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On the basin problem for Hénon-like attractors

By

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Abstract

The basin problem for a strange attractor asks the asymptotic distribution of Lebesgue almost every initial point in the basin of attraction. A solution to this problem for Hénon-like attractors was initially given by Benedicks-Viana, and later by Wang-Young, under certain assumptions on the Jacobian of the map, which are used in a crucial way to control the growth of volumes under iteration. The purpose of this paper is to remove the assumption on the Jacobian in their solutions, in a hope that the argument can be extended to a broader class of Hénon-like maps which are not necessarily invertible and possess singularities.

1. Introduction

In [10], Mora and Viana isolated a class of parameter families of diffeomorphisms which they call *Hénon-like*, as an abstract model of the renormarization in generic one-parameter families of surface diffeomorphisms unfolding homoclinic tangencies associated with dissipative saddles [11]. Recall that the Hénon-like family $(H_{a,b})$ is a two parameter family of planar diffeomorphisms such that

1. $(a,b,x,y) \to H_{a,b}(x,y)$ is continuous and $(a,x,y) \to H_{a,b}(x,y)$ is C^3 for any b.
2. there exists a constant J independent of b such that
   (a) $H_{a,b}$ has the following form:

$$H_{a,b}(x,y) = (1-ax^2,0) + R(x,y,a,b), \ \|R(x,y,a,b)\|_{C^3} \leq J\sqrt{b}.$$

   (b) for any (a,b), $b \neq 0$,

$$J^{-1}b \leq |\det DH_{a,b}| \leq Jb \text{ and } \|D\log|\det DH_{a,b}|\| \leq J.$$

They proved the abundance of strange attractors in this family around parameter values close to $(2,0)$, by extending the pioneering work of Benedicks-Carleson [2]. For this type of attractors Benedicks and Viana [3][*1] solved the *basin problem*, that is, the asymptotic distribution of Lebesgue almost every

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[*1] This paper appeared in 2001 but the result had been announced in 1995. See [18].

initial point in the basin of attraction coincides with the ergodic SRB measure, which is proved to exist by Benedicks-Young [4], [5]. In their argument on the basin problem, the assumption (b) which we call *homogeneity* is used at two crucial metric estimates: deducing that *unstable sides are roughly parallel*, and obtaining *area distortion bounds* which stay bounded as b tends to zero. The comprehensive paper of Wang-Young [19] on strange attractors also contains another solution to the basin problem in a similar but not the same context with a similar assumption on the Jacobian for the same purpose. We remark that all they actually need is that the condition (b) holds in a small neighborhood in which strange attractors potentially exist, i.e. in a neighborhood of the set $\{(x, 0) : |x| \leq 1\}$.

Our ultimate goal is to generalize these results on the basin problem [3] [19] to cases for non-invertible maps possessing singularities which deny the homogeneity of the Jacobian. This paper is an impetus to this goal; namely, we solve the basin problem for "Hénon-like attractors" generated by planar diffeomorphisms without recourse to the homogeneity. We do this in a hope that our argument can be combined with further parameter exclusions and be extended to cases where fold singularities are present. The author is currently working on this subject by using Tsujii's reconstruction of the Benedicks-Carleson theory [15].

One may ask whether families of diffeomorphisms which does not satisfy the homogeneity are naturally embedded in certain global bifurcations of dynamics. In a separate paper [12] we shall prove that such families bifurcate through critical saddle-node cycles [6].

1.1. The family

Throughout this paper we consider a two parameter family of planar diffeomorphisms of the following form:

$$F_{a,b}\colon \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} G(x,y,a) + bu(x,y,a,b) \\ bv(x,y,a,b) \end{pmatrix},$$

where $(a, x, y) \to u(x, y, a, b)$, $v(x, y, a, b)$, $G(x, y, a)$ are C^3 with bounded C^3 norms for any b. Letting $g_a = G(x, 0, a)$ we assume that g_a is a unimodal map defined on $[-1, 1]$. By this we mean g_a has a unique non-degenerate critical point $c \in (-1, 1)$, i.e. $g_a'(c) = 0$ and $g_a''(c) \neq 0$, and sends the boundary $\{-1, 1\}$ into itself. For simplicity we assume that the critical point of g_a does not change with parameter and it is 0, and that -1 is a fixed point of g_a. The map g_0 is a preperiodic Misiurewicz map, i.e. all periodic points are hyperbolic repelling and there exists $m \geq 2$ such that $g_0^m(0) = Q$ is a periodic point. Letting $D(a, n) = \frac{d(g_a^n(g_a(0)))}{da}$ we further assume the limit

$$\lim_{n\to\infty} \frac{D(0,n)}{g_0^n(g_0(0))}$$

which is known to exist [16] is nonzero. This assumption only concerns the parameter exclusion which we do not deal with in this paper. The point in the setting is that nothing particular is assumed on the Jacobian of the family $(F_{a,b})$.

We impose the following non-degeneracy conditions:

$$\partial_x v(0,0,0,0) \cdot g_0''(0) \neq 0. \tag{1.1}$$

(1.1) implies that if (a,b) is close to $(0,0)$ and $b \neq 0$, then $F_{a,b}$ maps a short segment in the x-axis containing $(0,0)$ to a curve which is C^2 close to the parabola $x = e \cdot y^2$ $(e \neq 0)$.

Denote by P the fixed point of g_0 which is not -1. We use the same letters P, Q to denote their continuations for $F_{a,b}$ with (a,b) close to $(0,0)$. If there is no fear of confusion, we write $F = F_{a,b}$ and $z_i = F_i(z)$ for $z \in \mathbf{R^2}$ and $i \in \mathbf{Z}$, when it makes sense. We maintain the same convention for an arbitrary set $A \subset \mathbf{R^2}$, i.e. $A_i = F^i(A)$.

The properties of $(F_{a,b})$ imply the existence of an F-forward invariant closed rectangle [*2] $D = D(F)$ which contains P, and is bounded by two horizontal curves and two vertical curves contained in $W^s(Q)$. The set D captures an important part of the dynamics of F. Put $\Omega = \bigcap_{n \geq 0} D_n$, where $D_n = F^n(D)$. The forward iterates of the horizontal boundaries of D are called *unstable sides*. The vertical boundaries of D play no role in our argument because they approach the fixed point Q under iteration.

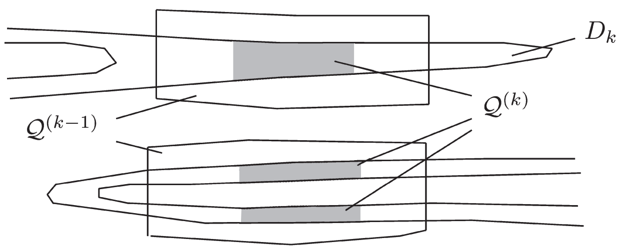

Figure 1. the geometry of the critical set.

1.2. The critical set of Wang-Young

Following [19], we present a geometric model called *critical set* which lies at the heart of our argument. For all our purposes, we arrange things in a slightly different way from the original paper [19].

Regarding nonzero positive constants $\alpha_0, \beta_0, \delta_0, \gamma_0, \Delta_0$, we assume the relations $\|g_0\|_{C^3} \leq e^{\Delta_0/2}$, $2.99\alpha_0/\Delta_0 < 1$, $\gamma_0 = \hat{\gamma}_0 - 5\alpha_0$, $0 < \beta_0 < 1$, and $\delta_0 < 1$. The constant $\hat{\gamma}_0$ will be chosen in $(0, \hat{\gamma})$, $\hat{\gamma} > 0$ depending only on g_0. See Proposition 2.1. The constant α_0 controls the speed of the recurrence of the critical points to the critical set, β_0 on the other hand determines the rate with which the critical regions decrease in size. Fix $\theta_0 > 0$ sufficiently small,

[*2] This D actually exists for such (a,b) belonging to a set whose intersection with any neighborhood of $(2,0)$ contains nonempty interior. See [10].

say $< 10^{-4}$, depending on g_0. Denote by $C > 0$ any auxiliary constant which appears in many places of our estimates. Keep in mind that the values of C are different in different places.

For two nonzero vectors u and $\tilde{u}$ in $\mathbf{R^2}$, $\text{angle}(u, \tilde{u}) \in [0, \pi/2]$ denotes the smaller angle which they make. Put $\text{slope}(v) = \tan \text{angle}(v, \binom{1}{0})$. For a C^1 curve γ and $z \in \gamma$, $t_\gamma(z)$ denotes any unit tangent vector of γ at z. If γ is contained in the unstable sides, we simply write $t(z)$. A nonzero vector v is called *horizontal* if $\text{slope}(v) \leq 10\theta_0$ holds. A C^2 curve γ is called *horizontal* if $\text{slope}(t_\gamma(z)) \leq 10\theta_0$ holds for all $z \in \gamma$, and the curvature of γ is smaller than θ_0^3 everywhere on γ.

1.2.1. Geometry of the critical set

Actually we only consider those (a, b) sufficiently close to $(0, 0)$ such that $\|DF_{a,b}\|_{C^3} \leq e^{\Delta_0}$ holds. Clearly, there exists $K > 0$ such that $|\det DF_{a,b}(z)| \leq K|b|$ holds for any $z \in D$ and (a, b). It implies no restriction to assume $b > 0$, and we do so. The critical set $\mathcal{C} = \mathcal{C}_{\delta_0} \subset \Omega$ is given by $\mathcal{C} = \bigcap_{k=0}^{\infty} \mathcal{C}^{(k)}$, where $\{\mathcal{C}^{(k)}\}_{k \geq 0}$ is a decreasing sequence called *critical regions* such that;

1. $\mathcal{C}^{(0)} = \{(x, y) \in D : |x| \leq \delta_0\}$.

2. $\mathcal{C}^{(k)}$ is a subset of $D_k = F^k(D)$ and has a finite number of components called $\mathcal{Q}^{(k)}$ each of which is diffeomorphic to a rectangle. The set $\mathcal{Q}^{(k)}$ is bounded by two vertical lines, and by two horizontal curves in the unstable sides of D_k. The Hausdorff distance between the two horizontal curves is $\mathcal{O}(b^{k/4})$, and their projection on the x-axis are intervals with length $\min\{\delta_0, e^{-\beta_0 k}\}$.

3. $\mathcal{C}^{(k)}$ is related to $\mathcal{C}^{(k-1)}$ as follows: $\mathcal{Q}^{(k-1)} \bigcap D_k$ has at most finitely many components. Each of them is bounded by the two vertical boundaries of $\mathcal{Q}^{(k-1)}$, and by two horizontal curves in the unstable sides of D_k. Each component of $\mathcal{Q}^{(k-1)} \bigcap D_k$ contains exactly one component of $\mathcal{C}^{(k)}$. See Figure 1.

1.2.2. Critical points

Around the mid point of each unstable side of $\mathcal{Q}^{(k)}$, there exists a unique point c such that

$$\left\| DF_{c_1}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| \geq e^{\gamma_0 n} \text{ and } \|DF_{c_1}^n t(c_1)\| \leq (Kb)^n$$

holds for all $n \geq 0$. The point c is called a *critical point of generation k*. By definition, $\mathcal{Q}^{(k)}$ contains infinitely many critical points. Letting $c = (c_x, c_y)$ be the critical point on the unstable side of $\mathcal{Q}^{(k)}$, we assume the relation $|c_x - c'_x| \leq (Kb/2)^k$ for any critical point $c' = (c'_x, c'_y) \in \mathcal{Q}^{(k)}$.

For $z = (x, y) \in D$, the distance to the critical set $d_{\mathcal{C}}(z)$ is defined as follows: $d_{\mathcal{C}}(z) = |x|$ for $z \notin \mathcal{C}^{(0)}$. Otherwise, letting $k_0 = \max\{k : z \in \mathcal{C}^{(k)}\}$ and $\mathcal{Q}^{(k_0)}$ be the component containing z, $d_{\mathcal{C}}(z)$ is defined to be the minimum of the horizontal distances between z and the two critical points on the unstable sides of $\mathcal{Q}^{(k_0)}$.

1.2.3. Dynamical assumptions

For the critical set $\mathcal{C}$ we put two assumptions:

($A1$) for any critical point c and $n \geq 0$,

$$\sum_{1 \leq j \leq n+1, c_j \in \bar{\mathcal{C}}^{(0)}} \log d_{\mathcal{C}}(c_j)^{-1} \leq \alpha_0 n,$$

where $\bar{\mathcal{C}}^{(0)} := \{(x,y) \in D : |x| \leq \delta_0^{2.99\alpha_0/\Delta_0}\}$. Notice the relation $\mathcal{C}^{(0)} \subset \bar{\mathcal{C}}^{(0)}$.

($A2$) For any critical point c and $n \geq 0$, there exists $\chi(n) \in [(1-10\alpha_0)n, n]$ such that

$$\mathrm{slope}\left(DF_{c_1}^{\chi(n)}\begin{pmatrix}1\\0\end{pmatrix}\right) \leq \theta_0.$$

The assumption ($A1$) states two things on the orbits of the critical points: they do not come too close to the critical set, This formulation is inspired by the *bounded recurrence condition* introduced by Luzzatto [8][*3]. He proved that the assumption ($A1$) in the one-dimensional situation is indeed realized with positive probability in parameter space. The reader should also refer to Luzzatto-Viana [9] in which a proof is given for the construction of a positive measure set of parameter values corresponding to the critical set[*4] satisfying ($A1$).

In [19], the critical set is defined only for those parameters which were selected by the huge inductive parameter exclusion argument. In contrast, we define the critical set explicitly from the beginning, and develop arguments assuming the existence of the critical set.

The assumption ($A1$) is stronger than the combination of the parameter exclusion rules (BA) and (FA), introduced by Benedicks-Carleson [2]. Wang and Young [19] only proved the abundance of parameter values corresponding to the critical set satisfying (BA) and (FA). Thus, the existence, let alone the abundance, of the critical set with ($A1$), ($A2$) does not immediately follow from [19]. However, we remark that one can reconstruct arguments of [19] in light of [9], and can show the abundance of parameter values possessing the critical set satisfying ($A1$). For these selected parameter values the assumption ($A2$), which is inspired by Tsujii [15], is necessarily satisfied. For this reason, it is fair to state the following

Theorem 1.1 (Wang-Young [19]). *Let $(F_{a,b})$ be as above. For any $\hat{\gamma}_0 \in (0, \hat{\gamma})$ and $\alpha_0 > 0$ sufficiently small, there exists $\delta > 0$ such that for any $b \neq 0$ sufficiently close to 0, there exists a set of a-values Δ_b with $\mathrm{Leb}(\Delta_b) > 0$ such that for any $a \in \Delta_b$, the corresponding $F_{a,b}$ has the critical set $\mathcal{C}_\delta$ satisfying ($A1$) and ($A2$), and admits an ergodic SRB measure $\mu_{a,b}$ supported on the closure of the unstable manifold of P.*

[*3] A similar condition implicitly appears in [13], [14].

[*4] By this we mean the geometric structure in dynamical space which is constructed in [9]. The term "critical set" is not used there, so we have slightly abused a language.

1.3. Statement of the result

We now introduce a constant $\mu_0 := -10^{-2} \cdot \log b$ and the following terminology to state our main theorem. We say $z \in F(\mathcal{C}^{(0)})$ is *controlled up to time* n if $d_{\mathcal{C}}(z_j) \geq e^{-3\mu_0 j}$ holds for all $0 \leq j \leq n$. We say $z \in D$ is *eventually controlled* if there exists some n_0 such that $z_{n_0} \in F(\mathcal{C}^{(0)})$ and z_{n_0} is controlled all the time.

Main theorem. *Let* $(F_{a,b})$ *be as above. For any* $\hat{\gamma}_0 \in (0, \hat{\gamma})$ *and any* $\alpha_0 > 0$, $\delta > 0$ *small, there exists* $\delta_0 \in (0, \delta]$ *such that for any* $b \neq 0$ *sufficiently close to* 0, *if* $F = F_{a,b}$ *has the critical set* $\mathcal{C}_{\delta_0}$ *satisfying* (A1) *and* (A2), *then Lebesgue almost every initial point* $z \in D$ *is eventually controlled. In particular,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \|DF^n(z)\| \geq \frac{\gamma_0}{3}$$

holds for Lebesgue almost every $z \in D$.

Three remarks: the lower estimate of the upper Lyapunov exponent directly follows from Corollary 4.1. The main theorem should be understood in conjunction with Wang-Young's theorem to be explained in the next paragraph. The author suspects that extending the main theorem to higher dimensions [17], [20] presents a serious difficulty.

We say $z \in D$ is generic with respect to a probability measure μ if the asymptotic distribution of the orbit of z exists and coincides with μ, i.e. $\lim_{n\to\infty} n^{-1} \sum_{i=0}^{n-1} \delta_{z_i} = \mu$ holds, δ_z being the Dirac measure. We claim that Wang-Young's theorem and the main theorem together imply that for any $(F_{a,b})$ as above and any (a, b) such that $a \in \Delta_b$, Lebesgue almost every initial point of D is generic with respect to the SRB measure $\mu_{a,b}$. Following [19], let us explain why this is so.

We begin with observing the relationship between the two theorems. According to Wang-Young's theorem, for any $a \in \Delta_b$ the corresponding $F_{a,b}$ has the critical set $\mathcal{C}(\delta)$. The relation $\delta_0 \leq \delta$, and the definition of the critical set then imply that $F_{a,b}$ has the critical set $\mathcal{C}(\delta_0)$ as well. Therefore, according to the main theorem, almost every $z \in D$ is eventually controlled under the iteration of $F_{a,b}$.

Take a small horizontal curve of length $\sim \delta_0/(-\log \delta_0)^2$, denoted by Δ_+, located near one of the vertical boundaries of $\mathcal{C}^{(0)}$. Imitating the parameter exclusion argument in one-dimensional systems [1], [2], we construct a positive measure subset $\tilde{\Delta}_+$ of Δ_+ such that all points of $F(\tilde{\Delta}_+)$ is controlled all the time. We do the same thing with respect to Δ_- and construct $\tilde{\Delta}_-$, where Δ_- is also a small horizontal curve of the same length as Δ_+, located near the other vertical boundary of $\mathcal{C}^{(0)}$. From a point which is controlled all the time emanates a stable leaf (Propositions 2.3, 4.1); by this we roughly mean a sufficiently long C^1 vertical curve such that any two points lying on it are future asymptotic to each other. The collection of the stable leaves through $F(\tilde{\Delta}_- \cup \tilde{\Delta}_+)$ forms a lamination with absolutely continuous holonomies. We denote by $\mathcal{H}$ its pull back by F. The leaves of $\mathcal{H}$ are still horizontal, since $\Delta_+ \cup \Delta_-$ is near the vertical boundaries of $\mathcal{C}^{(0)}$, and in particular they pass through the closure of $W^u(P)$. Suppose that there exists a positive Lebesgue measure set $B \subset D$ such that any point of B is not generic with respect to $\mu_{a,b}$. The SRB property of $\mu_{a,b}$, the Birkoff ergodic theorem, the absolute continuity of the holonomies along $\mathcal{H}$ altogether imply that the set $\{z \in B\colon \exists n \geq 0 \text{ s.t. } z_n \in \mathcal{H}\}$ has zero Lebesgue measure. Let $Y^{(i)}$ be the set of points $z \in D$ such that z_i is controlled all the time. According to the main theorem, there exists some i_0 such that $Y^{(i_0)} \bigcap B$ has positive Lebesgue measure. Let $\varepsilon > 0$ be an arbitrarily small number. By the Fubini theorem and the Lebesgue density theorem, one can take a horizontal curve γ in a way that $|\gamma \bigcap Y_{i_0}^{(i_0)} \bigcap B_{i_0}|_\gamma > 1-\varepsilon$ holds, where $|\cdot|_\gamma$ is the normalized arc length measure on γ. Let $\hat{\Delta}_+$ (resp. $\hat{\Delta}_-$) be a horizontal curve containing Δ_+ (resp. Δ_+) and extending to its both sides with length $\sim \delta_0/(-\log \delta_0)^2$. Define a return time function $R\colon \gamma \bigcap Y_{i_0}^{(i_0)} \bigcap B_{i_0} \to (0, \infty]$ in the following way; $R(z)$ is the first moment at which there exists a neighborhood V_z of z in γ such that $p_x(V_z) \supset p_x(\hat{\Delta}_+)$ or $p_x(V_z) \supset p_x(\hat{\Delta}_-)$ holds, where $p_x(x, y) = x$. Define $R(z) = \infty$ if no such $R(z)$ exists. By the main theorem, there exists a countable union of horizontal curves denoted by $\tilde{\gamma}$ such that $\tilde{\gamma} \subset \gamma$, $\tilde{\gamma} \supset \gamma \bigcap Y_{i_0}^{(i_0)} \bigcap B_{i_0}$, and R is well-defined on $\tilde{\gamma}$. The *return time estimate* of [2] or [5], including distortion estimates shows that the value of R is in fact finite for Lebesgue almost every $z \in \tilde{\gamma}$. Define a return map $T\colon \tilde{\gamma} \to \mathbf{R}^2$ by $T(z) = F^{R(z)}(z)$. By definition, $T(z)$ has a Markov-like structure with countably many branches with bounded distortions. Thus we obtain $|\{z \in \tilde{\gamma} \bigcap Y_{i_0}^{(i_0)} \bigcap B_{i_0} : \exists n \geq 0 \text{ s.t. } z_n \in \mathcal{H}\}|_{\tilde{\gamma}} \geq \min\{|\tilde{\Delta}_+|_{\Delta_+}, |\tilde{\Delta}_-|_{\Delta_-}\}/2$. Since the measure $|\tilde{\Delta}_\pm|_{\Delta_\pm}$ only depends on δ_0 and ε is arbitrary, this yields a contradiction if we choose $\varepsilon < \min\{|\tilde{\Delta}_+|_{\Delta_+}, |\tilde{\Delta}_-|_{\Delta_-}\}/2$ from the beginning. We lastly remark that the measure estimate of $\tilde{\Delta}_\pm$, and the return time estimate of course require a distortion argument which is not contained in this paper. For details, see [2], [10], [19].

