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Unrenormalized intersection local time of Brownian motion and its local time representation

By

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Abstract

We consider the intersection local time of Brownian motion without renormalization through Itô-Wiener expansions. In order to recognize the existence, we extend the Watanabe space. We also discuss how to substitute Wiener functionals for parameters of a generalized Wiener functional. As a consequence a relationship between the unrenormalized intersection local time and the local time is clarified.

1. Introduction

Intersection local times of Brownian motion assume different aspects according to the dimension of Brownian motion. Let $\{B_t\}$ be Brownian motion on $\mathbb{R}^N$ and $p_N(t,x)$ the N-dimensional Gaussian kernel. Then the intersection local time $\gamma(T)$ of planar Brownian motion is defined by the following limit in L^2 sense (cf. Le Gall [8]);

$$\gamma(T) = \lim_{\varepsilon\to 0}\left(\int_0^T\int_0^t p_2(\varepsilon, B_t - B_s)dsdt - E\left[\int_0^T\int_0^t p_2(\varepsilon, B_t - B_s)dsdt\right]\right).$$

It should be noticed that Nualart and Vives [10] showed that the above limit holds in the Watanabe space $\boldsymbol{D}_2^\alpha$ for all $\alpha < 1/2$ (Precise definition of the Watanabe space is stated in Section 2), and that Imkeller, Perez-Abreu and Vives [5] proved later $\gamma(T) \in \boldsymbol{D}_2^\alpha$ for all $\alpha < 1$.

In the case where $N \geq 3$,

$$\left\{\int_0^T\int_0^t p_N(\varepsilon, B_t - B_s)dsdt - E\left[\int_0^T\int_0^t p_N(\varepsilon, B_t - B_s)dsdt\right]; 0 < \varepsilon \leq 1\right\}$$

is no longer bounded in L^2. In these cases we need the renormalization.

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Imkeller, Perez-Abreu and Vives [5] showed that

$$\begin{aligned}(1.1)\quad &\left\{\frac{1}{\sqrt{\log(1/\varepsilon)}}\left(\int_0^T\int_0^t p_3(\varepsilon, B_t - B_s)dsdt\right.\right.\\ &\qquad\left.\left.-E\left[\int_0^T\int_0^t p_3(\varepsilon, B_t - B_s)dsdt\right]\right); 0 < \varepsilon \le 1\right\}\end{aligned}$$

is bounded in $\boldsymbol{D}_2^{\alpha}$ for all $\alpha < 1/2$ if $N = 3$, and that

$$\begin{aligned}&\left\{\varepsilon^{(N-3)/2}\left(\int_0^T\int_0^t p_N(\varepsilon, B_t - B_s)dsdt\right.\right.\\ &\qquad\left.\left.-\ E\left[\int_0^T\int_0^t p_N(\varepsilon, B_t - B_s)dsdt\right]\right); 0 < \varepsilon \le 1\right\}\end{aligned}$$

is bounded in $\boldsymbol{D}_2^{\alpha}$ for all $\alpha < (4-N)/2$ if $N \ge 4$. Unfortunately the uniqueness of limit points in weak topology has not yet been known in the case where $N \ge 3$. We should note that Yor [17] showed that the sequence (1.1) converges in distribution if we replace test functions $p_3(\varepsilon, x)$ by continuous functions of compact support which converge to delta function.

On the other hand, intersection local times without renormalization have also been studied. For example, De Faria, Hida, Streit and H. Watanabe [2] showed that the suitable subtracted counterpart of $\int_\varepsilon^T\int_0^t p_N(\varepsilon, B_t - B_s)dsdt$ converges in the Hida distribution space as $\varepsilon \to 0$. To state a little more precisely, let $\sum_{\boldsymbol{n}\in\mathbb{Z}_+^N} I_{\boldsymbol{n}}(f_{\boldsymbol{n}})$ be the Itô-Wiener expansion for some functional F, where $\mathbb{Z}_+$ denotes the totality of non-negative integers. Then its k th subtracted counterpart $F^{(k)}$ means $\sum_{|\boldsymbol{n}|\ge k} I_{\boldsymbol{n}}(f_{\boldsymbol{n}})$, where $|\boldsymbol{n}| = n_1 + \cdots + n_N$, $\boldsymbol{n} = (n_1, \ldots, n_N) \in \mathbb{Z}_+^N$. Let $L(\varepsilon) = \int_\varepsilon^T\int_0^t p_N(\varepsilon, B_t - B_s)dsdt$. Then De Faria, Hida, Streit and Watanabe [2] showed that $L^{(k)}(\varepsilon)$ has the limit in the Hida distribution space if $k > N - 2$.

