 3.2.], this space is included in the domain of δ.

Remark 1. Observe that $L^{1,2,Q}(H)$ can be defined also as the class of stochastic processes $(\Phi_s)_{s\in T}$ with values in $L_2(U_0, H)$ such that $\Phi_s \in \mathbb{D}^{1,2}(L_2(U_0, H))$ for every $s \in T$ and

$$\sum_{j\geq 1} \lambda_j \mathbb{E}\int_0^1 \|\Phi_s e_j\|_H^2 ds < \infty \tag{12}$$

and

$$\sum_{j\geq 1} \lambda_j \mathbb{E}\int_0^1\int_0^1 \|(D_\alpha \Phi_s)e_j\|_H^2 d\alpha ds < \infty. \tag{13}$$

The next result shows that the Skorohod integral of [10] can be defined, for enough regular non-adapted integrands, in the same way as the Itô integral (9).

Proposition 1. *Let $\Phi \in L^{1,2,Q}(H)$. The following properties hold:*
i) *For every $j \geq 1$, the following series converge in $L^2(\Omega; H)$*

$$S_j^\Phi = \sum_{k\geq 1} \left(\int_0^1 \langle \Phi_s e_j, h_k \rangle_H d\beta_j(s) \right) h_k \text{ and } S^\Phi = \sum_{j\geq 1} \sqrt{\lambda_j} S_j^\Phi; \tag{14}$$

ii) *S^Φ coincides with the Skorohod integral $\boldsymbol{\delta}(\Phi)$ given by (11).*

Proof. i) We note first that the real Skorohod integral exists. Indeed,

$$\mathbb{E} \int_0^1 \langle \Phi_s e_j, h_k \rangle_H^2 ds \leq \int_0^t \|\Phi_s e_j\|_H^2 ds \leq \frac{1}{\lambda_j} \mathbb{E} \int_0^1 \sum_{j\geq 1} \lambda_j \|\Phi_s e_j\|_H^2 ds < \infty$$

and

$$\mathbb{E} \int_0^1 \int_0^1 (D_\alpha \langle \Phi_s e_j, h_k \rangle_H)^2 d\alpha ds = \mathbb{E} \int_0^1 \int_0^1 \langle D_\alpha (\Phi_s e_j), h_k \rangle_H^2 d\alpha ds$$
$$= \mathbb{E} \int_0^1 \int_0^1 \langle (D_\alpha \Phi_s) e_j, h_k \rangle_H^2 d\alpha ds \leq \mathbb{E} \int_0^1 \int_0^1 \|(D_\alpha \Phi_s) e_j\|_H^2 d\alpha ds < \infty.$$

We will use the notation $\langle \Phi_s e_j, h_k \rangle_H = \Phi_{j,k}(s)$. The series (14) converges in $L^2(\Omega)$ because

$$\mathbb{E} \| \sum_k \int_0^1 \Phi_{j,k}(s) d\beta_j(s) h_k \|^2 = \sum_k \mathbb{E} \left(\int_0^t \Phi_{j,k}(s) d\beta_j(s) \right)^2$$
$$\leq \sum_k \mathbb{E} \int_0^1 \Phi_{j,k}(s)^2 ds + \sum_k \mathbb{E} \int_0^1 \int_0^1 (D_\alpha \Phi_{j,k}(s))^2 d\alpha ds.$$

and this is finite since

$$\sum_{k\geq 1} \mathbb{E} \int_0^1 \Phi_{j,k}(s)^2 ds = \mathbb{E} \int_0^1 \|\Phi_s e_j\|_H^2 ds < \infty$$

and

$$\sum_{k\geq 1} \mathbb{E} \int_0^1 \int_0^1 (D_\alpha \Phi_{j,k}(s))^2 d\alpha ds = \mathbb{E} \int_0^1 \int_0^1 \|(D_\alpha \Phi_s) e_j\|_H^2 d\alpha ds < \infty.$$