1.4. Arguments and techniques

The basic idea of the proof of the main theorem is the same as [3] and [19]. We construct a family of *bad regions*, and prove that the orbit of Lebesgue almost every initial point in the basin hits the regions only finitely many times. Although our argument is very much inspired by [3], [19], the absence of the homogeneity of the Jacobian significantly affects the two metric estimates as we said in the beginning: area distortions are going to explode as b tends to zero, and there is no bounded geometry of unstable sides, namely the two unstable sides forming the boundary of the bad regions may not be sufficiently parallel. As a result, their arguments does not work in our setting.

To overcome these serious difficulties, we develop new combinatorial and geometric constructions. First, using stable leaves constructed in Section 4, we define a family of bad regions which shrink with sufficiently high speed. The k-th bad region $\mathcal{B}^{(k)} \subset D_k$ contains all $z \in D$ such that z_{-k} is controlled up to time $k-1$, but not so up to time k. The bad region $\mathcal{B}^{(k)}$ has at most a finite number of connected components denoted by $\mathcal{A}^{(k)}$. Using stable leaves we divide each component $\mathcal{A}^{(k)}$ into countably many measurable rectangles and define a partition $\mathcal{P}^{(k)}$. For each rectangle $R \in \mathcal{P}^{(k)}$ we construct an infinite decreasing nested sequence $\{T^{(j)}(R)\}_{j\geq 0}$ and partitions $\mathcal{S}^{(j)}(R)$ of $T^{(j)}(R)$, again with stable leaves. The intersection of the nested sequence $\bigcap_{j\geq 0} T^{(j)}(R)$ contains all the points starting from R and not eventually controlled. The sequence of partitions $\{\mathcal{S}^{(j)}(R)\}_{j\geq 0}$ incorporates patterns of the recurrence of volumes to the critical set, and permits us to conclude that the set $\bigcap_{j\geq 0} T^{(j)}(R)$ has zero Lebesgue measure (Proposition 7.1), as long as k is sufficiently large, or equivalently, $\mathcal{B}^{(k)}$ is sufficiently small. Proposition 6.2 is the only place where the smallness of $\mathcal{B}^{(k)}$ becomes crucial. Since $\mathcal{B}^{(k)}$ has only a finite number of components $\mathcal{A}^{(k)}$, and since each $\mathcal{A}^{(k)}$ contains only countably many rectangles, the subadditivity in the measure theory claims that Lebesgue almost every point $z \in \mathcal{B}^{(k)}$ is eventually controlled. Actually, this completes the proof of the main theorem, by Corollary 7.1, which states that the orbits of points which are not eventually controlled intersect arbitrarily small bad regions.

On a technical level, we introduce a new definition of the *binding period*, taking advantage of the assumption $(A2)$. Our definition is simpler than the definition in [19], and as a by-product, considerably simplifies the construction of $\mathcal{P}^{(k)}$, and the area distortion estimates of Proposition 6.1.

1.5. Overview of the paper

This paper consists of seven sections. In Section 2, we collect some materials from [10], [19] needed for the construction of stable leaves. In Section 3, we introduce a new definition of the binding period and provide relevant basic estimates. In Section 4, we construct the bad region $\mathcal{B}^{(k)}$ and the associated partition $\mathcal{P}^{(k)}$. In Section 5, we analyze the behavior of rectangles $R \in \mathcal{P}^{(k)}$, and construct the nested sequence $\{T^{(j)}(R)\}_{j\geq 0}$ and their partitions $\{\mathcal{S}^{(j)}(R)\}_{j\geq 0}$. In Section 6, we prove the key metric estimates: area distortion bounds and bounded geometry of unstable sides. In Section 7, we put all the estimates and constructions together and complete the proof of the main theorem.

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2. Preliminaries

We begin with three propositions without proof. The arguments rely on the fact that F is regarded as a small perturbation of the map $F_{0,0}$, and do not require any structure of the critical set. Proposition 2.1 is intuitively obvious from the basic knowledge in one-dimensional dynamics. Propositions 2.2 and 2.3 are collections from [10], [19], with our slight modifications. We emphasize that our proof of the main theorem is self-contained, except these three propositions.

2.1. Dynamics outside of $\mathcal{C}^{(0)}$

Proposition 2.1. *There exists $\hat{\gamma} > 0$ depending only on g_0 such that for any $\hat{\gamma}_0 \in (0, \hat{\gamma})$, any small $\delta_0 > 0$, there exists a C^2 neighborhood $\mathcal{U}$ of $F_{0,0}$ such that the following hold for all $F \in \mathcal{U}$:*

1. *For an integer $m \in [1, 10]$, $z \notin \mathcal{C}^{(0)}$, and a tangent vector $v(z)$ with* $\mathrm{slope}(v(z)) \leq m\theta_0$, $\mathrm{slope}(DFv(z)) \leq m\theta_0$.
2. *For any $k \geq 1$, $z \in D$ such that $z, z_1, \ldots, z_{k-1} \notin \mathcal{C}^{(0)}$, and a horizontal vector $v(z)$, $\|DF_z^k v(z)\| \geq \delta_0 e^{\hat{\gamma}_0 k}\|v(z)\|$:*
3. *if moreover, $z_k \in \mathcal{C}^{(0)}$ or $z \in F(\mathcal{C}^{(0)})$, then $\|DF_z^k v(z)\| \geq e^{\hat{\gamma}_0 k}\|v(z)\|$.*
4. *If γ is a horizontal curve which does not intersect $\mathcal{C}^{(0)}$, then $F(\gamma)$ is a horizontal curve.*

2.2. Mostly contracting directions

Put $\lambda = b^{1/10} \gg b$. Following [10], we say $z \in D$ is *λ-expanding* up to time n if $\|DF_z^i\| \geq \lambda^i$ holds for all $1 \leq i \leq n$. We simply say z is λ-expanding if $\|DF_z^i\| \geq \lambda^i$ holds for all $i \geq 1$. Since $|\det DF| \leq Kb$, if z is λ-expanding up to time n then there exists a unit vector $e_n(z)$ which is mostly contracted by DF_z^n. We put

$$e^{(n)}(z_n) = \|DF^n e_n(z)\|^{-1} DF^n e_n(z).$$

We analogously define $f_n(z)$ and $f^{(n)}(z_n)$, where $f_n(z)$ is a unit vector mostly expanded by DF_z^n. Identifying each tangent space with its dual space under the Euclidean metric, we may regard $e^{(n)}(z_n)$ as the unit covector which is mostly contracted by $(DF^n)^*$.

Fix τ_0 such that $\Delta_0 \ll \tau_0 \ll -\log b$.

Proposition 2.2. *If $z \in F(\mathcal{C}^{(0)})$ is λ-expanding up to time $n \geq 1$, then the vector field e_n, f_n are defined in a neighborhood of z and satisfy*:

1. *e_n, f_n and $e^{(n)}$, $f^{(n)}$ are mutually orthogonal.*
2. $\|DF^n e_n\| \leq (Cb)^n$.
3. $\mathrm{angle}(e_n, e_{n-1})$, $\mathrm{angle}(e^{(n)}, e^{(n-1)}) \leq (Cb)^n$.
4. $\|D(\mathrm{angle}(e_n, e_{n-1}))\|$, $\|D(\mathrm{angle}(e^{(n)}, e^{(n-1)}))\| \leq (Cb)^n$.
5. $\|D(DF^n e_n)\|, \|D((DF^n)^* e^{(n)})\| \leq e^{2\Delta_0 n}$.
6. *The maximal integral curve $\Gamma^{(n)}(z)$ of e_n through z and contained in D has the form $\Gamma^{(n)}(z) = \{(x(y), y) : |y| \leq 1/10\}$.*
7. *Any point $\tilde{z} \in \{(x, y) : |x - x(y)| \leq e^{-\tau_0 n/2}, |y| \leq 1/10\}$ is $\lambda/2$-expanding up to time n. Moreover, if z is λ-expanding up to time $n+1$ then*

$$e^{-1} \leq \left\| DF_z^{n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| / \left\| DF_{\tilde{z}}^{n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| \leq e.$$

Part 5 of Proposition 2.2 can easily be checked by the chain rule of differentiation. This rough estimate does the job for our purpose[*5].

Corollary 2.1. *If $z \in F(\mathcal{C}^{(0)})$ is λ-expanding up to time n, then* $\mathrm{slope}(e_n) \geq Cb^{-1}$ *and* $\|De_n\| \leq Cb$. *Moreover, if z is λ-expanding, then the sequence $\{e_n(z)\}_{n\geq 1}$ converges to the limit $e_\infty(z)$, with* $\mathrm{angle}(e_n(z), e_\infty(z)) \leq (Cb)^n$.

Proof. Letting $e_0 = \binom{0}{1}$ we have the identity

$$\mathrm{angle}(e_n, e_0) = \sum_{k=1}^{n} \pm \,\mathrm{angle}(e_k, e_{k-1}),$$

where the signs are taken according to the orientation of the basis $\{e_k, e_{k-1}\}$. Then, part 3 of the proposition gives $\mathrm{angle}(e_n, e_0) \leq Cb$, or $\mathrm{slope}(e_n) \geq Cb^{-1}$. Differentiating the identity and using part 4 gives $\|De_n\| \leq Cb$. The last assertion is obvious from part 3. □

2.3. Stable leaves

Proposition 2.3. *If $z \in F(\mathcal{C}^{(0)})$ is λ-expanding, then its stable set $W^s(z) = \{y \in \mathbf{R^2} : \mathbf{y_n} \to \mathbf{z}\ (\mathbf{n} \to \infty)\}$ contains a segment of the form*

$$\Gamma(z) = \{(x(y), y) : |y| \leq 1/10\}$$

with $|x'| \leq C\sqrt{b}$, $|x''| \leq C\sqrt{b}$, $z \in \Gamma(z)$, and $d(\zeta_n, \tilde{\zeta}_n) \leq (Cb)^n d(\zeta, \tilde{\zeta})$ holds for all $\zeta, \tilde{\zeta} \in \Gamma(z)$ and $n \geq 1$. If both z and $\tilde{z}$ are λ-expanding and $\xi \in \Gamma(z)$, $\tilde{\xi} \in \Gamma(\tilde{z})$, then

$$\mathrm{angle}(t_{\Gamma(z)}(\xi), t_{\Gamma(\tilde{z})}(\tilde{\xi})) \leq C\sqrt{b}d(\xi, \tilde{\xi}).$$

Moreover, any point in the region $\{(x, y) : |x - x(y)| \leq e^{-\tau_0 n}, |y| \leq 1/10\}$ is $\lambda/2$-expanding up to time n.

Integrating the last inequality in Proposition 2.3, one can see that holonomies along stable leaves are (when they makes sense) Lipschitz continuous with Lipschitz constant smaller than $e^{C\sqrt{b}}$.

Corollary 2.2. *Let c be a critical point in the definition of the critical set. Then the stable leaf $\Gamma(c_1)$ passing through c_1 exists and is tangent to $t(c_1)$.*

Proof. The existence of $\Gamma(c_1)$ immediately follows from the definition of the critical point and Proposition 2.3. As for the last assertion, suppose that $\mathrm{angle}(e_n(z), t(z))$ does not tend to zero as $n \to \infty$. This contradicts the fact that $\|DF^n t(c_1)\| \leq (Kb)^n$ holds for all $n \geq 0$, which is also a part of the definition of the critical point. □

[*5]Mora and Viana [10] used the homogeneity to deduce a much stronger estimate $\|D(DF^n e_n)\| \leq (Cb)^n$. However, in this respect, the homogeneity is not crucial. Wang and Young [19] actually proved the same inequality without using the homogeneity.

3. Binding

The purpose of this section is to introduce a new definition of the binding period and provide relevant estimates. We remark that an essential part of the proof of Proposition 3.2 takes advantage of the fact that all the critical points remain in effect forever in our setting (recall the definition of the critical set). Therefore, the same argument does not directly carry over to the inductive construction of the critical set for a positive measure set of parameter values.

3.1. Folding period

We say a horizontal vector $v(z)$ is in *admissible position* relative to a critical point c, if $z \in \mathcal{C}^{(0)}$ and there exists a horizontal curve connecting c and z, tangent at z to $v(z)$. The geometry of the critical set implies that if $z \in D_k$ and $d_{\mathcal{C}}(z) \geq e^{-3\mu_0 k}$, then any horizontal vector at z is in admissible position. The converse is not true. This horizontal displacement of $v(z)$ relative to c implies partial resemblance to the one-dimensional dynamics, meaning that there is a chance of imitating the growth of tangent vectors along the orbit of c.

Proposition 3.1. *Suppose that a tangent vector $v(z)$ is in admissible position relative to a critical point c. Then* $\mathrm{slope}(DF^{\chi(q)+1}v(z)) \leq 2\theta_0$ *holds for any integer q such that*

$$q \in \left[\frac{\log d(z,c)^{4/(1-10\alpha_0)}}{\log |b|}, \frac{\log d(z,c)^{-1}}{\tau_0}\right] =: [q^-, q^+]. \tag{3.1}$$

For any $q \in [q^-, q^+]$, the integer $\chi(q)$ obtained by the assumption $(A2)$ is called the *folding period*. Notice that we have the freedom of choosing q within the interval $[q^-, q^+]$.

Proof of Proposition 3.1. Let γ be an admissible curve connecting c and z as in the definition. Put $DFt_\gamma(\zeta) = \xi(\zeta)t_{\gamma_1}(c_1)^\perp + \eta(\zeta)t_{\gamma_1}(c_1)$ for $\zeta = (x,y) \in \gamma$. For two positive numbers a and b, $a \sim b$ means $a/b, b/a \in [1-\theta_0, 1+\theta_0]$. We put $\ell = |g_0''(0)|$ to ease notation.

Lemma 3.1. $|\xi(\zeta)| \sim \ell d(c,\zeta)$ *holds for all* $\zeta \in \gamma$.

Proof. Since γ is a horizontal curve, the representation matrix $A(\zeta)$ of the derivative DF_ζ with respect to the bases $\{t_\gamma(\zeta), t_\gamma(\zeta)^\perp\}$, $\{t_{\gamma_1}(c_1)^\perp, t_{\gamma_1}(c_1)\}$ has the form

$$A(\zeta) = \begin{pmatrix} \partial_x G + * & \partial_y G + * \\ * & * \end{pmatrix},$$

where $*$ stands for the higher order terms whose C^1 norms are smaller than Kb. The matrix for $\zeta = c$ takes the form

$$A(c) = \begin{pmatrix} 0 & \cdots \\ * & * \end{pmatrix}.$$

The Taylor expansion of each element of $A(\zeta)$ around $c = (c_x, c_y)$ gives

$$A(\zeta) = \begin{pmatrix} Lg_0''(0)(x - c_x) + C(y - c_y) & \cdots \\ * & * \end{pmatrix},$$

as long as b is sufficiently small. The constant $L > 0$ can be made arbitrarily close to 1 by choosing small b. This implies $|\xi| = |Lg_a''(0)(x - c_x) + C(y - c_y)|$, and thus $1 - \theta_0^2 \le |\xi|/|g_0''(0)||x - c_x| \le 1 + \theta_0^2$, because γ is a horizontal curve. Moreover we have $1 \le d(c,\zeta)/|x - c_x| \le \sqrt{1 + 100\theta_0^2}$, and therefore $1 - \theta_0 \le |\xi|/|g_0''(0)|d(c,\zeta) \le 1 + \theta_0$. □

Choose $q \in [q^-, q^+]$. By the definition of q^+ and $\chi(q) \le q$, the mostly contracting direction $e_i(\zeta)$ is well-defined for all $1 \le i \le \chi(q)$ and all ζ which belongs to the region specified in Proposition 2.3. Clearly, this region contains γ and the stable leaf $\Gamma(c_1)$. Corollary 2.1 and the mean value theorem give

$$\text{angle}(e_{\chi(q)}(z_1), e_{\chi(q)}(c_1)) \le d(z_1, c_1)\|De_{\chi(q)}\| \le Cb \cdot d(c, z).$$

Corollary 2.1 again gives

$$\text{angle}(e_{\chi(q)}(c_1), e_\infty(c_1)) \le (Cb)^{\chi(q)}.$$

The definition of q^- in Proposition 3.1 gives

$$(Cb)^{\chi(q)} \le (Cb)^{(1-10\alpha_0)q} \le d(z, c)^2.$$

All these three and the triangle inequality yield

$$\text{angle}(e_\infty(c_1), e_{\chi(q)}(z_1)) \le Cb \cdot d(z, c),$$

which implies that regarding the expression $DFv(z) = \xi f_{\chi(q)}(z_1) + \eta e_{\chi(q)}(z_1)$, we have gives $|\xi| \ge \ell(1 - 2\theta_0)d(c, z)$. Moreover, $\|DF^{\chi(q)} f_{\chi(q)}(z_1)\| \ge 1$ holds because of Proposition 2.2–7 and the fact that c_1 is λ-expanding by the definition of the critical points. In addition we clearly have $|\eta| \le 1$ and $\chi(q) \ge 1$, and therefore

$$\frac{\|DF^{\chi(q)}\eta e_{\chi(q)}(z_1)\|}{\|DF^{\chi(q)}\xi f_{\chi(q)}(z_1)\|} \le \frac{(Cb)^{\chi(q)}}{\ell(1 - 2\theta_0)d(c, z)} \le (Cb)^{\chi(q)/2} \le C\sqrt{b}.$$

This implies

$$\text{angle}(DF^{\chi(q)+1}(v(z)), DF^{\chi(q)} f_{\chi(q)}(z_1)) \le C\sqrt{b},$$

by virtue of Proposition 2.2–1. By Corollary 2.1, $\text{slope}(f_{\chi(q)}) \le Cb$, and thus Proposition 2.2–1 gives

$$\text{angle}\left(DF^{\chi(q)} f_{\chi(q)}(c_1), DF_{c_1}^{\chi(q)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \le (Cb)^{\chi(q)} \le \sqrt{b}.$$

Corollary 2.1 again and the mean value theorem give

$$\text{angle}(DF^{\chi(q)} f_{\chi(q)}(z_1), DF^{\chi(q)} f_{\chi(q)}(c_1)) \le Cb \cdot d(z_{\chi(q)+1}, c_{\chi(q)+1}) \le \sqrt{b}.$$

Now, recall the assumption $(A2)$ which gives $\text{slope}(DF_{c_1}^{\chi(q)} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)) \le \theta_0$. Putting these four inequalities altogether we obtain

$$\mathrm{slope}(DF^{\chi(q)+1}v(z)) \le \theta_0 + C\sqrt{b} \le 2\theta_0.$$

□

3.2. Definition of the binding period

Let $v(z)$ be a horizontal vector in admissible position relative to a critical point c. We define the binding period $p(z)$ in the following way: Let $\hat{p}(z)$ be the maximum integer such that

$$d(z_{i+1}, c_{i+1}) \le e^{-2\alpha_0 i} \text{ holds for all } 1 \le i \le \hat{p} - 1. \tag{3.2}$$

Choose $q_0 \in [q_0^-, q_0^+]$ such that $|q_0 - (q_0^- + q_0^+)/2| < 1$, where $q_0^\pm$ are those in Proposition 3.1. There are two cases: whether $\chi(q_0) \ge \hat{p}(z)$, or not. In the first case, define

$$p(z) = \chi(q_0).$$

In the second case, which is a generic situation, there are further two cases: whether there is no return in between $[\chi(q_0), \hat{p}(z)]$, or not. Here, we say z_k or k is a *return* if $z_k \in \mathcal{C}^{(0)}$. In the first case, define

$$p(z) = \hat{p}(z).$$

In the second case, letting n_1 be the first return occurring after the expiration of the folding period $\chi(q_0)$, we observe that the vector v_{n_1+1}, which is horizontal by Propositions 3.1 and 2.1, is in admissible position. Indeed, we have $d(z_{n_1+1}, c_{n_1+1}) \le e^{-2\alpha_0 n_1}$, and moreover $(A1)$ implies $d_{\mathcal{C}}(c_{n_1+1}) \ge e^{-\alpha_0 n_1}$. We now again apply Proposition 3.1 to the vector v_{n_1} and obtain the corresponding folding period $\chi(q_1)$, where q_1 is again chosen in a way that $|q_1 - (q_1^- + q_1^+)/2| < 1$ holds. There are further two cases: whether $n_1 + \chi(q_1) \ge \hat{p}(z)$, or not. In the first case, define

$$p(z) = n_1 + \chi(q_1).$$

Now it is clear how to proceed with the argument. Repeating the same procedure we eventually end up with a sequence of returns $0 := n_0 < n_1 < \cdots < n_k \le \hat{p}(z)$ with the properties that v_{n_j+1} is a horizontal vector in admissible position for all $0 \le j \le k$, and $n_j + \chi(q_j) \le n_{j+1}$ for all $0 \le j \le k-1$, where $\chi(q_j)$ is the folding period corresponding to n_j, i.e. $|q_j - (q_j^- + q_j^+)/2| < 1$. Regarding n_k, either $n_k + \chi(q_k) \ge \hat{p}(z)$, or $n_k + \chi(q_k) \le \hat{p}(z)$ and no return exists up to time $\hat{p}(z)$. In the first case we have $p(z) = n_k + \chi(q_k)$, and in the second case we have $p(z) = \hat{p}(z)$. In both cases, $v_{p(z)+1}$ is a horizontal vector with $\mathrm{slope}(v_{p(z)+1}) \le 2\theta_0$.

