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Relative isoperimetric inequality on a curved surface

By

Keomkyo SEO*

Abstract

Let C be a closed convex set on a complete simply connected surface S whose Gaussian curvature is bounded above by a nonpositive constant K. For a relatively compact subset $\Omega \subset S \sim C$, we obtain the sharp relative isoperimeric inequality $2\pi \text{Area}(\Omega) - K\text{Area}(\Omega)^2 \leq \text{Length}(\partial\Omega \sim \partial C)^2$. And we also have a similar partial result with positive Gaussian curvature bound.

1. Introduction

The classical isoperimetric inequality says that $4\pi\text{Area}(\Omega) \leq \text{Length}(\partial\Omega)^2$ for a compact subset $\Omega \subset \mathbb{R}^2$. Equality holds if and only if Ω is a disk. Many mathematicians have generalized this inequality. For example, the following isoperimetric inequality is well-known. For a domain Ω in a complete surface S of Gaussian curvature bounded above by a constant K,

$$4\pi\text{Area}(\Omega) - K\text{Area}(\Omega)^2 \leq \text{Length}(\partial\Omega)^2. \tag{1.1}$$

Equality holds if and only if Ω is a geodesic disk of constant Gaussian curvature K ([1], [2], [3], [5], [9], [11], see [10] for more references.).

Now we consider the relative isoperimetric problem. It is to find an isoperimetric region outside a closed convex set C in a Euclidean space or in a Riemannian manifold M. We study this problem in a smooth category. On that account we assume ∂C and $\partial\Omega$ are smooth for a closed convex set C and a subset $\Omega \subset S \sim C$. One may then ask if the relative isoperimetric inequality similar to (1.1) holds. That is, given a complete simply connected surface S of Gaussian curvature bounded above by a constant K, a closed convex set C in S, and a relatively compact subset Ω of $S \sim C$, does the inequality

$$2\pi\text{Area}(\Omega) - K\text{Area}(\Omega)^2 \leq \text{Length}(\partial\Omega \sim \partial C)^2 \tag{1.2}$$

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hold? And does equality hold if and only if Ω is a geodesic half disk with constant Gaussian curvature K and $\partial\Omega \sim \partial C$ is a geodesic semicircle?

Choe ([6]) proved (1.2) for a disk type domain $\Omega \subset S \sim C$ with nonpositive Gaussian curvature, i.e. $K = 0$. In this article we will show that the inequality (1.2) holds for a relatively compact subset $\Omega \subset S \sim C$ of Gaussian curvature bounded above by a nonpositive constant K. And we also prove a similar partial result with positive Gaussian curvature bound.

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2. The case of constant Gaussian curvature

In this section we prove the relative isoperiemtric inequality on a complete simply connected surface with constant Gaussian curvature K, for completeness. First we consider the case of $K \leq 0$.

Theorem 2.1. *Let C be a closed convex set in a complete simply connected surface S with constant Gaussian curvature $K \leq 0$. Then, for a relatively compact subset Ω in $S \sim C$ we have*

$$2\pi \mathrm{Area}(\Omega) - K\mathrm{Area}(\Omega)^2 \leq \mathrm{Length}(\partial\Omega \sim \partial C)^2, \tag{2.1}$$

where equality holds if and only if Ω is a geodesic half disk and $\partial\Omega \sim \partial C$ is a geodesic semicircle.

Proof. If each connected component D of Ω is not a disk topologically, we fill it to get a disk type domain $\hat{D} \subset S \sim C$, using the simple connectivity of S. One can easily see that $\mathrm{Area}(D) \leq \mathrm{Area}(\hat{D})$ and $\mathrm{Length}(\partial\hat{D}_i \sim \partial C) \leq \mathrm{Length}(\partial D \sim \partial C)$. Once we have the inequality (2.1) for $\hat{D}$, we can prove (2.1) for a relatively compact subset $D \subset S \sim C$, since we have

$$\begin{aligned} 2\pi\mathrm{Area}(D) - K\mathrm{Area}(D)^2 &\leq 2\pi\mathrm{Area}(\hat{D}) - K\mathrm{Area}(\hat{D})^2 \\ &\leq \mathrm{Length}(\partial\hat{D} \sim \partial C)^2 \\ &\leq \mathrm{Length}(\partial D \sim \partial C)^2. \end{aligned} \tag{2.2}$$

And if D is an annular domain surrounding C, then $D \cup C$ satisfies the isoperimetric inequality (1.1) which automatically satisfies (2.1). Hence it is enough to show that the inequality (2.1) holds for a domain Ω which is a disjoint union of disk type D's. We assume that each D is a disk type domain. For each $D \subset S \sim C$, we obtain a new domain $\tilde{D} \subset S$ by reflecting the convex hull of D about its geodesic boundary inside C. Let

$$\tilde{\Omega} = \cup(\tilde{D} \cup \text{the convex hull of } D).$$

Then from the classical isoperimetric inequality for a constantly curved surface $\tilde{\Omega}$, we have

$$4\pi\mathrm{Area}(\tilde{\Omega}) - K\mathrm{Area}(\tilde{\Omega})^2 \leq \mathrm{Length}(\partial\overline{\tilde{\Omega}})^2. \tag{2.3}$$

Furthermore we know

$$2\mathrm{Area}(\Omega) \le \mathrm{Area}(\tilde{\Omega}) \text{ and } 2\mathrm{Length}(\partial\Omega \sim \partial C) \ge \mathrm{Length}(\partial\tilde{\Omega}).$$

Applying these equalities to (2.3), we can see that a domain Ω which is a disjoint union of disk type D satisfies the inequality (2.1). Equality occurs if and only if $\tilde{\Omega}$ satisfies equality in (2.3). Hence Ω is a geodesic half disk and $\partial\Omega \sim \partial C$ is a geodesic semicircle. Therefore we obtain the above theorem. $\square$

For $K > 0$, the proof of Theorem 2.1 doesn't work because the inequality (2.2) doesn't hold in this case. However, with more assumptions, we have a similar partial result as follows.

Theorem 2.2. *Let C be a closed convex set in a two dimensional sphere $S^2(\frac{1}{\sqrt{K}}) \subset \mathbb{R}^3$ of radius $\frac{1}{\sqrt{K}}$ with constant Gaussian curvature $K > 0$. Suppose that Ω is a disk type domain in $S^2(\frac{1}{\sqrt{K}}) \sim C$ and Ω is contained in a hemisphere. Then we have*

$$2\pi\mathrm{Area}(\Omega) - K\mathrm{Area}(\Omega)^2 \le \mathrm{Length}(\partial\Omega \sim \partial C)^2,$$

where equality holds if and only if Ω is a geodesic half disk and $\partial\Omega \sim \partial C$ is a geodesic semicircle.

Proof. Use the reflection arguments as in the proof of the Theorem 2.1. $\square$

3. The case of Gaussian curvature bounded above by a nonpositive constant

In 1933, Beckenbach and Radó [4] gave a proof of (1.1) by using subharmonic functions. This method was employed by Choe [6] to prove the relative isoperimetric inequality (1.2) holds for $K = 0$. We also apply this method to prove the following.

Theorem 3.1. *Let C be a closed convex set in a complete simply connected surface S with Gaussian curvature K_S bounded above by a nonpositive constant K. Then, for a relatively compact subset Ω in $S \sim C$ we have*

$$2\pi\mathrm{Area}(\Omega) - K\mathrm{Area}(\Omega)^2 \le \mathrm{Length}(\partial\Omega \sim \partial C)^2 \tag{3.1}$$

and equality holds if and only if Ω is a geodesic half disk with constant Gaussian curvature K and $\partial\Omega \sim \partial C$ is a geodesic semicircle.

Proof. As the same arguments of proof of Theorem 2.1, it is enough to show that the inequality (3.1) holds for a disk type $\Omega \subset S \sim C$. And we may assume that $\partial\Omega$ meets ∂C perpendicularly, otherwise for sufficiently small $\varepsilon > 0$, we approximate Ω with Ω_ε satisfying that $\partial\Omega_\varepsilon$ meets ∂C perpendicularly, $\mathrm{Area}(\Omega_\varepsilon) \to \mathrm{Area}(\Omega)$ and $\mathrm{Length}(\Omega_\varepsilon) \to \mathrm{Length}(\Omega)$ as $\varepsilon \to 0$. (In fact, we only approximate Ω with Ω_ε in an ε-ball $B(p,\varepsilon)$ for each point $p \in \partial(\partial\Omega \cap \partial C)$.)

Now let $D \subset \mathbb{R}^2$ be a half disk with the diameter $C_1 = \{(x, y) : x = 0, -1 \leq y \leq 1\}$ and the semicircle $C_2 = \{(x, y) : x \geq 0, x^2 + y^2 = 1\}$ such that $\partial D = C_1 \cup C_2$. We consider the isothermal coordinates (x, y) of Ω with a conformal map $\varphi : D \to \Omega$ such that $\varphi(C_1) = \partial\Omega \cap \partial C$. Then the metric of Ω is $g = e^{2f}(dx^2 + dy^2)$ for some smooth function f on Ω. Note that the Gaussian curvature $K_\Omega = -e^{-2f}\Delta f \leq K$ by assumption.

Let h be the solution of the mixed boundary value problem on D satisfying

$$\begin{aligned} \Delta h + Ke^{2h} &= 0 \quad \text{in the interior of D} \\ h &= f \quad \text{on} \quad C_2 \\ \frac{\partial h}{\partial \nu} &= 0 \quad \text{on} \quad C_1, \end{aligned}$$

where ν is the outward unit normal to C_1. We know the existence and regularity of the solution h of the above problem [7]. The convexity of C implies

$$\frac{\partial f}{\partial \nu} \leq 0 \quad \text{on} \quad C_1. \tag{3.2}$$

(See [6] for the proof.) It should be mentioned that the inequality (3.2) does not depend on the Gaussian curvature of the surface Ω.

Thus by invoking the maximum principle, we obtain

$$h \geq f \quad \text{on} \quad D. \tag{3.3}$$

Let $\tilde{D}$ be D equipped with the metric $g = e^{2h}(dx^2 + dy^2)$. Denote by $\tilde{C_1}$ and $\tilde{C_2}$ the parts of $\partial\tilde{D}$ corresponding to C_1, C_2, respectively. Hence by the above inequality (3.3),

$$\text{Area}(\Omega) \leq \text{Area}(\tilde{D}).$$

And $\text{Length}(\partial\Omega \sim \partial C) = \text{Length}(\tilde{C_2})$. On the other hands, by Theorem 2.1,

$$2\pi\text{Area}(\tilde{D}) - K\text{Area}(\tilde{D})^2 \leq \text{Length}(\tilde{C_2})^2$$

Using the above relations and the assumption $K \leq 0$, we have

$$\begin{aligned} 2\pi\text{Area}(\Omega) \leq 2\pi\text{Area}(\tilde{D}) &\leq \text{Length}(\tilde{C_2})^2 + K\text{Area}(\tilde{D})^2 \\ &\leq \text{Length}(\partial\Omega \sim \partial C)^2 + K\text{Area}(\Omega)^2 \end{aligned}$$

To have equality in (3.1), we notice that $h = f$ on D. In other words, D has constant Gaussian curvature K. Hence by Theorem 2.1, equality holds if and only if Ω is a geodesic half disk with the constant Gaussian curvature K and $\partial\Omega \sim \partial C$ is a geodesic semicircle. $\square$

4. The case of Gaussian curvature bounded above by positive constant

For a nonnegatively curved surface S, we cannot get the inequality (1.2) in general. For example, we consider a flat cylinder $S^1 \times [0, \infty)$ glued with a lower hemisphere S^2_- along its boundary S^1 in $\mathbb{R}^3$. If we take S^2_- as a convex set C and $\Omega = S^1 \times [0,t]$ for $t > 0$ satisfying $\Omega \cap C = S^1$. Then it is easy to see that Gauss curvature $K = 1$ but (1.2) does not hold. However, under additional conditions, we have the following theorem.

Theorem 4.1. *Let C be a closed convex set in a complete simply connected surface S with its Gaussian curvature K_S bounded above by a positive constant K. Suppose that for a disk type domain $\Omega \subset S \sim C$. Then we have*

$$2\pi \text{Area}(\Omega) - K\text{Area}(\Omega)^2 \leq \text{Length}(\partial\Omega \sim \partial C)^2$$

and equality holds if and only if Ω is a geodesic half disk with constant Gaussian curvature K and $\partial\Omega \sim \partial C$ is a geodesic semicircle.

Proof. We will use Bandle's method ([3]).
As in the proof of Theorem 3.1, we consider the isothermal coordinate of Ω with a conformal map $\varphi : D \to \Omega$ such that $\varphi(C_1)$ is contained in ∂C. The curvature assumption is

$$K_S = -e^{-2f}\Delta f \leq K.$$

Thus we have

$$\Delta f + Ke^{2f} \geq 0.$$

Put $k = f - h$, where h is the solution of the mixed boundary value problem on D:

$$\begin{aligned} \Delta h &= 0 \quad \text{in the interior of} \quad D \\ h &= f \quad \text{on} \quad C_2 \\ \frac{\partial h}{\partial \nu} &= 0 \quad \text{on} \quad C_1. \end{aligned} \tag{4.1}$$

We know the existence and smoothness of the solution h of the above problem ([7]). Then by the definition of k and (3.2) we get

$$\Delta k + Ke^{2k}e^{2h} \geq 0 \quad \text{in the interior of} \quad D \tag{4.2}$$

$$\begin{aligned} k &= 0 \quad \text{on} \quad C_2 \\ \frac{\partial k}{\partial \nu} &\leq 0 \quad \text{on} \quad C_1. \end{aligned} \tag{4.3}$$

Now let $\overline{D}$ be D equipped with the metric $ds^2 = e^{2h}(dx^2 + dy^2)$. Recall that $\overline{D}$ is flat by (4.1).

Let $D(t) = \{x \in D : k(x) > t\}$. Define

$$t_1 = \inf_{x \in D} k(x), \ t_2 = \sup_{x \in D} k(x), \ a(t) = \int_{D(t)} e^{2h} dv, \text{ and } A = \int_D e^{2h} dv \ ,$$

where dv is an area form in D.

Notice that $a(t)$ is a decreasing function with $a(t_1) = A$ and $a(t_2) = 0$. Since k is a smooth function and e^{2h} is a positive function, we have the function $t(a)$ which is the inverse function of $a(t)$. Then we define

$$H(a) = K \int_{D(t(a))} e^{2f} dv = K \int_0^a e^{2t(l)} dl.$$

By the co-area formula for $D(t) \subset D$,

$$\frac{da}{dt} = -\int_{\partial D(t) \sim C_1} \frac{e^{2h}}{|\nabla k|} ds \quad \text{for almost all} \quad t > t_1.$$

Applying the Schwarz inequality, we obtain

$$\left(\int_{\partial D(t) \sim C_1} e^h ds \right)^2 \leq \int_{\partial D(t) \sim C_1} \frac{e^{2h}}{|\nabla k|} ds \int_{\partial D(t) \sim C_1} |\nabla k| ds. \tag{4.4}$$

By (4) and (4.4) we have

$$-\frac{da}{dt} \geq \frac{(\int_{\partial D(t) \sim C_1} e^h ds)^2}{\int_{\partial D(t) \sim C_1} |\nabla k| ds}. \tag{4.5}$$

Now we use the divergence theorem to get

$$\int_{D(t)} \Delta k dv = \int_{\partial D(t)} \frac{\partial k}{\partial \nu} ds = \int_{C_1(t)} \frac{\partial k}{\partial \nu} ds + \int_{C_2(t)} \frac{\partial k}{\partial \nu} ds \leq \int_{C_2(t)} \frac{\partial k}{\partial \nu} ds,$$

where $C_1(t) = C_1 \cap \partial D(t)$, $C_2(t) = \partial D(t) \sim C_1(t)$ and ν is the outward unit normal to $\partial D(t)$. In the last inequality we applied the inequality (4.3). Note that $\dfrac{\partial k}{\partial \nu} = \langle \nabla k, \nu \rangle$ and $\nu = \dfrac{-\nabla k}{|\nabla k|}$ on $\partial D(t)$. Therefore,

$$\int_{D(t)} \Delta k dv \leq \int_{C_2(t)} \frac{\partial k}{\partial \nu} ds = -\int_{C_2(t)} |\nabla k| ds,$$

$$\begin{aligned} \int_{\partial D(t) \sim \partial C} |\nabla k| ds &= \int_{C_2(t)} |\nabla k| ds \\ &\leq -\int_{D(t)} \Delta k dv \leq H(a(t)) \text{ using (4.2).} \end{aligned} \tag{4.6}$$

Furthermore we know the classical relative isoperimetric inequality on a complete simply connected flat surface S_o, i.e., for any domain $\varphi(D(t)) \subset S_o$,

$$2\pi \int_{D(t)} e^{2h} \leq \left(\int_{\partial D(t) \sim \partial C_1} e^h ds \right)^2 \tag{4.7}$$

By the inequalities (4.5), (4.6), and (4.7),

$$-\frac{da}{dt} \geq \frac{2\pi a}{H(a)} \quad \text{for} \quad t_1 < t < t_2. \tag{4.8}$$

Note that in the above equalities, $t(a)$ is locally Lipschitz on $(0, a(t_1))$([3], Lemma 4). Therefore $H(a)$ is differentiable for almost all a in $(0, a(t_1))$. Since

$$\frac{dH(a)}{da} = H'(a) = \frac{d}{da}\left(K\int_{D(t(a))} e^{2f}dv\right) = \frac{d}{da}\left(K\int_0^a e^{2t(l)}dl\right) = Ke^{2t(a)}$$

$$\text{and } H''(a) = 2Ke^{2t(a)}\frac{dt}{da} \leq 0,$$

we get $\dfrac{H'}{H''} = \dfrac{1}{2}\dfrac{da}{dt}$. Thus by (4.8),

$$H'H \geq -\pi a H''.$$

In other words

$$\left(aH'(a) - H(a) + \frac{H^2(a)}{2\pi}\right)' \geq 0 \quad \text{for} \quad 0 < a < a(t_1).$$

We integrate above inequality between $a = 0$ and $a = a(0)$ to give

$$a(0)H'(a(0)) - H(a(0)) + \frac{H^2(a(0))}{2\pi} + H(0) - \frac{H^2(0)}{2\pi} \geq 0.$$

Moreover we have $H'(a(0)) = K$ and $H(0) = 0$. Therefore,

$$a(0) - M_0 + \frac{KM_0^2}{2\pi} \geq 0, \text{ where } M_0 = \frac{H(a(0))}{K}.$$

On the other hands, in the domain $D \sim D(0)$ where k is nonpositive we have the following inequality:

$$\int_{D\sim D(0)} e^{2h}dv \geq \int_{D\sim D(0)} e^{2f}dv$$

i.e.,

$$A - a(0) \geq M - M_0,$$

where $M = \displaystyle\int_D e^{2f}dv = \text{Area}(\Omega)$ and we recall $A = \displaystyle\int_D e^{2h}dv = \text{Area}(\overline{D})$.

We obtain

$$A - M + \frac{KM^2}{2\pi} \geq 0.$$

Hence

$$2\pi A - 2\pi M + KM^2 \geq 0.$$

And by the relative isoperimetric inequality in flat surfaces, we obtain

$$2\pi \text{Area}(\bar{D}) \leq \text{Length}(\partial\Omega \sim \partial C)^2.$$

Thus we have

$$2\pi \text{Area}(\Omega) \leq \text{Length}(\partial\Omega \sim \partial C)^2 + K\text{Area}(\Omega)^2.$$

Equality occurs only if k satisfies equality in (4.2). It follows that $\Delta f + Ke^{2f} = 0$. Hence Ω has constant Gaussian curvature K. By Theorem 2.2, equality holds if and only if Ω is a geodesic half disk with constant Gaussian curvature K and $\partial\Omega \sim \partial C$ is a geodesic semicircle. □

5. Remarks

Howard ([8]) proved the Sobolev inequality

$$4\pi \int_S f^2 + \left(\int_S |f|^2 dA\right)^2 \leq \left(\int_S \|\nabla f\| dA\right)^2, \tag{5.1}$$

where S is a complete simply connected surface with Gaussian curvature $K_S \leq -1$ and f is a compactly supported function of bounded variation on S. By the coarea formula ([8]) for functions of bounded variation on a surface S, the last term of (5.1) can be written as follows,

$$\int_S \|\nabla f\| dA = \int_0^\infty \mathcal{H}^1(\partial\{x \in S : |f(x)| \geq t\})dt,$$

where $\mathcal{H}^1$ is the one dimensional Hausdorff measure. We say that a function f on $S \sim C$ has *a relatively compact support* if the support of f is a compact subset in the relative topology on $S \sim C$. Using Howard's argument ([8]), we get the relative Sobolev inequality corresponding to the relative isoperimetric inequality (3.1) as follows.

Theorem 5.1. *Let S be a complete two dimensional simply connected Riemannian manifold with Gaussian curvature $K_S \leq K \leq 0$, and C its closed convex subset. Then*

$$2\pi \int_{S\sim C} f^2 dA - K\left(\int_{S\sim C} |f| dA\right)^2 \leq \left(\int_{S\sim C} \|\nabla f\| dA\right)^2$$

for every relatively compactly supported function f of bounded variation on $S \sim C$. Equality holds if and only if up to a set of measure zero, f is $c\chi_D$ where c is a constant and D is a geodesic half disk with constant Gaussian curvature K and $\partial D \sim \partial C$ is a geodesic semicircle.

Howard showed the inequality (5.1) for a compactly supported function f of bounded variation on S, but in our Theorem 5.1, the function f may not vanish on ∂C. It is sufficient that f is compactly supported in the relative topology on $S \sim C$ for a closed convex set $C \subset S$.

Department of Mathematical Sciences
Seoul National University
San56-1 Shinrim-dong Kwanak-gu
Seoul 151-747, Korea
e-mail: heretin@math.snu.ac.kr

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