In this paper we give another approach to recognize intersection local times of Brownian motion without renormalization. For this sake, we extend Watanabe spaces, and then show the existence of the intersection local time as an element of this extended Watanabe space.

The second aim of this paper is to clarify a relationship between the unrenormalized intersection local time of Brownian motion and the Brownian local time. The multidimensional Brownian local times $L(t, x)$ was introduced by Imkeller and Weisz [6] as a generalized Wiener functional. Roughly speaking, it holds that

$$L(t, x) = \int_0^t \delta_x(B_s)ds,$$

where δ_x denotes the Dirac delta function at $x \in \mathbb{R}^N$. The unrenormalized intersection local time of Brownian motion $\gamma(T)$ is formally represented as

$$\int_0^T\int_0^t \delta_0(B_t - B_s)dsdt - E\left[\int_0^T\int_0^t \delta_0(B_t - B_s)dsdt\right].$$

Therefore rough argument leads the following equations;

(1.2)

$$\gamma(T) = \int_0^T \int_0^t \delta_{B_t}(B_s) ds dt - E\left[\int_0^T \int_0^t \delta_{B_t}(B_s) ds dt\right] = \int_0^T L^{(1)}(t, B_t) dt,$$

where $L^{(1)}(t,x)$ denotes the first subtracted counterpart of the Brownian local time $L(t,x)$. In the expression above, we substitute B_t for x of $L^{(1)}(t,x)$. As $L^{(1)}(t,x)$ is not a Wiener functional if $N \geq 2$, this substitution is invalid in the pathwise sense.

We discussed in [15] how to substitute Wiener functionals for parameters of a generalized Wiener functional: Let $\{\Phi(x); x \in \mathbb{R}^N\}$ be a generalized Wiener functional parametrized by $x \in \mathbb{R}^N$ and F a non-degenerate Wiener functional in Malliavin's sense. Then it should be natural to understand the substitution $\Phi(F)$ as $\int_{\mathbb{R}^N} \Phi(x)\delta_x(F)dx$. In the case where $\Phi(x)$ is deterministic, the substitution admits the integral representation as above. In the case where $\Phi(x)$ is a generalized Wiener functional, however, the product of $\Phi(x)$ and $\delta_x(F)$ should be considered carefully. We apply the Wiener product to the product above, as is a natural extension of ordinary product. In this paper we show a relationship (1.2) with some modifications through this definition. Details are discussed in Sections 4 and 5.

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2. Extended Watanabe space

In this section we introduce the extended Watanabe space. At first we prepare some notation.

Let (W_0^N, P) be the N-dimensional standard Wiener space: $W_0^N = \{B_t = (B_t^1, B_t^2, \ldots, B_t^N) : [0,T] \to \mathbb{R}^N | B_t$ is continuous and $B_0 = 0\}$ and P is the standard Wiener measure. Let $\boldsymbol{n} = (n_1, n_2, \ldots, n_N) \in \mathbb{Z}_+^N$, where $\mathbb{Z}_+$ denotes the totality of non-negative integers, and set $|\boldsymbol{n}| = n_1 + n_2 + \cdots + n_N$. Let $I_{\boldsymbol{n}}(f_{\boldsymbol{n}})$ be the $\boldsymbol{n}$-ple Itô-Wiener integral with the kernel function $f_{\boldsymbol{n}}$,

$$\begin{cases} f_{\boldsymbol{n}} = f_{\boldsymbol{n}}(t_1, t_2, \ldots, t_n) = f_{\boldsymbol{n}}(t_1^{(1)}, \ldots, t_{n_1}^{(1)}; \cdots ; t_1^{(N)}, \ldots, t_{n_N}^{(N)}) \\ I_{\boldsymbol{n}}(f_{\boldsymbol{n}}) = \displaystyle\int_0^T \cdots \int_0^T f_{\boldsymbol{n}}(t_1^{(1)}, \ldots, t_{n_1}^{(1)}; \cdots ; t_1^{(N)}, \ldots, t_{n_N}^{(N)}) dB^1_{t_1^{(1)}} \cdots dB^1_{t_{n_1}^{(1)}} \\ \qquad\qquad\qquad\qquad\qquad\qquad \cdots dB^N_{t_1^{(N)}} \cdots dB^N_{t_{n_N}^{(N)}}, \end{cases}$$