Lemma 3.2. *$d(z_{i+1}, c_{i+1}) \le e^{-1.5\alpha_0 i}$ holds for all $i \le p(z)$.*

Proof. There is nothing to prove if $p(z) = \hat{p}(z)$. Otherwise, by definition there exists a return $n_k \le \hat{p}(z)$ with the folding period $\chi(q_k)$ such that $n_k +$

$\chi(q_k) > \hat{p}(z)$. First we observe $\chi(q_k) \le q_k \le \tau_0^{-1} \log d(c,z)^{-1} \le 2\alpha_0\tau_0^{-1} n_k$, where the first inequality is by $(A2)$, the second by Proposition 3.1, the last by $(A1)$ and (3.2). This gives $p(z) \le (1 + 2\alpha_0\tau_0^{-1})n_k$. Since α_0 is small, for all $1 \le i \le p(z) \le (1 + 2\alpha_0\tau_0^{-1})n_k$ we have

$$\begin{aligned}(\Delta_0 + 1.5\alpha_0)i &\le (\Delta_0 + 1.5\alpha_0)(1 + 2\alpha_0\tau_0^{-1})n_k \\ &\le (\Delta_0 + 2\alpha_0)n_k \le (\Delta_0 + 2\alpha_0)\hat{p}.\end{aligned}$$

This proves the lemma because $d(c_{i+1}, z_{i+1}) \le e^{-2\alpha_0\hat{p}} e^{\Delta_0(i-\hat{p})}$ holds for $i \ge \hat{p}(z)$. □

3.3. Binding period for individual points

Proposition 3.2. *Suppose that a tangent vector $v(z)$ is in admissible position relative to a critical point c, and $p(z)$ is the binding period. Then we have*

$$\frac{2\log d(c,z)^{-1}}{\Delta_0} \le p(z) \le \frac{2\log d(c,z)^{-1}}{\gamma_0}, \tag{3.3}$$

$$\|DF^{p(z)+1}v(z)\| \ge e^{\gamma_0(p(z)+1)/3}\|v(z)\|, \tag{3.4}$$

and moreover $\mathrm{slope}(DF^{p(z)+1}v(z)) \le 2\theta_0$.

Let $\{n_j\}_{j=0}^k$ and $\{\chi(q_j)\}_{j=0}^k$ be those in the definition of the binding period $p(z)$. Letting $c^{(0)} = c$, we suppose that the horizontal vector $DF_z^{n_j}v(z)$ is in admissible position relative to a critical point $c^{(j)}$ for all $1 \le j \le k$. Take a point z' on the stable leaf $\Gamma(c_1)$ which is connected with z_1 by a horizontal curve, say γ. We prove Proposition 3.2, given the following three lemmas.

Lemma 3.3. *For all $\zeta \in \gamma$ and $1 \le j \le k$, the vector $DF^{n_j}t_\gamma(\zeta)$ satisfies* $\mathrm{slope}(DF^{n_j}t_\gamma(\zeta)) \le 2\theta_0$*, and is in admissible position relative to the critical point $c^{(j)}$, and $\chi(q_j)$ is the folding period for ζ_{n_j}. Moreover, we have*

$$\|DF^{n_j+\chi(q_j)}t_\gamma(\zeta)\| \ge e^{(\gamma_0-3\alpha_0)(n_j+\chi(q_j))}.$$

Lemma 3.4. *For all $0 \le j \le k$, $\gamma_{n_j+\chi(q_j)}$ is a horizontal curve.*

Lemma 3.5. *For all ζ, $\tilde{\zeta} \in \gamma$, we have*

$$\frac{\|DF^p t_\gamma(\zeta)\|}{\|DF^p t_\gamma(\tilde{\zeta})\|} \le ((1+2\theta_0)e^2)^k 3e^{\Delta_0}\sum_{i=1}^{\infty} e^{-1.5\alpha_0 i/2}.$$

Proof of Proposition 3.2. (3.6), Lemma 3.3, and Proposition 2.1 give

$$\mathrm{length}(\gamma_p) \ge \delta_0 \ell e^{p(\gamma_0-3\alpha_0)} d(c,z)^2.$$

On the other hand, we have $\mathrm{length}(\gamma_p) \sim d(c_p, z_p) \le e^{-1.5\alpha_0 p}$ because all the tangent vectors of γ_p are horizontal according Lemma 3.3. Taking logs we obtain

$$p(z) \le \frac{\log d(c,z)^{-2} - \log\delta_0}{\gamma_0 - 1.5\alpha_0} \le \frac{2\log d(c,z)^{-1}}{\gamma_0}. \tag{3.5}$$

On the other direction, (3.6) and the definition of $p(z)$ and Lemma 3.5 give $e^{-2\alpha_0 p} \le d(c_p, z_p) \sim \text{length}(\gamma_p) \le \ell d(c,z)^2 e^{\Delta_0 p}$. Again taking logs of both sides gives the desired inequality.

Regarding the derivative estimate, the argument of Lemma 3.1 gives

$$\frac{\|DF^{p(z)+1}v(z)\|}{\|v(z)\|} \ge (1-Cb)|\xi(z)|\|DF^p t_\gamma(z_1)\|.$$

Therefore, Lemma 3.5, (3.6) and (3.5) together imply

$$\begin{aligned}\frac{\|DF^{p(z)+1}v(z)\|}{\|v(z)\|} &\ge \frac{|\xi(z)|}{C((1+2\theta_0)e^2)^k}\frac{\text{length}(\gamma_p)}{\text{length}(\gamma_0)} \\ &\ge C((1+2\theta_0)e^2)^{-k}\frac{e^{-2\alpha_0 p}}{d(c,z)} \\ &\ge C((1+2\theta_0)e^2)^{-k}e^{(\gamma_0/2-2\alpha_0)p}.\end{aligned}$$

Taking logs of both sides and using $k \le \alpha_0 p(z)/\log\delta_0^{-1}$ gives

$$\begin{aligned}\log\frac{\|DF^{p(z)+1}v(z)\|}{\|v(z)\|} &\ge (\gamma_0/2-2\alpha_0)p(z) - k\log((1+2\theta_0)e^2) \\ &\ge (\gamma_0/2-2\alpha_0)p(z) - \frac{\alpha_0 p(z)\log((1+2\theta_0)e^2)}{\log\delta_0^{-1}} \\ &\ge \gamma_0(p(z)+1)/3,\end{aligned}$$

because small δ_0 is chosen after α_0 is fixed. This competes the proof of Proposition 3.2. □

Proof of Lemma 3.3. We prove the lemma by induction on j, assuming that all the tangent vectors of γ_{n_1} are horizontal. Notice that this assumption is non-trivial because there may be a return before $\chi(q_0)$.

For $j = 1$: According to Lemma 3.1, $\text{length}(\gamma) \sim \ell\int_0^{d(c,z)} d(c,s)ds \sim \ell d(c,z)^2$, and in particular

$$d(c,z)^3 \le \text{length}(\gamma) \le d(c,z). \tag{3.6}$$

Proposition 2.2–7 and the definition of the critical points give $\|DF^{\chi(q_0)}t_\gamma(\zeta)\| \ge e^{-1}e^{\gamma_0\chi(q_0)}$, and thus $\|DF^{n_1}t_\gamma(\zeta)\| \ge e^{\gamma_0 n_1 - 1}$. Then, Proposition 3.1 gives

$$\begin{aligned}\text{length}(\gamma_{n_1}) &\ge d(c,z)^3 e^{\gamma_0 n_1-1} \ge d(c,z)^3 e^{\gamma_0\chi(q_0)-1} \\ &\ge e^{\frac{4q_0\log b}{1-10\alpha_0}+\gamma_0\chi(q_0)-1} \ge e^{\left(\frac{4\log b}{(1-10\alpha_0)^2}+\gamma_0\right)\chi(q_0)-1}.\end{aligned}$$

On the other hand, $d(z'_{n_1}, c_{n_1}) \le (Cb)^{n_1} \le (Cb)^{\chi(q_0)} = e^{\chi(q_0)\log Cb}$ holds since $z' \in \Gamma(c_1)$. These two inequalities imply $\text{length}(\gamma_{n_1}) \sim d(z_{n_1}, z'_{n_1}) \gg d(z'_{n_1}, c_{n_1})$, and therefore $d(z_{n_1}, z'_{n_1}) \le 2d(z_{n_1}, c_{n_1})$. In particular, we obtain

$$d(z_{n_1},\zeta_{n_1}) \le d(z_{n_1}, z'_{n_1}) \le 2d(z_{n_1}, c_{n_1}) \le 2e^{-1.5\alpha_0 n_1},$$

where the first inequality follows from the assumption that all the tangent vectors of γ_{n_1} are horizontal. Since $d(c^{(1)}, z_{n_1}) \geq d(c^{(1)}, c_{n_1}) - d(c_{n_1}, z_{n_1}) \geq e^{-1.3\alpha_0 n_1}$ holds by $(A1)$ and Lemma 3.2, there exists small $\tilde{\alpha}$ depending only on δ_0 which gives

$$d(c^{(1)}, \zeta_{n_1}) \leq d(c^{(1)}, z_{n_1}) + d(z_{n_1}, \zeta_{n_1}) \leq d(c^{(1)}, z_{n_1})^{1-\tilde{\alpha}} \tag{3.7}$$

and

$$d(c^{(1)}, \zeta_{n_1}) \geq d(c^{(1)}, z_{n_1}) - d(z_{n_1}, \zeta_{n_1}) \geq d(c^{(1)}, z_{n_1})^{1+\tilde{\alpha}}. \tag{3.8}$$

In particular, we obtain

$$1 - \tilde{\alpha} \leq \frac{\log d(c^{(1)}, \zeta_{n_1})}{\log d(c^{(1)}, z_{n_1})} \leq 1 + \tilde{\alpha}. \tag{3.9}$$

Clearly, $\tilde{\alpha}$ can be chosen arbitrarily small by choosing small δ_0. Now we appeal to the following sublemma which is obvious.

Sublemma 3.1. *If positive numbers a, b, d, $\tilde{d}$ satisfy $d \geq \tilde{d}$, $(b-a)\tilde{d} \geq 2$, and*

$$\frac{d}{\tilde{d}} \leq \min\left\{\frac{a+b}{2a}, \frac{2b}{a+b}\right\},$$

then the intersection of the two intervals $[ad, bd]$, $[a\tilde{d}, b\tilde{d}]$ contains any integer c satisfying $|c - (a+b)d/2| < 1$ or $|c - (a+b)\tilde{d}/2| < 1$.

(3.9), Lemma 3.1, and Proposition 3.1 make sure that all $\zeta \in \gamma$ share the common folding period $\chi(q_1)$ associated with the return n_1. This prove the first half of the statement of the lemma for $j = 1$.

By the definition of the folding period, the vector field $e_{\chi(q_1)}$ is defined in a domain containing γ_{n_1+1}. This allows us to use Proposition 2.2–1,7 and the argument of the proof of Lemma 3.1 with respect to the expression $DF^{n_1+1}t_\gamma(\zeta) = \xi f_{\chi(q_1)}(\zeta_{n_1+1}) + \eta e_{\chi(q_1)}(\zeta_{n_1+1})$, to obtain

$$\frac{\|DF^{n_1+\chi(q_1)}t_\gamma(\zeta)\|}{\|DF^{n_1}t_\gamma(\zeta)\|} \geq (1 - Cb)\|\xi DF^{\chi(q_1)} f_{\chi(q_1)}(\zeta_{n_1+1})\| \geq (1 - Cb)|\xi|e^{-2}. \tag{3.10}$$

(3.8), (3.10), $(A1)$, Lemma 3.1, and $n_1 \geq \log \delta_0^{-1}$ together yield

$$\begin{aligned}\log \|DF^{n_1+\chi(q_1)}t_\gamma(\zeta)\| &\geq \gamma_0 n_1 - 1 + (1+\theta_0)\log d(c^{(1)}, \zeta_{n_1}) + \log(1 - Cb) - 2 \\ &\geq \left(\gamma_0 - (1+\tilde{\alpha})(1+\theta_0)\alpha_0 + \log(1-Cb) - \frac{2}{\log \delta_0^{-1}}\right) n_1 \\ &\geq (\gamma_0 - 3\alpha_0)n_1.\end{aligned}$$

This completes the proof for $j = 1$.

Suppose that we have established the statement for all $1 \leq j \leq m-1$. We again aim at proving the inequality

$$1-\tilde{\alpha} \leq \frac{\log d(c^{(m)},\zeta_{n_m})}{\log d(c^{(m)},z_{n_m})} \leq 1+\tilde{\alpha}, \tag{3.11}$$

which makes sure that all $\zeta \in \gamma$ share the common folding period $\chi(q_m)$ associated with the return n_m. We skip details because the proof is almost identical to the case $j=1$, by virtue of the inductive assumption and Proposition 2.1 which together imply that all the tangent vectors of γ_{n_m} are horizontal, and

$$\|DF^{n_m}t_\gamma(\zeta)\| \geq \|DF^{n_{m-1}+\chi(q_{m-1})}t_\gamma(\zeta)\|e^{\hat{\gamma}_0(n_m-n_{m-1}-\chi(q_{m-1}))} \geq e^{\gamma_0 n_m}$$

for all $\zeta \in \gamma$.

We now move on to the derivative estimate. Regarding the expression $DF^{n_j+1}t_\gamma(\zeta) = \xi f_{\chi(q_j)}(\zeta_{n_j+1}) + \eta e_{\chi(q_j)}(\zeta_{n_j+1})$, once again we have

$$\|DF^{n_j+\chi(q_j)}t_\gamma(\zeta)\| \geq (1-Cb)\|\xi DF^{\chi(q_j)}f_{\chi(q_j)}(\zeta_{n_j+1})\| \geq (1-Cb)|\xi|e^{-2}, \tag{3.12}$$

by virtue of the assumption of the induction. We successively apply (3.12) for all $2 \leq j \leq m$ and obtain

$$\begin{aligned}\log\frac{\|DF^{n_m+\chi(q_m)}t_\gamma(\zeta)\|}{\|DF^{n_1+\chi(q_1)}t_\gamma(\zeta)\|} \geq \hat{\gamma}_0 &\left(n_m+\chi(q_m)-n_1-\sum_{j=1}^m\chi(q_j)\right)\\ &+(1+\theta_0)\sum_{2\leq j\leq m}\log d(c^{(j)},\zeta_{n_j})\\ &+(m-1)(\log(1-Cb)-2).\end{aligned}$$

Combining this with the previous estimate of $\|DF^{n_1+\chi(q_1)}t_\gamma(\zeta)\|$ yields

$$\begin{aligned}\log\|DF^{n_m+\chi(q_m)}t_\gamma(\zeta)\| \geq \hat{\gamma}_0 &\left(n_m+\chi(q_m)-\sum_{j=1}^m\chi(q_j)\right)+(\gamma_0-\hat{\gamma}_0)n_1\\ &+(1+\theta_0)\sum_{1\leq j\leq m}\log d(c^{(j)},\zeta_{n_j})+m(\log(1-Cb)-2).\end{aligned}$$

Regarding the inside of the parenthesis, (3.1) and (3.6) give

$$\sum_{j=1}^m\chi(q_j) \leq \sum_{j=1}^m \tau_0^{-1}\log d(c^{(j)},\zeta_{n_j})^{-1} \leq \alpha_0\tau_0^{-1}n_m.$$

On the second term, recall the relation $\hat{\gamma}_0-\gamma_0 = 5\alpha_0$. On the third term, recall the inequality $d(c^{(m)},\zeta_{n_m}) \geq d(c^{(m)},z_{n_m})^{1+\tilde{\alpha}}$, which is obtained in the course of the proof of (3.11), similarly to (3.8). The assumption $(A1)$ with respect to the orbit of c_1 and Lemma 3.2 together imply

$$\sum_{1\leq j\leq m}\log d(c^{(j)},\zeta_{n_j}) \geq (1+\tilde{\alpha})\sum_{1\leq j\leq m}\log d(c^{(j)},z_{n_j}) \geq -(1+\tilde{\alpha})2\alpha_0 n_m.$$

On the fourth term, we use $m \le \alpha_0 n_m / \log \delta_0^{-1}$ which follows from $(A1)$. In all, we obtain

$$\begin{aligned}\log \|DF^{n_m+\chi(q_m)} t_\gamma(\zeta)\| &\ge \hat{\gamma}_0 \left(1 - 2\alpha_0 \tau_0^{-1}\right)(n_m + \chi(q_m)) - 10\alpha_0 n_m \\ &\quad - (1+\theta_0)\alpha_0 n_m - 2\alpha_0 n_m / \log \delta_0^{-1} \\ &\ge (\hat{\gamma}_0 - 3\alpha_0)(n_m + \chi(q_m)),\end{aligned}$$

which restores the assumption of the induction.

We are left to prove that all the tangent vectors of γ_{n_1} are horizontal. It is easy to see this, similarly to the final part of the proof of Proposition 3.1, using the following two facts: the mostly contracting direction $e_i(\zeta)$ is well defined for all $1 \le i \le n_1$ and $\zeta \in \gamma$; the vector $DF_{c_1}^{n_1}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ is horizontal, because of Proposition 2.1 and $DF_{c_1}^{\chi(q_0)}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ being horizontal by $(A2)$. This completes the proof of Lemma 3.3. □

Proof of Lemma 3.4. To ease notation we put $w_i = DF^i t_\gamma(\zeta)$. We begin with claiming that all $\zeta \in \gamma$ is λ-expanding up to time $p(z)$, namely $\|w_i\| \ge \lambda^i$. There are three cases to consider: $1 \le i \le \chi(q_0)$, $n_j + \chi(q_j) < i \le n_{j+1}$ for some $j \ge 0$, or $n_j \le i \le n_j + \chi(q_j)$ for some $j \ge 0$.

The first case: Proposition 2.2–7 gives that ζ is λ-expanding up to time $\chi(q_0)$. The second case: Lemma 3.3 gives $\|w_{n_j+\chi(q_j)}\| \ge e^{(\gamma_0 - 3\alpha_0)(n_j+\chi(q_j))}$. Then, Proposition 2.1 gives $\|w_i\| \ge e^{(\gamma_0-3\alpha_0)i}$. By Lemma 3.2 and $(A1)$, we have $d(c^{(j)}, z_{n_j}) \ge e^{-2\alpha_0 n_j}$. The structure of the critical set on the other hand gives $d(c^{(j)}, z_{n_j}) \le \delta_0$, which together yield $\|w_i\| \ge \delta_0 e^{(\gamma_0-3\alpha_0)i} \ge e^{(\gamma_0-3\alpha_0)i - 2\alpha_0 n_j} \ge e^{(\gamma_0 - 5\alpha_0)i} \ge \lambda^i$. The third case: Lemma 3.3 and $\|DF\| \le e^{\Delta_0}$ give

$$\|w_i\| \ge e^{(\gamma_0-3\alpha_0)(n_j+\chi(q_j)) - \Delta_0(n_j+\chi(q_j)-i)} = e^{-(\Delta_0-(\gamma_0-3\alpha_0))(n_i+\chi(q_i))+\Delta_0 i}.$$

Substituting $n_j \le i$ and $\chi(q_j) \le 2\alpha_0 n_j/\tau_0$, which follows from Proposition 3.1 and $(A1)$, yields

$$\begin{aligned}\|w_i\| &\ge e^{-(\Delta_0-(\gamma_0-3\alpha_0))(1+2\alpha_0/\tau_0)n_j+\Delta_0 i} \\ &\ge e^{-(\Delta_0-(\gamma_0-3\alpha_0))(1+12\alpha_0/\tau_0)i+\Delta_0 i} \\ &\ge \lambda^i.\end{aligned}$$

By the above claim, $e_i(\zeta)$ is well defined for all $\zeta \in \gamma$ and all $1 \le i \le p$. We introduce a local coordinate transformation $\phi(\xi, \eta) = (x, y)$ around $\zeta = (\zeta_x, \zeta_y) \in \gamma$ by the formula

$$\begin{pmatrix} x - \zeta_x \\ y - \zeta_y \end{pmatrix} = \int_0^\xi f_{n_j+\chi(q_j)}(s)ds + \int_0^\eta e_{n_j+\chi(q_j)}(s)ds.$$

Define analogously a local coordinate transformation $\phi'(\xi', \eta') = (x, y)$ around $\zeta_{n_j+\chi(q_j)}$, using $e^{(n_j+\chi(q_j))}$, $f^{(n_j+\chi(q_j))}$ instead of $e_{n_j+\chi(q_j)}$, $f_{n_j+\chi(q_j)}$. Notice that ϕ and ϕ' are isometries. Denote by s an arc length parameter and by $\kappa_i(s)$ the curvature of γ_i at $F^i(\gamma(s))$. Denote by $\overline{\kappa}(s)$ (resp. $\overline{\kappa}_{n_j+\chi(q_j)}(s)$) the curvature of γ (resp. $\gamma_{n_j+\chi(q_j)}$) at $\gamma(s)$ (resp. $F^{n_j+\chi(q_j)}(\gamma(s))$) with respect to the coordinate (ξ,η) (resp. (ξ',η')). Corollary 2.1 immediately gives $\|D\phi-\mathrm{Id}\|,\|D\phi'-\mathrm{Id}\|\leq Cb$ and $\|D^2\phi\|,\|D^2\phi'\|\leq Cb$. This implies that $\overline{\kappa}(s)$ (resp. $\overline{\kappa}_p(s)$) differs from $\kappa(s)$ (resp. $\kappa_p(s)$) only by at most $\sqrt{b}$. Thus, the proof completes if we show $\overline{\kappa}_{n_j+\chi(q_j)}(s)\leq\theta_0^4$ for all s.

Since γ is a horizontal vector, we have $\kappa(s)\leq\theta_0^3$. Put $\overline{F}(\xi,\eta)=(\phi')^{-1}\circ F^{n_j+\chi(q_j)}\circ\phi(\xi,\eta)$. Let $J\colon s\to J(s)\in\mathbf{R^2}$ be the local expression of γ with respect to the coordinate (ξ,η), i.e. $J(s)$ is defined for sufficiently small s, with $J(0)=(0,0)$ and $\phi_\zeta(J(s))\subset\gamma$. The remark on ϕ and ϕ' imply $\|\overline{F}(J(s))'\|\geq 1$, and therefore

$$\frac{\|J(s)'\|^3}{\|\overline{F}(J(s))'\|^3}|\det D\overline{F}|\overline{\kappa}(0)\leq(\theta_0^3+Cb)(Kb)^{n_j+\chi(q_j)}\leq b^{\frac{n_j+\chi(q_j)}{2}}.$$

Put $A=\|DF^{n_j+\chi(q_j)}f_{n_j+\chi(q_j)}\|$ and $B=\|DF^{n_j+\chi(q_j)}e_{n_j+\chi(q_j)}\|$. We clearly have

$$D\overline{F}J'(s)=\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}J(s)',$$

and

$$D^2\overline{F}(J'(s),J'(s))=\begin{pmatrix}\langle\nabla_{\xi,\eta}A,J(s)'\rangle, & 0\\ 0, & \langle\nabla_{\xi,\eta}B,J(s)'\rangle\end{pmatrix}\cdot J(s)',$$

and therefore

$$\|D\overline{F}J'(s)\times D^2\overline{F}(J'(s),J'(s))\|=|A\langle\nabla_{\xi,\eta}B,J(s)'\rangle-B\langle\nabla_{\xi,\eta}A,J(s)'\rangle|\leq\sqrt{b},$$

where the inequality follows from Proposition 2.2. Altogether these imply

$$\begin{aligned}\overline{\kappa}_{n_j+\chi(q_j)}(s)&=\frac{\|\overline{F}(J(s))'\times\overline{F}(J(s))''\|}{\|\overline{F}(J(s))'\|^3}\\&\leq\frac{\|J(s)'\|^3}{\|\overline{F}(J(s))'\|^3}|\det D\overline{F}|\overline{\kappa}(0)+\frac{\|D\overline{F}J'(s)\times D^2\overline{F}(J'(s),J'(s))\|}{\|\overline{F}(J(s))'\|^3}\\&\leq b^{\frac{n_j+\chi(q_j)}{2}}+\sqrt{b}\\&\leq\theta_0^4,\end{aligned}$$

where all the derivatives are taken at $s=0$. □

Proof of Lemma 3.5. To estimate the distortion, we firstly consider the contribution from the concatenation of the orbit segments contained in $D\setminus\mathcal{C}^{(0)}$. To ease notation, we denote by $t(\zeta_i)$ the unit tangent vector of γ_i at $\zeta_i\in\gamma_i$. Denote by $\tilde{d}$ the distance on the tangent bundle $T\mathbf{R^2}$ defined by

$$\tilde{d}(v(z),\tilde{v}(\tilde{z}))=d(z,\tilde{z})+\|v-\tilde{v}\|.$$

Proposition 2.1 gives $\|DF^it(\zeta_{n_{j+1}-i})\|\geq 1$ for all $1\leq i\leq n_{j+1}-n_j-\chi(q_j)$. On the other hand, (3.14) gives $d(\zeta_{n_{j+1}},\tilde{\zeta}_{n_{j+1}})\leq\delta_0^2$, and therefore we have

$d(\zeta_i,\tilde{\zeta}_i)\le\delta_0^2$ for all $n_j+\chi(q_j)\le i\le n_{j+1}$. Using this inequality and $d(\zeta_i,\tilde{\zeta}_i)\le 2(1+\theta_0)e^{-1.5\alpha_0 i}$, we have $d(\zeta_i,\tilde{\zeta}_i)\le\delta_0\sqrt{2}e^{-1.5\alpha_0 i/2}$. Since γ_i is a horizontal curve for all $i\in[n_j+\chi(q_j),n_{j+1}]$ by Lemma 3.4, we have $\tilde{d}(t(\zeta_i),t(\tilde{\zeta}_i))\le d(\zeta_i,\tilde{\zeta}_i)(1+10\theta_0)\le(1+10\theta_0)\delta_0\sqrt{2}e^{-1.5\alpha_0 i/2}$. The mean value theorem then gives

$$\begin{aligned}|\log\|DFt(\zeta_i)\|-\log\|DFt(\tilde{\zeta}_i)\||&\le\frac{2e^{\Delta_0}}{\delta_0}\tilde{d}(t(\zeta_i),t(\tilde{\zeta}_i))\\&\le 2\sqrt{2}e^{\Delta_0}(1+10\theta_0)e^{-1.5\alpha_0 i/2},\end{aligned}$$

and moreover

$$\begin{aligned}&\left|\log\frac{\|DF^{n_{j+1}-n_j-\chi(q_j)}t(\zeta_{n_j+\chi(q_j)})\|}{\|DF^{n_{j+1}-n_j-\chi(q_j)}t(\tilde{\zeta}_{n_j+\chi(q_j)})\|}\right|\\&\le\sum_{i=n_j+\chi(q_j)}^{n_{j+1}}|\log\|DFt(\zeta_i)\|-\log\|DFt(\tilde{\zeta}_i)\||\\&\le 3e^{\Delta_0}\sum_{i=n_j+\chi(q_j)}^{n_{j+1}}e^{-1.5\alpha_0 i/2}.\end{aligned}$$

On the other hand, (3.11) gives

$$\frac{|\xi(\zeta_{n_j+1})|}{|\xi(\zeta_{n_j+1})|}\le(1+\tilde{\alpha})^2,$$

where $\xi(\cdot)$ are those obtained in the orthogonal decomposition in Lemma 3.1. Using Proposition 2.2–7 in addition yields

$$\begin{aligned}\frac{\|DF^{\chi(q_j)+1}t(\zeta_{n_j})\|}{\|DF^{\chi(q_j)+1}t(\tilde{\zeta}_{n_j})\|}&\le(1+\theta_0)\frac{|\xi(\zeta_{n_j+1})|}{|\xi(\zeta_{n_j+1})|}\cdot\frac{\|DF^{\chi(q_j)}f_{\chi(q_j)}(\zeta_{n_j+1})\|}{\|DF^{\chi(q_j)}f_{\chi(q_j)}(\tilde{\zeta}_{n_j+1})\|}\\&\le(1+2\theta_0)e^2.\end{aligned}$$

Letting $n_{k+1}=p(z)$ we obtain

$$\begin{aligned}\frac{\|DF^p t_\gamma(\zeta)\|}{\|DF^p t_\gamma(\tilde{\zeta})\|}&\le\prod_{j=1}^{k}\frac{\|DF^{\chi(q_j)+1}t(\zeta_{n_j})\|}{\|DF^{\chi(q_j)+1}t(\tilde{\zeta}_{n_j})\|}\cdot\prod_{j=0}^{k}\frac{\|DF^{n_{j+1}-n_j-\chi(q_j)}t(\zeta_{n_j+\chi(q_j)})\|}{\|DF^{n_{j+1}-n_j-\chi(q_j)}t(\tilde{\zeta}_{n_j+\chi(q_j)})\|}\\&\le((1+2\theta_0)e^2)^k 3e^{\Delta_0}\sum_{i=1}^{\infty}e^{-1.5\alpha_0 i/2}.\end{aligned}$$

□

3.4. Binding period for host intervals

Fix a positive number g satisfying

$$g\ge\Delta_0/\gamma_0+2. \tag{3.13}$$

We say a horizontal curve $\gamma \subset \mathcal{C}^{(0)}$ is a *host interval* if there exist $z \in \gamma$ such that the tangent vector of γ at z is in admissible position relative to a critical point c, and $\text{length}(\gamma) \leq d(c,z)^g$ holds. In this case, all the tangent vectors of γ are in admissible position relative to c, and we say γ is centered around z. Define by $p_\gamma = p(z)$ the binding period of the host interval γ.

Proposition 3.3. *If γ is a host interval centered around $z \in \gamma$, then $\|DF_\zeta^{p_\gamma} t_\gamma(\zeta)\| \geq e^{\gamma_0 p_\gamma/3}$ holds for all $\zeta \in \gamma$. Moreover, γ_{p_γ} is a horizontal curve.*

Proof of Proposition 3.3. Let $\{n_j\}_{j=0}^k$, $\{q_j\}_{j=0}^k$, $\{c^{(j)}\}_{j=0}^k$ be the same as before: the sequence of returns, the corresponding folding periods and the critical points concerning the definition of $p(z)$.

Lemma 3.6. *For all $0 \leq j \leq k$ and all $\zeta \in \gamma$, $DF^{n_j} t_\gamma(\zeta)$ is a horizontal vector in admissible position relative to $c^{(j)}$, and there exists small $\tilde{\alpha} > 0$ depending only on δ_0 such that we have*

$$1 - \tilde{\alpha} \leq \frac{\log d(c^{(j)}, \zeta_{n_j})}{\log d(c^{(j)}, z_{n_j})} \leq 1 + \tilde{\alpha}. \tag{3.14}$$

Proof. For $j = 0$, the first half of the claim follows from the definition of the host interval. On the second half, since $g \geq 2$, and γ is centered around z, we have $\text{length}(\gamma) \leq d(c,z)^g \leq d(c,z)^2$. Therefore there exists small $0 < \tilde{\alpha} < 1$ depending only on δ_0 which gives $d(c,z)^{1+\tilde{\alpha}} \leq d(c,z) - \text{length}(\gamma)$ and $d(c,z) + \text{length}(\gamma) \leq d(c,z)^{1-\tilde{\alpha}}$. Since all the tangent vectors of γ are horizontal, these two inequalities yield

$$d(c,z)^{1+\tilde{\alpha}} \leq d(c,\zeta) \leq d(c,z)^{1-\tilde{\alpha}},$$

and therefore (3.14) for $j = 0$.

Suppose that the claim is true for all $0 \leq j \leq i-1$. By Proposition 3.1, all the tangent vectors of $\gamma_{n_{i-1}+\chi(q_{i-1})}$ are horizontal, and the same is true to for γ_{n_i} by Proposition 2.1. The definition of g and $n_i \leq p$ give $\text{length}(\gamma_{n_i}) \leq d(c,z)^g e^{\Delta_0 n_i} \leq d(c,z)^g e^{\Delta_0 p} \leq d(c,z)^2$. On the other hand, Proposition 3.2 gives $p \leq 4 \log d(c,z)^{-1}/\gamma_0$, and therefore $d(c^{(i)}, z_{n_i}) \geq e^{-2\alpha_0 n_i} \geq d(c,z)^{8\alpha_0/\gamma_0}$ $\gg d(c,z)$ holds. Altogether these imply that all the tangent vectors of γ_{n_i} are in admissible position relative to the same critical point $c^{(i)}$. This proves the first half of the statement. Proving (3.14) for $j = i$ is almost identical to the case $j = 0$, by $\text{length}(\gamma_{n_i}) \leq d(c,z)^2$ and $d(c^{(i)}, z_{n_i}) \gg d(c,z)$, which follows from $(A1)$. □

By Lemmas 3.7, 3.1, and Proposition 3.1, the orbits of all points of γ share the same sequence of folding periods $\{q_j\}_{j=0}^k$, which implies $\|DF^{p_\gamma} t_\gamma(\zeta)\| \geq e^{\gamma_0 p_\gamma/3}$. This finishes the proof of the first half of the statement.

It follows from the previous argument that all the tangent vectors of γ_p are horizontal. Therefore, it is enough to give a curvature estimate to complete the proof of Proposition 3.3. We can do this similarly to the proof of the claim contained in Lemma 3.4.

Lemma 3.7. *Any $\zeta \in \gamma_1$ is λ-expanding up to time p.*

Proof. Let $u \in \gamma$ be one of the edges of γ which is further away from c. Let $\tilde{\gamma}$ be a horizontal curve connecting u_1 and the stable leaf $\Gamma(c_1)$. By Proposition 2.2–7, it is enough to show that all $\tilde{\zeta} \in \tilde{\gamma}$ is λ-expanding up to time $p(z)$, which has already been established in the proof of Lemma 3.4. □

The rest of the curvature estimate proceeds along the line with Lemma 3.4. We define two local coordinates $\phi(\xi,\eta) = (x,y)$ around $\zeta = (\zeta_x, \zeta_y) \in \gamma_1$, and $\phi'(\xi',\eta') = (x,y)$ and ζ_p, using $e^{(p)}$, $f^{(p)}$ and e_p, f_p. Denote by s an arc length parameter and by $\kappa_i(s)$ the curvature of γ_i at $F^i(\gamma(s))$. For $i = 1$ and p, denote by $\overline{\kappa}_i(s)$ the curvature of γ_i at $F^i(\gamma(s))$ in terms of the coordinate (ξ,η). Corollary 2.1 implies that $\overline{\kappa}_1(s)$ (resp. $\overline{\kappa}_p(s)$) differs from $\kappa_1(s)$ (resp. $\kappa_p(s)$) only by at most $\sqrt{b}$, and therefore the proof completes by showing $\overline{\kappa}_p(s) \leq \theta_0^4$ for all s. The only distinction from Lemma 3.4 is the curvature estimate of γ_1 which is not a horizontal curve. Lemma 3.1 gives

$$\begin{aligned}\kappa_1(s) &= \frac{\|F(\gamma(s))' \times F(\gamma(s))''\|}{\|F(\gamma(s))'\|^3} \\ &= \frac{\|DF_{\gamma(s)}\gamma'(s) \times (DF_{\gamma(s)}\gamma''(s) + D^2F_{\gamma(s)}(\gamma'(s),\gamma'(s)))\|}{\|DF_{\gamma(s)}\gamma'(s)\|^3} \\ &\leq 3e^{\Delta_0} d(c,u)^{-3},\end{aligned}$$

where u is the edge of γ closer to c. Put $\overline{F}(\xi,\eta) = (\phi')^{-1} \circ F^p \circ \phi(\xi,\eta)$. Let $J\colon s \to J(s) \in \mathbf{R^2}$ be a local expression of γ_1 in terms of the coordinate (ξ,η), i.e. $J(s)$ is defined for sufficiently small s, with $J(0) = (0,0)$ and $\phi_\zeta(J(s)) \subset \gamma_1$. We clearly have $\|\overline{F}(J(s))'\| \geq 1$, and therefore

$$\frac{\|J(s)'\|^3}{\|\overline{F}(J(s))'\|^3} |\det D\overline{F}| \overline{\kappa}_1(0) \leq (3e_0^{\Delta} d(c,u)^{-3} + Cb)(Kb)^p \leq \sqrt{b^p},$$

where we have used Proposition 3.2 to deal with the term $d(c,u)^{-3}$. From this, it is obvious that we can continue the curvature estimate of γ_p exactly in the same way as before. □

4. Construction of stable leaves, $\mathcal{B}^{(k)}$ and $\mathcal{P}^{(k)}$

The purpose of this section is to construct the bad region $\mathcal{B}^{(k)}$, and the associated partition $\mathcal{P}^{(k)}$. We begin with the construction of stable leaves in light of Proposition 2.3.

4.1. Controlled points are expanding

According to Proposition 2.3, stable leaves pass through expanding points. We now give a criterion for points to be expanding.

Proposition 4.1. *If $z \in F(\mathcal{C}^{(0)})$ is controlled up to time n, then z is λ-expanding up to time n.*

We put $w_n = DF_z^n \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ to ease notation.

Lemma 4.1. *Suppose that $z \in F(\mathcal{C}^{(0)})$ is controlled up to time n and n_0 is the smallest integer such that $z_{n_0} \in \mathcal{C}^{(0)}$. There exists a sequence $n_0 < n_1 < \cdots < n_k \le n$ such that $n_{i+1} = \min\{j \ge n_i + p_i + 1 : z_j \in \mathcal{C}^{(0)}\}$, w_{n_i} is a horizontal vector in admissible position, p_i the corresponding binding period.*

Proof. By Propositions 2.1 and 3.2, it is enough to show that if $z_j \in \mathcal{C}^{(0)}$ and w_j is a horizontal vector, then w_j is in admissible position. Suppose that $z_j \in \mathcal{C}^{(0)}$. Since z is controlled up to time n we have $d_{\mathcal{C}}(z_j) \ge e^{-3\mu_0 j}$, which implies there exists a critical point c of generation g such that the horizontal distance between c and z_j is larger than $e^{-3\mu_0 j}$. By the structure of the critical set, the width of the component $\mathcal{Q}^{(g)}$ containing z and c is smaller than $e^{-\beta_0 g}$. Thus we have $e^{-3\mu_0 j} \le d_{\mathcal{C}}(z_j) \le e^{-\beta_0 g}$, and therefore $g \le 3\beta_0^{-1}\mu_0 j$. Again, by the structure of the critical set, the height of $\mathcal{Q}^{(g)}$ is smaller than $b^{g/4} \le b^{3\beta_0^{-1}\mu_0 j/4} \ll e^{-3\mu_0 j}$, which implies that w_j is in admissible position relative to c. □

Corollary 4.1. *If $z \in F(\mathcal{C}^{(0)})$ is controlled all the time, then*

$$\limsup_{n\to\infty} \frac{1}{n} \log \|DF_z^n\| \ge \frac{\gamma_0}{3}.$$

Proof. By Proposition 2.1, the statement clearly holds if the forward orbit of z enters $\mathcal{C}^{(0)}$ only finitely many times. Otherwise, let $\{n_i\}_{i\ge0}$ be the infinite sequence obtained by Lemma 4.1. It is enough to show $\|w_{n_i+p_i+1}\| \ge e^{\gamma_0(n_i+p_i+1)/3}$ for all $i \ge 0$. We prove that if $\|w_{n_i}\| \ge e^{\gamma_0 n_i/3}$, then $\|w_{n_i+p_i+1}\| \ge e^{\gamma_0(n_i+p_i+1)/3}$ and $\|w_{n_{i+1}}\| \ge e^{\gamma_0 n_{i+1}/3}$ hold. This is indeed enough since Proposition 2.1 gives $\|w_{n_0}\| \ge e^{\gamma_0 n_0/3}$.

By Lemma 4.1, w_{n_i} is a horizontal vector in admissible position. Proposition 3.2 gives $\frac{\|w_{n_i+p_i+1}\|}{\|w_{n_i}\|} \ge e^{\gamma_0(p_i+1)/3}$, and therefore $\|w_{n_i+p_i+1}\| \ge e^{\gamma_0(n_i+p_i+1)/3}$. Since $w_{n_i+p_i+1}$ is a horizontal vector, Proposition 2.1 and $n_{i+1} \ge n_i + p_i + 1$ yield

$$\begin{aligned}\|w_{n_{i+1}}\| &\ge \frac{\|w_{n_{i+1}}\|}{\|w_{n_i+p_i+1}\|}\|w_{n_i+p_i+1}\| \\ &\ge e^{\gamma_0(n_{i+1}-(n_i+p_i+1))}e^{\gamma_0(n_i+p_i+1)/3} \ge e^{\gamma_0 n_{i+1}/3}.\end{aligned}$$

□

Proof of Proposition 4.1. We are going to establish the inequality $\|w_j\| \ge \lambda^j$ for all $1 \le j \le n$. There are three cases: $1 \le j \le n_0$, $n_i + p_i + 1 < j \le n_{i+1}$, or $n_i \le j \le n_i + p_i + 1$.

The inequality for the first case easily follows from $z \in F(\mathcal{C}^{(0)})$ and Proposition 2.1. The second case: the proof of Corollary 4.1 gives $\|w_{n_i+p_i+1}\| \ge e^{\gamma_0(n_i+p_i+1)/3}$. Proposition 2.1 gives

$$\|w_j\| \ge \frac{\|w_j\|}{\|w_{n_i+p_i+1}\|}\|w_{n_i+p_i+1}\| \ge \delta_0 e^{\gamma_0(j-(n_i+p_i+1))}e^{\gamma_0(n_i+p_i+1)/3} \ge \delta_0 e^{\gamma_0 j/3}$$

for all $n_i + p_i + 1 \leq j \leq n_{i+1}$. Suppose that w_{n_i} is in admissible position relative to a critical point c. Then we have $d(c, z_{n_i}) \geq e^{-3\mu_0 n_i}$. We also have $d(c, z_{n_i}) \leq \delta_0$ according to the structure of the critical set, and therefore

$$\|w_j\| \geq \delta_0 e^{\gamma_0 j/3} \geq e^{\gamma_0 j/3 - 3\mu_0 n_i} \geq e^{(\gamma_0/3 - 3\mu_0)j} \geq \lambda^j.$$

The third case: $\|w_{n_i+p_i+1}\| \geq e^{\gamma_0(n_i+p_i+1)/3}$ and $\|DF\| \leq e^{\Delta_0}$ gives

$$\|w_j\| \geq e^{\gamma_0(n_i+p_i+1)/3 - \Delta_0(n_i+p_i+1-j)} = e^{-(\Delta_0 - \gamma_0/3)(n_i+p_i)+\Delta_0 j}.$$

Substituting $n_i \leq j$ and $p_i \leq 12\mu_0 n_i/\gamma_0$ into this inequality gives

$$\|w_j\| \geq e^{-(\Delta_0-\gamma_0/3)(1+12\mu_0/\gamma_0)n_i+\Delta_0 j} \geq e^{-(\Delta_0-\gamma_0/3)(1+12\mu_0/\gamma_0)j+\Delta_0 j} \geq \lambda^j.$$

□

4.2. Finding controlled points

We show that there exist sufficiently many points which are controlled all the time. Moreover, there points are characterized by a sufficiently slow recurrence property to the critical set.

A host interval γ centered around z is called a *long host interval* if $1/10 \cdot d(c, z)^g \leq \text{length}(\gamma))$ holds. Otherwise, γ is called a *short host interval.*

Proposition 4.2. *For a long host interval γ contained in the unstable side of D_k, there exists $z \in \gamma$ such that*

$$d_{\mathcal{C}}(z_{n+1}) \geq e^{-\mu_0/2 \cdot n} \tag{4.1}$$

holds for all $n \geq 1$.

