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On the Bott suspension map for non-compact Lie groups

By

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1. Introduction

The Bott suspension map ([1]) is a map from the suspension of a symmetric space H/K into another symmetric space G/H, and its adjoint $H/K \to \Omega(G/H)$ for compact classical groups G is well known. But its analogue for non-compact Lie groups has not been so studied. In this paper we present two such maps.

A construction of the Bott suspension map in [6] can be applied to a non-compact group G. Precisely, we take two automorphisms σ, τ of G which commute and satisfy certain conditions. Then we have maps

$$b_0 : \boldsymbol{GL}(n, \mathbb{C})/\boldsymbol{O}(n, \mathbb{C}) \to \Omega(\boldsymbol{Sp}(n, \mathbb{C})/\boldsymbol{GL}(n, \mathbb{C})),$$
$$b_0 : \boldsymbol{Sp}(2n, \mathbb{R})/\boldsymbol{GL}(2n, \mathbb{R}) \to \Omega(\boldsymbol{SL}(4n, \mathbb{R})/\boldsymbol{Sp}(2n, \mathbb{R}))$$

and show that a certain diagram involving b_0 is homotopy-commutative. Such a diagram appeared in a proof of the Bott periodicity theorems ([3]).

In Section 2, we revise some argument of [6] which we need. In Section 3, a key lemma is proved. In Section 4, main results are shown.

2. Preliminaries

For the argument of this section, we refer to Section 1 of [6].

Throughout this paper, G will be a connected Lie group that is not necessarily compact, and $e \in G$ will be the identity element of G.

Let $\sigma : G \to G$ be an automorphism. We denote by G^σ the subgroup of G left fixed by σ, i.e.,

$$G^\sigma = \{g \in G \mid \sigma(g) = g\}.$$

Let $\tau : G \to G$ be another automorphism. Consider the following six conditions:

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(1) σ and τ commute, i.e.,

$$\sigma \circ \tau = \tau \circ \sigma.$$

This condition implies that $\sigma(G^\tau) \subset G^\tau$ and $\tau(G^\sigma) \subset G^\sigma$. If we write $G^{\sigma\tau}$ for

$$(G^\sigma)^\tau = \{g \in G^\sigma \mid \tau(g) = g\},$$

this condition also implies that $G^{\sigma\tau} = G^\sigma \cap G^\tau$.

(2) τ is inner and of order 2. That is, there exists an element $x_\tau \in G$ such that

$$\tau(g) = x_\tau\, g\, x_\tau^{-1} \quad \text{and} \quad x_\tau^2\, g\, x_\tau^{-2} = g$$

for all $g \in G$. The last equality is equivalent to

$$x_\tau\, g\, x_\tau^{-1} = x_\tau^{-1}\, g\, x_\tau$$

for all $g \in G$. Note that $g \in G$ belongs to G^τ if and only if $g\, x_\tau = x_\tau\, g$.

(3) There is a one-parameter subgroup

$$v_\tau : \mathbb{R} \to G$$

such that $v_\tau(1) = x_\tau$. (This is not a trivial condition, because in non-compact groups G, the exponential map $\exp : T_eG \to G$ is usually not surjective; see [2, p. 74].) Note that $v_\tau(t) \in G^\tau$ for all $t \in \mathbb{R}$.

(4) If $g \in G^{\sigma\tau}$, the relation

$$g\, v_\tau(t) = v_\tau(t)\, g$$

holds for all $t \in \mathbb{R}$. In other words, $G^{\sigma\tau}$ is contained in the centralizer of $\operatorname{Im} v_\tau = \{v_\tau(t) \mid t \in \mathbb{R}\}$:

$$G^{\sigma\tau} \subset C_G(\operatorname{Im} v_\tau).$$

(5) G^σ is not contained in $C_G(\operatorname{Im} v_\tau)$. That is, there are elements $g_0 \in G^\sigma$ and $t_0 \in \mathbb{R}$ such that