where $f_{\boldsymbol{n}}$ belongs to $L^2([0,T]^{|\boldsymbol{n}|} \to \mathbb{R})$, and is symmetric with respect to $t_1^{(j)}, \ldots, t_{n_j}^{(j)}$ for all fixed j $(j = 1, \ldots, N)$. We denote the totality of such functions by $L^2_{\boldsymbol{n}}$ or $L^2_{\boldsymbol{n}}(d\boldsymbol{t})$. $I_{\boldsymbol{0}}(f_{\boldsymbol{0}})$ represents a constant and we also use the notation $f_{\boldsymbol{0}}$ together with $I_{\boldsymbol{0}}(f_{\boldsymbol{0}})$. With the notation above, the Watanabe spaces $\boldsymbol{D}_2^s$ of square integrable type are defined as follows:

Definition 2.1. Let $s \in \mathbb{R}$. We set

(2.1) $\boldsymbol{D}^{ser} = \{\boldsymbol{I}(\boldsymbol{f}) = (I_{\boldsymbol{0}}(f_{\boldsymbol{0}}), \ldots, I_{\boldsymbol{n}}(f_{\boldsymbol{n}}), \ldots) : f_{\boldsymbol{n}} \in L^2_{\boldsymbol{n}}, \, \boldsymbol{n} \in \mathbb{Z}_+^N\}$

and

$$\boldsymbol{D}_2^s = \left\{ \boldsymbol{I}(\boldsymbol{f}) \in \boldsymbol{D}^{ser} : \|\boldsymbol{I}(\boldsymbol{f})\|_s^2 \equiv \sum_{n=0}^{\infty} (1+n)^s \sum_{|\boldsymbol{n}|=n} \boldsymbol{n}! \|f_{\boldsymbol{n}}\|^2 < \infty \right\}, \tag{2.2}$$

where $\boldsymbol{n}! = n_1! \times \cdots \times n_N!$ and $\|f\|$ denotes the L^2-norm of f.

Note that $\boldsymbol{D}_2^s$ above coincides with $\boldsymbol{D}_{2,s}$ in Ikeda and Watanabe [4] or $\mathbb{D}^{s,2}$ in Nualart [9].

Let $w_{\boldsymbol{n}} = w_{\boldsymbol{n}}(t_1^{(1)}, \ldots, t_{n_1}^{(1)}; \cdots; t_1^{(N)}, \ldots, t_{n_N}^{(N)})$ be a symmetric function whose essential infimum is positive. Since $L^2_{\boldsymbol{n}}(w_{\boldsymbol{n}}\, d\boldsymbol{t}) \subset L^2_{\boldsymbol{n}}(d\boldsymbol{t})$, the $\boldsymbol{n}$-ple Itô-Wiener integral $I_{\boldsymbol{n}}(f_{\boldsymbol{n}})$ of $f_{\boldsymbol{n}} \in L^2_{\boldsymbol{n}}(w_{\boldsymbol{n}}\, d\boldsymbol{t})$ is well-defined. Therefore we understand the $\boldsymbol{n}$-ple Itô-Wiener integral $I_{\boldsymbol{n}}(f_{\boldsymbol{n}})$ of $f_{\boldsymbol{n}} \in L^2_{\boldsymbol{n}}(w_{\boldsymbol{n}}^{-1}\, d\boldsymbol{t})$ as a generalized Wiener functional satisfying $\langle I_{\boldsymbol{n}}(f_{\boldsymbol{n}}), I_{\boldsymbol{n}}(g_{\boldsymbol{n}})\rangle_W = \boldsymbol{n}! \langle f_{\boldsymbol{n}}, g_{\boldsymbol{n}}\rangle_2$ for any $g_{\boldsymbol{n}} \in L^2(w_{\boldsymbol{n}}\, d\boldsymbol{t})$, where $\langle *, \star\rangle_W$ denotes the pairing of Wiener functionals and generalized ones, and $\langle *, \star\rangle_2$ the $L^2(d\boldsymbol{t})$-inner product. Noticing the above, we extend Watanabe spaces:

Definition 2.2. Let $\mathcal{W} = \{w_{\boldsymbol{n}}; \boldsymbol{n} \in \mathbb{Z}_+^N\}$ be a set of symmetric positive functions each of whose essential infimum is positive or each of which is bounded. Let $\mathcal{A} = \{a_n; n \in \mathbb{Z}_+\}$ be a sequence of non-negative numbers. We set

$$\mathcal{D}_{\mathcal{W}}^{ser} = \{\boldsymbol{I}(\boldsymbol{f}) = (I_{\boldsymbol{0}}(f_{\boldsymbol{0}}), \ldots, I_{\boldsymbol{n}}(f_{\boldsymbol{n}}), \ldots) : f_{\boldsymbol{n}} \in L^2_{\boldsymbol{n}}(w_{\boldsymbol{n}}\, d\boldsymbol{t}),\ \boldsymbol{n} \in \mathbb{Z}_+^N\} \tag{2.3}$$

and

$$\mathcal{D}_{\mathcal{W}}^{\mathcal{A}} = \left\{ \boldsymbol{I}(\boldsymbol{f}) \in \mathcal{D}_{\mathcal{W}}^{ser} : \|\boldsymbol{I}(\boldsymbol{f})\|^2_{\mathcal{W},\mathcal{A}} \equiv \sum_{n=0}^{\infty} a_n \sum_{|\boldsymbol{n}|=n} \boldsymbol{n}! \|f_{\boldsymbol{n}}\|^2_{w_{\boldsymbol{n}}} < \infty \right\}, \tag{2.4}$$

where $\|f_{\boldsymbol{n}}\|_{w_{\boldsymbol{n}}}$ denotes the $L^2(w_{\boldsymbol{n}}\, d\boldsymbol{t})$-norm of $f_{\boldsymbol{n}}$. We call this Banach space $(\mathcal{D}_{\mathcal{W}}^{\mathcal{A}}, \|\cdot\|_{\mathcal{W},\mathcal{A}})$ the extended Watanabe space.

Let $\delta \in \mathbb{R}$. We denote $\mathcal{D}_{(\delta)}^{ser}$ and $\mathcal{D}_{(\delta)}^{\mathcal{A}}$ instead of $\mathcal{D}_{\mathcal{W}}^{ser}$ and $\mathcal{D}_{\mathcal{W}}^{\mathcal{A}}$, respectively, in the case where $w_{\boldsymbol{n}}(t_1, \ldots, t_n) = (t_1 \vee \cdots \vee t_n - t_1 \wedge \cdots \wedge t_n)^{-\delta}$, where $s \vee t = \max\{s,t\}$ and $s \wedge t = \min\{s,t\}$. In this case we use the notation $\|f_{\boldsymbol{n}}\|_{(\delta)}$ and $\|\cdot\|_{(\delta),\mathcal{A}}$ instead of $\|f_{\boldsymbol{n}}\|_{w_{\boldsymbol{n}}}$ and $\|\cdot\|_{\mathcal{W},\mathcal{A}}$, respectively. Moreover we denote $\mathcal{D}_{(\delta)}^s$ and $\|\cdot\|_{(\delta),s}$ instead of $\mathcal{D}_{(\delta)}^{\mathcal{A}}$ and $\|\cdot\|_{(\delta),\mathcal{A}}$, respectively, in the case where $a_n = (1+n)^s$, $s \in \mathbb{R}$.

Remark 1. (1) Assume $a_n > 0$ for all n. Set $\mathcal{W}^{-1} = \{w_{\boldsymbol{n}}^{-1}; \boldsymbol{n} \in \mathbb{Z}_+^N\}$ and $\mathcal{A}^{-1} = \{a_n^{-1}; n \in \mathbb{Z}_+\}$. Then $\mathcal{D}_{\mathcal{W}^{-1}}^{\mathcal{A}^{-1}}$ can be identified with the dual space of $\mathcal{D}_{\mathcal{W}}^{\mathcal{A}}$.