Concerning the second sum in (14), we observe first that, for $j \neq l$, $\mathbb{E}[S_j^\Phi(t) S_l^\Phi(t)] = 0$ and then, due to the nuclearity of Q

$$\mathbb{E} \| \sum_j \sqrt{\lambda_j} S_j^\Phi(t) \|_H^2 = \sum_j \lambda_j \mathbb{E}(S_j^\Phi(t))^2 \leq \|\Phi\|_{L^{1,2,Q}}^2 < \infty.$$

ii) The next step is to prove that, for $\Phi \in L^{1,2,Q}$ and $F \in \mathbf{S}_H$, S^Φ and F satisfy the duality relation

$$\mathbb{E} \langle S^\Phi, F \rangle_H = \mathbb{E} \int_0^1 \langle \Phi_s, \mathbf{D}_s F \rangle_{HS} ds. \tag{15}$$

For simplicity, take $F = f(\mathbf{W}(h))u$, with $h \in L^2(T; U_0)$ and $u \in H$. We have

$$\mathbf{D}_t F = f'(\mathbf{W}(h)) h_t \otimes u. \tag{16}$$

Therefore it holds

$$\begin{aligned}
\langle \mathbf{D}_s F, \Phi_s \rangle_{L_2(U_0,H)} &= \sum_j \lambda_j \langle (\mathbf{D}_s F)(e_j), \Phi_s e_j \rangle_H \\
&= \sum_j \lambda_j \langle f'(\mathbf{W}(h))(h_s \otimes u)(e_j), \Phi_s e_j \rangle_H \\
&= \sum_j \lambda_j f'(\mathbf{W}(h)) \langle h_s, e_j \rangle_{U_0} \langle u, \Phi_s e_j \rangle_H.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\mathbb{E}\langle S^\Phi, F \rangle_H &= \sum_{j,k\geq 1} \sqrt{\lambda_j}\, \mathbb{E}\left(\int_0^1 \langle \Phi_s e_j, h_k \rangle_H d\beta_j(s) \right) f(\mathbf{W}(h)) \langle u, h_k \rangle_H \\
&= \sum_{j,k\geq 1} \sqrt{\lambda_j}\, \mathbb{E}\left(\int_0^1 D_s^j f(\mathbf{W}(h)) \langle \Phi_s e_j, h_k \rangle ds \right) \langle u, h_k \rangle_H \\
&= \sum_{j\geq 1} \sqrt{\lambda_j}\, \mathbb{E}\left(\int_0^1 D_s^j f(\mathbf{W}(h)) \langle \Phi_s e_j, u \rangle_H ds \right),
\end{aligned}$$

where D^j denotes the Malliavin derivative with respect to the Brownian motion β_j. Thus the duality follows observing that

$$D_s^j f(\mathbf{W}(h)) = f'(\mathbf{W}(h)) D_s^j(\mathbf{W}(h)) = f'(\mathbf{W}(h)) \sqrt{\lambda_j} \langle h_s, e_j \rangle_{U_0}.$$

The relation (15), implies that Φ belongs to $Dom(\boldsymbol{\delta})$ and thus $L^{1,2,Q} \subset Dom(\boldsymbol{\delta})$ and S^Φ coincides with $\boldsymbol{\delta}(\Phi)$. □

4. Infinite-dimensional fBm and Itô formula

With the notation of Section 3 we introduce a fBm $\mathbf{B}^h$ on U as follows.

Definition 2. We say that the U-valued process $(\mathbf{B}_t^h)_{t\in T}$ is an infinite-dimensional fractional Brownian motion (or a $Q - fbm$) if $\mathbf{B}^h$ is a centered Gaussian process with covariance

$$C_{\mathbf{B}^h}(t,s) = R(t,s)Q.$$

Proposition 2. *$\boldsymbol{B}^h$ is a $Q - fBm$ if and only if there exists a sequence $(\beta_j^h)_{j\geq 1}$ of real and independent fBm such that*

$$\boldsymbol{B}_t^h = \sum_{j\geq 1} \sqrt{\lambda_j} \beta_j^h(t) e_j \tag{17}$$

where the series converges in $L^2(\Omega; U)$.

Proof. We refer to [7] or [9] for the fact that $\sum_{j\geq 1}\sqrt{\lambda_j}\beta_j^h(t)e_j$ is a fBm in the sense of Definition 2. Conversely, let $\mathbf{B}^h$ be a centered Gaussian process on U with covariance $R(t,s)Q$. Put $\beta_j^h(t)=\frac{1}{\sqrt{\lambda_j}}\langle \mathbf{B}_t^h, e_j\rangle_U$. Then β_j^h is a Gaussian process since

$$\sum_{l=1}^n \alpha_l\beta_j^h(t_l)=\sum_{l=1}^n\langle \mathbf{B}_{t_l}^h,\frac{\alpha_l}{\sqrt{\lambda_j}}e_j\rangle$$

for every $\alpha_1,\dots,\alpha_n\in\mathbb{R}$ and $t_1,\dots,t_n\in T$. Moreover, for every $i,j\geq 1$, $s,t\in T$, we have

$$\mathbb{E}\left(\beta_i^h(s)\beta_j^h(t)\right)=\frac{1}{\sqrt{\lambda_i\lambda_j}}\mathbb{E}\left(\langle \mathbf{B}_s^h,e_i\rangle_U\langle \mathbf{B}_t^h,e_j\rangle_U\right)=R(t,s)\delta_{ij}$$

and it implies that $\mathbb{E}\left(\beta_i^h(s)\beta_i^h(t)\right)=R(t,s)$ and for $i\neq j$, the random variables $\beta_i^h(s)$ and $\beta_j^h(t)$ are uncorrelated, thus independent. We finish the proof by noting that the sum $\sum_{j\geq 1}\sqrt{\lambda_j}\beta_j^h(t)e_j$ converges in $L^2(\Omega;U)$. □

Remark 2. Note that the process $\mathbf{B}^h$ always has a continuous version (see [15, Prop. 3.15.]).

Remark 3. When the covariance operator Q is not nuclear we can still introduce an infinite dimensional fBm with covariance Q. As before, let U real separable Hilbert space and $Q\in L(U)$, $Q=Q^*>0$ (Q is not necessary nuclear, in particular Q may be the identity operator). Let $U_1\supset U_0$ be an other real and separable Hilbert space and $(g_j)_j$ an orthonormal basis in U_0 such that the mapping $J:(U_0,\|\cdot\|_0)\to(U_1,\|\cdot\|_1)$ is a Hilbert-Schmidt operator, i.e. $\sum_{j\geq 1}\|Jg_j\|_1^2<\infty$. Consider the operator $Q_1=JJ^*:U_1\to U_1$ which is nuclear, positive and self-adjoint and let $(\beta_j^h)_{j\geq 1}$ be real and independent fBm with $h\in(0,1)$. Then the U_1- valued process

$$B_t^h=\sum_{j\geq 1}(Jg_j)\beta_j^h(t) \tag{18}$$

is an Q_1-fBm, provided that the series (18) converges in $L^2(\Omega)$.

Let us fix $h\in(0,1)$ and consider $\mathbb{D}_K^{1,2,Q}(H)$ the class of processes Φ with values in $L_2(U_0;H)$ such that

$$\sum_{j\geq 1}\lambda_j\mathbb{E}\,\|\|\Phi e_j\|_H\|_{\mathcal{H}}^2<\infty$$

and

$$\sum_{j\geq 1}\lambda_j\mathbb{E}\,\|\|(D\Phi)e_j\|_H\|_{\mathcal{H}\otimes\mathcal{H}}^2<\infty.$$

Definition 3. If $\mathbf{B}^h$ is an infinite dimensional fBm in the form (17) and Φ is a process in the space $\mathbb{D}_K^{1,2,Q}(H)$, then we define

$$\int_0^1 \Phi_s d\mathbf{B}_s^h = \sum_{j\geq 1} \sqrt{\lambda_j} T^j(\Phi) \tag{19}$$

where

$$T^j(\Phi) = \sum_{k\geq 1} \left(\int_0^t \langle \Phi_s e_j, h_k \rangle d\beta_j^h \right) h_k. \tag{20}$$

As in the proof of Proposition 1, one can show that the sums (19) and (20) are finite for $\Phi \in \mathbb{D}_K^{1,2,Q}(H)$.

We prove now the Itô formula for the U-valued fBm with nuclear covariance Q.

Theorem 2. *Let $(\boldsymbol{B}_t^h)_{t\in T}$ a U-valued fBm with $h \in (\frac{1}{2}, 1)$ and let $F : U \to \mathbb{R}$, $F \in C_b^3(U)$ such that F', F'' are uniformly continuous. Then it holds*

$$F(\boldsymbol{B}_t^h) = F(0) + \int_0^t F'(\boldsymbol{B}_s^h) d\boldsymbol{B}_s^h + h \int_0^t F''(\boldsymbol{B}_s^h) s^{2h-1} ds$$

where the last term is defined as

$$\int_0^t F''(\boldsymbol{B}_s^h) s^{2h-1} ds := \sum_{j\geq 1} \lambda_j \int_0^t F''(\boldsymbol{B}_s^h)(e_j)(e_j) s^{2h-1} ds.$$

Proof. Let $\pi : 0 = t_0 < t_1 < \cdots < t_n = t$ denote a partition of the interval $[0, t]$. We write the following version of the Taylor formula in the differential calculus

$$\begin{aligned} F(\mathbf{B}_t^h) = F(0) &+ \sum_{i=0}^{n-1} F'(\mathbf{B}_{t_i}^h)(\mathbf{B}_{t_{i+1}}^h - \mathbf{B}_{t_i}^h) \\ &+ \frac{1}{2} \sum_{i=0}^{n-1} F''(\bar{\mathbf{B}}^h{}_{t_i})(\mathbf{B}_{t_{i+1}}^h - \mathbf{B}_{t_i}^h)(\mathbf{B}_{t_{i+1}}^h - \mathbf{B}_{t_i}^h) \end{aligned}$$

where $\bar{\mathbf{B}}^h{}_{t_i}$ is located between $\mathbf{B}_{t_i}^h$ and $\mathbf{B}_{t_{i+1}}^h$. Using the definition (17) of $\mathbf{B}^h$

and the linearity of $F'(x)$, we obtain

$$F(\mathbf{B}^h_t) = F(0) + \sum_{i=0}^{n-1} F'(\mathbf{B}^h_{t_i}) \left(\sum_{j\geq 1} \sqrt{\lambda_j} (\beta^h_j(t_{i+1}) - \beta^h_j(t_i)) e_j \right)$$
$$+ \frac{1}{2} \sum_{i=0}^{n-1} F''(\bar{\mathbf{B}}^h{}_{t_i})(\mathbf{B}^h_{t_{i+1}} - \mathbf{B}^h_{t_i})(\mathbf{B}^h_{t_{i+1}} - \mathbf{B}^h_{t_i})$$
$$= F(0) + \sum_{j\geq 1} \sqrt{\lambda_j} \sum_{i=0}^{n-1} (\beta^h_j(t_{i+1}) - \beta^h_j(t_i)) F'(\mathbf{B}^h_{t_i})(e_j)$$
$$+ \frac{1}{2} \sum_{i=0}^{n-1} F''(\bar{\mathbf{B}}^h{}_{t_i})(\mathbf{B}^h_{t_{i+1}} - \mathbf{B}^h_{t_i})(\mathbf{B}^h_{t_{i+1}} - \mathbf{B}^h_{t_i}).$$

By the integration by parts formula (7), it holds

$$F(\mathbf{B}^h_t) = F(0) + \sum_{j\geq 1} \sqrt{\lambda_j} \sum_{i=0}^{n-1} \delta^{h,j} \left((1_{(t_i,t_{i+1}]}(\cdot)) F'(\mathbf{B}^h_{t_i})(e_j) \right)$$
$$+ \sum_{j\geq 1} \sqrt{\lambda_j} \sum_{i=0}^{n-1} \langle D_\cdot^{h,j} F'(\mathbf{B}^h_{t_i})(e_j), 1_{(t_i,t_{i+1}]}(\cdot) \rangle_{\mathcal{H}}$$
$$+ \frac{1}{2} \sum_{i=0}^{n-1} F''(\bar{\mathbf{B}}^h{}_{t_i})(\mathbf{B}^h_{t_{i+1}} - \mathbf{B}^h_{t_i})(\mathbf{B}^h_{t_{i+1}} - \mathbf{B}^h_{t_i})$$