$$g_0\, v_\tau(t_0) \neq v_\tau(t_0)\, g_0.$$

(6) $\operatorname{Im} v_\tau$ is not contained in G^σ. That is, there is an element $t_1 \in \mathbb{R}$ such that

$$\sigma(v_\tau(t_1)) \neq v_\tau(t_1).$$

Let us assume these conditions. Then by (3) we define a map

$$\hat{b}_0 : \Sigma(G^\sigma/G^{\sigma\tau}) \to G/G^\sigma$$

by

$$\hat{b}_0([gG^{\sigma\tau}, t]) = v_\tau(t)^{-1} g v_\tau(t) G^\sigma$$

for $gG^{\sigma\tau} \in G^\sigma/G^{\sigma\tau}$ and $t \in [0,1]$, where Σ denotes the reduced suspension. By (4) this map is well defined.

Remark 1. If (5) or (6) is not satisfied, then b_0 becomes a constant map. We need these two conditions only for excluding such a trivial case. Notice that, when we prove Lemma 2.1 below, the conditions (5) and (6) are not used.

In general, for any automorphism $\sigma : G \to G$, we have a map

$$\xi_\sigma : G/G^\sigma \to G$$

defined by $gG^\tau \mapsto g\,\sigma(g)^{-1}$. By virtue of (1), the map $\xi_\tau : G/G^\tau \to G$ can be restricted to $\xi_\tau : G^\sigma/G^{\sigma\tau} \to G^\sigma$. We have a fiber sequence

$$G^\sigma \xrightarrow{i_\sigma} G \xrightarrow{p_\sigma} G/G^\sigma \xrightarrow{q_\sigma} BG^\sigma,$$

where BG denotes a classifying space for G, and there is a (weak) homotopy equivalence between G and ΩBG. The following result due to Harris [4] was given as Lemma 1 of [6]. But its proof was omitted there. We will give its details in the next section.

Lemma 2.1. *Under the above conditions* (1) *to* (6)*, the diagram*

$$\begin{array}{ccc} G^\sigma/G^{\sigma\tau} & \xrightarrow{b_0} & \Omega(G/G^\sigma) \\ {\scriptstyle \xi_\tau}\downarrow & & \downarrow{\scriptstyle \Omega q_\sigma} \\ G^\sigma & \xrightarrow[\simeq]{} & \Omega BG^\sigma, \end{array} \tag{2.1}$$

is homotopy-commutative.

3. Proof of Lemma 2.1

For a space X with base point $x_0 \in X$, let CX be the reduced cone of X, i.e.,

$$CX = (X \times [0,1])/(X \times \{0\} \cup \{x_0\} \times [0,1]).$$

There is an inclusion $i : X \to CX$ defined by $i(x) = [x, 1]$ for $x \in X$. Let ΣX be the reduced suspension of X, i.e.,

$$\Sigma X = CX/X = (X \times [0,1])/(X \times \{0,1\} \cup \{x_0\} \times [0,1]).$$

We have a cofiber sequence

$$X \xrightarrow{i} CX \xrightarrow{\pi} \Sigma X \xrightarrow{\simeq} \Sigma X.$$

Define a map $\Xi : C(G^\sigma/G^{\sigma\tau}) \to G$ by

$$\Xi([gG^{\sigma\tau}, t]) = v_\tau(1-t)^{-1} g v_\tau(1-t) \tau(g)^{-1}$$

for $gG^{\sigma\tau} \in G^\sigma/G^{\sigma\tau}$ and $t \in [0,1]$. This is well defined. For, first we show that, if $g \in G^{\sigma\tau}$, then $\Xi([gG^{\sigma\tau},t]) = \Xi([G^{\sigma\tau},t])$ for all $t \in [0,1]$. Indeed, suppose that $g \in G^{\sigma\tau}$. Then we have

$$\begin{aligned}\Xi([gG^{\sigma\tau},t]) &= v_\tau(1-t)^{-1} g v_\tau(1-t)\tau(g)^{-1} \\ &= v_\tau(1-t)^{-1} v_\tau(1-t) g\tau(g)^{-1} \qquad \text{by (4)} \\ &= g\tau(g)^{-1} = gg^{-1} = e \qquad \text{since } \tau(g) = g \\ &= v_\tau(1-t)^{-1} e v_\tau(1-t)\tau(e)^{-1} \\ &= \Xi([G^{\sigma\tau},t]).\end{aligned}$$