(2) Unless $\delta = 0$, we neglect the terms $\{I_{\boldsymbol{n}}(f_{\boldsymbol{n}}); |\boldsymbol{n}| = 1\}$ in the definition of $\mathcal{D}_{(\delta)}^{ser}$. If $\delta = 0$, we set $(t_1 - t_1)^0 = 1$. Therefore $\|f_{\boldsymbol{n}}\|_{(0)} = \|f_{\boldsymbol{n}}\|$ and $\mathcal{D}_{(0)}^s = \boldsymbol{D}_2^s$ hold.

(3) Setting $w_{\boldsymbol{n}}(t_1,\dots,t_n) = (t_1 \vee \cdots \vee t_n)^{-\gamma}$ and $a_n = c^n(1+n)^\rho$, $\mathcal{D}^{\mathcal{A}}_{\mathcal{W}}$ coincides with $\mathcal{D}^{(c,\rho)}_{\gamma}$, which appears in Uemura [13], [14].

3. Unrenormalized intersection local times of Brownian motion

Let $\varphi \in C_b^\infty(\mathbb{R}^N)$, an $\mathbb{R}^N$-valued smooth function which and whose derivatives of any orders are bounded, be positive. Suppose $\int \varphi(x)dx = 1$. For $0 < \varepsilon \le 1$, set $\varphi_\varepsilon(x) = \varepsilon^{-N}\varphi(x/\varepsilon)$. Then we recognize the unrenormalized intersection local time $\gamma(T)$ through the following theorem;

Theorem 3.1. *Assume $N \ge 2$. Let $\alpha < 2 - N/2$ and $\delta < 3 - N$. Then there exists $\gamma(T) \in \mathcal{D}^\alpha_{(\delta)}$ satisfying*

$$\int_0^T \int_0^t \varphi_\varepsilon(B_t - B_s)dsdt - E\left[\int_0^T \int_0^t \varphi_\varepsilon(B_t - B_s)dsdt\right] \to \gamma(T) \quad \text{as } \varepsilon \to 0 \quad \text{in} \quad \mathcal{D}^\alpha_{(\delta)}. \tag{3.1}$$

Remark 2. In the case where $N = 2$ we can choose δ to be positive. Therefore our result improves that of Imkeller, Perez-Abreu and Vives [5] mentioned in introduction.

Proof. At first we expand the left hand side of (3.1) into the Itô-Wiener chaos;

(3.2)

$$\int_0^T \int_0^t \varphi_\varepsilon(B_t - B_s)dsdt - E\left[\int_0^T \int_0^t \varphi_\varepsilon(B_t - B_s)dsdt\right] = \sum_{|\boldsymbol{n}|\ge 1} I_{\boldsymbol{n}}(f^\varepsilon_{\boldsymbol{n}}),$$

(3.3)

$$f^\varepsilon_{\boldsymbol{n}} = \frac{1}{\boldsymbol{n}!}\int_{s_1\vee\cdots\vee s_n}^T \int_0^{s_1\wedge\cdots\wedge s_n} \left(\frac{1}{\sqrt{t-s}}\right)^n \int_{\mathbb{R}^N} H_{\boldsymbol{n}}\left(\frac{\varepsilon x}{\sqrt{t-s}}\right) p_N(t-s,\varepsilon x) \varphi(x)\,dx \times \mathbf{1}_{[0,T]}(s_1\vee\cdots\vee s_n)\,dsdt,$$

where $n = |\boldsymbol{n}|$, and for $x = (x_1,\dots,x_N) \in \mathbb{R}^N$ and $\boldsymbol{n} = (n_1,\dots,n_N) \in \mathbb{Z}^N_+$,

$$H_{\boldsymbol{n}}(x) = \prod H_{n_i}(x_i),$$

H_n denoting the Hermite polynomial;

$$H_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}, \qquad x \in \mathbb{R}. \tag{3.4}$$

Hermite polynomials admit the uniform estimate (see, for instance, Imkeller, Perez-Abreu and Vives [5, Proposition 3]),

$$\sup_x \left|H_n(x)e^{-x^2/2}\right| \le C\sqrt{n!}(n\vee 1)^{-1/4},$$

C being a constant independent of n. