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Spectrum perturbations of operators on tensor products of Hilbert spaces

By

M. I. Gil'*

Abstract

We investigate the spectrum perturbations and spectrum localization of a class of operators on a tensor product of separable Hilbert spaces. In particular, estimates for the spectral radius and norm of the resolvent are derived. Applications to partial integral and integro-differential operators are also discussed.

1. Introduction and notation

Operators on tensor products of Hilbert spaces arise in various problems of pure and applied mathematics, cf. [4], [11], and references therein. In many applications, for example, in numerical mathematics and stability analysis, bounds for the spectrum of operators on tensor products are very important. But for the best of our knowledge, the bounds are not investigated. In the present paper we consider a class of linear operators on tensor products of Hilbert spaces. The spectrum perturbations and localization are investigated. In particular, we suggest estimates for the spectral radius and the norm of the resolvent. Applications to partial integral operators and integro-differential operators are also discussed.

A few words about the contents. In Section 2, estimates for quasinilpotent operators are derived. They are needed to prove *the main result of the paper-Theorem* 3.3 on an estimate for the resolvent. By virtue of Theorem 3.3, in Section 4, we establish bounds for the spectrum. Section 5 deals with partial integral operators. Section 6 is devoted to integro-differential operators.

Let E_1 and E_2 be separable Hilbert spaces with the scalar products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$, respectively and norms $\|\cdot\|_j = \sqrt{\langle\cdot,\cdot\rangle_j}$ $(j = 1, 2)$. Let $H = E_1 \otimes E_2$ be a tensor product of E_1 and E_2. This means that H is the collection of all

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formal sums of the form

$$u = \sum_j y_j \otimes h_j \qquad (y_j \in E_1, h_j \in E_2) \tag{1.1}$$

with the understanding that

$$\lambda(y \otimes h) = (\lambda y) \otimes h = y \otimes (\lambda h),\ (y + y_1) \otimes h = y \otimes h + y_1 \otimes h,$$
$$y \otimes (h + h_1) = y \otimes h + y \otimes h_1.$$

Here $y, y_1 \in E_1$; $h, h_1 \in E_2$, and λ is a number. The scalar product in H is defined as

$$\langle y \otimes h, y_1 \otimes h_1 \rangle_H = \langle y, y_1 \rangle_1 \ \langle h, h_1 \rangle_2 \qquad (y, y_1 \in E_1, h, h_1 \in E_2)$$

and the cross norm $\| \cdot \|_H = \sqrt{\langle \cdot, \cdot \rangle_H}$. From the theory of tensor products we only need the basic definition and elementary facts which can be found in [4].

For a linear operator A, $\sigma(A)$ is the spectrum, *Dom* (A) is the domain, $r_s(A)$ denotes the spectral radius, $\alpha\ (A) = \sup Re\ \sigma(A)$ and

$$\rho(A, \lambda) := \inf_{t \in \sigma(A)} |t - \lambda|$$

is the distance between $\sigma(A)$ and a $\lambda \in \mathbf{C}$.

A linear operator V is said to be *quasinilpotent if* $\sigma(V) = \{0\}$. V *is called a Volterra operator, if it is quasinilpotent and compact.* In addition, $I = I_H$ and I_j mean the unit operator in H and E_j, respectively.

Let us consider the operator

$$A = D + V_1 \otimes I_2 + I_1 \otimes V_2, \tag{1.2}$$

where D is a normal operator, V_1 and V_2 are Volterra operators in E_1 and E_2, respectively. A wide classes of linear operators on tensor products of Hilbert spaces can be represented as perturbations of operators of type (1.2).

Recall that *a maximal resolution of the identity (m.r.i.)* $\tilde{P}(t)$ $(-\infty \le t \le \infty)$ is a left continuous orthogonal resolution of the identity, such that any gap $\tilde{P}(t_0 + 0) - \tilde{P}(t_0)$ of $\tilde{P}(t)$ (if it exists) is one-dimensional, cf. the books by Brodskii [3], Gohberg and Krein [9] and Gil' [5, p. 69]. It is assumed that there are m.r.i. $P_j(t)$ $(j = 1, 2)$ in E_j, such that

$$P_j(t) V_j P_j(t) = V_j P_j(t) \qquad (-\infty \le t \le \infty) \tag{1.3}$$

and

$$D = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(t, s) dP(t, s), \tag{1.4}$$

where

$$P(t, s) := P_1(t) \otimes P_2(s) \qquad (t, s \in \mathbf{R}) \tag{1.5}$$

and w is a P-measurable scalar-valued function defined on $\mathbf{R}^2$. Below we will check that

$$V_A := V_1 \otimes I_2 + I_1 \otimes V_2 \tag{1.6}$$

is a quasinilpotent operator. In the sequel, $P(t,s)$, D and V_A will be called *the spectral measure, diagonal part and nilpotent part of* A*, respectively.* In addition, the equality

$$A = D + V_A \tag{1.7}$$

is said to be *the triangular representation of* A.

2. Powers of quasinilpotent operators

Everywhere below, $ni(V)$ *denotes the nilpotency index* of a nilpotent operator V, so that $V^{ni(V)} = 0 \neq V^{ni(V)-1}$; if V is quasinilpotent but not nilpotent we write $ni(V) = \infty$. Recall the following formula for the spectral radius of an operator A, cf. [4]

$$r_s(A) = \lim_{m\to\infty} \sqrt[m]{\|A^m\|}.$$

Thus a quasinilpotent operator V satisfies the relation

$$\lim_{m\to\infty} \sqrt[m]{\|V^m\|} = 0.$$

Let W_1, W_2 be commuting operators in H. Then, clearly,

$$(W_1 + W_2)^n = \sum_{k=0}^{n} C_n^k W_1^k W_2^{n-k}. \tag{2.1}$$

Here and below $C_n^k = n!/k!(n-k)!$ are the binomial coefficients. Let $c_{jk} := \|W_j^k\|$ and

$$\sqrt[k]{c_{jk}} \to 0 \qquad (j = 1, 2;\ k = 1, 2, \ldots).$$

So W_1, W_2 are quasinilpotent operators. Then $W_1 + W_2$ is a quasinilpotent operator. Indeed, due to (2.1),

$$\|(W_1 + W_2)^n\| \le c_{3n} := \sum_{k=0}^{n} C_n^k c_{1k} c_{2,n-k}$$

since W_1, W_2 commute. Since, $c_{1k}, c_{2,k}$ are coefficients of some entire functions $f_1(z)$ and $f_2(z)$, and

$$\sum_{k=0}^{n} c_{1k} c_{2,n-k}$$

are coefficients of the entire function $f_1(z) f_2(z)$, taking into account that $C_n^k \le 2^n$ $(k \le n)$, we can assert that $\sqrt[n]{c_{3n}} \to 0$. So $W_1 + W_2$ is really a quasinilpotent operator.

Recall that *a norm ideal* Y_j $(j=1,2)$ *of compact operators acting* in a E_j is algebraically a two-sided ideal, which is complete in an auxiliary norm $|\cdot|_{Y_j}$ for which $|CB|_{Y_j}$ and $|BC|_{Y_j}$ are both dominated by $\|C\|_j|B|_{Y_j}$ for a bounded operator C in E_j and a $B\in Y_j$, cf. [9]. Assume, in addition, that there are positive constants $\theta_k^{(j)}$ $(k\in\mathbf{N})$, with

$$\sqrt[k]{\theta_k^{(j)}}\to 0,$$

for which, for an arbitrary Volterra operator $\tilde V\in Y_j$

$$\|\tilde V^k\|_j\le\theta_k^{(j)}|\tilde V|^k_{Y_j}\qquad (k=1,2,\dots,ni(\tilde V)-1;\ j=1,2). \tag{2.2}$$

Below we will check that the Neumann-Schatten ideal has the property (2.2). Let us suppose that

$$V_j\in Y_j\qquad (j=1,2) \tag{2.3}$$

and

$$W_1=V_1\otimes I_2\quad\text{and}\quad W_2=I_1\otimes V_2. \tag{2.4}$$

Then

$$\|W_j^k\|_H=\|V_j^k\|_j\le\theta_k^{(j)}|V_j|^k_{Y_j}\qquad (k=1,2,\dots,ni(V_j)-1;\ j=1,2).$$

Thus,

$$\|(W_1+W_2)^n\|_H\le\sum_{k=n_2}^{n_1}C_n^k\theta_k^{(1)}\theta_{n-k}^{(2)}|V_1|^k_{Y_1}|V_2|^{n-k}_{Y_2}, \tag{2.5}$$

where

$$n_1=\min\{n,ni(V_1)-1\},\ n_2=\max\{0,n-ni(V_2)+1\}. \tag{2.6}$$

Here we have $(W_1+W_2)^n=0$ if $n_1<n_2$. We thus have proved

Lemma 2.1. *Let W_1 and W_2 be quasinilpotent and commuting operators. Then the operator W_1+W_2 is quasinilpotent. Moreover, conditions* (2.3) *and* (2.4) *imply inequality* (2.5).

In particular, let

$$V_j\in\tilde C_2\qquad (j=1,2), \tag{2.7}$$

where $\tilde C_2=C_2(E_j)$ is the ideal of Hilbert-Schmidt operators in E_j with the Hilbert-Schmidt norm

$$N_2(K)\equiv\ [Trace\ K^*K]^{1/2}\qquad (K\in C_2).$$

The asterisk means the adjoint operation. Due to Lemma 2.3.1 from [5], any quasinilpotent operator $\tilde{V} \in C_2$ in E_j satisfies the inequality

$$\|\tilde{V}^k\|_j \leq \frac{N_2^k(\tilde{V})}{\sqrt{k!}} \qquad (k = 1, 2, \ldots, ni(\tilde{V}) - 1). \tag{2.8}$$

Now Lemma 2.1 implies

Corollary 2.2. *Under conditions,* (2.4) *and* (2.7)*, we have*

$$\|(W_1 + W_2)^n\|_H \leq \sum_{k=n_2}^{n_1} C_n^k \frac{N_2^k(V_1) N_2^{n-k}(V_2)}{\sqrt{(n-k)!k!}}.$$

Since, $C_n^k \leq 2^n \ \ (k \leq n)$, we have

$$\begin{aligned} \|(W_1 + W_2)^n\|_H &\leq \frac{1}{\sqrt{n!}} \sum_{k=0}^{n} C_n^k \sqrt{C_n^k} N_2^k(V_1) N_2^{n-k}(V_2) \\ &\leq \frac{2^{n/2}}{\sqrt{n!}} \sum_{k=0}^{n} C_n^k N_2^k(V_1) N_2^{n-k}(V_2) \\ &= \frac{[\sqrt{2}(N_2(V_1) + N_2(V_2))]^n}{\sqrt{n!}} \qquad (V_1, V_2 \in \tilde{C}_2). \end{aligned} \tag{2.9}$$

Let now $\tilde{C}_p = C_p(E_j)$ be the Neumann-Schatten ideal in E_j with some $p > 0$. That is,

$$N_p(K) := \ [Trace\ (K^* K)^{p/2}]^{1/p} < \infty \qquad (K \in \tilde{C}_p).$$

Recall that for an arbitrary natural $r \geq 1$,

$$N_{p/r}(K^r) \leq N_p^r(K) \qquad (K \in \tilde{C}_p),$$

(cf. [8, Section III.7]). According to this inequality and (2.8), for any quasinilpotent operator $V \in \tilde{C}_{2p}(E_j)$ with a natural $p > 1$, we have

$$\|V^{mp}\|_j \leq \frac{N_2^m(V^p)}{\sqrt{m!}} \leq \frac{N_{2p}^{pm}(V)}{\sqrt{m!}} \qquad (m = 1, 2, \ldots).$$

Hence, for any $k = i + mp$ $(i = 0, \ldots, p-1;\ m = 0, 1, 2, \ldots)$, we have

$$\|V^k\|_j = \|V^{i+pm}\|_j \leq \frac{\|V^i\|_j N_2^m(V^p)}{\sqrt{m!}} \leq \frac{N_{2p}^{i+pm}(V)}{\sqrt{m!}}.$$

This inequality can be written as

$$\|V^k\|_j \leq \frac{N_{2p}^k(V)}{\sqrt{[k/p]!}} \qquad (V \in \tilde{C}_{2p};\ k = 1, 2, \ldots), \tag{2.10}$$

where $[x]$ means the integer part of a number $x > 0$.

Corollary 2.3. *Under the conditions* (2.4) *and*

$$V_j \in \tilde{C}_{2p} \qquad (j = 1, 2;\ p = 1, 2, \dots) \tag{2.11}$$

we have

$$\|(W_1 + W_2)^n\|_H \le \sum_{k=0}^{n} C_n^k \frac{N_{2p}^k(V_1) N_{2p}^{n-k}(V_2)}{\sqrt{[k/p]![(n-k)/p]!}}. \tag{2.12}$$

Under condition (2.11) we can also derive estimate similar to (2.9).

3. Estimates for resolvents

Simple calculations show that

$$\|A_1 \otimes A_2\|_H = \|A\|_1 \|A_2\|_2 \tag{3.1}$$

for all bounded operators A_j acting in E_j $(j = 1, 2)$. Again consider the operator A defined by (1.2) under conditions (1.3), (1.4). Due to the triangular representation (1.7) we have

$$(A - \lambda I)^{-1} = (D + V_A - \lambda I)^{-1} = (I + Q_\lambda)^{-1}(D - \lambda I)^{-1} \qquad (\lambda \notin \sigma(A)), \tag{3.2}$$

where

$$Q_\lambda = (D - \lambda I)^{-1} V_A.$$

According to (1.4),

$$(D - I\lambda)^{-1} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (w(t, s) - \lambda)^{-1} dP(t, s) \qquad (\lambda \notin \sigma(D)).$$

Or

$$(D - I\lambda)^{-1} = \int_{-\infty}^{\infty} dP_1(t) \otimes T_2(t, \lambda) = \int_{-\infty}^{\infty} T_1(s, \lambda) \otimes dP_2(s),$$

where

$$T_1(s, \lambda) = \int_{-\infty}^{\infty} (w(t, s) - \lambda)^{-1} dP_1(t)$$

and

$$T_2(t, \lambda) = \int_{-\infty}^{\infty} (w(t, s) - \lambda)^{-1} dP_2(s).$$

Then $Q_\lambda = B_1(\lambda) + B_2(\lambda)$, where

$$B_1(\lambda) := (D - \lambda)^{-1}(V_1 \otimes I_2) = \int_{-\infty}^{\infty} T_1(s, \lambda) V_1 \otimes dP_2(s)$$

and

$$B_2(\lambda) := (D - \lambda)^{-1}(I_1 \otimes V_2) = \int_{-\infty}^{\infty} dP_1(t) \otimes T_2(t, \lambda) V_2.$$

It can be directly checked that operators $B_1(\lambda)$ and $B_2(\lambda)$ commute and that

$$B_1^n(\lambda) = \int_{-\infty}^{\infty} (T_1(s,\lambda)V_1)^n \otimes dP_2(s) \qquad (n = 1, 2, \ldots).$$

Since $T_j(s,\lambda)$ and V_j have the same m.r.i. P_j, due to Lemma 3.2.4 from [5] $T_j(s,\lambda)V_j$ are qausinilpotent operators. So

$$\|(T_j(s,\lambda)V_j)^n\|_j \le \theta_n^{(j)} |V_j|_{Y_j}^n \|T_j(s,\lambda)\|_j^n \le \frac{\theta_n^{(j)} |V_j|_{Y_j}^n}{\rho^n(D,\lambda)} \qquad (j = 1, 2). \tag{3.3}$$

Let $\{e_k\}$ be an orthogonal normal basis in E_1 and $\{d_k\}$ an orthogonal normal basis in E_2. Vectors of the form

$$\tilde{h} = \sum_{j=1}^{s} \sum_{k=1}^{s} c_{kj} e_k \otimes d_j = \sum_{k=1}^{s} e_k \otimes v_k \tag{3.4}$$

are dense in H. Here

$$v_k = \sum_{k=1}^{s} c_{kj} d_j.$$

Now let $w \in E_2$ be a generating vector. That is, for any $h_2 \in E_2$ and $\epsilon > 0$, there are numbers $c_k \in \mathbf{C}$ and

$$-\infty < t_0 < t_1 < \cdots < t_s < \infty,$$

such that

$$\left\| h_2 - \sum_{k=1}^{s} c_k \Delta P_2(t_k) w \right\|_2 \le \epsilon \;\; (\Delta P_2(t_k) = P_2(t_k) - P_2(t_{k-1}))$$

(cf. [1, Section VI.83]). Thus, there are coefficients b_{kj}, $j = 1, \ldots, l$, such that

$$v_k = \sum_{j=1}^{l} b_{kj} \Delta P_2(t_j) w + \alpha_k \qquad (\alpha_k \in E_2)$$

with $\|\alpha_k\|_2 \le \epsilon \|v_k\|_2$. So

$$\sum_{k=1}^{s} e_k \otimes v_k = \sum_{k=1}^{s} \sum_{j=1}^{l} e_k \otimes b_{kj} \Delta P_2(t_j) w + \sum_{k=1}^{s} e_k \otimes \alpha_k.$$

But

$$\left\| \sum_{k=1}^{s} e_k \otimes \alpha_k \right\|_H^2 = \sum_{k=1}^{s} \|\alpha_k\|_2^2 \le \epsilon^2 \sum_{k=1}^{s} \|v_k\|_2^2.$$

Thus vectors of the form

$$h_0 = \sum_{k=1}^{s} \sum_{j=1}^{l} e_k \otimes b_{kj} \Delta P_2(t_j) w \tag{3.5}$$

are dense in H. Furthermore, due to (3.3),

$$\|B_1^n(\lambda)h_0\|_H^2$$
$$= \sum_{k=1}^{s}\sum_{j=1}^{l}|b_{kj}|^2\int_{-\infty}^{\infty}\|(T_1(s,\lambda)V_1)^n e_k\|_1^2 d\langle P_2(s)\Delta P_2(t_j)w, \Delta P_2(t_j)w\rangle_2$$
$$\le \sum_{k=1}^{s}\sum_{j=1}^{l}|b_{kj}|^2\frac{(\theta_n^{(1)}|V_1|_{Y_1}^n)^2}{\rho^{2n}(D,\lambda)}\int_{-\infty}^{\infty}d\langle P_2(s)\Delta P_2(t_j)w, \Delta P_2(t_j)w\rangle_2$$
$$= \frac{(\theta_n^{(1)}|V_1|_{Y_1}^n)^2}{\rho^{2n}(D,\lambda)}\sum_{k=1}^{s}\sum_{j=1}^{l}|b_{kj}|^2\|\Delta P_2(t_j)w\|_2^2.$$

But according to (3.5)

$$\|h_0\|_H^2 = \sum_{k=1}^{s}\sum_{j=1}^{l}|b_{kj}|^2\|\Delta P_2(t_j)w\|_2^2.$$

Thus

$$\|B_1^n(\lambda)h_0\|_H \le \frac{\theta_{1n}^{(1)}|V_1|_{Y_1}^n}{\rho^n(D,\lambda)}\|h_0\|_H.$$

Since vectors of the form (3.5) are dense in H, we have

$$\|B_1^n(\lambda)\|_H \le \frac{\theta_{1n}^{(1)}|V_1|_{Y_1}^n}{\rho^n(D,\lambda)}.$$

Similarly,

$$\|B_2^n(\lambda)\|_H \le \frac{\theta_n^{(2)}|V_2|_{Y_2}^n}{\rho^n(D,\lambda)}.$$

Now (2.1) implies

$$\|(B_1(\lambda)+B_2(\lambda))^n\|_H = \|Q_\lambda^n\|_H \le \frac{b_n(A,Y)}{\rho^n(D,\lambda)}, \tag{3.6}$$

where

$$b_n(A,Y) := \sum_{k=0}^{n} C_n^k \theta_{n-k}^{(1)}\theta_k^{(2)}|V_1|_{Y_1}^{n-k}|V_2|_{Y_2}^k. \tag{3.7}$$

Relations (3.2) imply

$$\|(A-\lambda I)^{-1}\|_H \le \|(D-\lambda I)^{-1}\|_H \sum_{n=0}^{\infty}\|Q_\lambda^n\|_H.$$

According to (3.6) we get

Lemma 3.1. *Under conditions* (1.2) *through* (1.4) *and* (2.3), *the inequality*

$$\|(A-\lambda I)^{-1}\| \le \sum_{n=0}^{\infty} \frac{b_n(A,Y)}{\rho^{n+1}(D,\lambda)}$$

is valid for any regular point λ of D.

Lemma 3.2. *Under conditions* (1.2) *through* (1.4) *and* (2.3) *the relation $\sigma(D)=\sigma(A)$ is true.*

Proof. Let λ be a regular point of D. Then due to the previous lemma λ is a regular point of A.