Secondly we can show that $\Xi([gG^{\sigma\tau},0]) = e$ for all $g \in G^\sigma$. Indeed,

$$\begin{aligned}\Xi([gG^{\sigma\tau},0]) &= v_\tau(1)^{-1} g v_\tau(1)\tau(g)^{-1} \\ &= x_\tau^{-1} g x_\tau \tau(g)^{-1} \qquad \text{since } v_\tau(1) = x_\tau \\ &= x_\tau g x_\tau^{-1} \tau(g)^{-1} \qquad \text{by the last equality in (2)} \\ &= \tau(g)\tau(g)^{-1} \qquad \text{by (2)} \\ &= e.\end{aligned}$$

Lastly we have to show that $\Xi([G^{\sigma\tau},t]) = e$ for all $t \in [0,1]$. But we have already seen it .

Consider the diagram

$$\begin{array}{ccccccc} G^\sigma/G^{\sigma\tau} & \xrightarrow{\ i\ } & C(G^\sigma/G^{\sigma\tau}) & \xrightarrow{\ \pi\ } & \Sigma(G^\sigma/G^{\sigma\tau}) & \xrightarrow{\ \simeq\ } & \Sigma(G^\sigma/G^{\sigma\tau}) \\ {\scriptstyle \xi_\tau}\downarrow & & \downarrow{\scriptstyle \Xi} & & \downarrow{\scriptstyle \hat{b}_0} & & \downarrow{\scriptstyle \hat{\xi}_\tau} \\ G^\sigma & \xrightarrow[i_\sigma]{} & G & \xrightarrow[p_\sigma]{} & G/G^\sigma & \xrightarrow[q_\sigma]{} & BG^\sigma, \end{array}$$

where $\hat{b}_0$ is the map whose adjoint is b_0, and $\hat{\xi}_\tau$ is the map whose adjoint is the composite

$$G^\sigma/G^{\sigma\tau} \xrightarrow{\ \xi_\tau\ } G^\sigma \xrightarrow{\ \simeq\ } \Omega BG^\sigma.$$

To prove Lemma 2.1 it is enough to show that the right-hand square is homotopy-commutative.

The left-hand square is commutative, i.e., $i_\sigma \circ \xi_\tau = \Xi \circ i$. In fact,

$$i_\sigma \circ \xi_\tau(gG^{\sigma\tau}) = i_\sigma(g\tau(g)^{-1}) = g\tau(g)^{-1}$$

for $gG^{\sigma\tau} \in G^\sigma/G^{\sigma\tau}$. On the other hand,

$$\begin{aligned}\Xi \circ i(gG^{\sigma\tau}) = \Xi([gG^{\sigma\tau},1]) &= v_\tau(0)^{-1} g v_\tau(0)\tau(g)^{-1} \\ &= e^{-1} g e\tau(g)^{-1} \qquad \text{since } v_\tau(0) = e \\ &= g\tau(g)^{-1}\end{aligned}$$

for $gG^{\sigma\tau} \in G^\sigma/G^{\sigma\tau}$.

The middle square is homotopy-commutative, i.e., $p_\sigma \circ \Xi \simeq \hat{b}_0 \circ \pi$. In fact,

$$\begin{aligned} p_\sigma \circ \Xi([gG^{\sigma\tau}, t]) &= p_\sigma(v_\tau(1-t)^{-1} g v_\tau(1-t)\tau(g)^{-1}) \\ &= v_\tau(1-t)^{-1} g v_\tau(1-t)\tau(g)^{-1} G^\sigma \end{aligned}$$

for $gG^{\sigma\tau} \in G^\sigma/G^{\sigma\tau}$ and $t \in [0,1]$. On the other hand,

$$(\hat{b}_0 \circ \pi)([gG^{\sigma\tau}, t]) = \hat{b}_0([gG^{\sigma\tau}, t]) = v_\tau(t)^{-1} g v_\tau(t) G^\sigma$$

for $gG^{\sigma\tau} \in G^\sigma/G^{\sigma\tau}$ and $t \in [0,1]$. Define a map $H : C(G^\sigma/G^{\sigma\tau}) \times [0,1] \to G/G^\sigma$ by

$$\begin{aligned} &H([gG^{\sigma\tau}, t], u) \\ &= v_\tau((1-t)(1-u)+tu)^{-1} g v_\tau((1-t)(1-u)+tu) v_\tau(1-u)^{-1} g^{-1} v_\tau(1-u) G^\sigma \end{aligned}$$

for $gG^{\sigma\tau} \in G^\sigma/G^{\sigma\tau}$ and $t, u \in [0,1]$. This H is a well-defined, desired homotopy. For, first we show that, if $g \in G^{\sigma\tau}$, then $H([gG^{\sigma\tau}, t], u) = H([G^{\sigma\tau}, t], u)$ for all $t, u \in [0,1]$. Indeed, suppose that $g \in G^{\sigma\tau}$. Then we can use (4) and have

$$\begin{aligned} &H([gG^{\sigma\tau}, t], u) \\ &= v_\tau((1-t)(1-u)+tu)^{-1} v_\tau((1-t)(1-u)+tu) g g^{-1} v_\tau(1-u)^{-1} v_\tau(1-u) G^\sigma \\ &= G^\sigma \\ &= v_\tau((1-t)(1-u)+tu)^{-1} e v_\tau((1-t)(1-u)+tu) v_\tau(1-u)^{-1} e^{-1} v_\tau(1-u) G^\sigma \\ &= H([G^{\sigma\tau}, t], u). \end{aligned}$$

Secondly we have

$$\begin{aligned} &H([gG^{\sigma\tau}, t], u) \\ &= v_\tau((1-t)(1-u)+tu)^{-1} e v_\tau((1-t)(1-u)+tu) v_\tau(1-u)^{-1} e^{-1} v_\tau(1-u) G^\sigma \\ &= G^\sigma \end{aligned}$$

for all $t, u \in [0,1]$. Thirdly we have

$$\begin{aligned} H([gG^{\sigma\tau}, 0], u) &= v_\tau(1-u)^{-1} g v_\tau(1-u) v_\tau(1-u)^{-1} g^{-1} v_\tau(1-u) G^\sigma \\ &= G^\sigma \end{aligned}$$

for all $u \in [0,1]$. Fourthly, since $\tau(g) = x_\tau g x_\tau^{-1} = x_\tau^{-1} g x_\tau$ by (2), we have

$$\begin{aligned} H([gG^{\sigma\tau}, t], 0) &= v_\tau(1-t)^{-1} g v_\tau(1-t) v_\tau(1)^{-1} g^{-1} v_\tau(1) G^\sigma \\ &= v_\tau(1-t)^{-1} g v_\tau(1-t)\tau(g)^{-1} G^\sigma \\ &= p_\sigma \circ \Xi([gG^{\sigma\tau}, t]) \end{aligned}$$

for all $t \in [0,1]$. Lastly, since $v_\tau(0) = e$ and $g^{-1} \in G^\sigma$, we have

$$\begin{aligned} H([gG^{\sigma\tau}, t], 1) &= v_\tau(t)^{-1} g v_\tau(t) v_\tau(0)^{-1} g^{-1} v_\tau(0) G^\sigma \\ &= v_\tau(t)^{-1} g v_\tau(t) g^{-1} G^\sigma \\ &= v_\tau(t)^{-1} g v_\tau(t) G^\sigma \\ &= \hat{b}_0 \circ \pi([gG^{\sigma\tau}, t]) \end{aligned}$$

for all $t \in [0, 1]$.

Consequently the right-hand square is homotopy-commutative, and the proof is completed. □

4. Main results

Let I_n denote the unit $n \times n$ matrix. We put

$$I_{n,n} = \begin{pmatrix} -I_n & O \\ O & I_n \end{pmatrix} \quad \text{and} \quad J_n = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}.$$

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$, and let $M_n(\mathbb{K})$ be the set of all $n \times n$ matrices $g = (g_{ij})$ with entries g_{ij} in $\mathbb{K}$. The transpose of $g \in M_n(\mathbb{K})$ is denoted by ${}^t g$. According to [5], the real and complex symplectic groups are defined by

$$\boldsymbol{Sp}(n, \mathbb{K}) = \{ g \in M_{2n}(\mathbb{K}) \mid {}^t g \, J_n \, g = J_n \}$$

for $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$, respectively.

One of our main results is

Theorem 4.1. *The diagram*

$$\begin{array}{ccc} \boldsymbol{GL}(n,\mathbb{C})/\boldsymbol{O}(n,\mathbb{C}) & \xrightarrow{b_0} & \Omega(\boldsymbol{Sp}(n,\mathbb{C})/\boldsymbol{GL}(n,\mathbb{C})) \\ \downarrow{\scriptstyle \xi_\tau} & & \downarrow{\scriptstyle \Omega q_\sigma} \\ \boldsymbol{GL}(n,\mathbb{C}) & \xrightarrow[\simeq]{} & \Omega B\boldsymbol{GL}(n,\mathbb{C}) \end{array}$$

is homotopy-commutative.

Proof. Consider the case $G = \boldsymbol{Sp}(n, \mathbb{C})$. Then it is easy to see that J_n belongs to G, but $I_{n,n}$ does not. However, if $\boldsymbol{i}$ denotes the imaginary unit, then $\boldsymbol{i} I_{n,n}$ belongs to G. We take $\sigma : G \to G$ to be

the inner automorphism of G defined by $g \mapsto (\boldsymbol{i} I_{n,n})\, g \,(\boldsymbol{i} I_{n,n})^{-1}$

and $\tau : G \to G$ to be

the inner automorphism of G defined by $g \mapsto J_n \, g \, J_n{}^{-1}$.

In this case we shall show that the conditions (1) to (6) are satisfied.

Since

$$I_{n,n} \, J_n = -J_n \, I_{n,n}, \tag{4.1}$$

it follows that $\sigma \circ \tau = \tau \circ \sigma$. Thus the condition (1) is satisfied.

Since ${J_n}^2 = -I_{2n}$, the condition (2) is satisfied.

Define a map $v_\tau : \mathbb{R} \to M_{2n}(\mathbb{C})$ by

$$v_\tau(t) = \begin{pmatrix} \left(\cos \frac{\pi}{2} t\right) I_n & \left(\sin \frac{\pi}{2} t\right) I_n \\ \left(-\sin \frac{\pi}{2} t\right) I_n & \left(\cos \frac{\pi}{2} t\right) I_n \end{pmatrix} \tag{4.2}$$

for $t \in \mathbb{R}$. Then $v_\tau(1) = J_n$. Since

$$v_\tau(t)^{-1} = \begin{pmatrix} \left(\cos \frac{\pi}{2} t\right) I_n & \left(-\sin \frac{\pi}{2} t\right) I_n \\ \left(\sin \frac{\pi}{2} t\right) I_n & \left(\cos \frac{\pi}{2} t\right) I_n \end{pmatrix} = {}^t(v_\tau(t))$$

for all $t \in \mathbb{R}$, we see that $\operatorname{Im} v_\tau$ is contained in G. It is clear that the relation

$$v_\tau(t_1)\, v_\tau(t_2) = v_\tau(t_1 + t_2)$$

holds for all $t_1, t_2 \in \mathbb{R}$. Thus v_τ may be viewed as a one-parameter subgroup of G. Thus the condition (3) is satisfied.

Writing $g \in G$ in the form

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, D are complex $n \times n$ matrices, by definition we have

$${}^tA\, C = {}^tC\, A, \qquad {}^tB\, D = {}^tD\, B, \qquad {}^tA\, D - {}^tC\, B = I_n.$$

In terms of these block matrices, $\sigma : G \to G$ is given by

$$\sigma\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}. \tag{4.3}$$

So we find that

$$G^\sigma = \left\{ \begin{pmatrix} A & O \\ O & D \end{pmatrix} \middle| A, D \in M_n(\mathbb{C}),\ {}^tA\, D = I_n \right\}.$$

Therefore G^σ may be identified with the general linear group $\boldsymbol{GL}(n, \mathbb{C})$ by

$$\begin{pmatrix} A & O \\ O & D \end{pmatrix} \longleftrightarrow A.$$

Similarly $\tau : G \to G$ is given by

$$\tau\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}.$$

So we find that

$$G^{\tau} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \middle| \, A, B \in M_n(\mathbb{C}), \ {}^tA\,B = {}^tB\,A, \ {}^tA\,A + {}^tB\,B = I_n \right\}.$$

Since $G^{\sigma\tau} = G^{\sigma} \cap G^{\tau}$ by (1), it follows that

$$G^{\sigma\tau} = \left\{ \begin{pmatrix} A & O \\ O & A \end{pmatrix} \middle| \, A \in M_n(\mathbb{C}), \ {}^tA\,A = I_n \right\}.$$

Therefore $G^{\sigma\tau}$ may be identified with the complex orthogonal group $\boldsymbol{O}(n, \mathbb{C})$ by

$$\begin{pmatrix} A & O \\ O & A \end{pmatrix} \longleftrightarrow A.$$

Suppose that $g \in G^{\sigma\tau}$. Then

$$g = \begin{pmatrix} A & O \\ O & A \end{pmatrix}$$

for some $A \in M_n(\mathbb{C})$. Using this and (4.2), one readily checks that $g\,v_{\tau}(t) = v_{\tau}(t)\,g$ for all $t \in \mathbb{R}$. Thus the condition (4) is satisfied.

Consider the case when $g_0 = \boldsymbol{i}I_{n,n}$ and $t_0 = 1$ in (5). Then by (4.1) we see that the condition (5) is satisfied.

Consider the case when $t_1 = 1$ in (6). Then by (4.3) we see that the condition (6) is satisfied.

In this way we can apply Lemma 2.1 to obtain a desired homotopy-commutative diagram, and the proof is completed. □

The other of our main results is

Theorem 4.2. *The diagram*

$$\begin{array}{ccc} \boldsymbol{Sp}(2n,\mathbb{R})/\boldsymbol{GL}(2n,\mathbb{R}) & \xrightarrow{b_0} & \Omega(\boldsymbol{SL}(4n,\mathbb{R})/\boldsymbol{Sp}(2n,\mathbb{R})) \\ {\scriptstyle \xi_{\tau}}\downarrow & & \downarrow{\scriptstyle \Omega q_{\sigma}} \\ \boldsymbol{Sp}(2n,\mathbb{R}) & \xrightarrow[\simeq]{} & \Omega B\boldsymbol{Sp}(2n,\mathbb{R}) \end{array}$$

is homotopy-commutative.

Proof. Consider the case $G = \boldsymbol{SL}(4n, \mathbb{R})$. Then it is easy to see that both $I_{2n,2n}$ and J_{2n} belong to G. We take $\sigma : G \to G$ to be

the outer automorphism of G defined by $g \mapsto J_{2n}({}^tg^{-1}){J_{2n}}^{-1}$

and $\tau : G \to G$ to be

the inner automorphism of G defined by $g \mapsto (I_{2n,2n})\,g\,(I_{2n,2n})^{-1}$.

In this case we shall show that the conditions (1) to (6) are satisfied.

Since ${}^{t}I_{2n,2n} = I_{2n,2n} = {I_{2n,2n}}^{-1}$, it follows from (4.1) that $\sigma \circ \tau = \tau \circ \sigma$. Thus the condition (1) is satisfied.

Since $(I_{2n,2n})^2 = I_{4n}$, the condition (2) is satisfied.

Define a map $u_\tau : \mathbb{R} \to M_{2n}(\mathbb{R})$ by

$$u_\tau(t) = \begin{pmatrix} (\cos \pi t) I_n & (\sin \pi t) I_n \\ (-\sin \pi t) I_n & (\cos \pi t) I_n \end{pmatrix}$$

for $t \in \mathbb{R}$, and define a map $v_\tau : \mathbb{R} \to M_{4n}(\mathbb{R})$ by

$$v_\tau(t) = \begin{pmatrix} u_\tau(t) & O \\ O & I_{2n} \end{pmatrix} \tag{4.4}$$

for $t \in \mathbb{R}$. Then $v_\tau(1) = I_{2n,2n}$. Since $\det u_\tau(t) = 1$ for all $t \in \mathbb{R}$, we see that $\operatorname{Im} v_\tau$ is contained in G. It is clear that the relation

$$v_\tau(t_1)\, v_\tau(t_2) = v_\tau(t_1 + t_2)$$

holds for all $t_1, t_2 \in \mathbb{R}$. Thus v_τ may be viewed as a one-parameter subgroup of G. Thus the condition (3) is satisfied.

An element $g \in G$ belongs to G^σ if and only if $J_{2n}({}^{t}g^{-1}){J_{2n}}^{-1} = g$, which is equivalent to ${}^{t}g\, J_{2n}\, g = J_{2n}$. So we find that

$$\begin{aligned} G^\sigma &= \boldsymbol{Sp}(2n, \mathbb{R}) \\ &= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle|\ \begin{matrix} A, B, C, D \in M_{2n}(\mathbb{R}),\ {}^{t}A\, C = {}^{t}C\, A, \\ {}^{t}B\, D = {}^{t}D\, B,\ {}^{t}A\, D - {}^{t}C\, B = I_{2n} \end{matrix} \right\}. \end{aligned}$$

Since $\tau : G \to G$ is given by

$$\tau\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix},$$

we find that

$$G^\tau = \left\{ \begin{pmatrix} A & O \\ O & D \end{pmatrix} \middle|\ A, D \in M_{2n}(\mathbb{R}),\ (\det A)(\det D) = 1 \right\}.$$

Similarly it follows from (1) that

$$G^{\sigma\tau} = \left\{ \begin{pmatrix} A & O \\ O & D \end{pmatrix} \middle|\ A, D \in M_{2n}(\mathbb{R}),\ {}^{t}A\, D = I_{2n} \right\}.$$

Therefore $G^{\sigma\tau}$ may be identified with the general linear group $\boldsymbol{GL}(2n, \mathbb{R})$ by

$$\begin{pmatrix} A & O \\ O & D \end{pmatrix} \longleftrightarrow A.$$

Suppose that $g \in G^{\sigma\tau}$. Then

$$g = \begin{pmatrix} A & O \\ O & D \end{pmatrix}$$

for some $A, D \in M_{2n}(\mathbb{R})$. Using this and (4.4), one readily checks that $g\, v_\tau(t) = v_\tau(t)\, g$ for all $t \in \mathbb{R}$. Thus the condition (4) is satisfied.

Using (4.1), we easily see that the conditions (5) and (6) are satisfied.

In this way we can apply Lemma 2.1 to obtain a desired homotopy-commutative diagram, and the proof is completed. □

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