Lemma 4.2. *Let γ be a host interval contained in the unstable side of D_k, p its binding period, and $\xi = \min\{i \geq p : \gamma_i \bigcap \mathcal{C}^{(0)} \neq \emptyset\}$. Then all the tangent vectors of γ_ξ are in admissible position relative to the same critical point.*

Proof. By Propositions 2.1 and 3.3, γ_ξ is a horizontal curve. Let $\tilde{\gamma}$ be the maximal horizontal curve in the unstable side and containing γ_ξ. There are two cases; either $\tilde{\gamma}$ contains a critical point, or not. We are done in the first case and thus consider the second case. According to the structure of the critical set, it makes sense to speak about whether $\tilde{\gamma}$ is at the right or the left of the critical points on the unstable sides of $\mathcal{C}^{(k+\xi)}$. Without loss of generality we may assume that $\tilde{\gamma}$ is at the right of the critical points, and let $c \in D_g$ be the critical point of generation g whose forward iterate $c_{k+\xi-g} \in D_{k+\xi}$ is contained in the folded part at the left of $\tilde{\gamma}$. The assumption $(A1)$ implies $d_{\mathcal{C}}(c_{k+\xi-g}) \geq e^{-\alpha_0(k+\xi-g)}$. Thus the leftmost point z of $\tilde{\gamma}$ satisfies $d_{\mathcal{C}}(z) \geq e^{-\alpha_0(k+\xi-g)} \geq e^{-\alpha_0(k+\xi)}$, which implies that all the tangent vectors of γ_ξ are in admissible position. □

To ease notation, we put $d(\gamma) = -\log d(c, z)$ for a host interval γ centered around z.

Lemma 4.3. *Let γ be a long host interval with the binding period p. Then we have*

$$\mathrm{length}(\gamma_p) \geq e^{(\frac{4\gamma_0}{3\Delta_0}-g)d(\gamma)} \geq e^{p(\frac{\gamma_0}{3}-\frac{\Delta_0 g}{4})}.$$

Proof. Proposition 3.3 gives $\|DF^p t_\gamma(z)\| \geq e^{\gamma_0/3\cdot p}$ for all $z \in \gamma$, and thus $\mathrm{length}(\gamma_p) \geq \mathrm{length}(\gamma)e^{\gamma_0/3\cdot p}$. The definition of the long host interval gives

$$\mathrm{length}(\gamma)e^{\gamma_0/3\cdot p} \geq e^{\gamma_0/3\cdot p-gd(\gamma)} \geq e^{(\frac{4\gamma_0}{3\Delta_0}-g)d(\gamma)}.$$

The last inequality in the statement follows from (3.3). □

Lemma 4.4. *Let γ be a long host interval with the binding period p_0, and $p_0 \leq \eta_1 < \eta_1+p_1 \leq \eta_2 \leq \eta_2+p_2 \leq \eta_3 < \cdots < \eta_j < \eta_j+p_j$ be such that for all $1 \leq i \leq j$, γ_{η_i} is a short host interval, p_i the corresponding binding period, and $\gamma_k \bigcap \mathcal{C}^{(0)} = \emptyset$ for all $\eta_i + p_i \leq k \leq \eta_{i+1}-1$. Then we have*

$$d(\gamma_{\eta_i}) \leq g\Delta_0\gamma_0^{-1}\cdot d(\gamma).$$

Proof. Proposition 3.3 gives $\|DF_z^{\eta_j+p_j}t_\gamma(z)\| \geq e^{\gamma_0/3\cdot\sum_{i=0}^{j}p_i}$ for all $z \in \gamma$, and therefore $\mathrm{length}(\gamma_{\eta_j+p_j}) \geq \mathrm{length}(\gamma)e^{\gamma_0/3\cdot\sum_{i=0}^{j}p_i}$. Since $\gamma_{\eta_j+p_j}$ is a horizontal curve, we have $\mathrm{length}(\gamma_{\eta_j+p_j}) \leq 2$. On the other hand, the definition of the long host interval gives $\mathrm{length}(\gamma) \geq 1/10\cdot e^{-gd(\gamma)}$. Combining these three inequalities and taking logs of the both sides, we obtain

$$\frac{4gd(\gamma)}{\gamma_0} \geq \sum_{i=0}^{j}p_j \geq p_i \geq \frac{4d(\gamma_{\eta_i})}{\Delta_0},$$

where the last inequality follows from Proposition 3.2. □

Proof of Proposition 4.2. Let γ be bound to a critical point c and p_0 the binding period. By the relation $e^{-\mu_0} < \delta_0$, it is enough to establish the inequality (4.1) for all z and $n \geq 1$ such that $z_n \in \mathcal{C}^{(0)}$.

Suppose that there exist $z \in \gamma$ and $0 \leq i \leq p_0$ such that $z_{i+1} \in \mathcal{C}^{(0)}$. If $c_{i+1} \in \bar{\mathcal{C}}^{(0)}$, then the assumption $(A1)$, Lemma 3.2, and $\mu_0 \gg \alpha_0$ give $d_{\mathcal{C}}(z_{i+1}) \geq e^{-\mu_0 i/2}$. If $c_{i+1} \notin \bar{\mathcal{C}}^{(0)}$, put $M_0 := \min\{m \geq 1 : F^m(\mathcal{C}^{(0)}) \cap \mathcal{C}^{(0)} \neq \emptyset\}$. A simple calculation on the map g_0 gives $M_0 > -2/\Delta_0 \cdot \log\delta_0$. We have $i \geq M_0$ since $z \in \mathcal{C}^{(0)}$, and therefore Lemma 3.2 gives

$$\begin{aligned} d_{\mathcal{C}}(z_{i+1}) &\geq \delta_0^{2.99\alpha_0/\Delta_0} - e^{-1.5\alpha_0 i} \geq \delta_0^{2.99\alpha_0/\Delta_0} - e^{-1.5\alpha_0 M_0} \\ &\geq \delta_0^{2.99\alpha_0/\Delta_0} - \delta_0^{3\alpha_0/\Delta_0} \geq 1/2\cdot\delta_0^{2.99\alpha_0/\Delta_0} > \delta_0, \end{aligned}$$

which yields a contradiction to $z_{i+1} \in \mathcal{C}^{(0)}$. In all, we have established (4.1) for all $z \in \gamma$ and $0 \leq i \leq p_0$.

We say a host interval γ centered around $z \in \gamma$ is an l-host interval ($0 < l \leq 1$) if $\mathrm{length}(\gamma) \leq l \cdot d(c,z)^g$ holds. The following holds by definition: any short host interval is a 1/10-host interval; the union of two adjacent l-host intervals

($l \le 1/3$) $\gamma^{(1)}$ and $\gamma^{(2)}$ which are centered around $z^{(1)}$ and $z^{(2)}$ respectively is a $3l$-host interval which is centered around both $z^{(1)}$ and $z^{(2)}$.

By Proposition 3.3 and Lemma 4.3, γ_{p_0} is a horizontal curve with $\text{length}(\gamma_{p_0}) \ge e^{(\frac{4\gamma_0}{3\Delta_0}-g)d(\gamma)}$. Let n_1 be the smallest integer $n_1 \ge p_0$ such that $\gamma_{n_1} \bigcap \mathcal{C}^{(0)} \ne \emptyset$. According to Lemma 4.2, all the tangent vectors of γ_{n_1} is in admissible position relative to the same critical point.

There are two cases to consider: either γ_{n_1} contains at least one long host interval, or not. In the first case, we cut γ_{n_1} into 1/3-host intervals. If there are (at most two) short host intervals left, we glue them to the adjacent 1/3-long host intervals and regard the union as one 1-host interval. In this way, we define a partition of γ into host intervals containing 1/3-long host intervals. Pick up one element from the partition of γ, say, $\gamma^{(1)}$, satisfying (4.1) for all $z \in \gamma^{(1)}$ and $1 \le i \le n_1$. The existence of such $\gamma^{(1)}$ is guaranteed by

$$\begin{aligned}\text{length}(\gamma_{n_1}) &\ge \text{length}(\gamma_{p_0}) \ge e^{(\frac{4\gamma_0}{3\Delta_0}-g)d(\gamma)}\\ &\ge 2e^{-\frac{2\mu_0}{\Delta_0}\cdot d(\gamma)} \ge 2e^{-p_0\mu_0/2} \ge 2e^{-n_1\mu_0/2},\end{aligned}$$

where the first inequality is by Proposition 2.1, the second already mentioned, the third because of large μ_0, the fourth by Proposition 3.2, the last by $n_1 \ge p_0$.

In the second case, γ_{n_1} is a short host interval by definition. We keep γ_{n_1} as it is and consider its further iterate. We now claim (4.1) for all $z \in \gamma^{(1)} := \gamma$ and $1 \le i \le n_1$. Lemma 4.4 gives $d(\gamma_{n_1}) \le g\Delta_0\gamma_0^{-1} \cdot d(\gamma)$. According to Proposition 3.2 and $p_0 \le n_1$, we have

$$g\Delta_0\gamma_0^{-1} \cdot d(\gamma) \le g\Delta_0\gamma_0^{-1} \cdot \frac{\Delta_0 p_0}{4} \le \frac{\mu_0 n_1}{2},$$

which proves the claim.

In both cases, (4.1) holds for all $z \in \gamma^{(1)}$ and $1 \le i \le n_1$. Let p_1 be the binding period of the host interval $\gamma^{(1)}_{n_1}$. Arguing similarly to the beginning of the proof, we can easily check that (4.1) holds for all $z \in \gamma^{(1)}$ and all $i \le n_1+p_1$. Clearly, we can repeat this procedure infinitely many times and eventually end up with an infinite nested sequence $\gamma =: \gamma^{(0)} \supset \gamma^{(1)} \supset \gamma^{(2)} \cdots$, and an infinite sequence $0 = n_0 < n_1 < n_2 \cdots$, with the properties that $\gamma^{(j)}_{n_j}$ is a 1/4-host interval containing an 1/8-long host interval with the binding period p_i, and (4.1) holds for for all $z \in \gamma^{(j)}$ and all $0 \le i \le n_j + p_j$. By construction, the point of the intersection $\bigcap_{j\ge 0}\gamma^{(j)}$ satisfies (4.1) for all $n \ge 1$. □

4.3. Itineraries

Let $z \in \mathcal{C}^{(0)}$ be a point on the unstable side of $\mathcal{Q}^{(k)}$ whose forward orbit does not hit the critical points, and p the binding period of a long host interval $\gamma^{(0)}$ containing z. Such $\gamma^{(0)}$ always exists according to the structure of the critical set. The proof of Proposition 4.2 leads us to the following combinatorial description of the pattern of the recurrence to the critical set. There exists an infinite sequence of integers $p \le \xi_1 < \xi_2 < \cdots$ and an infinite nested sequence $\gamma^{(0)} \supset \gamma^{(1)} \supset \gamma^{(2)} \supset \cdots$ of neighborhoods of z such that $\gamma^{(i)}_{\xi_i}$ is a long host

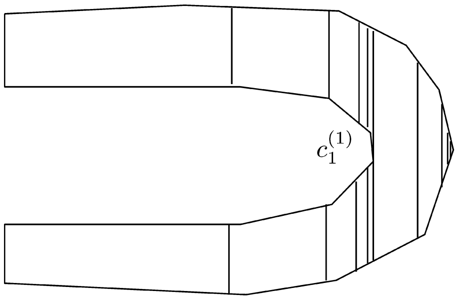

Figure 2. Partition of E_1

interval with the binding period p_i satisfying $\xi_i + p_i \le \xi_{i+1}$. There possibly exists a finite sequence of integers $\xi_i + p_i \le \eta_{i1} \le \cdots \le \eta_{i,m(i)} < \xi_{i+1}$ such that $\gamma^{(i)}_{\xi_{ij}}$ is a short host interval with the binding period p_{ij} satisfying $\eta_{ij} + p_{ij} \le \eta_{i,j+1}$. Moreover we have $\gamma^{(i)}_k \bigcap \mathcal{C}^{(0)} = \emptyset$ for all $\eta_{ij} + p_{ij} \le k \le \eta_{i,j+1} - 1$. An *itinerary* for z is an infinite sequence $I(z) = \{I_0, I_1, \cdots\}$ such that each I_t consists of the triple $(\gamma^{(i)}, \xi_i, p_i)$, or $(\gamma^{(i)}, \eta_{ij}, p_{ij})$. Both ξ_i and η_{ij} are called *free return.* For a free return ν with the binding period p, the interval $[\nu, \nu + p]$ is called *a bound state.* We say $n \in \mathbf{N}$ is *free* if n does not belong to any bound state of the preceding free returns. The notion of itinerary becomes important in the proof of Proposition 6.1.

4.4. Construction of $\mathcal{B}^{(k)}$

We fix one component $\mathcal{Q}^{(k)}$ of $\mathcal{C}^{(k)}$, and define its subset $\mathcal{A}^{(k)}$ in the following way. Let $U^{(u)}$, $U^{(l)}$ be the upper and lower unstable side of $\mathcal{Q}^{(k)}$. According to the structure of the critical set, $U^{(u)}$ contains the unique critical point c. On the unstable side $U^{(u)}$ there are exactly two points z, z' whose distance to c is equal to $e^{-2\mu_0 k}$. Let γ, γ' be two long host intervals containing z, z' respectively. By Proposition 4.2, there exists $\zeta \in \gamma_1$ such that $d_{\mathcal{C}}(\zeta_i) \ge e^{-\mu_0 i/2}$ holds for all $i \ge 0$. In particular, ζ is controlled all the time, and therefore the stable leaf $\Gamma(\zeta)$ through ζ exists. The same goes with respect to γ'_1; there exists $\zeta' \in \gamma'_1$ with the same property and the stable leaf $\Gamma(\zeta')$ through ζ'. We define $\mathcal{A}^{(k)}$ to be the region bounded by $U^{(u)}$, $U^{(l)}$, and two paraboras $F^{-1}(\Gamma(\zeta))$, $F^{-1}(\Gamma(\zeta'))$. The bad region $\mathcal{B}^{(k)}$ is the collection of all these $\mathcal{A}^{(k)}$. By construction, $\mathcal{B}^{(k)}$ contains all $z \in D_k$ such that any horizontal vector $v(z) \in T_z\mathbf{R}^2$ is not in admissible position relative to the critical set.

4.5. Penetrating set

We say a closed set $E \subset \mathcal{A}^{(k)}$ diffeomorphic to a rectangle is a *penetrating set*, or more precisely, a penetrating set through $\mathcal{A}^{(k)}$, if E is bounded by the parabolas $F^{-1}(\Gamma(\zeta))$, $F^{-1}(\Gamma(\zeta'))$ forming the boundary of $\mathcal{A}^{(k)}$, and by two horizontal curves contained in the unstable sides. According to the structure of the critical set, there exists a unique critical point on each unstable side of any penetrating set E. By definition, $\mathcal{A}^{(k)}$ itself is also a penetrating set.

4.6. Construction of a partition of penetrating sets

We define a partition $\mathcal{P}_E^{(k)}$ of a penetrating set $E \subset \mathcal{A}^{(k)}$ in the following way. Let $c^{(1)}$, $c^{(2)}$ be the two critical points on the unstable sides of E such that the location of their images is as in Fig. 2. We call the unstable side of E containing $c^{(1)}$ *near side*, and the other *far side*. As before, we cut the unstable sides of E at the right of the two critical points into a collection of 1/3-host intervals which contain a 1/9-long host interval. There exists at least one stable leaf which is associated to each of these host intervals in the sense of Proposition 4.2. Among those stable leaves associated to each of the host intervals in the near side, we pick up only one leaf, and define a collection of stable leaves $A^{(1)}$. Do the same thing with respect to the far side: pick up only one leaf from those associated to each 1/3 host interval in the far side, and define a collection of leaves $A^{(2)}$. Subtract from $A^{(2)}$ those leaves which lie at the left of $\Gamma(c_1^{(1)})$, and denote by $\tilde{A}^{(2)}$ the remaining collection. Define a pre-partition of E by the collection of stable leaves $A^{(1)} \cup \tilde{A}^{(2)} \cup \Gamma(c_1^{(1)})$. Define a partition $\mathcal{P}_E^{(k)}$ of E by gluing[*6] the two adjacent elements of the pre-partition. By construction, $\mathcal{P}_E^{(k)}$ contains countably many elements which we call *rectangles*. A rectangle is bounded by two pieces of the unstable sides of E and two stable leaves. If $E = \mathcal{A}^{(k)}$, we simply write $\mathcal{P}^{(k)} = \mathcal{P}_{\mathcal{A}^{(k)}}^{(k)}$. For any rectangle $R \in \mathcal{P}_E^{(k)}$, if R_1 is at the right (resp. left) of $\Gamma(c_1^{(1)})$, then $\gamma_R^{(3)}$ denotes the unstable side of R which is at the right of $c^{(2)}$ (resp. $c^{(2)}$). Denote by $\gamma_R^{(4)}$ the unstable side of R which is different from $\gamma_R^{(3)}$.

The following lemma is obvious.

Lemma 4.5. *Let γ be a horizontal curve, and suppose that $\gamma = \gamma' \cup \gamma'' \cup \gamma'''$ is a partition of γ into three interior disjoint 1/3-host intervals. Suppose that γ'' is centered around ζ. For any $z' \in \gamma'$ and $z''' \in \gamma'''$, the horizontal curve connecting z' and z''' inside γ is a host interval centered around ζ and contains γ''.*

By construction and Lemma 4.5, $\gamma_R^{(3)}$ is a host interval which contains a 1/9-long host interval centered around z. Moreover, $\gamma_R^{(3)}$ itself is also centered around z.

5. Construction of $T^{(j)}(R)$ and $\mathcal{S}^{(j)}(R)$ for $R \in \mathcal{P}^{(k)}$

The purpose of this section is to construct for each rectangle $R \in \mathcal{P}^{(k)}$ a nested sequence $\{T^{(j)}(R)\}_{j \geq 0}$ and an associated sequence of partitions $\{\mathcal{S}^{(j)}(R)\}_{j \geq 0}$. We begin with defining a suitable binding period for each rectangle contained in penetrating sets, and analyze the geometry of its forward iterates.

[*6] We do this because two adjacent stable leaves may be too close to each other.

5.1. Binding period for rectangles

We continue using the same notation as before: $E \subset \mathcal{A}^{(k)}$ is a penetrating set through $\mathcal{A}^{(k)}$, with critical points $c^{(1)}$ and $c^{(2)}$ on its unstable sides. The binding period p_R of a rectangle $R \in \mathcal{P}_E^{(k)}$ is defined by $p_R = p_{\gamma^{(3)}}$. Let us record the inequality

$$p_R \geq 4\mu_0 k/\Delta_0, \tag{5.1}$$

which follows from the construction of $\mathcal{P}_E^{(k)}$ and Proposition 3.2.

Proposition 5.1. *If $z \in R$ on the far side, then $p_R \geq \log d(c^{(2)}, z)^{-2/\Delta_0}$. If $z \in R$ is on the near side, then $p_R \geq \log d(c^{(1)}, z)^{-2/\Delta_0}$.*

Proof. There are three cases: (i) R_1 is at the right of $\Gamma(c_1^{(1)})$ and z is on the far side, (ii) R_1 is at the left of $\Gamma(c_1^{(1)})$ and z is on the far side, (iii) R_1 is at the left of $\Gamma(c_1^{(1)})$ and z is on the near side. The proofs of (i) and (iii) are identical, so it is enough to give a proof for (i) and (ii).

Case (i): suppose that $\gamma^{(3)}$ is centered around $\tilde{z}$ which defines the binding period $p_{\gamma^{(3)}}$, namely $p_R = p_{\gamma^{(3)}} = p(\tilde{z}) \geq \log d(c^{(2)}, \tilde{z})^{-4/\Delta_0}$. This immediately implies $p_R \geq \log d(c^{(2)}, z)^{-3/\Delta_0}$ for all $z \in \gamma^{(3)}$, due to the upper bound of the length of the host intervals. If $z \notin \gamma^{(3)}$, let z' be the edge of $\gamma^{(3)}$ which is closer to $c^{(2)}$. Let z'' be the edge of $\gamma^{(4)}$ which is closer to $c^{(2)}$. By the construction of $\mathcal{P}_E^{(k)}$, there exists a stable leaf connecting z_1' and z_1''. By Proposition 2.3, the holonomies associated with the lamination of the stable leaves are Lipschitz continuous with Lipshcitz constant $\leq e^{C\sqrt{b}}$. Therefore, denoting by $d_h(\cdot)$ the distance to the stable leaf $\Gamma(c_1^{(2)})$, we have $d_h(z_1'') \leq e^{C\sqrt{b}} d_h(z_1')$. On the other hand, (3.6) gives $d(z', c^{(2)})^3 \leq d_h(z_1') \leq d(z', c^{(2)})$ and $d(z'', c^{(2)})^3 \leq d_h(z_1'') \leq d(z'', c^{(2)})$, and therefore $d(z'', c^{(2)})^3 \leq e^{C\sqrt{b}} d(z', c^{(2)})$. Taking logs of both sides and plugging them into the inequality $p_R \geq \log d(c^{(2)}, z')^{-3/\Delta_0}$ gives the desired inequality.

Case (ii): let z' be the edge of $\gamma^{(3)}$ closer to $c^{(1)}$. Let z'' be the edge of $\gamma^{(4)}$ closer to $c^{(2)}$. Since the stable sides of R has a parabolic shape, we clearly have $d(c^{(2)}, z'') \geq (1-\theta_0) d(c^{(1)}, z')$. Substituting this into $p_R \geq \log d(c^{(1)}, z')^{-3/\Delta_0}$ gives the desired inequality. □

5.2. Close return times

Suppose that $z \in F(\mathcal{C}^{(0)})$ is controlled up to time $\nu - 1$. We say ν is a *close return time* for z if $z_\nu \in \mathcal{B}^{(\nu)}$. Remark that $z \in F(\mathcal{C}^{(0)})$ may be controlled up to time ν even if ν is a close return time for z. We say a close return time ν for $z \in F(\mathcal{C}^{(0)})$ is the *first close return time* for z if there is no close return time before ν.

Proposition 5.2. *Any $n < p_R$ is not a close return time for any $z \in R_1$.*

Proof. We first observe that any $n < p_R$ is not a close return time for any $z \in \gamma_1^{(3)}$, which is because the forward iterates of $\gamma^{(3)}$ up to time p_R is bound to the critical orbit which satisfies the condition $(A1)$. Moreover, any $n < p_R$ is not a close return time for any point on the stable sides of R_1, because there exists a point on each stable side whose speed of recurrence to the critical set is sufficiently slow (Proposition 4.2), and the distance between any two points on the same stable side is exponentially contracted at a rate faster than Cb (Proposition 2.2).