where $\delta^{h,j}$ and $D^{h,j}$ denotes respectively the Skorohod integral and the Malliavin derivative with respect to the real fractional Brownian motion β^h_j. Since, by (3)

$$D^{h,j}_\alpha F'(\mathbf{B}^h_{t_i})(e_j) = F''(\mathbf{B}^h_{t_i}) D^{h,j}_\alpha \mathbf{B}^h_{t_i}(e_j),$$
$$D^{h,j}_\alpha \mathbf{B}^h_{t_i} = \sum_{k\geq 1} \sqrt{\lambda_k} D^{h,j}_\alpha \beta^h_k(t_i)) e_k = \sqrt{\lambda_j} 1_{[0,t_i]}(\alpha) e_j$$

we will have

$$F(\mathbf{B}^h_t) = F(0) + \sum_{j\geq 1} \sqrt{\lambda_j} \sum_{i=0}^{n-1} \delta^{h,j} \left(1_{(t_i,t_{i+1}]}(\cdot) F'(\mathbf{B}^h_{t_i})(e_j) \right)$$
$$+ \sum_{j\geq 1} \lambda_j \sum_{i=0}^{n-1} \langle 1_{(0,t_i]}, 1_{(t_i,t_{i+1}]} \rangle_{\mathcal{H}} F''(\mathbf{B}^h_{t_i})(e_j)(e_j)$$
$$+ \frac{1}{2} \sum_{i=0}^{n-1} F''(\bar{\mathbf{B}}^h{}_{t_i})(\mathbf{B}^h_{t_{i+1}} - \mathbf{B}^h_{t_i})(\mathbf{B}^h_{t_{i+1}} - \mathbf{B}^h_{t_i}).$$

Notice that, if $R(t) := R(t,t)$,

$$\langle 1_{(0,t_i]}, 1_{(t_i,t_{i+1}]} \rangle_{\mathcal{H}} = R(t_i, t_{i+1}) - R(t_i) = \frac{1}{2}(t^{2h}_{i+1} - t^{2h}_i) - \frac{1}{2}(t_{i+1} - t_i)^{2h}$$

and, therefore, the last sum becomes

$$
\begin{aligned}
F(\mathbf{B}^h_t) \;=\; & F(0) + \sum_{j\geq 1}\sqrt{\lambda_j}\sum_{i=0}^{n-1}\delta^{h,j}\left(1_{(t_i,t_{i+1}]}(\cdot)F'(\mathbf{B}^h_{t_i})(e_j)\right)\\
&+\frac{1}{2}\sum_{j\geq 1}\lambda_j\sum_{i=0}^{n-1}F''(\mathbf{B}^h_{t_i})(e_j)(e_j)\left(R_{t_{i+1}}-R_{t_i}\right)\\
&-\frac{1}{2}\sum_{j\geq 1}\lambda_j\sum_{i=0}^{n-1}F''(\mathbf{B}^h_{t_i})(e_j)(e_j)(t_{i+1}-t_i)^{2h}\\
&+\frac{1}{2}\sum_{i=0}^{n-1}F''(\bar{\mathbf{B}}^h{}_{t_i})(\mathbf{B}^h_{t_{i+1}}-\mathbf{B}^h_{t_i})(\mathbf{B}^h_{t_{i+1}}-\mathbf{B}^h_{t_i})\\
:=\; & F(0)+T_1+T_2+T_3+T_4.
\end{aligned}
$$

Step 1: We regard the stochastic integral term

$$
T_1 = \sum_{j\geq 1}\sqrt{\lambda_j}\sum_{i=0}^{n-1}\delta^{h,j}\left(1_{(t_i,t_{i+1}]}(\cdot)F'(\mathbf{B}^h_{t_i})(e_j)\right) \tag{21}
$$

and we will show its convergence to

$$
\sum_{j\geq 1}\sqrt{\lambda_j}\delta^{h,j}\left(F'(\mathbf{B}^h_\cdot)(e_j)1_{[0,t]}(\cdot)\right) = \sum_{j\geq 1}\sqrt{\lambda_j}\int_0^t F'(\mathbf{B}^h_\cdot)(e_j)d\beta^h_j(s)
$$

in $L^2(\Omega)$ as $|\pi|\to 0$. First, we prove the $L^2(\Omega)\otimes\mathcal{H}$-convergence to 0, as $|\pi|\to 0$ of the sum

$$
\sum_{i=0}^{n-1}1_{(t_i,t_{i+1}]}(s)\left(F'(\mathbf{B}^h_{t_i})-F'(\mathbf{B}^h_s)\right)(e_j).
$$

We compute

$$
\begin{aligned}
&\mathbb{E}\left\|\sum_{i=0}^{n-1}1_{(t_i,t_{i+1}]}(s)\left(F'(\mathbf{B}^h_{t_i})-F'(\mathbf{B}^h_s)\right)(e_j)\right\|^2_{\mathcal{H}}\\
&=\mathbb{E}\left\langle\sum_{i=0}^{n-1}1_{(t_i,t_{i+1}]}(s)\left(F'(\mathbf{B}^h_{t_i})-F'(\mathbf{B}^h_s)\right)(e_j),\right.\\
&\qquad\left.\sum_{l=0}^{n-1}1_{(t_l,t_{l+1}]}(s)\left(F'(\mathbf{B}^h_{t_l})-F'(\mathbf{B}^h_s)\right)(e_j)\right\rangle_{\mathcal{H}}.
\end{aligned}
$$

Since

$$
\left|\left(F'(\mathbf{B}^h_{t_i})-F'(\mathbf{B}^h_s)\right)(e_j)\right|\leq\|F''\|_\infty\|\mathbf{B}^h_{t_i}-\mathbf{B}^h_s\|_U
$$

using the form of the scalar product (5), the last sum will be lesser than

$$\|F''\|_\infty^2 \sup_{|a-b|\le|\pi|} \|\mathbf{B}_b^h - \mathbf{B}_a^h\|_U^2 \sum_{i,l=0}^{n-1} \langle 1_{(t_i,t_{i+1}]}, 1_{(t_l,t_{l+1}]}\rangle_{\mathcal{H}}$$

and that goes to 0 using the continuity of $\mathbf{B}^h$ and $\sum_{i,l=0}^{n-1}\langle 1_{(t_i,t_{i+1}]}, 1_{(t_l,t_{l+1}]}\rangle_{\mathcal{H}} = t^{2h}$.

Now, let's regard the convergence of the derivative of the sum

$$\sum_{i=0}^{n-1} 1_{(t_i,t_{i+1}]}(s)\left(F'(\mathbf{B}_{t_i}^h) - F'(\mathbf{B}_s^h)\right)(e_j)$$

in $L^2(\Omega)\otimes\mathcal{H}\otimes\mathcal{H}$-convergence to 0, as $|\pi|\to 0$. We have, by (3)

$$\begin{aligned} &D_\alpha^{h,j}\left(\sum_{i=0}^{n-1} 1_{(t_i,t_{i+1}]}(s)F'(\mathbf{B}_{t_i}^h)(e_j)\right)\\ &\quad= \sqrt{\lambda_j}\sum_{i=0}^{n-1} 1_{(t_i,t_{i+1}]}(s)1_{(0,t_i)}(\alpha)F''(\mathbf{B}_{t_i}^h)(e_j)(e_j),\\ &D_\alpha^{h,j}\left(F'(\mathbf{B}_s^h)(e_j)\right) = \sqrt{\lambda_j}1_{(0,s)}(\alpha)F''(\mathbf{B}_s^h)(e_j)(e_j). \end{aligned}$$

Therefore

$$\begin{aligned} &\mathbb{E}\left\| D^{h,j}\left(\sum_{i=0}^{n-1} 1_{(t_i,t_{i+1}]}(s)\left(F'(\mathbf{B}_{t_i}^h) - F'(\mathbf{B}_s^h)\right)(e_j)\right)\right\|_{\mathcal{H}\otimes\mathcal{H}}^2\\ &\le 2\lambda_j\mathbb{E}\left\|\sum_{i=0}^{n-1} 1_{(t_i,t_{i+1}]}(s)1_{(0,t_i)}(\alpha)\left(F''(\mathbf{B}_{t_i}^h) - F''(\mathbf{B}_s^h)\right)(e_j)(e_j)\right\|_{\mathcal{H}\otimes\mathcal{H}}^2\\ &\quad + 2\lambda_j\mathbb{E}\left\|\sum_{i=0}^{n-1} 1_{(t_i,t_{i+1}]}(s)1_{(t_i,s)}(\alpha)\left(F''(\mathbf{B}_s^h)\right)(e_j)(e_j)\right\|_{\mathcal{H}\otimes\mathcal{H}}^2 = A_1 + A_2. \end{aligned}$$

The first summand A_1 goes to 0 from the following bounds

$$\begin{aligned} A_1 &= 2\lambda_j\|F'''\|_\infty^2 \sup_{|a-b|\le|\pi|}\|\mathbf{B}_b^h - \mathbf{B}_a^h\|_U^2 \sum_{i,l=0}^{n-1}\langle 1_{(t_i,t_{i+1}]}, 1_{(t_l,t_{l+1}]}\rangle_{\mathcal{H}}\langle 1_{(0,t_i]}, 1_{(0,t_l]}\rangle_{\mathcal{H}}\\ &\le 2\lambda_j t^{2h}\|F'''\|_\infty^2 \sup_{|a-b|\le|\pi|}\|\mathbf{B}_b^h - \mathbf{B}_a^h\|_U^2 \end{aligned}$$

since $\langle 1_{(0,t_i]}, 1_{(0,t_l]}\rangle_{\mathcal{H}} \le 1$ and $\sum_{i,l=0}^{n-1}\langle 1_{(t_i,t_{i+1}]}, 1_{(t_l,t_{l+1}]}\rangle_{\mathcal{H}} = t^{2h}$. For the second summand A_2, using the definition of the scalar product in $\mathcal{H}\otimes\mathcal{H}$, we have

$$A_2 \le 2\lambda_j\sum_{i,l=0}^{n-1}\langle 1_{(t_i,t_{i+1}]}, 1_{(t_l,t_{l+1}]}\rangle_{\mathcal{H}}^2 \le 2\lambda_j\left(\sum_{i=0}^{n-1}|t_{i+1}-t_i|^{2h}\right)^2 \to 0.$$

Step 2: Clearly the term T_2 converges in $L^1(\Omega)$ to

$$\int_0^t F''(\mathbf{B}^h_s)dR_s = \sum_{j\geq 1}\lambda_j \int_0^t F''(\mathbf{B}^h_s)(e_j)(e_j)dR_s.$$

Step 3: Recall that $h > \frac{1}{2}$. It is easy to observe that T_3 converges to 0 since

$$\mathbb{E}|T_3| \leq \frac{1}{2}(TrQ)\|F''\|_\infty \sum_{i=0}^{n-1}(t_{i+1}-t_i)^{2h} \leq \frac{1}{2}(TrQ)\|F''\|_\infty |\pi|^{2h-1}.$$

We finally study the term T_4. We can write, since by hypothesis for every $x \in U$, $F'(x)$ is a bounded continuous operator in $L_2(U^2;\mathbb{R})$,

$$\begin{aligned}\mathbb{E}\,|T_4| &\leq \frac{1}{2}\|F''\|_\infty \mathbb{E}\sum_{i=0}^{n-1}\|\mathbf{B}^h_{t_{i+1}} - \mathbf{B}^h_{t_i}\|^2_U \\ &\leq \frac{1}{2}\|F''\|_\infty Tr(Q)\sum_{i=0}^{n-1}(t_{i+1}-t_i)^{2h} \to_{|\pi|\to 0} 0.\end{aligned}$$

This finishes the proof. □

Remark 4. The proof of Theorem 2 can be also applied to the one-dimensional case and it is an alternative proof to the one given in [2]. The study of the indefinite integral process (continuity of the paths, Itô formula) can be done in the Hilbert space-valued situation without difficulty, following the lines of the one-dimensional case. Recently, in [4], the authors extended the divergence integral with respect to fBm for any parameter $h \in (0,1)$. We think that their approach can be used in the infinite-dimensional context.

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