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# Global existence on nonlinear Schrödinger-IMBq equations

Dedicated to Professor Masatake Miyake on the occasion of his sixtieth birthday

By

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**Abstract**

In this paper, we consider the Cauchy problem of Schrödinger-IMBq equations in $\mathbb{R}^n, n \geq 1$. We first show the global existence and blowup criterion of solutions in the energy space for the 3 and 4 dimensional system without power nonlinearity under suitable smallness assumption. Secondly the global existence is established to the system with $p$-powered nonlinearity in $H^s(\mathbb{R}^n), n = 1, 2$ for some $\frac{n}{2} < s < \min(2, p)$ and some $p > \frac{n}{2}$. We also provide a blowup criterion for $n = 3$ in Triebel-Lizorkin space containing BMO space naturally.

## 1. Introduction

We consider the Cauchy problem to the following system of equations (nonlinear Schrödinger-IMBq equations):

$$(1.1)\qquad \begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = vu \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}, \\ \partial_t^2 v - \Delta v - \Delta\partial_t^2 v = \Delta(f(v) + |u|^2) \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}, \\ u(0) = u_0, \quad (v(0), \partial_t v(0)) = (v_0, v_1) \quad \text{in} \quad \mathbb{R}^n, \end{cases}$$

where $u$ is a complex-valued function of $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, $v$ is a real-valued function of $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, and $f(v) = \lambda|v|^{p-1}v$ for a fixed real number $\lambda$. In this paper, we restrict our attention to positive time for simplicity since the case of negative time is treated analogously. The system is regarded as a substitute for the Zakharov system:

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = vu, \\ \partial_t^2 v - \Delta v = \Delta|u|^2. \end{cases}$$

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See [17], [18] for further details. Concerning the Zakharov system, see [10], [11], [23]–[25] for instance.

The local existence of system (1.1) was studied in [5] in $H^k, k \geq 5$ setting in multidimensional space and recently the global existence has been studied in [31] in $H^2$ setting in one space dimension.

The case $\lambda = 0$ for $1 \leq n \leq 4$ has been studied in [22], where the local well-posedness is proved in $L^2$ setting in space dimension $n \leq 4$ and global well-posedness is also proved in the energy class in space dimensions $n \leq 2$. Here the energy class stands for $H^1 \oplus L^2 \oplus \omega L^2$ for $(u, v, \partial_t v)$, where $v_1 \in \omega L^2$ means $\omega^{-1} v_1 \in L^2$ and $H^s = (1-\Delta)^{-\frac{s}{2}} L^2$ is the usual Sobolev space with the norm $\|\psi\|_{H^s} = \|(1-\Delta)^{\frac{s}{2}}\psi\|_{L^2}$. We use similar notation $v \in \omega^\alpha X$ to mean $\omega^{-\alpha} v \in X$ for a function space $X$ and a nonnegative number $\alpha$.

In this paper, we study the global existence of solutions to the system (1.1), extending results in [22], [31]. By Duhamel's principle, (1.1) is rewritten as

$$
\begin{aligned}
u(t) &= U(t)u_0 - i\int_0^t U(t-t')(vu)(t')\,dt', \\
v(t) &= (\partial_t K)(t)v_0 + K(t)v_1 + \int_0^t K(t-t')\omega^2(f(v) + |u|^2)(t')\,dt',
\end{aligned}
\tag{1.2}
$$

where $U(t) = e^{i(t/2)\Delta}$, $K(t) = \omega^{-1}\sin t\omega$, $\partial_t K(t) = \cos\omega t$ and $\omega = (-\Delta)^{\frac{1}{2}}(1-\Delta)^{-\frac{1}{2}}$. The second integral equation of (1.2) is also written as

$$
\begin{aligned}
\begin{pmatrix} v(t) \\ \omega^{-1}\partial_t v(t) \end{pmatrix} &= V(t)\begin{pmatrix} v_0 \\ \omega^{-1} v_1 \end{pmatrix} \\
&\quad + \int_0^t V(t-t')\begin{pmatrix} 0 \\ \omega(f(v) + |u|^2)(t') \end{pmatrix} dt',
\end{aligned}
\tag{1.3}
$$

where

$$
V(t) = \exp\left(t\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}\right) = \begin{pmatrix} \cos t\omega & \sin t\omega \\ -\sin t\omega & \cos t\omega \end{pmatrix}.
$$

The local and global existence results for IMBq equation, namely, $u = 0$, can be found in [6]–[8], [16], [28]–[30] and the references therein.

In [22], the Strichartz estimates for Schrödinger evolution group $U$ for (1.2) and conservation laws were the basic tools for the local or global well-posedness in case that $\lambda = 0$.

The Strichartz estimate on $U$ can be stated as follows: For any admissible pairs $(q, r)$ and $(\widetilde{q}, \widetilde{r})$,

$$
\begin{aligned}
&\|U(\cdot)\phi\|_{L_T^q L^r} \leq C\|\phi\|_{L^2}, \\
&\left\| \int_0^t U(t-t')F(t')\,dt' \right\|_{L_T^q L^r} \leq C\|F\|_{L_T^{\widetilde{q}'} L^{\widetilde{r}'}},
\end{aligned}
\tag{1.4}
$$

where $C$ is independent of $T > 0$ and $\|G\|_{L_T^q L^r} = \|G\|_{L_t^q(0,T;\,L_x^r)}$. Here we say the pair $(q, r)$ is *admissible*, if $\frac{1}{q} + \frac{n}{2r} = \frac{n}{4}$, $2 \leq q, r \leq \infty$ and $(q, r, n) \neq (2, \infty, 2)$.

The estimate (1.4) hold even when $T = \infty$. We use the notation $\|u\|_{L^qL^r} = \|u\|_{L^q_t(0,\infty;\, L^r)}$ when $T = \infty$. For the details of Strichartz estimates for $U$, see [14], [1].

The solution $(u, v)$ with sufficient regularity for the system (1.1) or (1.2) satisfies the basic physical laws, $L^2$ and energy conservations. Let $[0, T]$ be a existence time interval. Then for all $t \in [0, T]$

$$\|u(t)\|_{L^2} = \|u(0)\|_{L^2}, \tag{1.5}$$

$$\begin{aligned} E(t) \equiv \frac{1}{2}(\|\nabla u(t)\|^2_{L^2} + \|v(t)\|^2_{L^2} + \|\omega^{-1}\partial_t v(t)\|_{L^2}) \\ + (v(t), |u(t)|^2) + \frac{\lambda}{p+1}\|v(t)\|^{p+1}_{L^{p+1}} \\ = E(0). \end{aligned} \tag{1.6}$$

Using the standard regularizing argument for $f(v)$, the conservation laws can be shown with $(u_0, v_0, v_1) \in H^1 \oplus L^2 \oplus \omega L^2$ for $\lambda = 0$ and $H^1 \oplus H^1 \oplus H^1$ for $\lambda \neq 0$. See for instance [3], [11]. See also [21].