Suppose that $n < p_R$ is the first occurrence of a close return. Then, by the above observations, $\gamma_{n+1}^{(4)}$ must have a hook, namely, $\gamma_{n+1}^{(4)} \bigcap \mathcal{A}^{(n)}$ consists of two horizontal curve, and $\gamma_{n+1}^{(4)}$ has a folded part on the opposite side of $\gamma_{n+1}^{(3)}$ with respect to $\mathcal{A}^{(n)}$. This implies that there exists some $i < n$ such that $\gamma_i^{(4)}$ contains a critical point. This contradicts the assumption that n is the first occurrence of a close return time. □

Proposition 5.3. *Let $R \in \mathcal{P}_E^{(k)}$ and p the binding period. Then the unstable side of R_p consists of two horizontal curves.*

Proof. By Proposition 5.2 and Proposition 4.2, all points $z \in R_1$ is λ-expanding up to time p, and therefore $e_i(z)$ is defined for all $z \in R_1$ and $1 \leq i \leq p$. Let $\sigma^{(1)}$, $\sigma^{(2)}$ be the two stable sides of R. Proposition 2.2 gives $\text{length}(\sigma_p^{(i)}) \leq (Cb)^p$. On the other hand, $\gamma_{p+1}^{(3)}$ is a horizontal curve with $\text{length}(\gamma_{p+1}^{(3)}) \geq e^{-2\alpha_0 d(c,z)} \gg (Cb)^p$. By Proposition 2.2, F^p sends the integral curves of the vector field e_p on R_1 to the integral curves of the vector field $e^{(p)}$ defined on R_p. Moreover these integral curves are nearly vertical. Thus if we take a mid point $z^{(3)}$ of $\gamma_{p+1}^{(3)}$, then the integral curve passing through $z^{(3)}$ does not intersect the stable sides, and intersects $\gamma_{p+1}^{(4)}$ at a point, denoted by $z^{(4)}$.

Proposition 2.2 gives

$$\text{angle}(DF^p f_p(z_{-p}^{(3)}), DF^p f_p(z_{-p}^{(4)})) \leq Cb \cdot d(z_{-p}^{(3)}, z_{-p}^{(4)}) \leq Cb.$$

Consider the decomposition $t(z_{-p}^{(i)}) = \xi f_p(z_{-p}^{(i)}) + \eta e_p(z_{-p}^{(i)})$. Propositions 5.1 and 3.2 give $|\xi| \geq e^{-\Delta_0 p/4}$. Since $z^{(i)}$ is λ-expanding up to time p, using Proposition 2.2 we have

$$\text{angle}(t(z^{(i)}), DF^p f_p(z_{-p}^{(i)})) \leq (Cb)^p \lambda^p e^{\Delta_0 p/4}.$$

Using the above for $i = 3, 4$ and the triangle inequality we obtain

$$\text{angle}(t(z^{(3)}), t(z^{(4)})) \leq Cb + 2(Cb)^p \lambda^p e^{\Delta_0 p/4} \leq \sqrt{b}.$$

Propositions 4.1 and 2.1 give $\text{slope}(t(z^{(3)})) \leq 2\theta_0$, and therefore the above inequality implies $\text{slope}(t(z^{(4)})) \leq 3\theta_0$. On the other hand, essentially the same reasoning as the curvature estimate in Proposition 3.3 yields that the curvature of $\gamma_{p+1}^{(4)}$ is smaller than θ_0^3 everywhere. Altogether these imply that the slope of all the tangent vectors of $\gamma_{p+1}^{(4)}$ is smaller than $4\theta_0$. □

5.3. Local geometry of rectangles at close return times

Denote by $\tilde{\mathcal{A}}^{(k)} \supset \mathcal{A}^{(k)}$ the region consisting of all points which is contained in $\mathcal{Q}^{(k)}$ and its distance to the critical point on the unstable side of $\mathcal{Q}^{(k)}$ is smaller than $e^{-\mu_0 k}$. The collection of $\tilde{\mathcal{A}}^{(k)}$ is denoted by $\tilde{\mathcal{B}}^{(k)}$. For a set $A \subset \mathbf{R^2}$ and $z \in A$, denote by $C(A; z)$ the component of A containing z.

Proposition 5.4. *Let $R \in \mathcal{P}_E^{(k)}$, and suppose that there exists $z \in R_1$ and $\nu \geq p$ such that ν is the first close return time for z. Then the set $C(R_{\nu+1} \bigcap \mathcal{B}^{(\nu)}; z_\nu)$ is a penetrating set through $C(\mathcal{B}^{(\nu)}; z_\nu)$. Moreover, the unstable sides of $C(R_{\nu+1} \bigcap \tilde{\mathcal{B}}^{(\nu)}; z_\nu)$ consist of two horizontal curves.*

Proof. Let $T^{(1)}(R)$ be the subset of R consisting of all points having their first close return times. Define a function ν which corresponds to each $z \in T^{(1)}(R)$ its first close return time $\nu(z)$. The range of ν is a sequence of integers $a_1 < a_2 < \cdots$ which is finite or infinite as the case may be. Define

$$C^{(i)} := \{z \in T^{(1)}(R) : \ \nu(z) = a_i\}.$$

Clearly $T^{(1)}(R) = \bigsqcup_i C^{(i)}$ holds. The set $C^{(i)}$ is not necessarily connected. Define a sequence of sets $\{E^{(i)}\}$ by $E^{(1)} = R_{a_1} \setminus C^{(1)}_{a_1}$, and $E^{(i)} = E^{(i-1)}_{a_i - a_{i-1}} \setminus C^{(i)}_{a_i}$ for $i \geq 2$. We prove the proposition by induction on i: (i) we first prove the statement for all $z \in C^{(1)}$. (ii) Next, suppose that we have established the statement for all $z \in C^{(i)}$ $(i \geq 1)$. This implies that we do not need to consider further forward iterates of $C^{(i)}_{a_i}$. Thus we restrict ourselves to forward iterates of $E^{(i)}$, and prove the statement for all $z \in C^{(i+1)}$.

Let c be one of the critical point on the unstable side of $\mathcal{A}^{(\nu)}$. We first observe that the forward iterates of stable sides of R do not come into $\tilde{\mathcal{A}}^{(\nu)}$ for the same reason as in Proposition 5.2. On the other hand, it is obvious from $(A1)$ that if z is a point on the unstable sides of R and $z_\nu \in \tilde{\mathcal{A}}^{(\nu)}$, then ν is a free return. In this case, Proposition 3.3 claims that there exists a horizontal curve which contains z. Since z is arbitrary, this finishes the proof of step (i).

The proof of step (ii) is analogous. The only distinction is that we need to care the forward iterates of stable leaves arising from the successive inductive construction of the sequence $\{E^{(i)}\}$. Again, by Propositions 4.2 and 2.2, they do not come inside $\tilde{\mathcal{A}}^{(\nu)}$. This completes the proof of Proposition 5.4. □

Since g_0 is unimodal, the set $E^{(i)}$ has at most $2^{a_i - p}$ components. Moreover, Proposition 5.4 claims that each of the components of $E^{(i)}$ gives rise to at most only one component of $C^{(i)}$. Therefore, the following corollary is obvious.

Corollary 5.1. *The number of the components of $C^{(i)}$ does not exceed $2^{a_i - p}$.*

5.4. Construction of $T^{(j)}(R)$ and $\mathcal{S}^{(j)}(R)$ for $R \in \mathcal{P}^{(k)}$

We have defined the set $T^{(1)}(R)$ in the proof of Proposition 5.4. By definition, for each $C^{(i)}$ there exists $k(i) \in \mathbf{N}$ such that each connected component of the set $C^{(i)}_{a_i}$ is a penetrating set through one of the components of $\mathcal{B}^{(k(i))}$.

Therefore, the partitions $\mathcal{P}^{(k(i))}$ of $\mathcal{A}^{(k(i))}$ induces a partition of $C^{(i)}$. As a result we obtain a partition of $T^{(1)}(R)$ which we denote by $\mathcal{S}^{(1)}(R)$. Each element of $\mathcal{S}^{(1)}(R)$ is bounded by the unstable side of R and backward iterates of stable leaves.

A successive use of Proposition 5.4 permits us to construct a nested sequence $\{T^{(j)}(R)\}_{j\geq 0}$ and a sequence of partitions $\{\mathcal{S}^{(j)}(R)\}_{j\geq 0}$ in the following way: put $\nu_0(z)=0$ for all $z\in T^{(1)}(R)$. For $j\geq 2$ we inductively define

$$\nu_{j-1}(z)=j-2+\sum_{i=1}^{j-2}\nu(z_{\nu_i(z)+1}) \text{ for } z\in T^{(j-1)}(R),$$

and

$$T^{(j)}(R):=\{z\in T^{(j-1)}(R): z_{\nu_{j-1}(z)+1} \text{ has the first close return time}\}.$$

The function ν_{j-1} take on different values $\tilde{a}_1<\tilde{a}_2<\cdots$. By Corollary 5.1, the level set $\tilde{C}^{(i)}$ corresponding to each $\tilde{a}_i$ has at most $2^{\tilde{a}_i-p}$ number of components, and there exists $\tilde{k}(i)\in\mathbf{N}$ such that each component of $\tilde{C}^{(i)}_{\tilde{a}_i}$ is a penetrating set through one of the component of $\mathcal{B}^{(\tilde{k}(i))}$. The situation being exactly the same as when we constructed $\mathcal{S}^{(1)}(R)$, we obtain a partition $\mathcal{S}^{(j)}(R)$ of $T^{(j)}(R)$ by the same procedure as before. For convenience we put $T^{(0)}(R)=R$ and $\mathcal{S}^{(0)}(R)=\{R\}$. This completes the inductive construction of $\{T^{(j)}(R)\}_{j\geq 0}$ and $\{\mathcal{S}^{(j)}(R)\}_{j\geq 0}$. Clearly, $\{T^{(j)}(R)\}_{j\geq 0}$ is a decreasing sequence of Borel sets, and $\mathcal{S}^{(j+1)}(R)$ is a refinement of $\mathcal{S}^{(j)}(R)|_{T^{(j+1)}(R)}$. By construction, for all $j\geq 1$ and $z\in T^{(j)}(R)$ there exists a sequence $0=\nu_0(z)<\nu_1(z)<\nu_2(z)<\cdots<\nu_j(z)$ with the property that for all $1\leq i\leq j$, $\nu^{(i)}(z)=\nu_i(z)-\nu_{i-1}(z)-1$ is a close return time for $z_{\nu_{i-1}(z)+1}$. Let us record the inequality

$$\nu^{(i)}\geq (4\mu_0/\Delta_0)^i\cdot k, \tag{5.2}$$

which is an immediate consequence of Proposition 5.2.

6. Metric estimates

In this section we prove the two key metric estimates mentioned in the very beginning of this paper. In what follows we use the following notation: for any $R\in\mathcal{P}^{(k)}$, $\{R^{(j)}\}_{j\geq 0}$ denotes any nested sequence contained in the product $\prod_{j\geq 0}\mathcal{S}^{(j)}(R)$, with $\{\nu_j\}_{j\geq 0}$ the corresponding sequence of close return times and the binding period $p^{(j)}$ of the rectangle $R^{(j)}_{\nu_j}$.

6.1. Area distortion bounds

Proposition 6.1. *There exists $\mathcal{D}=\mathcal{D}(F)$ such that*

$$\frac{|\det DF_z^{\nu^{(j)}}|}{|\det DF_{\tilde{z}}^{\nu^{(j)}}|}\leq\mathcal{D}$$

holds for all z, $\tilde{z}$ belonging to the same component of $F^{-\nu^{(j)}-1}(R^{(j-1)}_{\nu_j}\bigcap\tilde{\mathcal{B}}^{(\nu^{(j)})})$.

The distortion constant $\mathcal{D}$ does depend on F, and it does not matter for our argument. In the corresponding ([3], Proposition 4.6), the homogeneity is used to obtain a uniform distortion constant independent of F.

Proof of Proposition 6.1. We take full advantage of the notion of itineraries, and considerably save the amount of necessary calculations and constructions, compared with the proof of ([3], Proposition 4.6). Let $n \geq p^{(j-1)}$ be the smallest integer such that $R^{(j-1)}_{\nu_{j-1}+n+1} \bigcap \mathcal{C}^{(0)} \neq \emptyset$. By Propositions 2.1 and 5.3, the unstable sides of $R^{(j-1)}_{\nu_{j-1}+n+1}$ are horizontal curves. Moreover, by Lemma 4.2, all the tangent vectors of $\gamma^{(3)}_{n+1}$ are in admissible position relative to the same critical point c, where $\gamma^{(3)}$ the unstable side of $R^{(j-1)}_{\nu_{j-1}}$ defined in Section 4.6. Without loss of generality we may assume that n is not a close return time. Then, all the tangent vectors of the unstable sides of $R^{(j-1)}_{\nu_{j-1}+n+1}$ are in admissible position relative to c. Thus, for any z is the unstable side of $F^{-\nu^{(j)}-1}(R^{(j-1)}_{\nu_j} \bigcap \mathcal{B}^{(\nu^{(j)})})$, we can associate its itinerary $I(z) = \{I_1(z), \cdots\}$ introduced in Section 4.3 which begins from the time no earlier than $p^{(j-1)}$, namely, $I_1(z)$ records the first return of the orbit of z after time $p^{(j-1)}$.

Recall that we have a certain degree of freedom in encoding itineraries, namely, the same point can have different itineraries, depending on the choice of l-host intervals involved ($0 < l \leq 1$). In some relevant cases, this freedom allows us to adjust given itineraries of all the points in a small horizontal curve, say γ, and let all of them share the same itineraries, say, up to time n. If this is so, we say all the points on γ share the same itinerary up to time n.

Lemma 6.1. *All the points on any component γ of the unstable side of the set $F^{-\nu^{(j)}-1}(R^{(j-1)}_{\nu_j} \bigcap \mathcal{B}^{(\nu^{(j)})})$ share the same itinerary up to time $\nu^{(j)}$.*

Proof. We define for all $z \in \gamma$ its itinerary by considering only l-long host intervals with $l \leq 1/3$. We claim that there are only two cases.

Sublemma 6.1. *All the points of $\gamma_{-\nu^{(j)}-1}$ share the same itinerary, or there exists a decomposition $\gamma_{-\nu^{(j)}-1} = \gamma' \cup \gamma''$ into two interior disjoint host intervals such that all the points belonging to the same host interval share the same itinerary.*

Proof. If the first case does not occur, then there exists $z \in \gamma_{-\nu^{(j)}-1}$ and $p^{(j-1)} \leq n \leq \nu^{(j)}$ such that n is a free return for z and there exists an l-long host interval in the unstable side of $R^{(j-1)}_{\nu_{j-1}+n+1}$ which contains z_n. Suppose that n is the last free return among those with the same property and p the corresponding binding period. Proposition 3.2 gives $2\Delta_0^{-1} \log d(c, z_n)^{-2} \leq p$. By $(A1)$, the time $\nu^{(j)}$ remains free for any $z \in \gamma_{-\nu^{(j)}}$ regardless of any adjustment of itineraries. Therefore we have $p \leq \nu^{(j)} - p^{(j-1)}$, and in particular $2\Delta_0^{-1} \log(c, z_n)^{-2} \leq \nu^{(j)}$, or equivalently $d(c, z_n) \geq e^{-\Delta_0 \nu^{(j)}/4}$. By Lemma 4.3, we have

$$\text{length}(\tilde{\gamma}_p) \geq e^{-(g - 4\gamma_0/2\Delta_0)\Delta_0 \nu^{(j)}/4} \gg e^{-\mu_0 \nu^{(j)}},$$

where $\tilde{\gamma}$ is the l-long host interval containing z_n and taking part in the definition of the itineraries. Dividing the rest of the itineraries into free and bound states and successively applying Propositions 2.1, 3.2 give length$(\tilde{\gamma}_{\nu^{(j)}-n}) \geq$ length$(\tilde{\gamma}_p)$. These two inequalities rule out the possibility that three host intervals with different itineraries are contained in γ. □

Back to the proof of the lemma, there is nothing to prove in the first case of Sublemma 6.1. In the second case, there exists $z' \in \gamma'$ and $z'' \in \gamma''$ such that $I(z') \neq I(z'')$. Let ν be the smallest integer such that $I_\nu(z') \neq I_\nu(z'')$. There are only one possibility: $I_\nu(z')$ and $I_\nu(z'')$ record free returns contained in different l-host intervals. Recalling that the union of two adjacent l-host interval is a $3l$-host interval, we glue the two l-host intervals together and redefine the itineraries. Even if the itineraries do not still coincide, there are once again left only two possibilities according to Sublemma 6.1. Therefore, the same procedure can be continued and we eventually end up with coincident itineraries. □

Back to the proof of the Proposition, the mean value theorem gives

$$\begin{aligned}\left|\log \frac{|\det DF_z^{\nu^{(j)}}|}{|\det DF_{\tilde{z}}^{\nu^{(j)}}|}\right| &\leq \sum_{i=0}^{\nu^{(j)}-1}\left|\log \frac{|\det DF_{z_i}|}{|\det DF_{\tilde{z}_i}|}\right| \\ &\leq \sup_{z\in D}\|D\log|\det DF_z|\| \sum_{i=0}^{\nu^{(j)}-1} d(z_i,\tilde{z}_i).\end{aligned}$$

We now suppose that z and $\tilde{z}$ are on the same unstable side of $R^{(j-1)}_{\nu_{j-1}}$. If $R^{(j-1)}_{\nu_{j-1}+1}$ is at the right of the stable leaf $\Gamma(c_1^{(1)})$, then, clearly $d(c_{i+1}^{(1)}, z_{i+1}) \leq e^{-1.5\alpha_0 i}$ and $d(c_{i+1}^{(1)}, \tilde{z}_{i+1}) \leq e^{-1.5\alpha_0 i}$ hold for all $1 \leq i \leq p^{(j-1)}$, and thus $d(z_{i+1}, \tilde{z}_{i+1}) \leq 2e^{-1.5\alpha_0 i}$. If $R^{(j-1)}_{\nu_{j-1}+1}$ is at the left of $\Gamma(c_1^{(1)})$ and z and $\tilde{z}$ are on the near side of $R^{(j-1)}_{\nu_{j-1}+1}$, then we again have $d(z_{i+1}, \tilde{z}_{i+1}) \leq 2e^{-1.5\alpha_0 i}$ for all $1 \leq i \leq p^{(j-1)}$ for the same reason. If z and $\tilde{z}$ are on the far side, let z' and $\tilde{z}'$ be the points of the intersection between the near side and the stable leaves $\Gamma^{\nu^{(j)}}(z_{\nu^{(j)}})$ and $\Gamma^{\nu^{(j)}}_{\nu^{(j)}}(\tilde{z}_{\nu^{(j)}})$. Then we have $d(z'_{i+1}, \tilde{z}'_{i+1}) \leq e^{-1.4\alpha_0 i}$ for all $1 \leq i \leq p^{(j-1)}$, and therefore $d(z_{i+1}, \tilde{z}_{i+1}) \leq 2e^{-1.4\alpha_0 i}$ for all $1 \leq i \leq p^{(j-1)}$.

Denote by γ the horizontal curve connecting z and $\tilde{z}$ in the unstable side of $R^{(j-1)}_{\nu_{j-1}}$. By Lemma 6.1, all points of γ share the same itinerary up to time $\nu^{(j)}-1$. If γ_i is free, then we clearly have $\|DF^{\nu^{(j)}-i}t(\zeta_i)\| \geq 1$ for all $\zeta \in \gamma$, and moreover $d(z_i, \tilde{z}_i) \leq e^{-\mu_0\nu^{(j)}}$, which is because of $d(z_{\nu^{(j)}}, \tilde{z}_{\nu^{(j)}}) \leq e^{-\mu_0\nu^{(j)}}$. If γ_i is not free, then there exists a free return $\nu \geq p^{(j-1)}$ with the binding period p such that $\nu < i < \nu+p$. Since $\|DF^{\nu^{(j)}-\nu}t(\zeta_\nu)\| \geq e^{\gamma_0/3\cdot(\nu^{(j)}-\nu)}$ holds for all $\zeta \in \gamma$, we have $\|DF^{\nu+p-i}t(\zeta_i)\| \geq e^{\gamma_0/3\cdot(\nu^{(j)}-\nu)-(i-\nu)\Delta_0} \geq e^{\gamma_0/3\cdot(\nu^{(j)}-\nu)-p\Delta_0}$, and moreover $d(z_i, \tilde{z}_i) \leq e^{-\gamma_0/3\cdot(\nu^{(j)}-\nu)+p\Delta_0-\mu_0\nu^{(j)}} \leq e^{-0.5\mu_0\nu^{(j)}}$, which is because of $\Delta_0 \ll \mu_0$.

In all, we obtain

$$\sum_{i=0}^{\nu^{(j)}-1} d(z_i, \tilde{z}_i) \leq \nu^{(j)} e^{-0.5\mu_0 \nu^{(j)}} + \sum_{i=0}^{p^{(j-1)}} 2e^{-1.4\alpha_0 i}.$$

The right hand side is bounded by a universal constant C depending on α_0 and μ_0, because $\lim_{n\to\infty} nr^n = 0$ holds for $r \in (0,1)$.

On the other hand, by the exponential contraction along stable leaves, the same inequality as above is true for z, $\tilde{z}$ which are not on the same unstable side but are connected by an integral curve of the vector field $e_{\nu^{(n)}}$ defined on $R^{(n-1)}_{\nu_{n-1}+1}$. This completes the proof of Proposition 6.1. □

6.2. Local bounded geometry of unstable sides

Proposition 6.2. *There exists large $k_0 \in \mathbf{N}$ such that for any $k \geq k_0$ and any component $\mathcal{A}^{(k)}$ of $\mathcal{B}^{(k)}$, there exists $\mathcal{G} = \mathcal{G}(\mathcal{A}^{(k)})$ such that for any $R \in \mathcal{P}^{(k)}$, the following holds for all nested sequence $\{R^{(j)}\}_{j\geq 0} \in \prod_{j\geq 0} \mathcal{S}^{(j)}(R)$: for any $j \geq 1$ and any z, $\tilde{z}$ belonging to the unstables side of the same component of $R^{(j-1)}_{\nu_j} \bigcap \tilde{\mathcal{B}}^{(\nu^{(j)})}$, we have*

$$\text{angle}(t(z), t(\tilde{z})) \leq \mathcal{G} e^{10\Delta_0 \nu^{(j)}} d(z, \tilde{z}). \tag{6.1}$$

This proposition is a key estimate in the entire argument. If one starts with a rectangle R contained in a sufficiently small bad region, then the two unstable sides of any penetrating set associated with any close return time are roughly parallel. Due to the lack of the homogeneity of the Jacobian, the estimate does not have any global character, i.e. we need to restrict ourselves to sufficiently small bad regions, and the estimate contains the constant $\mathcal{G}$ which depends on the component of the bad region. In other words, we make a compromise to tolerate the absence of the homogeneity assumption. In [3] and [19], the homogeneity is used in a crucial way to deduce a global version of this proposition.

We prove the proposition by induction on j. In the sequel is given only a proof of the generic step of the induction, i.e. a proof of the statement for $j = n \geq 2$, assuming the statement for $j = n-1$. It is easy to see that arguments for $j = 1$ are almost identical to those for the generic step of the induction.

Proof of Proposition 6.2. Choose $\mathcal{G}$ in a way that $\text{angle}(t(z), t(\tilde{z})) \leq \mathcal{G} d(z, \tilde{z})$ holds for any $z, \tilde{z}$ on the unstable side of $\mathcal{A}^{(k)}$. Assume that (6.1) holds for $j = n-1$.

Lemma 6.2. *For any z and $\tilde{z}$ on the unstable side of of $R^{(n-1)}_{\nu_{n-1}}$,*

$$\text{angle}(DFt(z), DFt(\tilde{z})) \leq \mathcal{G} e^{\Delta_0 \nu^{(n)}} d(z_1, \tilde{z}_1).$$

Proof. We begin with two elementary observations. Recall that $\tilde{d}$ is the distance on the tangent bundle $T\mathbf{R^2}$ defined by

$$\tilde{d}(v(z), \tilde{v}(\tilde{z})) = d(z, \tilde{z}) + \|v - \tilde{v}\|.$$

Then, for any nonzero vectors v, $\tilde{v} \in T\mathbf{R}^2$ such that $\text{angle}(v,\tilde{v}) \ll 1$ and $\|v\|/\|\tilde{v}\| \sim 1$, there exists $C \sim 1$ independent of v, $\tilde{v}$ such that

$$\text{angle}(v,\tilde{v}) \leq \frac{C\tilde{d}(v,\tilde{v})}{\|v\|}. \tag{6.2}$$

The proof is left as an easy exercise.

Let DDF be the derivative of the tangent map $DF : (z,v) \to (Fz, DF_z v)$. Define

$$m(DDF_{(z,v)}) = \min\{\|DDF_{(z,v)}w\| : w \in TT\mathbf{R}^2, \|\mathbf{w}\| = \mathbf{1}\},$$

and put

$$\mathcal{L} = \min_{(z,v)\in S^1\mathcal{C}^{(0)}} \min\{m(DDF_{(z,v)}), \|DF_z v\|\},$$

where $S^1\mathcal{C}^{(0)}$ is the unit tangent bundle on $\mathcal{C}^{(0)}$. Since $S^1\mathcal{C}^{(0)}$ is compact and F is a diffeomorphism, $\mathcal{L}$ is nonzero. The mean value theorem gives

$$\mathcal{L}\tilde{d}(v,\tilde{v}) \leq \tilde{d}(DFv, DF\tilde{v}) \leq e^{\Delta_0}\tilde{d}(v,\tilde{v}). \tag{6.3}$$

The assumption of the induction for $j = n-1$ gives $\text{angle}(t(z), t(\tilde{z})) \leq \mathcal{G}e^{10\Delta_0\nu^{(n-1)}}d(z,\tilde{z})$, and therefore $\tilde{d}(t(z),t(\tilde{z})) \leq (1 + \mathcal{G}e^{10\Delta_0\nu^{(n-1)}})d(z,\tilde{z})$. Choose sufficiently large k_0 so that $\text{angle}(DFt(z), DFt(\tilde{z})) \ll 1$ and $\frac{\|DFt(z)\|}{\|DFt(\tilde{z})\|} \sim 1$ hold for any two points z, $\tilde{z}$ contained in any component $\mathcal{A}^{(k)}$ of $\mathcal{B}^{(k)}$ $(\forall k \geq k_0)$. Using (6.2) and (6.3), we have

$$\text{angle}(DFt(z), DFt(\tilde{z})) \leq \frac{\tilde{d}(DFt(z), DFt(\tilde{z}))}{\|DFt(z)\|}.$$

The definition of $\mathcal{L}$ gives

$$\frac{\tilde{d}(DFt(z), DFt(\tilde{z}))}{\|DFt(z)\|} \leq \mathcal{L}^{-1}e^{\Delta_0}\tilde{d}(t(z), t(\tilde{z})).$$

The assumption of the induction gives

$$\mathcal{L}^{-1}e^{\Delta_0}\tilde{d}(t(z),t(\tilde{z})) \leq \mathcal{G}\mathcal{L}^{-1}e^{\Delta_0}e^{10\Delta_0\nu^{(n-1)}}d(z,\tilde{z}),$$

Again the definition of $\mathcal{L}$ gives

$$\mathcal{G}\mathcal{L}^{-1}e^{\Delta_0}e^{10\Delta_0\nu^{(n-1)}}d(z,\tilde{z}) \leq \mathcal{G}\mathcal{L}^{-2}e^{\Delta_0}e^{10\Delta_0\nu^{(n-1)}}d(z_1,\tilde{z}_1).$$

We absorb the term $\mathcal{G}\mathcal{L}^{-2}e^{\Delta_0}$ in the following way. Choose large k_0 such that $k_0 \geq \log(\mathcal{L}^{-2}e^{\Delta_0})$ holds. Then we have $(4\mu_0/\Delta_0)^{j-1}k \geq \log(\mathcal{L}^{-2}e^{\Delta_0})$ for all $j \geq 1$ and $k \geq k_0$. (5.2) gives the relation $\nu^{(i)} \geq (4\mu_0/\Delta_0)^i k$ for all $i \geq 1$, and therefore $\Delta_0\nu^{(n)} - 10\Delta_0\nu^{(n-1)} \geq (\Delta_0 \cdot 4\mu_0/\Delta_0 - 10\Delta_0)\,\nu^{(n-1)} \geq \log(\mathcal{L}^{-2}e^{\Delta_0})$, which implies $e^{\Delta_0\cdot\nu^{(n)} - 10\Delta_0\nu^{(n-1)}} \geq \mathcal{L}^{-2}e^{\Delta_0}$. Eventually we obtain

$$\mathcal{G}\mathcal{L}^{-2}e^{\Delta_0}e^{10\Delta_0\nu^{(n-1)}}d(z_1,\tilde{z}_1) \leq \mathcal{G}e^{\Delta_0\cdot\nu^{(n)}}d(z_1,\tilde{z}_1),$$

which gives the desired inequality. □

Lemma 6.3. *For any z, $\tilde{z}$ which belong to the different unstable sides of $R^{(n-1)}_{\nu_{n-1}+1}$ and satisfy $z \in \Gamma^{(\nu^{(n)})}(\tilde{z})$, we have*

$$1 \leq \frac{\|DF^{\nu^{(n)}}t(z)\|}{\|DF^{\nu^{(n)}}f_{\nu^{(n)}}(z)\|}, \quad \frac{\|DF^{\nu^{(n)}}t(\tilde{z})\|}{\|DF^{\nu^{(n)}}f_{\nu^{(n)}}(\tilde{z})\|} \leq 1+\theta_0^2, \tag{6.4}$$

and

$$\mathrm{angle}(DF^{\nu^{(n)}}t(z), DF^{\nu^{(n)}}t(\tilde{z})) \ll 1. \tag{6.5}$$

Proof. The inequality of the left hand side of (6.4) is obvious. On the right hand side, let c be the critical point which belongs to the unstable side of $R^{(n-2)}_{\nu_{n-1}} \bigcap \tilde{\mathcal{B}}^{(\nu^{(n-1)})}$ containing z_{-1}. Regarding the decomposition $t(z) = \xi(z)f_{\nu^{(n)}}(z)+\eta(z)e_{\nu^{(n)}}(z)$ Lemma 3.1 gives $\xi(z) \geq d(c, z_{-1})$. On the other hand, Proposition 5.1 gives $p^{(n-1)} \geq -\log d(c, z_{-1})^{4/\Delta_0}$. Thus we have $d(c, z_{-1}) \geq e^{-\Delta_0 p^{(n-1)}/4} \geq e^{-\Delta_0 \nu^{(n)}/4}$. This implies

$$\|DF^{\nu^{(n)}}t(z)\| \geq \|\xi(z)DF^{\nu^{(n)}}f_{\nu^{(n)}}(z)\| \gg (Cb)^{\nu^{(n)}} \geq \|\eta(z)DF^{\nu^{(n)}}e_{\nu^{(n)}}(z)\|, \tag{6.6}$$

which implies the desired inequality. The same goes with respect to $t(\tilde{z})$.

We move on to proving (6.5).

Sublemma 6.2. *Suppose that the vector field e_i is defined on an arcwise connected open set Ω. If $\|DF^i f_i(z)\| \geq 1$ holds for all $z \in \Omega$, then for any rectifiable arc $\gamma \subset \Omega$ connecting z and $\tilde{z}$, we have*

$$\frac{\|DF^i f_i(z)\|}{\|DF^i f_i(\tilde{z})\|} \leq \exp(e^{2\Delta_0 i}\,\mathrm{length}(\gamma_i)).$$

Proof. Without loss of generality we may assume $\|DF^i f_i(z)\| \geq \|DF^i f_i(\tilde{z})\|$. The fact that $\log(1+x) \leq x$ for $x \geq 0$ gives

$$\begin{aligned}\log\|DF^i f_i(z)\| - \log\|DF^i f_i(\tilde{z})\| &= \log\left(1+\frac{\|DF^i f_i(z)\| - \|DF^i f_i(\tilde{z})\|}{\|DF^i f_i(\tilde{z})\|}\right)\\ &\leq \|DF^i f_i(z)\| - \|DF^i f_i(\tilde{z})\|\\ &= \|(DF^i)^* f^{(i)^*}(z_i)\| - \|(DF^i)^* f^{(i)^*}(\tilde{z}_i)\|.\end{aligned}$$

We estimate the derivative of the function $(DF^i)^* f^{(i)^*}$ defined on Ω_i. The chain rule gives $\|D((DF^i)^*)\cdot f^{(i)^*}\| \leq ie^{\Delta_0(i-1)} \leq e^{1.9\Delta_0 i}$. By Proposition 2.2 we have $Df^{(i)^*} = e^{(i)}$, which implies $\|(DF^i)^*\cdot Df^{(i)^*}\| \leq Cb$. Therefore we obtain

$$\|D((DF^i)^* f^{(i)^*})\| \leq \|D((DF^i)^*)\cdot f^{(i)^*}\| + \|(DF^i)^*\cdot Df^{(i)^*}\| \leq e^{2\Delta_0 i}.$$

The mean value theorem yields

$$\begin{aligned}\|(DF^i)^* f^{(i)^*}(z_i)\| - \|(DF^i)^* f^{(i)^*}(\tilde{z}_i)\| &\leq \|(DF^i)^* f^{(i)^*}(z_i) - (DF^i)^* f^{(i)^*}(\tilde{z}_i)\|\\ &\leq e^{2\Delta_0 i}\,\mathrm{length}(\gamma_i).\end{aligned}$$

□

Back to the lemma, (6.4) gives

$$\frac{\|DF^{\nu^{(n)}}t(z)\|}{\|DF^{\nu^{(n)}}t(\tilde{z})\|} \sim \frac{\|DF^{\nu^{(n)}}f_{\nu^{(n)}}(z)\|}{\|DF^{\nu^{(n)}}f_{\nu^{(n)}}(\tilde{z})\|}.$$

Sublemma 6.2 with $\gamma = \Gamma^{(\nu^{(n)})}(\tilde{z})$ gives

$$\frac{\|DF^{\nu^{(n)}}f_{\nu^{(n)}}(z)\|}{\|DF^{\nu^{(n)}}f_{\nu^{(n)}}(\tilde{z})\|} \leq 1 + \sqrt{b}, \tag{6.7}$$

because of the inequality length$(\Gamma^{(\nu^{(n)})}_{\nu^{(n)}}(\tilde{z})) \leq (Cb)^{\nu^{(n)}}$ which follows from Proposition 2.2. Therefore, we have

$$\frac{\|DF^{\nu^{(n)}}t(z)\|}{\|DF^{\nu^{(n)}}t(\tilde{z})\|} \leq 1 + 2\theta_0.$$

On the other hand, Proposition 2.2 and Corollary 2.1 give

$$\begin{aligned}\text{angle}(DF^{\nu^{(n)}}f_{\nu^{(n)}}(z), DF^{\nu^{(n)}}f_{\nu^{(n)}}(\tilde{z})) &\leq \|De^{(\nu^{(n)})}\| d(z_{\nu^{(n)}}, \tilde{z}_{\nu^{(n)}}) \\ &\leq Cb \cdot d(z_{\nu^{(n)}}, \tilde{z}_{\nu^{(n)}}) \ll 1.\end{aligned}$$

Altogether these imply (6.5). This completes the proof of Lemma 6.3. □

For two nonzero vectors u and $\tilde{u}$, define $\angle(u, \tilde{u}) = \frac{\langle u, \tilde{u}\rangle}{\|u\|\|\tilde{u}\|}$, where $\langle\cdot,\cdot\rangle$ denotes the scholar product in the Euclidean space. Suppose that the vector fields $e_{\nu^{(n)}}$, $f_{\nu^{(n)}}$ on $R^{(n-1)}_{\nu_{n-1}+1}$ satisfy $\langle\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), e_{\nu^{(n)}}\rangle > 0$ and $\langle\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), f_{\nu^{(n)}}\rangle < 0$. Consider the orthogonal decomposition of unit tangent vectors $t(z) = \xi(z)e_{\nu^{(n)}}(z) + \eta(z)f_{\nu^{(n)}}(z)$ and $t(\tilde{z}) = \xi(\tilde{z})e_{\nu^{(n)}}(\tilde{z}) + \eta(\tilde{z})f_{\nu^{(n)}}(\tilde{z})$, with $\eta(z)$, $\eta(\tilde{z}) > 0$.

Lemma 6.4. *For any z, $\tilde{z}$ which belong to the different unstable sides of $R^{(n-1)}_{\nu_{n-1}+1}$ and satisfy $z \in \Gamma^{(\nu^{(n)})}(\tilde{z})$, we have*

$$\text{angle}(DF^{\nu^{(n)}}t(z), DF^{\nu^{(n)}}t(\tilde{z})) \leq \mathcal{G}e^{9\Delta_0\nu^{(j)}} d(z_{\nu^{(n)}}, \tilde{z}_{\nu^{(n)}}).$$

To prove the lemma, we appeal to (6.2) which recasts the angle estimate in question into an estimate of the distance between two tangent vectors, given that the two prerequisites are satisfied: angle $\ll 1$ and ratio of length ~ 1. (6.5) supplies the first one regardless of the geometry of the rectangle $R^{(n-1)}_{\nu_{n-1}+1}$. However, regarding the second, the ratio $\frac{\|DF^{\nu^{(n)}}t(z)\|}{\|DF^{\nu^{(n)}}t(\tilde{z})\|}$ depends on the geometry of $R^{(n-1)}_{\nu_{n-1}+1}$, and is unbounded in general. See Figure 3. To fix this problem, we properly choose two vectors $v(z)$, $\tilde{v}(z)$, according to the geometry of $R^{(n-1)}_{\nu_{n-1}+1}$, in a way that they are collinear to $t(z)$, $t(\tilde{z})$ respectively, and the ratio $\frac{\|DF^{\nu^{(n)}}v(z)\|}{\|DF^{\nu^{(n)}}v(\tilde{z})\|}$ is close to 1.

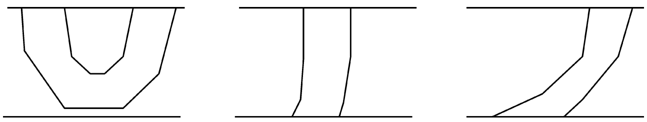

Figure 3. The geometry of $R^{(n-1)}_{\nu_{n-1}}$. The right one corresponds to the first case. The left and the middle correspond to the second case.

Proof of Lemma 6.4. Let c (resp. $\tilde{c}$) be the critical point on the unstable side of $R^{(n-2)}_{\nu_{n-1}} \bigcap \tilde{\mathcal{B}}^{(\nu^{(n-1)})}$ containing z_{-1} (resp. $\tilde{z}_{-1}$). Notice that $c = \tilde{c}$ or $c \neq \tilde{c}$, according to the geometry of $R^{(n-1)}_{\nu_{n-1}+1}$. There are two cases: whether

$$\max\left\{\frac{d(c,z_{-1})}{d(\tilde{c},\tilde{z}_{-1})}, \frac{d(\tilde{c},\tilde{z}_{-1})}{d(c,z_{-1})}\right\} \geq \frac{1+\theta_0}{1-\theta_0}, \tag{6.8}$$

or not. For the moment we put aside the second and consider the first case.

Proof in the first case. Lemma 3.1 gives

$$1-\theta_0 \leq \frac{\eta(z)}{d(c,z_{-1})}, \quad \frac{\eta(\tilde{z})}{d(\tilde{c},\tilde{z}_{-1})} \leq 1+\theta_0, \tag{6.9}$$

(6.9) implies $|\eta(z)-\eta(\tilde{z})| \leq |d(c,z_{-1})-d(\tilde{c},\tilde{z}_{-1})| + \theta_0(d(c,z_{-1})+d(\tilde{c},\tilde{z}_{-1})) \leq 2|d(c,z_{-1})-d(\tilde{c},\tilde{z}_{-1})|$. On the other hand, it is easy to see $|d(c,z_{-1})-d(\tilde{c},\tilde{z}_{-1})| \leq d(z_{-1},\tilde{z}_{-1})$, regardless of $c=\tilde{c}$ or $c \neq \tilde{c}$ (draw a picture!). Therefore we have

$$|\eta(z)-\eta(\tilde{z})| \leq 2d(z_{-1},\tilde{z}_{-1}). \tag{6.10}$$

Choose $\bar{\xi}(z) \in \mathbf{R}$ in a way that the vector $v(z) = \bar{\xi}(z)e_{\nu^{(n)}}(z) + f_{\nu^{(n)}}(z)$ is tangent at z to the unstable side. We also choose $\bar{\xi}(\tilde{z})$ and define $v(\tilde{z}) = \bar{\xi}(\tilde{z})e_{\nu^{(n)}}(\tilde{z}) + f_{\nu^{(n)}}(\tilde{z})$ which is tangent at $\tilde{z}$ to the unstable side. Since $v(z)$ is collinear to $t(z)$, we have $\bar{\xi}(z) = \xi(z)/\eta(z)$, and also $\bar{\xi}(\tilde{z}) = \xi(\tilde{z})/\eta(\tilde{z})$. The triangle inequality gives

$$|\bar{\xi}(z)-\bar{\xi}(\tilde{z})| \leq \frac{|\xi(z)-\xi(\tilde{z})|}{\eta(z)} + \frac{|\xi(\tilde{z})||\eta(z)-\eta(\tilde{z})|}{\eta(z)\eta(\tilde{z})},$$

where we clearly have $p^{(n-1)} \leq \nu^{(n)}$, and (6.9) gives $\eta(z),\ \eta(\tilde{z}) \geq e^{-\Delta_0 p^{(n-1)}}$. The estimate of $|\xi(z)-\xi(\tilde{z})|$ is as follows. By definition, we clearly have $\xi(z) = \cos\angle(e_{\nu^{(n)}}(z), t(z))$. The same applies to $\tilde{z}$, and therefore the mean value theorem with respect to $\cos^{-1}$ gives $|\xi(z)-\xi(\tilde{z})| \leq |\cos^{-1}(\xi(z)) - \cos^{-1}(\xi(\tilde{z}))|$. Then we have

$$\begin{aligned}
|\cos^{-1}(\xi(z)) - \cos^{-1}(\xi(\tilde{z}))| &= |\angle(e_{\nu^{(n)}}(z), t(z)) - \angle(e_{\nu^{(n)}}(\tilde{z}), t(\tilde{z}))| \\
&= |\angle(e_{\nu^{(n)}}(z), t(z)) - \angle(e_{\nu^{(n)}}(z), t(\tilde{z})) \\
&\quad \pm \text{angle}(e_{\nu^{(n)}}(z), e_{\nu^{(n)}}(\tilde{z}))| \\
&\leq \angle(t(z), t(\tilde{z})) + \text{angle}(e_{\nu^{(n)}}(z), e_{\nu^{(n)}}(\tilde{z})).
\end{aligned}$$

Notice the identity $\angle(t(z),t(\tilde z))=\mathrm{angle}(t(z),t(\tilde z))$, following from $\eta(z)$, $\eta(\tilde z)>0$. Proposition 2.2 and Lemma 6.2 give

$$\mathrm{angle}(t(z),t(\tilde z))+\mathrm{angle}(e_{\nu^{(n)}}(z),e_{\nu^{(n)}}(\tilde z))\leq\left(\mathcal{G}e^{\Delta_0\nu^{(n)}}+Cb\right)d(z,\tilde z).$$

Put $\Gamma=\Gamma^{(\nu^{(n)})}(z)=\Gamma^{(\nu^{(n)})}(\tilde z)$ to ease notation. By Corollary 2.1, Γ is a nearly vertical curve, and in particular $d(z,\tilde z)\leq(1+Cb)\,\mathrm{length}(\Gamma)$. Substituting this into the above inequality we obtain

$$|\xi(z)-\xi(\tilde z)|\leq\left(\mathcal{G}e^{\Delta_0\nu^{(n)}}+Cb\right)(1+Cb)\,\mathrm{length}(\Gamma). \tag{6.11}$$

As a result, we obtain

$$|\bar\xi(z)-\bar\xi(\tilde z)|\leq e^{\nu^{(n)}}\left(\mathcal{G}e^{\Delta_0\nu^{(n)}}+Cb\right)(1+Cb)\,\mathrm{length}(\Gamma)+e^{2\Delta_0\nu^{(n)}}2d(z_{-1},\tilde z_{-1}).$$

Using the definition of $\mathcal{L}$ and $d(z,\tilde z)\leq(1+Cb)\,\mathrm{length}(\Gamma)$, we obtain

$$\begin{aligned}|\bar\xi(z)-\bar\xi(\tilde z)|&\leq e^{\nu^{(n)}}\left(\mathcal{G}e^{\Delta_0\nu^{(n)}}+Cb+2e^{\Delta_0\nu^{(n)}}\mathcal{L}^{-1}\right)(1+Cb)\,\mathrm{length}(\Gamma)\\&\leq\mathcal{G}e^{3\Delta_0\nu^{(n)}}\,\mathrm{length}(\Gamma).\end{aligned}$$

Define a real-valued function $\bar\xi(t)$ on $[0,\mathrm{length}(\Gamma)]$ by

$$\bar\xi(t)=\bar\xi(\tilde z)+\frac{\bar\xi(z)-\bar\xi(\tilde z)}{\mathrm{length}(\Gamma)}\cdot t.$$

The above inequality gives $|d\bar\xi/dt|\leq\mathcal{G}e^{3\Delta_0\nu^{(n)}}$. Moreover, $|\bar\xi(z)|\leq\eta(z)^{-1}\leq e^{\Delta_0p^{(n-1)}}$ gives $|\bar\xi(t)|\leq e^{\Delta_0p^{(n-1)}}$. We parametrize Γ by arc length t, i.e. $\Gamma(t)=\tilde z+\int_0^t e_{\nu^{(n)}}(\Gamma(s))ds$ for $t\in[0,\mathrm{length}(\Gamma)]$, and introduce a variable ω by the following implicit formula:

$$F^{\nu^{(n)}}(\Gamma(t))=\int_0^\omega e^{(\nu^{(n)})}(\Gamma_{\nu^{(n)}}(s))ds+\tilde z_{\nu^{(n)}}.$$

This equation can easily be solved with respect to t, using the fact that Γ and $\Gamma_{\nu^{(n)}}$ are nearly vertical curves, by Proposition 2.2. We immediately have

$$t=(p_y\circ\Gamma)^{-1}\circ p_y\circ F^{-\nu^{(n)}}\left(\int_0^\omega e^{(\nu^{(n)})}(s)ds+\tilde z_{\nu^{(n)}}\right),$$

where $p_y\colon(x,y)\to y$. The chain rule gives

$$\begin{aligned}\left|\frac{dt}{d\omega}\right|&\leq\left|\frac{d(p_y\circ\Gamma)(t)}{dt}\right|^{-1}\|DF^{-\nu^{(n)}}_{F^{\nu^{(n)}}(\Gamma(t))}\|\\&\leq\|DF^{-\nu^{(n)}}e^{(\nu^{(n)})}(F^{\nu^{(n)}}(\Gamma(t)))\|,\end{aligned}$$

where the last inequality follows from $\|d\Gamma(t)\| = 1$ and the nature of the mostly contracting direction. Put $\phi(t) = \bar{\xi}(t) DF^{\nu^{(n)}} e_{\nu^{(n)}}(\Gamma(t))$. The chain rule gives

$$\frac{d\phi}{d\omega} = \frac{d\bar{\xi}(t)}{d\omega} \cdot DF^{\nu^{(n)}} e_{\nu^{(n)}}(\Gamma(t)) + \bar{\xi}(t) \frac{d}{d\omega} DF^{\nu^{(n)}} e_{\nu^{(n)}}(\Gamma(t)).$$

By Proposition 2.2–5, the norm of the second term is smaller than $\|\bar{\xi}(t)\|_{C^0} e^{2\Delta_0 \nu^{(n)}} \le e^{3\Delta_0 \nu^{(n)}}$. On the first term, we have

$$\begin{aligned} \|\frac{d\bar{\xi}(t)}{d\omega} \cdot DF^{\nu^{(n)}} e_{\nu^{(n)}}(\Gamma(t))\| &\le \mathcal{G} e^{3\Delta_0 \cdot \nu^{(n)}} \left| \frac{dt}{d\omega} \right| \cdot \|DF^{\nu^{(n)}} e_{\nu^{(n)}}(\Gamma(t))\| \\ &\le \mathcal{G} e^{3\Delta_0 \cdot \nu^{(n)}}. \end{aligned}$$

We regard ϕ as a function of ω, and use the mean value theorem to obtain

$$\begin{aligned} &\|\bar{\xi}(z) DF^{\nu^{(n)}} e_{\nu^{(n)}}(z) - \bar{\xi}(\tilde{z}) DF^{\nu^{(n)}} e_{\nu^{(n)}}(\tilde{z})\| \\ &\quad \le (\mathcal{G} e^{3\Delta_0 \cdot \nu^{(n)}} + e^{2\Delta_0 \nu^{(n)}}) \operatorname{length}(\Gamma_{\nu^{(n)}}). \end{aligned}$$

On the other hand, Proposition 2.2 gives

$$\operatorname{angle}(DF^{\nu^{(n)}} f_{\nu^{(n)}}(z), DF^{\nu^{(n)}} f_{\nu^{(n)}}(\tilde{z})) \le Cb \cdot d(z_{\nu^{(n)}}, \tilde{z}_{\nu^{(n)}}),$$

and therefore Lemma 6.3 implies

$$\begin{aligned} &\|DF^{\nu^{(n)}} f_{\nu^{(n)}}(z) - DF^{\nu^{(n)}} f_{\nu^{(n)}}(\tilde{z})\| \\ &\quad \le (1 + \sqrt{b}) \|DF^{\nu^{(n)}} f_{\nu^{(n)}}(z)\| Cb \cdot d(z_{\nu^{(n)}}, \tilde{z}_{\nu^{(n)}}) \\ &\quad \le e^{\Delta_0 \nu^{(n)}} Cb \cdot d(z_{\nu^{(n)}}, \tilde{z}_{\nu^{(n)}}). \end{aligned}$$

These two inequalities yield

$$\begin{aligned} \tilde{d}(DF^{\nu^{(n)}} v(z), DF^{\nu^{(n)}} v(\tilde{z})) &\le \left(\mathcal{G} e^{3\Delta_0 \nu^{(n)}} + e^{\Delta_0 \nu^{(n)}} Cb\right) d(z_{\nu^{(n)}}, \tilde{z}_{\nu^{(n)}}) \\ &\le \mathcal{G} e^{5\Delta_0 \nu^{(n)}} d(z_{\nu^{(n)}}, \tilde{z}_{\nu^{(n)}}). \end{aligned}$$

Since $\nu^{(n)}$ is a free return, we have $\|DF^{\nu^{(n)}+1} t(z_{-1})\| \ge e^{\gamma_0 \nu^{(n)}/3}$, and therefore

$$\|DF^{\nu^{(n)}} v(z)\| \ge \|DF^{\nu^{(n)}} t(z)\| \ge e^{\gamma_0 \nu^{(n)}/3 - \Delta_0} \ge 1.$$

By (6.2), we obtain

$$\operatorname{angle}(DF^{\nu^{(n)}} v(z), DF^{\nu^{(n)}} v(\tilde{z})) \le \mathcal{G} e^{9\Delta_0 \nu^{(n)}} d(z_{\nu^{(n)}}, \tilde{z}_{\nu^{(n)}}).$$

This completes the proof of Lemma 6.4 for the first case in (6.8).

Proof in the second case. We are left to consider the case

$$\max\left\{ \frac{d(c, z_{-1})}{d(\tilde{c}, \tilde{z}_{-1})}, \frac{d(\tilde{c}, \tilde{z}_{-1})}{d(c, z_{-1})} \right\} \le \frac{1 + \theta_0}{1 - \theta_0}. \tag{6.12}$$

We only use $t(z)$ and $t(\tilde{z})$ to estimate the angle in question. Define a real-valued function $\xi(t)$ on $[0, \mathrm{length}(\Gamma)]$ by

$$\xi(t) = \xi(\tilde{z}) + \frac{\xi(z) - \xi(\tilde{z})}{\mathrm{length}(\Gamma)} \cdot t.$$

(6.11) gives $|d\xi/dt| \leq \mathcal{G}e^{3\Delta_0 \nu^{(n)}}$. Therefore, almost the same argument works and we obtain

(6.13)
$$\|\xi(z)DF^{\nu^{(n)}}e_{\nu^{(n)}}(z) - \xi(\tilde{z})DF^{\nu^{(n)}}e_{\nu^{(n)}}(z)\| \leq (\mathcal{G}e^{3\Delta_0 \cdot \nu^{(n)}} + e^{2\Delta_0 \nu^{(n)}})\,\mathrm{length}(\Gamma_{\nu^{(n)}}).$$

On the other hand, Sublemma 6.2 and (6.12) give

$$\frac{\|\eta(z)DF^{\nu^{(n)}}f_{\nu^{(n)}}(z)\|}{\|\eta(\tilde{z})DF^{\nu^{(n)}}f_{\nu^{(n)}}(\tilde{z})\|} \sim 1.$$

Thus, similarly to the first case we obtain

$$(6.14)\quad \|\eta(z)DF^{\nu^{(n)}}f_{\nu^{(n)}}(z) - \eta(\tilde{z})DF^{\nu^{(n)}}f_{\nu^{(n)}}(\tilde{z})\| \leq e^{\Delta_0 \nu^{(n)}} Cb \cdot d(z_{\nu^{(n)}}, \tilde{z}_{\nu^{(n)}}).$$

Combining (6.13) and (6.14), and using (6.2) yield

$$\mathrm{angle}(DF^{\nu^{(n)}}t(z), DF^{\nu^{(n)}}t(\tilde{z})) \leq \mathcal{G}e^{9\Delta_0 \nu^{(j)}} d(z_{\nu^{(n)}}, \tilde{z}_{\nu^{(n)}}).$$

This completes the proof of Lemma 6.4. □

To complete the proof of Proposition 6.2, we are left to show that (6.1) holds for any two points z, $\tilde{z}$ on the unstable side of $R^{(j-1)}_{\nu_j} \bigcap \tilde{\mathcal{B}}^{(\nu^{(j)})}$. By the definition of a horizontal curve, (6.1) clearly holds if z, $\tilde{z}$ are on the same unstable side. Otherwise, take a point z' which is connected with z by the integral curve of the vector field $e^{(\nu^{(n)})}$. Since z' and $\tilde{z}$ belong to the same unstable side, we have $\mathrm{angle}(t(\tilde{z}), t(z')) \leq 10\theta_0 d(\tilde{z}, z')$. By Lemma 6.4 and the triangle inequality we obtain

$$\begin{aligned}\mathrm{angle}(t(z), t(\tilde{z})) &\leq \mathcal{G}e^{\Delta_0 \nu^{(n)}} d(z, \tilde{z}) + 10\theta_0 d(\tilde{z}, z') \\ &\leq \mathcal{G}e^{9\Delta_0 \nu^{(n)}}(d(z, \tilde{z}) + d(\tilde{z}, z')).\end{aligned}$$

Since the integral curve connecting z and z' is perpendicular to the unstable side containing $\tilde{z}$ and z', we have $d(z, \tilde{z}) + d(\tilde{z}, z') \leq 2d(z, \tilde{z})$. The proof of Proposition 6.2 is complete. □

Denote by $|\cdot|$ the two dimensional Lebesgue measure.

Corollary 6.1. *For any component B of $R^{(j-1)}_{\nu_j} \bigcap \mathcal{B}^{(\nu^{(j)})}$, we have*

$$\frac{|B|}{|R^{(j-1)}_{\nu_j}|} \leq e^{2\mathcal{G}} e^{-\mu_0 \nu^{(j)}}.$$

Proof. Denote by $\gamma^{(1)}$ and $\gamma^{(2)}$ the two unstable sides of the same component of $R^{(j-1)}_{\nu_j} \bigcap \mathcal{B}^{\nu^{(j)}}$, and by $c^{(i)}$ the critical point on $\gamma^{(i)}$. It suffices to show that

$$\exp(-\mathcal{G}) d(c^{(1)}, c^{(2)}) \le d(z^{(1)}, z^{(2)}) \le \exp(\mathcal{G}) d(c^{(1)}, c^{(2)})$$

holds for any two points $z^{(1)} = (x, y^{(1)}) \in \gamma^{(1)}$ and $z^{(2)} = (x, y^{(2)}) \in \gamma^{(2)}$.

We parametrize $\gamma^{(1)}$ and $\gamma^{(2)}$ by arc length s and assume $\gamma^{(1)}(0) = c^{(1)}$, $\gamma^{(2)}(0) = c^{(2)}$. Using Proposition 6.2 we have

$$\begin{aligned} d(\gamma^{(1)}(s), \gamma^{(2)}(s)) &\le d(c^{(1)}, c^{(2)}) + \int_0^s \|t(\gamma^{(1)}(u)) - t(\gamma^{(2)}(u))\| du \\ &\le d(c^{(1)}, c^{(2)}) + \int_0^s \mathrm{angle}(t(\gamma^{(1)}(u)), t(\gamma^{(2)}(u))) du \\ &\le d(c^{(1)}, c^{(2)}) + \mathcal{G} e^{10\Delta_0 \nu^{(j)}} \int_0^s d(\gamma^{(1)}(u), \gamma^{(2)}(u)) du \\ &\le d(c^{(1)}, c^{(2)}) + s \cdot \mathcal{G} e^{10\Delta_0 \nu^{(j)}}. \end{aligned}$$

Substituting the result into the inequality

$$d(\gamma^{(1)}(s), \gamma^{(2)}(s)) \le d(c^{(1)}, c^{(2)}) + \mathcal{G} e^{10\Delta_0 \nu^{(j)}} \int_0^s d(\gamma^{(1)}(u), \gamma^{(2)}(u)) du,$$

which has been obtained already, we have

$$d(\gamma^{(1)}(s), \gamma^{(2)}(s)) \le d(c^{(1)}, c^{(2)}) + \mathcal{G} e^{10\Delta_0 \nu^{(j)}} d(c^{(1)}, c^{(2)}) s + (\mathcal{G} e^{10\Delta_0 \nu^{(j)}} s)^2 / 2!.$$

Repeating the same procedure recursively for m times yields

$$d(\gamma^{(1)}(s), \gamma^{(2)}(s)) \le d(c^{(1)}, c^{(2)}) \sum_{k=0}^{m} \frac{(\mathcal{G} e^{10\Delta_0 \nu^{(j)}} s)^k}{k!}.$$

The definition of $\tilde{\mathcal{B}}^{(\nu^{(j)})}$ gives $s \le e^{-\mu_0 \nu^{(j)}}$, and in particular $e^{10\Delta_0 \nu^{(j)}} s \le 1$. Substituting this into the above inequality and passing $m \to \infty$ we obtain the desired upper bound for $d(z^{(1)}, z^{(2)})$. The lower bound is obtained by the same reasoning with the role of $c^{(1)}$, $c^{(2)}$ and that of $z^{(1)}$, $z^{(2)}$ interchanged. □

7. Conclusion

7.1. The reduction

In the rest of this paper we fix sufficiently large k, fix one component $\mathcal{A}^{(k)}$ of $\mathcal{B}^{(k)}$, fix $R = R^{(0)} \in \mathcal{P}^{(k)}$, and prove that the set of points consisting of all $z \in R$ which is not eventually controlled has zero Lebesgue measure. In fact, this completes the proof of the main theorem. To see why this is so, suppose that this assertion is true. Since the partition $\mathcal{P}^{(k)}$ contains only countably many rectangles all of which are measurable, and since the set $\mathcal{B}^{(k)}$ has only finitely many components, the subadditivity in the measure theory claims that the set of points of $\mathcal{B}^{(k)}$ which are not eventually controlled has zero Lebesgue measure.

Suppose that there exists a positive Lebesgue measure set E of points which are not eventually controlled. Then, the set $E \setminus \bigcup_{k,n\geq 0} F^{-n}(\partial\mathcal{B}^{(k)})$ also has positive measure. Let z be a Lebesgue density point of the set and suppose that $z_{n_0} \in \mathcal{C}^{(0)}$. Using repeatedly the following Corollary 7.1, we can construct an arbitrarily long controlled orbit which is a subset of the forward orbit of z_{n_0+1}. Since z is not eventually controlled, there must exist an arbitrarily large k such that $z_{k+n_0+1} \in \mathcal{B}^{(k)}$. In particular, we have $z_{k+n_0+1} \in \operatorname{Int}\mathcal{B}^{(k)}$ according to the choice of z. This yields a contradiction to the assertion in the previous paragraph because F maps a positive measure set to a positive measure set.

Corollary 7.1. *Let $R \in \mathcal{P}^{(k)}$ and $z \in R$. If z is controlled up to time $k-1$, but not so up to time k, then z_{k+1} is controlled up to time $4\mu_0 k/\Delta_0$.*

Proof. See Proposition 5.1 and Lemma 5.2. □

7.2. Measure estimate

Proposition 7.1. *The set $\bigcap_{j=0}^{\infty} T^{(j)}(R)$ has zero Lebesgue measure.*

Proof. We clearly have

$$\frac{|T^{(j)}(R)|}{|R|} = \sum_{R^{(j)}} \frac{|R^{(j)}|}{|R|},$$

where the sum runs over all j-th element of all the nested sequences $\{R^{(i)}\}_{i=0}^{j}$ contained in $\prod_{i=0}^{j} \mathcal{S}^{(j)}(R)$. Rearranging gives

$$\sum_{R^{(j)}} \frac{|R^{(j)}|}{|R|} = \sum_{R^{(j-1)}} \sum_{R^{(j)} \subset R^{(j-1)}} \frac{|R^{(j-1)}|}{|R|} \frac{|R^{(j)}|}{|R^{(j-1)}|},$$

where the ranges of the sums are now obvious. Further rearranging gives

$$\sum_{R^{(j-1)}} \sum_{R^{(j)} \subset R^{(j-1)}} \frac{|R^{(j-1)}|}{|R|} \frac{|R^{(j)}|}{|R^{(j-1)}|} = \sum_{R^{(j-1)}} \frac{|R^{(j-1)}|}{|R|} \sum_{R^{(j)} \subset R^{(j-1)}} \frac{|R^{(j)}|}{|R^{(j-1)}|}.$$

Put

$$\tilde{\mathcal{D}} = \mathcal{D} \cdot \sup_{z,\tilde{z} \in \mathcal{C}^{(0)}} \frac{|\det DF_z|}{|\det DF_{\tilde{z}}|},$$

where $\mathcal{D}$ is the area distortion constant in Proposition 6.1. A successive use of Proposition 6.1 gives

$$\sum_{R^{(j)} \subset R^{(j-1)}} \frac{|R^{(j)}|}{|R^{(j-1)}|} \leq \tilde{\mathcal{D}}^j \sum_{R^{(j)} \subset R^{(j-1)}} \frac{|R^{(j)}_{\nu_j}|}{|R^{(j-1)}_{\nu_j}|}.$$

We rearrange the sum of the right hand side and obtain

$$\sum_{R^{(j)} \subset R^{(j-1)}} \frac{|R^{(j)}_{\nu_j}|}{|R^{(j-1)}_{\nu_j}|} = \sum_{\nu=\nu_{j-1}+p^{(j-1)}}^{\infty} \sum_{\nu_j=\nu} \frac{|R^{(j)}_{\nu_j}|}{|R^{(j-1)}_{\nu_j}|}.$$

The lower bound of ν of the right hand side comes from Proposition 5.2, claiming that all points of $R^{(j-1)}_{\nu_{j-1}+1}$ does not experience any close return up to time $p^{(j-1)} - 1$. Proposition 5.1 and Corollary 6.1 together yield

$$\begin{aligned}\sum_{\nu=\nu_{j-1}+p^{(j-1)}}^{\infty} \sum_{\nu_j=\nu} \frac{|R^{(j)}_{\nu_j}|}{|R^{(j-1)}_{\nu_j}|} &\leq e^{2\mathcal{G}} \sum_{i=p^{(j-1)}}^{\infty} 2^{i-p^{(j-1)}} e^{-\mu_0 i} \\ &\leq e^{2\mathcal{G}} \sum_{i=p^{(j-1)}}^{\infty} e^{-\mu_0 i/2}.\end{aligned}$$

By (5.2), we have $p^{(j-1)} \geq (4\mu_0/\Delta_0)^{j-1}k$, and therefore

$$\begin{aligned}e^{2\mathcal{G}} \sum_{i=p^{(j-1)}}^{\infty} e^{-\mu_0 i/2} &\leq e^{2\mathcal{G}} \sum_{i=(4\mu_0/\Delta_0)^j k}^{\infty} e^{-\mu_0 i/2} \\ &\leq e^{2\mathcal{G}} e^{-\mu_0 (4\mu_0/\Delta_0)^j k/2}.\end{aligned}$$

In all, we obtain

$$\frac{|T^{(j)}(R)|}{|R|} \leq \frac{|T^{(j-1)}(R)|}{|R|} \cdot \tilde{\mathcal{D}}^j e^{2\mathcal{G}} e^{-\mu_0 (4\mu_0/\Delta_0)^j k/2}.$$

A recursive use of the above inequality for j-times yields

$$\frac{|T^{(j)}(R)|}{|R|} \leq e^{2\mathcal{G}j} \tilde{\mathcal{D}}^{j(j+1)/2} \exp\left(-\mu_0 k/2 \cdot \sum_{i=1}^{j-1} (4\mu_0/\Delta_0)^i \right),$$

which clearly goes to zero as $j \to \infty$, and therefore $|\bigcap_{j=0}^{\infty} T^{(j)}(R)| = 0$. □

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