Now we are going to prove that from $\mu \in \sigma(D)$ it follows that $\mu \in \sigma(A)$.

First, let μ be the eigenvalue of D and h the corresponding eigenvector. Then according to (1.4), $P(t,s)$ has a jump ΔP corresponding the eigenspace, such that $D\Delta P = \mu \Delta P$ and $\Delta P h = h$. In addition, V_A can have the zero eigenvalues, only, since it is quasinilpotent. So $\Delta P V_A \Delta P = 0$. We thus, have $Dh = \mu h$, $\langle V_A h, V_A h\rangle_H = 0$ and due to (1.7),

$$\begin{aligned}\langle (A-\mu)h,(A-\mu)h\rangle_H &= \langle (D+V_A-\mu)h,(D+V_A-\mu)h\rangle_H\\ &= \langle V_A h, V_A h\rangle_H = 0.\end{aligned}$$

Therefore $\mu \in \sigma(A)$.

Let now $\mu \in \sigma(D)$ be a point of the continuous spectrum. Then according to (1.4) $\mu = w(t_1,s_1)$ for some real t_1, s_1. For a $\delta > 0$, put

$$\tilde{\Delta} P = P(t_1+\delta, s_1+\delta) - P(t_1,s_1).$$

Then

$$(D-\mu)\tilde{\Delta} P v = \int_{s_1}^{s_1+\delta}\int_{t_1}^{t_1+\delta} w(t,s)dP(t,s).$$

Since P is continuous in a neighborhood of point (t_1,s_1), for any $\epsilon > 0$, there is a δ, such that

$$\|(D-\mu)\tilde{\Delta} P\|_H \le \epsilon \quad \text{and} \quad \|\tilde{\Delta} P V_A \tilde{\Delta} P\|_H \le \epsilon,$$

since V_A is quasinilpotent. Hence,

$$\begin{aligned}|\langle (D-\mu)\tilde{\Delta} P v, V_A \tilde{\Delta} P v\rangle_H| &= |\langle (D-\mu)\tilde{\Delta} P v, \tilde{\Delta} P V_A \tilde{\Delta} P v\rangle_H|\\ &\le \epsilon^2 \|v\|_H^2 \qquad (v \in H)\end{aligned}$$

and according to (1.7),

$$\begin{aligned}\langle (A-\mu)\tilde{\Delta} P v,(A-\mu)\tilde{\Delta} P v\rangle_H &= \langle (D+V_A-\mu)\tilde{\Delta} P v,(D+V_A-\mu)\tilde{\Delta} P v\rangle_H\\ &\le 2\epsilon^2 + \langle (D-\mu)\tilde{\Delta} P v,(D-\mu)\tilde{\Delta} P v\rangle_H\\ &\quad + (\langle V_A \tilde{\Delta} P v, V_A \tilde{\Delta} P v\rangle_H \le 4\epsilon^2.\end{aligned}$$

Take $v \in \tilde{\Delta}PH$. Then $\|(A-\mu)v\|_H^2 \le 4\epsilon^2\|v\|_H^2$. Since ϵ is arbirary, this proves that $\mu \in \sigma(A)$. Since we also have proved that any regular point of D is a regular point of A, the proof of the lemma is complete. □

Lemmas 3.1 and 3.2 imply the main result of the paper.

Theorem 3.3. *Under conditions* (1.2) *through* (1.4) *and* (2.3), *the inequality*

$$\|(A-\lambda I)^{-1}\| \le \sum_{n=0}^{\infty} \frac{b_n(A,Y)}{\rho^{n+1}(A,\lambda)} \tag{3.8}$$

is valid for any regular point λ of A.

If $A = D$ is normal, that is, $V_1 = V_2 = 0$, then we have the exact relation

$$\|(A-\lambda I)^{-1}\|_H = \frac{1}{\rho(A,\lambda)} \qquad (\lambda \notin \sigma(A)).$$

Note that according to (2.5), we can replace $b_n(A,Y)$ in (3.8) by

$$\tilde{b}_n(Y) := \sum_{k=n_2}^{n_1} C_n^k \theta_{n-k}^{(1)} \theta_k^{(2)} |V_1|_{Y_1}^{n-k} |V_2|_{Y_2}^k,$$

where n_1, n_2 are defined by (2.6).

Theorem 3.3 and Corollary 2.3 imply

Corollary 3.4. *Under conditions* (1.2) *through* (1.4) *and* (2.11), *the inequality*

$$\|(A-\lambda I)^{-1}\| \le \sum_{n=0}^{\infty} \frac{b_n(A,\tilde{C}_{2p})}{\rho^{n+1}(A,\lambda)} \qquad (\lambda \notin \sigma(A)) \tag{3.9}$$

is valid with

$$b_n(A,\tilde{C}_{2p}) := \sum_{k=0}^{n} \frac{C_n^k N_{2p}^k(V_1) N_{2p}^{n-k}(V_2)}{\sqrt{[(n-k)/p]![k/p]!}}. \tag{3.10}$$

Note that according to (2.5), in (3.9) we can replace $b_n(A,\tilde{C}_{2p})$ by

$$\tilde{b}_n(\tilde{C}_{2p}) := \sum_{k=n_2}^{n_1} \frac{C_n^k N_{2p}^k(V_1) N_{2p}^{n-k}(V_2)}{\sqrt{[(n-k)/p]![k/p]!}}.$$

Moreover, if V_1, V_2 are Hilbert-Schmidt operators, due to (2.9) we have

$$\|(A-\lambda I)^{-1}\| \le \sum_{n=0}^{\infty} \frac{[\sqrt{2}(N_2(V_1)+N_2(V_2))]^n}{\sqrt{n!}\rho^{n+1}(A,\lambda)} \qquad (\lambda \notin \sigma(A)). \tag{3.11}$$

By the Schwarz inequality

$$\sum_{n=0}^{\infty} \frac{b^n}{\sqrt{n!}x^n} = \sum_{n=0}^{\infty} \frac{(\sqrt{2}b)^n}{2^{n/2}\sqrt{n!}x^n}$$
$$\leq \left[\sum_{n=0}^{\infty} \frac{2^n b^{2n}}{n! x^{2n}}\right]^{1/2} \left[\sum_{n=0}^{\infty} 2^{-n}\right]^{1/2} = \sqrt{2}\exp\left[\frac{b^2}{x^2}\right] \qquad (b, x > 0).$$

This relation and (3.11) imply

(3.12)
$$\|(A-\lambda I)^{-1}\| \leq \frac{\sqrt{2}}{\rho(A,\lambda)} \exp\left[\frac{2(N_2(V_1)+N_2(V_2))^2}{\rho^2(A,\lambda)}\right] \qquad (V_1, V_2 \in \tilde{C}_2;\ \lambda \notin \sigma(A)).$$

4. Spectrum of perturbed operators

Let us consider the perturbed operator $B = A + Z$, where operator A has the form (1.2) and Z is a bounded operator in H with a “sufficiently small” norm $q := \|Z\|$. So

(4.1)
$$B = D + V_1 \otimes I_2 + I_1 \otimes V_2 + Z.$$

Denote

(4.2)
$$\psi(A, x) := \sum_{k=0}^{\infty} \frac{b_k(A, Y)}{x^{k+1}} \qquad (x > 0),$$

where $b_k(A, Y)$ are defined by (3.7).

Theorem 4.1. *Under conditions* (1.2) *through* (1.4) *and* (2.3), *let*

$$q\psi(A, \rho(D, \lambda)) < 1.$$

Then λ is a regular point of B. Moreover,

$$\|R_\lambda(B)\|_H \leq \frac{\psi(A, \rho(A, \lambda))}{1 - q\psi(A, \rho(D, \lambda))}.$$

Proof. It is simple to check that under conditions $q\|R_\lambda(A)\| < 1$, λ is a regular point of operator $B = A + Z$ and

$$\|R_\lambda(B)\|_H \leq \frac{\|R_\lambda(A)\|_H}{1 - q\|R_\lambda(A)\|_H}.$$

Now the result is due to Theorem 3.3. □

Furthermore, under (2.11), set

$$\psi_p(A, x) := \sum_{k=0}^{\infty} \frac{b_k(A, \tilde{C}_{2p})}{x^{k+1}}.$$

Recall that $b_k(A, \tilde{C}_{2p})$ are defined by (3.10). Now Theorem 4.1 and Corollary 3.4 imply

Corollary 4.2. *Under conditions* (1.2) *through* (1.4) *and* (2.11), *let*

$$q\psi_p(A, \rho(D, \lambda)) < 1.$$

Then λ is a regular point of B. Moreover,

$$\|R_\lambda(B)\| \leq \frac{\psi_p(A, \rho(D, \lambda))}{1 - q\psi_p(A, \rho(D, \lambda))}.$$

Let A and B be arbitrary linear operators in H. The quantity

$$sv_A(B) := \sup_{\mu \in \sigma(B)} \inf_{\lambda \in \sigma(A)} |\mu - \lambda|$$

is said to be *the spectral variation of a linear operator B with respect to a linear operator A.*

Theorem 4.3. *Let conditions* (1.2) *through* (1.4), (2.3) *and* (4.1) *hold. Then, $sv_D(B) \leq z(A, q)$, where $z(A, q)$ is the extreme right-hand* (*nonnegative*) *root of the equation*

$$1 = q\psi(A, x). \tag{4.3}$$

In particular, $\alpha(B) \leq \alpha(D) + z(A, q)$. If, in addition, D is bounded, then $r_s(B) \leq r_s(D) + z(A, q)$.

Proof. This result follows from [5, Lemma 4.1.4] and Theorem 3.3 with Lemma 3.2 taken into account. □

If $V_1 = V_2 = 0$, then $z(A, q) = q$ and $sv_D(B) \leq q$.
To estimate $z(A, q)$, let us consider the equation

$$\sum_{k=1}^{\infty} a_k z^k = 1, \tag{4.4}$$

where the coefficients a_k are nonnegative and have the property

$$\gamma_0 \equiv 2 \max_k \sqrt[k]{a_k} < \infty.$$

Due to Lemma 3.4 from [7], the unique nonnegative root z_0 of equation (4.4) satisfies the estimate

$$z_0 \geq 1/\gamma_0. \tag{4.5}$$

Hence it follows

$$z(A, q) \leq \delta(A, q) := 2 \max_k \sqrt[k+1]{q b_k(A, Y)}. \tag{4.6}$$

Now Theorem 4.1 implies

Corollary 4.4. *Let conditions* (1.2) *through* (1.4), (2.3) *and* (4.1) *hold. Then* $sv_D(B) \leq \delta(A,q)$. *In particular,* $\alpha(B) \leq \alpha(D) + \delta(A,q)$. *If in addition, D is bounded, then* $r_s(B) \leq r_s(D) + \delta(A,q)$.

Furthermore, due to Corollary 3.4, Theorem 4.1 and inequality (4.5) imply.

Corollary 4.5. *Let conditions* (1.2) *through* (1.4), (2.11) *and* (4.1) *hold. Let* $z_p(A,q)$ *be the extreme right-hand root of the equation*

$$1 = q\psi_p(A,x). \tag{4.7}$$

Then, $sv_D(B) \leq z_p(A,q) \leq \delta_p(A,q)$, *where*

$$\delta_p(A,q) := 2\max_k \sqrt[k+1]{q b_k(A,\tilde{C}_{2p})}.$$

In particular, $\alpha(B) \leq \alpha(D) + z_p(A,q) \leq \alpha(D) + \delta_p(A,q)$. *If in addition, D is bounded, then*

$$r_s(B) \leq r_s(D) + z_p(A,q) \leq r_s(D) + \delta_p(A,q).$$

Let us assume that V_1, V_2 are Hilbert-Schmidt operators. According to (3.12), $z_2(A,q) \leq \tilde{z}_2(A,q)$, where $\tilde{z}_2(A,q)$ is the extreme right-hand root of the equation

$$1 = q\sqrt{2}x^{-1} exp\left[\frac{2(N_2(V_1)+N_2(V_2))^2}{x^2}\right]. \tag{4.8}$$

Let us use the following

Lemma 4.6. *The unique positive root* z_0 *of the equation*

$$ze^z = a \qquad (a = const > 0) \tag{4.9}$$

satisfies the estimate

$$z_0 \geq ln\,[1/2 + \sqrt{1/4 + a}]. \tag{4.10}$$

If, in addition, the condition $a \geq e$ *holds, then* $z_0 \geq ln\ a - ln\ ln\ a$.

For the proof see [7, Lemma 4.3]. Equation (4.8) is equivalent to the following one:

$$1 = 2q^2x^{-2} exp\left[\frac{4(N_2(V_1)+N_2(V_2))^2}{x^2}\right].$$

Substituting

$$y = \frac{4(N_2(V_1)+N_2(V_2))^2}{x^2}$$

we have equation (4.9). Now (4.10) gives us the inequality $\tilde{z}_2(A,q) \leq \tilde{\delta}(A,q)$, where

$$\tilde{\delta}(A,q) := \frac{2(N_2(V_1)+N_2(V_2))}{ln^{1/2}\left[\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(N_2(V_1)+N_2(V_2))^2}{q^2}}\right]}. \tag{4.11}$$

Now the previous Corollary yields.

Corollary 4.7. *Let the conditions* (1.2) *through* (1.4), (4.1) *and* $V_1, V_2 \in \tilde{C}_2$ *hold. Then,* $sv_D(B) \leq \tilde{\delta}_2(A,q)$*. In particular,* $\alpha(B) \leq \alpha(D) + \tilde{\delta}_2(A,q)$*. If in addition, D is bounded, then* $r_s(B) \leq r_s(D) + \tilde{\delta}_2(A,q)$*.*

5. Example 1. A partial integral operator

Let us consider in the complex space $H \equiv L^2([0,1]\times[0,1])$ the operator B defined by

$$(Bu)(x,y) = a(x,y)u(x,y) + \int_0^1 K_1(x,x_1)u(x_1,y)dx_1 + \int_0^1 K_2(y,y_1)u(x,y_1)dy_1, \tag{5.1}$$

where K_1, K_2 are scalar Hilbert-Schmidt kernels, and $a(x,y)$ is scalar bounded measurable function defined on $[0,1]^2$. Such operators arose in various applications, (cf. [2], [10]). In the considered case $E_1 = E_2 = L^2[0,1]$.

Rewrite B as $B = A + Z$, where

$$(Au)(x,y) = a(x,y)u(x,y) + \int_0^x K_1(x,x_1)u(x_1,y)dx_1 + \int_0^y K_2(y,y_1)u(x,y_1)dy_1 \tag{5.2}$$

and

$$(Zu)(x,y) = \int_x^1 K_1(x,x_1)u(x_1,y)dx_1 + \int_y^1 K_2(y,y_1)u(x,y_1)dy_1.$$

In this case (1.2) holds with D defined by $(Du)(x,y) = a(x,y)u(x,y)$ and

$$(V_j v)(x) = \int_0^x K_j(x,x_1)v(x_1)dx_1 \qquad (j=1,2;\ v \in L^2[0,1]). \tag{5.3}$$

So

$$N_2(V_j) \equiv \left[\int_0^1 \int_0^x |K_j(x,x_1)|^2 dx_1\, dx\right]^{1/2} < \infty. \tag{5.4}$$

For $0 \leq t \leq 1$, define $P_1(t)$ and $P_2(t)$ by

$$(P_1(t)u)(x) = (P_2(t)u)(x) = \begin{cases} 0 & \text{if} \quad t < x \leq 1, \\ u(x) \text{ for } 0 \leq x < t & \text{if} \quad 0 \leq x < t. \end{cases} \tag{5.5}$$

In addition, put $P_j(t) = I_j$ for $t > 1$ and $P_j(t) = 0$ for $t < 0; j = 1,2$. Clearly,

$$\sigma(D) = \{z \in \mathbf{C} : z = a(x,y),\ 0 \leq x,y \leq 1\}.$$

Then due to Corollary 4.7

$$\sigma(B) \subset \{z \in \mathbf{C} : |z - a(x,y)| \leq z_2(A,q) \leq \delta_2(A,q),\ 0 \leq x, y \leq 1\},$$

where $q = \|Z\|$, $\tilde{z}_2(A,q)$ is the unique positive root of the equation (4.8) and $\tilde{\delta}_2(A,q)$ is defined by (4.11) with (5.4) taken into account. Simple calculations show that

$$q \leq \left[\int_0^1 \int_x^1 |K_1(x,x_1)|^2 dx_1\, dx\right]^{1/2} + \left[\int_0^1 \int_x^1 |K_2(x,x_1)|^2 dx_1\, dx\right]^{1/2}.$$

In particular Corollary 4.7 gives us the inequality

$$r_s(B) \leq \max_{x,y} |a(x,y)| + \tilde{z}_2(A,q) \leq \max_{x,y} |a(x,y)| + \tilde{\delta}_2(A,q) \tag{5.6}$$

and

$$\alpha(B) \leq \max_{x,y} Re\ a(x,y) + \tilde{z}_2(A,q) \leq \max_{x,y} Re\ a(x,y) + \tilde{\delta}_2(A,q).$$

An arbitrary linear operator A is said to be *stable*, if $\alpha(A) < 0$.

Thus, the operator defined by (5.1) is stable, provided $a(x,y)+\tilde{\delta}_2(A,q) < 0$ for all $x, y \in [0,1]$.

Clearly, instead of (5.2), we can take

$$(Au)(x,y) = a(x,y)u + \int_x^1 K_1(x,x_1)u(x_1,y)dx_1 + \int_y^1 K_2(y,y_1)u(x,y_1)dy_1$$

and

$$(Zu)(x,y) = \int_0^x K_1(x,x_1)u(x_1,y)dx_1 + \int_0^y K_2(y,y_1)u(x,y_1)dy_1.$$

Similarly, we can consider operators of the type

$$(Bu)(x,y) = a(x,y)u + \int_0^1 K_1(x,x_1)u(x_1,y)dx_1 + \int_0^1 K_2(y,y_1)u(x,y_1)dy_1 + \int_0^1 \int_0^1 K_2(x,x_1,y,y_1)u(x_1,y_1)dy_1\, dx_1.$$

Moreover, Theorem 4.3 allows us to investigate operators with unbounded $a(\cdot,\cdot)$.

6. Example 2. An integro-differential operator

Let us consider in $H \equiv L^2([0,1] \times [0,1])$ the operator

$$(Bu)(x,y) := \frac{\partial^2 u(x,y)}{\partial y^2} + \int_0^1 K_1(x,x_1)u(x_1,y)dx_1 \qquad (u \in Dom(B)) \tag{6.1}$$

with

$$Dom\ (B) = \left\{ u \in H : \frac{\partial^2 u}{\partial y^2} \in H;\ u(x,0) = u(x,1) = 0 \right\}.$$

Here K_1 is a Hilbert-Schmidt kernel. We can write out $B = A + Z$, where

$$(6.2)\quad (Au)(x,y) = \frac{\partial^2 u(x,y)}{\partial y^2} + \int_0^x K_1(x,x_1)u(x_1,y)dx_1 \qquad (u \in Dom(B))$$

and

$$(Zu)(x,y) = \int_x^1 K_1(x,x_1)u(x_1,y)dx_1.$$

In this case (1.2) holds with V_1 defined by (5.3), $V_2 = 0$ and

$$(Du)(x,y) = \frac{\partial^2 u(x,y)}{\partial y^2} \qquad (u \in Dom\ (B)).$$

Take P_1 as in (5.5) and

$$(P_2(t)v)(y) = (P_2(n)v))(y) = \sum_{k=1}^{n} sin\ (k\pi y) \int_0^1 v(y_1) sin\ (k\pi y_1) v(y_1) dy_1.$$

$(n = 1, 2, \ldots)$. Clearly, $\sigma(D) = \{-\pi^2 k^2;\ k = 1, 2, \ldots\}$. Then due to Corollary 4.7

$$\sigma(B) \subset \{z \in \mathbf{C} : |z + \pi^2 m^2| \leq \tilde{z}_2(A,q) \leq \tilde{\delta}_2(A,q),\ m = 1, 2, \ldots\},$$

where

$$q = \|Z\|_H \leq \left[\int_0^1 \int_x^1 |K_1(x,x_1)|^2 dx_1\ dx \right]^{1/2},$$

$\tilde{z}_2(A,q)$ is the unique positive root of the equation (4.8) and $\tilde{\delta}_2(A,q)$ is defined by (4.11) with (5.4) taken into account. In particular,

$$\alpha(B) \leq -\pi^2 + \tilde{z}_2(A,q) \leq -\pi^2 + \tilde{\delta}_2(A,q).$$

Thus, B is stable, provided $-\pi^2 + \tilde{\delta}_2(A,q) < 0$.

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Department of Mathematics
Ben Gurion University of the Negev
P.O. Box 653, Beer-Sheva 84105, Israel
e-mail: gilmi@cs.bgu.ac.il

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