The first two results are extensions of the global existence in [22] for the case of $n = 1, 2$ to the case $n = 3, 4$.

**Theorem 1.1.** *Let $n = 3$. Assume that $\lambda = 0$. Then*

(i) *There exists a constant $\varepsilon_0$ such that for any $(u_0, v_0, v_1) \in H^1 \oplus L^2 \oplus \omega L^2$ with*

$$(\|\nabla u_0\|_{L^2} + \|v_0\|_{L^2} + \|\omega^{-1}\|_{L^2})\|u_0\|_{L^2} \leq \varepsilon_0$$

*the system* (1.1) *has a unique global solution $(u, v)$ such that for any admissible pair $(q, r)$*

$$\begin{aligned} u \in C_b([0, \infty); H^1) \cap L^q_{loc}([0, \infty); L^r), \\ v \in C^2_b([0, \infty); L^2), \quad \partial_t v \in C_b([0, \infty); \omega L^2). \end{aligned} \tag{1.7}$$

(ii) *Let $T^*$ be the maximal existence time to the Cauchy problem* (1.1) *with general initial data and it be finite. Then we have*

$$\int_0^{T^*} (T^* - t)\|u(t)\|^2_{L^4}\, dt = \infty. \tag{1.8}$$

Here, we denote $C^k_b([0, \infty)), k \geq 0$ by the space of bounded $C^k$ functions on $[0, \infty)$.

**Remark 1.** The smallness condition in $(i)$ of Theorem 1.1 is satisfied for data for the following forms:

(1) $(u_0, v_0, v_1) = (\varepsilon\phi, \varepsilon\psi_0, \varepsilon\psi_1)$ with $(\phi, \psi_0, \psi_1) \in H^1 \oplus L^2 \oplus \omega L^2$ and $\varepsilon > 0$ sufficiently small.

(2) $(u_0, v_0, v_1) = (\phi_\varepsilon, \psi_0, \psi_1)$, where $\phi_\varepsilon(x) = \varepsilon^{-\frac{1}{2}}\phi(\varepsilon^{-1}x)$, $(\phi, \psi_1, \psi_1) \in H^1 \oplus L^2 \oplus \omega L^2$, and $\varepsilon > 0$ sufficiently small.

**Theorem 1.2.** *Let $n=4$. Assume that $\lambda=0$. Then*

(i) *There exists a constant $\varepsilon_0>0$ such that for any $(u_0,v_0,v_1)\in H^1\oplus L^2\oplus\omega L^2$ with*

$$\|\nabla u_0\|_{L^2}+\|v_0\|_{L^2}+\|\omega^{-1}v_1\|_{L^2}\le\varepsilon_0$$

*the system* (1.1) *has a unique global solution $(u,v)$ satisfying* (1.7) *replaced by $L^qL^r$.*

(ii) *There exist a pair of functions $(u_0^+,v_0^+,v_1^+)\in L^2\oplus L^2\oplus\omega L^2$ such that*

$$\begin{aligned}&\|u(t)-U(t)u_0^+\|_{L^2}\to 0,\\ &\|v(t)-\partial_tK(t)v_0^+-K(t)v_1^+\|_{L^2}\to 0,\\ &\|\omega^{-1}\left(\partial_tv(t)+\omega^2K(t)v_0^+-\partial_tK(t)v_1^+\right)\|_{L^2}\to 0\end{aligned}\tag{1.9}$$

*as $t\to+\infty$.*

**Remark 2.** The smallness condition in Theorem 1.2 is satisfied for data of the following forms:

(1) $(u_0,v_0,v_1)=(\varepsilon\phi,\varepsilon\psi_0,\varepsilon\psi_1)$ with $(\phi,\psi_0,\psi_1)\in H^1\oplus L^2\oplus\omega L^2$ and $\varepsilon>0$ sufficiently small.

(2) $(u_0,v_0,v_1)=(\phi_\varepsilon,\varepsilon\psi_0,\varepsilon\psi_1)$, where $\phi_\varepsilon(x)=\varepsilon^2\phi(\varepsilon x)$, $(\phi,\psi_0,\psi_1)\in H^1\oplus L^2\oplus\omega L^2$, and $\varepsilon>0$ sufficiently small. Note that $\|\phi_\varepsilon\|_{L^2}=\|\phi\|_{L^2}$ and that the size of the $L^2$ norm may not be small.

(3) $(u_0,v_0,v_1)=(\phi^\varepsilon,\psi_0^\varepsilon,\psi_1^\varepsilon)$, where $\phi^\varepsilon(x)=\varepsilon^{-a}\phi(\varepsilon^{-1}x)$, $\psi_j^\varepsilon(x)=\varepsilon^{-b}\psi_j(\varepsilon^{-1}x)$, $j=0,1$, $(\phi,\psi_0,\psi_1)\in H^1\oplus L^2\oplus\omega L^2$, $0<a<1$, $0<b<2$, and $\varepsilon>0$ sufficiently small.

**Remark 3.** There is no general result on the global existence for Zakharov system for $n=3,4$, except special problem (see [10], [12], [23]–[25]).

Now we consider the case of nonzero nonlinearity. For the simplicity of presentation, we assume that for some positive number $s$

$$(u_0,v_0,v_1)\in H^s\oplus H^s\oplus\omega H^s,\quad 1\le n\le 3,\quad \lambda>0,\quad 1<p<\infty.\tag{1.10}$$

The second result is the following.

**Theorem 1.3.** (i) *If $n=1$, $1<p<\infty$ and $1\le s<\min(2,p)$, then there exists a unique global solution $(u,v)$ satisfying*

$$u\in C([0,\infty);H^s),\quad v\in C^2([0,\infty);H^s),\quad \partial_tv\in C([0,\infty);\omega H^s).\tag{1.11}$$

(ii) *If $n=2$, $1<p\le 3$ and $1<s<\min(2,p)$, then there exists a unique global solution $(u,v)$ satisfying* (1.11).

**Remark 4.** Part $(i)$ is an extension of result in [31], where the global existence of solutions in $H^2$ is studied for odd integer $p\ge 3$.

**Remark 5.** In the case $n = 1$, we use the conservation laws for the control of $L^\infty$ through the Sobolev embedding. But when $n = 2$, we cannot use such embedding any more. Instead, we use a version of Brezis-Gallouet-Wainger inequality in Sobolev space (see [20]). In view of these Sobolev inequalities, we can obtain a global existence in $H^s(\mathbb{R}^n)$ even for all $s > 1$, if $n = 1$ and $p$ is odd integer, and if $n = 2$ and $p = 3$.

The next result is on the local existence and blowup criterion for $n = 3$.

**Theorem 1.4.** (i) *If $n = 3$, $1 < p < \infty$ and $\frac{3}{2} < s < \min(2, p)$, then there exists a positive time $T_*$ and unique solution $(u, v)$ such that*

$$u \in C([0, T_*]; H^s), \quad v \in C^2([0, T_*]; H^s), \quad \partial_t v \in C([0, T_*]; \omega H^s). \tag{1.12}$$

(ii) *Let $T^*$ be the maximal existence time of solution $(u, v)$ to Cauchy problem* (1.1) *and it be finite. Then*

$$\int_0^{T^*} \left( \|u(t)\|_{\dot{F}^0_{\infty,\infty}} + \|v(t)\|_{\dot{F}^0_{\infty,\infty}} \right)^{p-1} dt = \infty$$

*for $\frac{3}{2} < p \le 2$. Furthermore, if*

$$(\|\nabla u_0\|_{L^2} + \|v_0\|_{L^2} + \|\omega^{-1} v_1\|_{L^2}) \|u_0\|_{L^2}$$

*is sufficiently small, then*

$$\int_0^{T^*} \|u(t)\|_{\dot{F}^0_{\infty,\infty}}^{\frac{4(p-1)}{p+1}} dt = \infty$$

*for $\frac{3}{2} < p \le \frac{5}{3}$.*

Here, $\dot{F}^0_{\infty,\infty}$ is the Triebel-Lizorkin space defined as follows. Let $\varphi$ be a Littlewood-Paley function such that $\sum_{j\in\mathbb{Z}} \varphi\left(\frac{\xi}{2^j}\right) = 1$, if $\xi \ne 0$. Let $\Delta_j$ be a frequency projection operator such that $\widehat{\Delta_j \psi}(\xi) = \varphi(\xi/2^j)\widehat{\psi}(\xi)$, where $\widehat{\psi}$ is the Fourier transform of $\psi$.

$$\dot{F}^0_{\infty,\infty} = \left\{ \psi \in \mathcal{S}' : \|\psi\|_{\dot{F}^0_{\infty,\infty}} \equiv \left\| \sup_{j\in\mathbb{Z}} |\Delta_j \psi| \right\|_{L^\infty} < \infty \right\}.$$

It should be noted that BMO $= \dot{F}^0_{\infty,2} \hookrightarrow \dot{F}^0_{\infty,\infty}$ (for the details, see [26], [27]).

**Remark 6.** The blowup criterion in Theorem 1.4 can be extended to some value of $p$ such that $2 < p \le 3$ for large data and $\frac{5}{3} < p \le 5$ for small data. For the details, see Remark 9 below.

**Remark 7.** For the local existence in Theorems 1.3 and 1.4, the $H^s$-regularity on data for $s > \frac{n}{2}$ is used to control the $L^\infty$ norm of $u$ and $v$ for estimation of the bilinear term $vu$ and of the nonlinear term $\lambda|v|^{p-1}v$ via Sobolev embedding. But by deriving dispersive estimates on $K$ and $\partial_t K$, we can control $L^\infty$ without resort to the Sobolev embedding. For one-dimensional and multi-dimensional arguments, see for instance the papers [7] and [8], respectively.

If not specified, throughout this paper, we denote $C$ by a generic constant varying line by line and depending only on the norms of initial data, $s$, $\lambda$, $p$, admissible pair $(q,r)$ and absolute constant.

## 2. Case $\lambda = 0$: Proof of Theorems 1.1 and 1.2

In this section, we consider the global existence of the system with $n = 3, 4$ in case that $\lambda = 0$ based on the conservation laws (1.5) and (1.6). Since the local existence was studied in [22], we have only to consider global a priori estimates of solutions in the 3 and 4 dimensional energy space. That is to say, it suffices to show that for all $T > 0$

$$\|u\|_{L_T^\infty H^1} + \|v\|_{L_T^\infty L^2} + \|\omega^{-1}\partial_t v\|_{L_T^\infty L^2} \le C. \tag{2.1}$$

### 2.1. Proof of Theorem 1.1

**2.1.1. Global existence** By the Hölder inequality, the standard Sobolev inequality $\|u\|_{L^6} \le C_0\|\nabla u\|_{L^2}$, and the $L^2$ conservation (1.5), we have

$$|(v,|u|^2)| \le \|v\|_{L^2}\|u\|_{L^2}^{\frac12}\|u\|_{L^6}^{\frac32} \le C_0^{\frac32}\|u_0\|_{L^2}^{\frac12}\|v\|_{L^2}\|\nabla u\|_{L^2}^{\frac32}. \tag{2.2}$$

This implies upper and lower bounds on $E(0)$ in terms of Cauchy data in the energy space. Regarding lower bounds, we have

$$\begin{aligned} E(0) &= \frac12\left(\|\nabla u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \|\omega^{-1}v_1\|_{L^2}^2\right) + (v_0,|u_0|^2)\\ &\ge \frac12\left(\|\nabla u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \|\omega^{-1}v_1\|_{L^2}^2\right)\\ &\quad - \frac12\|v_0\|_{L^2}^2 - \frac{C_0^3}{2}\|u_0\|_{L^2}\|\nabla u_0\|_{L^2}^3\\ &\ge \frac12(1 - C_0^3\|u_0\|_{L^2}\|\nabla u_0\|_{L^2})\|\nabla u_0\|_{L^2}^2 + \frac12\|\omega^{-1}v_1\|_{L^2}^2\\ &\ge 0, \end{aligned} \tag{2.3}$$

provided $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} \le C_0^{-3}$. Now we set

$$M(t) = \frac12\left(\|\nabla u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\omega^{-1}\partial_t v\|_{L^2}^2\right).$$

Then the RHS of the last inequality in (2.2) is bounded by

$$C_0^{\frac32}\|u_0\|_{L^2}^{\frac12}M(t)\|\nabla u\|_{L^2}^{\frac12} \le 2^{\frac14}C_0^{\frac32}\|u_0\|_{L^2}^{\frac12}M(t)^{\frac54}.$$

By the energy conservation (1.6), this implies

$$\begin{aligned} M(t) &= E(t) - (v, |u|^2) \\ &= E(0) - (v, |u|^2) \\ &\le E(0) + 2^{\frac{1}{4}} C_0^{\frac{3}{2}} \|u_0\|_{L^2}^{\frac{1}{2}} M(t)^{\frac{5}{4}}. \end{aligned} \tag{2.4}$$

We see from (2.4) that

$$M(t) \le 5E(0), \tag{2.5}$$

provided

$$E(0)(2C_0^6 \|u_0\|_{L^2}^2) < \frac{1}{5}\left(\frac{4}{5}\right)^4. \tag{2.6}$$

The required a priori estimate (2.1) follows from (2.5) under smallness condition given by (2.3), (2.4) and (2.6).

An application of (2.5) is the Strichartz estimate of $u$. For any admissible pair $(q, r)$, using (1.4), we have

$$\|u\|_{L_T^q L^r} \le C T^{\frac{1}{q}} \|u\|_{L_T^\infty H^1} \le C(T),$$

where $C(T)$ is a constant depending only on $C$ and $T$. Therefore, we deduce that $u \in L_{loc}^q L^r$.

**2.1.2. Blow-up criterion** Assume that

$$\int_0^{T^*} (T^* - t) \|u(t)\|_{L^4}^2 \, dt < \infty. \tag{2.7}$$

Then let us observe from the equation (1.2) that

$$\|\partial_t v(t)\|_{L^2} \le C(\|v_0\|_{L^2} + \|v_1\|_{L^2}) + C \int_0^t \|u(t')\|_{L^4}^2 \, dt'.$$

and hence that the finiteness of (2.7) implies that of $\|\partial_t v\|_{L^1(0,T^*;\, L^2)}$.

Now we assume that

$$\int_0^{T^*} \|\partial_t v\|_{L^2} \, dt \equiv M < \infty. \tag{2.8}$$

Then taking $L^2$ norm of $v(t) = v_0 + \int_0^t \partial_t v(t') \, dt'$, we have

$$\sup_{0 \le t < T^*} \|v(t)\|_{L^2} \le \|v_0\|_{L^2} + M. \tag{2.9}$$

By conservation laws (1.5), (1.6), and the estimates (2.2) and (2.9),

$$\begin{aligned} \frac{1}{2}\|\nabla u\|_{L^2}^2 &= E(t) - \frac{1}{2}\left(\|v\|_{L^2}^2 + \|\omega^{-1}\partial_t v\|_{L^2}^2\right) - (v, |u|^2) \\ &\le E(0) + \|v\|_{L^2} \|u\|_{L^4}^2 \\ &\le E(0) + (\|v_0\|_{L^2} + M) \|u_0\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{3}{2}}. \end{aligned} \tag{2.10}$$

This implies

$$\sup_{0\le t<T^*} \|\nabla u(t)\|_{L^2}^2 \le CE(0) + CE(\|v_0\|_{L^2} + M)^4 \|u_0\|_{L^2}^2 \tag{2.11}$$

Moreover, we obtain

$$\begin{aligned}\|v\|_{L^2}^2 + \|\omega^{-1}\partial_t v\|_{L^2}^2 &= 2E(0) - \|\nabla u\|_{L^2}^2 - 2(v, |u|^2)\\ &\le 2E(0) - \|\nabla u\|_{L^2}^2 + \frac{1}{2}\|v\|_{L^2}^2 + 8\|u\|_{L^4}^4\\ &\le 2E(0) - \|\nabla u\|_{L^2}^2 + \frac{1}{2}\|v\|_{L^2}^2 + 8C_0^3\|u_0\|_{L^2}\|\nabla u\|_{L^2}^3.\end{aligned}$$

This implies

$$\begin{aligned}&\sup_{0\le t<T^*} \left(\|v(t)\|_{L^2}^2 + \|\omega^{-1}\partial_t v\|_{L^2}^2\right)\\ &\qquad \le 4E(0) + 16C_0^3\|u_0\|_{L^2}\left(\sup_{0\le t<T^*}\|\nabla u(t)\|_{L^2}\right)^3.\end{aligned} \tag{2.12}$$

Estimates (2.11) and (2.12) contradict the maximality of $T^*$.

### 2.2. Proof of Theorem 1.2

**2.2.1. Global existence** By the Hölder inequality and the standard Sobolev inequality $\|u\|_{L^4} \le C_0\|\nabla u\|_{L^2}$, we have

$$|(v, |u|^2)| \le \|v\|_{L^2}\|u\|_{L^4}^2 \le C_0^2\|v\|_{L^2}\|\nabla u\|_{L^2}^2.$$

Therefore,

$$\begin{aligned}E(0) &= \frac{1}{2}\left(\|\nabla u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \|\omega^{-1}v_1\|_{L^2}^2\right) + (v, |u|^2)\\ &\le \frac{1}{2}\left(\|\nabla u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \|\omega^{-1}v_1\|_{L^2}^2\right) + C_0^2\|v_0\|_{L^2}\|\nabla u_0\|_{L^2}^2\end{aligned}$$

and

$$\begin{aligned}E(0) &\ge \frac{1}{2}\left(\|\nabla u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2\right) - \varepsilon C_0^2\|v_0\|_{L^2}\|\nabla u_0\|_{L^2}\\ &\ge \frac{1-\varepsilon C_0^2}{2}\left(\|\nabla u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2\right),\end{aligned}$$

provided $\|\nabla u_0\|_{L^2} \le \varepsilon < C_0^{-2}$. Now we set

$$M(t) = \frac{1}{2}\left(\|\nabla u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 + \|\omega^{-1}\partial_t v(t)\|_{L^2}^2\right).$$

Then

$$\begin{aligned}M(t) &= E(0) - (v, |u|^2)\\ &\le E(0) + C_0^2\|v\|_{L^2}\|\nabla u\|_{L^2}^2\\ &\le E(0) + C_0^2 M(t)\|\nabla u\|_{L^2}\\ &\le E(0) + 2^{\frac{1}{2}}C_0^2 M(t)^{\frac{3}{2}},\end{aligned}$$

from which the required a priori estimate on $M$ follows, provided $\|\nabla u_0\|_{L^2} + \|v_0\|_{L^2} + \|\omega^{-1} v_1\|_{L^2}$ is sufficiently small.

By the endpoint estimate $(q,r) = (2,4)$ of (1.4), we have

$$\begin{aligned}\|u\|_{L^2_T L^4} &\le C\|u_0\|_{L^2} + C\|vu\|_{L^2_T L^{\frac{4}{3}}} \\ &\le C\|u_0\|_{L^2} + C\|v\|_{L^\infty_T L^2}\|u\|_{L^2_T L^4},\end{aligned}$$

where $C$ is independent of $T > 0$. Since $M$ may be taken sufficiently small by the smallness assumption on the data,

$$\|u\|_{L^2_T L^2} \le C\|u_0\|_{L^2},$$

where $C$ is independent of $T > 0$. By Fatou's lemma, $u \in L^2 L^4$. Since $u \in L^\infty L^2$, for any admissible pair $(q,r)$, $\|u\|_{L^q L^r} \le C\|u_0\|_{L^2}$.

**2.2.2. Scattering** By the integral equation, we have

$$U(-t)u(t) - U(-s)u(s) = -i\int_s^t U(-t')(vu)(t')\,dt'.$$

By the endpoint Strichartz estimate, we have

$$\begin{aligned}\|U(-t)u(t) - U(-s)u(s)\|_{L^2} &= \left\|\int_s^t U(-t')(vu)(t')\,dt'\right\|_{L^2} \\ &\le C\|vu\|_{L^2(s,t;L^{\frac{4}{3}})} \\ &\le C\|v\|_{L^\infty L^2}\|u\|_{L^2(s,t;L^4)} \\ &\to 0\end{aligned}$$

as $t > s \to +\infty$. This gives a unique asymptotic state $u_0^+ \in L^2$.

Similarly, the existence of unique asymptotic sates $(v_0^+, v_1^+) \in L^2 \oplus \omega L^2$ follows by the integral equation (1.3).

## 3. Case $\lambda \neq 0$: Proof of Theorems 1.3 and 1.4

### 3.1. Local existence

In this section, we discuss the local existence theory of the system (1.1) with general nonlinear term $f(v)$. To do this, let us first introduce a lemma for the nonlinear estimates (see [7], [9], [13], [19] for instance).

**Lemma 3.1.** (i) *Let $f \in C^1(\mathbb{R}, \mathbb{R})$ with $f(0) = f'(0) = 0$ and assume that*

$$(3.1) \qquad |f'(t_1) - f'(t_2)| \le C \left\{ \begin{array}{ll} |t_1 - t_2|^{p-1}, & \text{if} \quad 1 \le p \le 2 \\ (|t_1|^{p-2} + |t_2|^{p-2})|t_1 - t_2|, & \text{if} \quad p > 2 \end{array} \right\}$$

*for all* $t_1, t_2 \in \mathbb{R}$. *Let* $0 \le s < \min(2, p)$. *Then* $f$ *satisfies the estimate*

$$\|f(v)\|_{\dot{H}^s} \le C\|v\|_{L^\infty}^{p-1}\|v\|_{\dot{H}^s} \tag{3.2}$$

*for any* $v \in L^\infty \cap \dot{H}^s$.

(ii) *Let* $f \in C^k(\mathbb{R}, \mathbb{R})(k \ge 2)$ *with*

$$|f^{(j)}(t)| \le C|t|^{p-j} \tag{3.3}$$

*for all* $0 \le j \le k \le p$ *and* $t \in \mathbb{R}$. *Then* $f$ *satisfies the estimate*

$$\|f(v)\|_{\dot{H}^s} \le C\|v\|_{L^\infty}^{p-1}\|v\|_{\dot{H}^s} \tag{3.4}$$

*for any* $s$ *with* $0 \le s \le k$ *and any* $v \in L^\infty \cap \dot{H}^s$.

(iii) *For any* $s \ge 0$, *we have*

$$\|uv\|_{\dot{H}^s} \le C(\|u\|_{L^\infty}\|v\|_{\dot{H}^s} + \|v\|_{L^\infty}\|u\|_{\dot{H}^s}) \tag{3.5}$$

*for any* $v \in L^\infty \cap \dot{H}^s$

Using the lemma above, we obtain the following.

**Proposition 3.1.** *Let* $f \in C^k(\mathbb{R}, \mathbb{R})$ *satisfy* (3.1) *for* $k = 1$ *and* (3.3) *for* $k \ge 2$. *Suppose that* $(u_0, v_0, v_1) \in H^s \oplus H^s \oplus \omega H^s$ *for* $s < p$, *if* $k = 1$ *and* $s \le k$, *if* $k \ge 2$. *Then there exists a positive time* $T_*$ *and unique solution* $(u, v)$ *satisfying the regularity* (1.12).

*Proof of Proposition* 3.1. Let us define nonlinear functionals $N_1$ and $N_2$ by

$$N_1(u,v)(x,t) = U(t)u_0 - i\int_0^t U(t-t')(vu)(t')\,dt',$$
$$N_2(u,v)(x,t) = (\partial_t K)(t)v_0 + K(t)v_1 + \int_0^t K(t-t')\omega^2(f(v)+|u|^2)(t')\,dt'.$$

We also define a complete metric space $X_R(T)$ with metric $d_T$ by

$$X_R(T) = \{(u,v) : \|(u,v)\|_{X(T)} \equiv \|u\|_{L_T^\infty H^s} + \|v\|_{L_T^\infty H^s} \le R\},$$
$$d_T((u,v),(\widetilde{u},\widetilde{v})) = \|(u,v) - (\widetilde{u},\widetilde{v})\|_{L_T^\infty L^2}.$$

Then from Sobolev embedding $H^s \hookrightarrow L^\infty$ for $s > \frac{n}{2}$ and Lemma 3.1, we have for any $(u,v) \in X_R(T)$

$$\|N_1(t)\|_{H^s} \le \|u_0\|_{H^s} + C\int_0^t \|v(t')\|_{H^s}\|u(t')\|_{H^s}\,dt' \le C + CR^2T,$$
$$\|N_2(t)\|_{H^s} \le \|v_0\|_{H^s} + \|\omega^{-1}v_1\|_{H^s} + C\int_0^t \left(\|v\|_{H^s}^p + \|u\|_{H^s}^2\right)dt'$$
$$\le C + C(M(R)R + R^2)T.$$

If $R \geq 3C$ and $T < T_1$ for some small $T_1$ such that

$$CR^2T_1 + C(M(R)R + R^2)T_1 \leq C,$$

then $(N_1(u,v), N_2(u,v)) \in X_R(T)$.

On the other hand, if $(u,v), (\widetilde{u}, \widetilde{v}) \in X_R(T)$ and $T < T_2$ for some small $T_2$, then from Lemma 3.1

$$\begin{aligned}
&d_T((N_1(u,v), N_2(u,v)), (N_1(\widetilde{u},\widetilde{v}), N_2(\widetilde{u},\widetilde{v}))) \\
&\quad \leq C \int_0^T (\|(u,v)\|_{X(t')} + \|(\widetilde{u},\widetilde{v})\|_{X(t')}) d_{t'}((u,v),(\widetilde{u},\widetilde{v}))\, dt' \\
&\qquad + \int_0^T \left( \|v\|_{H^s}^{p-1} + \|\widetilde{v}\|_{H^s}^{p-1} + \|u\|_{H^s} + \|\widetilde{u}\|_{H^s} \right) d_{t'}((u,v),(\widetilde{u},\widetilde{v}))\, dt' \\
&\quad \leq (2CRT + 2M(R)T) d_T((u,v),(\widetilde{u},\widetilde{v})) \\
&\quad \leq \frac{1}{2} d_T((u,v),(\widetilde{u},\widetilde{v})).
\end{aligned}$$

Therefore by contraction mapping theorem, there exists a solution $(u,v) \in X_R(T_*)$ of (1.2), where $T_* = \min(T_1, T_2)$. The uniqueness follows immediately from the above argument. Using the original equation, the time regularity is readily obtainable. So we leave them to the readers. This completes the proof of proposition. □

**Remark 8.** (1) Since $s$ can be chosen to be greater than equal to 1, as stated in the introduction, the $L^2$ norm of the solution $u$ and the energy of $(u,v)$ are conserved up to the existence time of solution.

(2) If $f(v) = \lambda |v|^{p-1} v$ with $p > 1$, then $f$ satisfies the condition (3.1).

(3) If $f(v) = \lambda v^k$ for some fixed $\lambda \in \mathbb{R}$ and integer $k \geq 2$, then we do not need the restriction on $n$, $s$ and $p$ for the local existence of solutions.

### 3.2. Proof of Theorem 1.3

Now we prove the first and second parts of main theorem. Since $\lambda > 0$, from the $L^2$ and energy conservation, we can get

$$\begin{aligned}
&\|u\|_{L_T^\infty H^1} + \|v\|_{L_T^\infty L^{p+1}} \leq C \quad \text{for} \quad n = 1, \\
&\|u\|_{L_T^\infty H^1} + \|v\|_{L_T^\infty L^{p+1}} \leq C(T) \quad \text{for} \quad n = 2.
\end{aligned}$$

These estimate follow from the observation that

$$\begin{aligned}
&|(v, |u|^2)| \leq \frac{1}{4}\|v\|_{L^2}^2 + \frac{1}{4}\|\nabla u\|_{L^2}^2 + C\|u\|_{L^2}^6, \quad \text{if} \quad n = 1, \\
&|(v, |u|^2)| \leq \frac{1}{4}\|\nabla u\|_{L^2}^2 + C\|u\|_{L^2}^2 \|v\|_{L^2}^2 \quad \text{if} \quad n = 2
\end{aligned}$$

and the estimate $\|v(t)\|_{L^2} \leq \|v_0\|_{L^2} + \int_0^t \|\partial_t v(t')\|_{L^2}\, dt'$. For the details, see Section 4 of [22].

As in [29], from the regularity of the local solution we can observe that

$$\partial_t \left( |\partial_t v|^2 + v^2 + \frac{2\lambda}{p+1}|v|^{p+1} \right) = \big( (1-\Delta)^{-1}(v + f(v)) + \omega^2(|u|^2) \big)\, \partial_t v. \tag{3.6}$$

Here we note that $(1-\Delta)^{-1}$ is bounded from $L^r$ to $L^\infty$ for $1 \le r \le \infty$ if $n = 1$, for $1 < r \le \infty$ if $n = 2$, and for $\frac{n}{2} < r \le \infty$ if $n \ge 3$ and that $\omega^2$ is bounded in $L^r$ for any $r$ with $1 \le r \le \infty$. Thus if $n = 1, 2$, then we have

$$\begin{aligned}
&\|\partial_t v(t)\|_{L^\infty}^2 + \|v(t)\|_{L^\infty}^2 + \lambda \|v(t)\|_{L^\infty}^{p+1} \\
&\quad \le C + C\int_0^t \|u\|_{L^\infty}^4\, dt' + C\int_0^t \|(1-\Delta)^{-1} f(v)\|_{L^\infty}^2\, dt' \\
&\qquad\qquad + C\int_0^t (\|\partial_t v\|_{L^\infty}^2 + \|v\|_{L^\infty}^2)\, dt' \\
&\quad \le C + C\int_0^t \|u\|_{L^\infty}^4\, dt' + C\int_0^t \|v\|_{L^{p+1}}^{2p}\, dt' + C\int_0^t (\|\partial_t v\|_{L^\infty}^2 + \|v\|_{L^\infty}^2)\, dt' \\
&\quad \le C + C\int_0^t \|u\|_{L^\infty}^4\, dt' + CT\|v\|_{L_T^\infty L^{p+1}}^{2p} + C\int_0^t (\|\partial_t v\|_{L^\infty}^2 + \|v\|_{L^\infty}^2)\, dt'.
\end{aligned}$$

Hence Gronwall's inequality yields

$$\sup_{0 \le t \le T} \left( \|\partial_t v(t)\|_{L^\infty}^2 + \|v(t)\|_{L^\infty}^2 + \|v\|_{L^\infty}^{p+1} \right) \le C(T) \int_0^T \|u\|_{L^\infty}^4\, dt. \tag{3.7}$$

To obtain the global existence, we have only to prove that

$$\|u(t)\|_{H^s} + \|v(t)\|_{H^s} + \|\omega^{-1}\partial_t v(t)\|_{H^s} \le C(T) \tag{3.8}$$

for all $t \in [0, T]$ and any $T > 0$.

If $n = 1$, then by Sobolev embedding and energy conservation (1.6), we see that $\|u\|_{L^\infty([0,T]\times\mathbb{R})} \le C$ and hence by (3.7), $\|v\|_{L^\infty([0,T]\times\mathbb{R})} \le C(T)$. Therefore we have

$$\begin{aligned}
\|u(t)\|_{H^s} &\le C + C\int_0^t (\|v\|_{L^\infty}\|u\|_{H^s} + \|v\|_{H^s}\|u\|_{L^\infty})\, dt' \\
&\le C + C(T)\int_0^t (\|v\|_{H^s} + \|u\|_{H^s})\, dt', \\
\|v(t)\|_{H^s} &\le C + C\int_0^t (\|v\|_{L^\infty}^{p-1}\|v\|_{H^s} + \|u\|_{L^\infty}\|u\|_{H^s})\, dt' \\
&\le C + C(T)\int_0^t (\|v\|_{H^s} + \|u\|_{H^s})\, dt', \\
\|\omega^{-1}\partial_t v(t)\|_{H^s} &\le C + C(T)\int_0^t (\|v\|_{H^s} + \|u\|_{H^s})\, dt'.
\end{aligned} \tag{3.9}$$

Combining above three inequalities and using Gronwall's inequality, we get (3.8) for all $t \in [0, T]$.

Since the embedding $H^1 \hookrightarrow L^\infty$ does not hold for $n = 2$, instead we use the Brezis-Gallouet-Wainger inequality (for instance see [2] and [20]). More precisely, we use the following inequality

$$\|u(t)\|_{L^\infty} \le C\|u(t)\|_{H^1}\left(1+\log\left(1+\frac{\|u(t)\|_{H^s}}{\|u(t)\|_{H^1}}\right)\right)^{\frac{1}{2}}. \tag{3.10}$$

Since $\|u(t)\|_{H^1} \le C(T)$ for all $t \in [0, T]$, from (3.10) we have

$$\|u(t)\|_{L^\infty} \le C(T)(1+\log(1+\|u(t)\|_{H^s}))^{\frac{1}{2}}. \tag{3.11}$$

Now using (3.7), (3.9) and the fact that $\frac{2(p-1)}{p+1} \le 1$ for $p \le 3$, we have

$$\begin{aligned}&\|u\|_{H^s} + \|v\|_{H^s}\\ &\quad \le C + C(T)\int_0^t (1+\log(1+\|u\|_{H^s}+\|v\|_{H^s}))(\|v\|_{H^s}+\|u\|_{H^s})\,dt'.\end{aligned}$$

Thus by Gronwall's inequality, we finally get the bound (3.8) for all $t \in [0, T]$. This completes the proof of $(i)$ and $(ii)$ of Theorem 1.3.

### 3.3. Proof of Theorem 1.4

Since the local existence was already established in the previous section, we consider only blowup criterion in this section.

Using (3.7), we first have for $1 < p \le 2$

$$\begin{aligned}&\|u(t)\|_{H^s} + \|v(t)\|_{H^s}\\ &\quad \le C + \int_0^t (1+\|u\|_{L^\infty}+\|v\|_{L^\infty})^{p-1}(\|u\|_{H^s}+\|v\|_{H^s})\,dt'.\end{aligned} \tag{3.12}$$

Let us invoke the Brezis-Gallouet-Wainger inequality in Triebel-Lizorkin space. For any $s > \frac{3}{2}$,

$$\|\psi\|_{L^\infty} \le C(1+\|\psi\|_{\dot F^0_{\infty,\infty}}(1+\log(1+\|\psi\|_{H^s}))). \tag{3.13}$$

For the proof, see [4] and Remark 8 below. Now we set

$$M(t) = \|u(t)\|_{\dot F^0_{\infty,\infty}} + \|v(t)\|_{\dot F^0_{\infty,\infty}}.$$

Then by (3.13), we obtain for all $t \in [0, T^*)$

$$\begin{aligned}&\|u(t)\|_{H^s} + \|v(t)\|_{H^s} \le C(T^*)\\ &\quad + \int_0^t (1+M(t'))^{p-1}(1+\log(1+\|u\|_{H^s}+\|v\|_{H^s}))^{p-1}(\|u\|_{H^s}+\|v\|_{H^s})\,dt'.\end{aligned}$$

Hence by Gronwall's inequality we have

$$\|u(t)\|_{H^s} + \|v(t)\|_{H^s} \le C(T^*)\exp\left[C(T^*)\exp\left(C(T^*)\int_0^t M(t')^{p-1}\,dt'\right)\right].$$

Since left had side of the above inequality tend to infinity as $t \to T^*$, we can obtain the first part of blowup criterion.

On the other hand, if $(\|\nabla u_0\|_{L^2} + \|v_0\|_{L^2} + \|\omega^{-1}v_1\|_{L^2})\|u_0\|_{L^2}$ is small, then by the same argument as in the proof of part $(i)$ of Theorem 1.1, we can obtain that

$$\|u\|_{L^\infty_T H^1} + \|v\|_{L^\infty_T L^{p+1}} \le C(T)$$

for all $T < T^*$. Hence using (3.6), we have

$$\|\partial_t v\|_{L^\infty}^2 + \|v\|_{L^\infty}^2 + \lambda\|v\|_{L^\infty}^{p+1} \le C + C\int_0^t \|u\|_{L^\infty}^4\,dt' + C\int_0^t \|v\|_{L^{pr}}^{2p}\,dt' + C\int_0^t (\|\partial_t v\|_{L^\infty}^2 + \|v\|_{L^\infty}^2)\,dt'.$$

For the third term, we use the estimate $\|(1-\Delta)^{-1}f(v)\|_{L^\infty} \le C\|f(v)\|_{L^r}$ for $\frac{3}{2} < r \le \infty$. If $p < 2$, then we take $r = \frac{p+1}{p}$. If $2 \le p \le 5$, then we choose $r$ such that $\frac{p+1}{p} < r < \frac{p+1}{p-1}$ and hence

$$pr > p+1, \quad p\left(1 - \frac{p+1}{pr}\right) < 1. \tag{3.14}$$

Since $p + 1 < pr < \infty$,

$$\|v\|_{L^{pr}} \le \|v\|_{L^{p+1}}^\theta \|v\|_{L^\infty}^{1-\theta},$$

where $\theta = \frac{p+1}{pr}$. Thus by (3.14) and Young's inequality, we get

$$\|\partial_t v\|_{L^\infty}^2 + \|v\|_{L^\infty}^2 + \lambda\|v\|_{L^\infty}^{p+1} \le C(T) + CT\|u\|_{L^\infty}^4 + C\int_0^t (\|\partial_t v\|_{L^\infty}^2 + \|v\|_{L^\infty}^2)\,dt'.$$

By Gronwall's inequality, we obtain the estimate (3.7) and substituting this into (3.12), we can deduce from the fact $\frac{4(p-1)}{p+1} \le 1$ for $p \le \frac{5}{3}$ that

$$\|u(t)\|_{H^s} + \|v(t)\|_{H^s} \le C(T^*) + \int_0^t (1 + \|u\|_{\dot F^0_{\infty,\infty}})^{\frac{4(p-1)}{p+1}}(1 + \log(1 + \|u\|_{H^s} + \|v\|_{H^s}))(\|u\|_{H^s} + \|v\|_{H^s})\,dt'.$$

Hence the Gronwall's inequality yields the second blowup criterion. For the case that $\frac{5}{3} < p \le 5$, see Remark 9 below.

**Remark 9.** Concerning the Brezis-Gallouet-Wainger inequality in Triebel-Lizorkin space, let us introduce a slightly modified version. We first define a homogeneous Triebel-Lizorkin type space $\dot{\mathfrak{F}}^0_{\infty,q}(0 < q < \infty)$ as follows.

$$\dot{\mathfrak{F}}^0_{\infty,q} \equiv \left\{\psi : \psi = \sum_{j\in\mathbb{Z}} \Delta_j\psi_j \quad \text{in} \quad \mathcal{S}' \quad \text{for some} \quad \psi_j \in \mathcal{S}' \quad \text{with} \quad \{\Delta_j\psi_j\}_{j\in\mathbb{Z}} \in L^\infty\ell^q \right\},$$

(3.15) $$\|\psi\|_{\dot{\mathfrak{F}}^0_{\infty,\,q}} \equiv \inf_{\psi=\sum \Delta_j\psi_j} \|\{\Delta_j\psi_j\}\|_{L^\infty \ell^q}.$$

If $q = \infty$, then we define $\dot{\mathfrak{F}}^0_{\infty,\,\infty}$ by $\dot{F}^0_{\infty,\,\infty}$. The usual Triebel-Lizorkin space $\dot{F}^0_{\infty,q}(0 < q < \infty)$ is defined by

$$\dot{F}^0_{\infty,\,q} \equiv \{\psi : \psi = \sum_{j\in\mathbb{Z}} \Delta_j\psi_j \quad \text{for some} \quad \{\psi_j\} \in L^\infty\ell^q\},$$
$$\|\psi\|_{\dot{F}^0_{\infty,\,q}} \equiv \inf_{\psi=\sum \Delta_j\psi_j} \|\{\psi_j\}\|_{L^\infty\ell^q}.$$

One can easily see that $\dot{B}^0_{\infty,q} \hookrightarrow \dot{\mathfrak{F}}^0_{\infty,\,q} \hookrightarrow \dot{F}^0_{\infty,\,q}$ for $q < \infty$, while in general it is likely that the converse inclusion $\dot{F}^0_{\infty,\,q} \hookrightarrow \dot{\mathfrak{F}}^0_{\infty,\,q}$ is an open question. If $\psi \in H^s(\mathbb{R}^n)$ for $s > \frac{n}{2}$, then

$$\begin{aligned}\psi &= \sum_{j<-N} \Delta_j\psi + \sum_{-N\le j\le N} \Delta_j\psi + \sum_{j>N} \Delta_j\psi \\ &\equiv \psi_- + \psi_0 + \psi_+.\end{aligned}$$

Revisiting the proof in [4], for the first and second terms, we obtain

(3.16) $$|\psi_-(x)| + |\psi_+(x)| \le C2^{-N}\|\psi\|_{H^s}.$$

On the other hand, as for $\psi_0$, we can find $\psi_j \in \mathcal{S}'$ such that $\psi = \sum_j \Delta_j\psi_j$ and $\{\Delta_j\psi_j\} \in L^\infty\ell^q$, and hence

$$|\psi_0(x)| \le |\widetilde{\psi}_N(x)| + \sum_{-N+1\le j\le N-1} |\Delta_j\psi_j(x)|,$$

where

$$\widetilde{\psi}_N = \sum_{-N\le j\le N} \sum_{k\in\mathbb{Z}} \Delta_j\Delta_k\psi_k - \sum_{-N+1\le j\le N-1} |\Delta_j\psi_j|.$$

Since the number of sum consisting of $\widetilde{\psi}_N$ is finite, independently of $N$, and hence $\|\widetilde{\psi}_N\|_{L^\infty} \le C\|\psi\|_{\dot{\mathfrak{F}}^0_{\infty,\infty}}$, we deduce from the definition (3.15) that

(3.17) $$|\psi_0(x)| \le C\|\psi\|_{\dot{\mathfrak{F}}^0_{\infty,\infty}} + N^{\frac{1}{q'}}\|\psi\|_{\dot{\mathfrak{F}}^0_{\infty,\,q}}.$$

Combining (3.16) and (3.17), we obtain

$$\|\psi\|_{L^\infty} \le C2^{-N}\|\psi\|_{H^s} + N^{\frac{1}{q'}}\|\psi\|_{\dot{\mathfrak{F}}^0_{\infty,\,q}}.$$

Thus choosing the optimal $N$, we can obtain

(3.18) $$\|\psi\|_{L^\infty} \le C(1 + \|\psi\|_{\dot{\mathfrak{F}}^0_{\infty,\,q}}(1 + \log(1 + \|\psi\|_{H^s}))^{\frac{1}{q'}}).$$

Applying the estimate (3.18) to (3.7) and (3.12), we see that for $2 < p \le 3$ and $q = \frac{p-1}{p-2}$

$$\begin{aligned}\|u(t)\|_{H^s} &+ \|v(t)\|_{H^s} \\ &\le C(T^*) + \int_0^t (1 + \|u\|_{\dot{\mathfrak{F}}^0_{\infty,q}} + \|v\|_{\dot{\mathfrak{F}}^0_{\infty,q}})^{p-1} \\ &\qquad \times (1 + \log(1 + \|u\|_{H^s} + \|v\|_{H^s}))(\|u\|_{H^s} + \|v\|_{H^s})\, dt'\end{aligned}$$

and hence

$$\int_0^{T^*} (\|u\|_{\dot{\mathfrak{F}}^0_{\infty,q}} + \|v\|_{\dot{\mathfrak{F}}^0_{\infty,q}})^{p-1}\, dt = \infty.$$

Similarly, for $\frac{5}{3} < p \le 5$ and $q = \frac{4p-4}{3p-5}$

$$\begin{aligned}\|u(t)\|_{H^s} &+ \|v(t)\|_{H^s} \\ &\le C(T^*) + \int_0^t (1 + \|u\|_{\dot{\mathfrak{F}}^0_{\infty,q}})^{\frac{4(p-1)}{p+1}} (1 + \log(1 + \|u\|_{H^s} + \|v\|_{H^s})) \\ &\qquad \times (\|u\|_{H^s} + \|v\|_{H^s})\, dt'.\end{aligned}$$

Hence Gronwall's inequality yields the blowup criterion

$$\int_0^{T^*} \|u\|_{\dot{\mathfrak{F}}^0_{\infty,q}}^{\frac{4(p-1)}{p+1}}\, dt = \infty,$$

provided the maximal existence time $T^*$ is finite and

$$(\|\nabla u_0\|_{L^2} + \|v_0\|_{L^2} + \|\omega^{-1} v_1\|_{L^2})\|u_0\|_{L^2}$$

is sufficiently small.

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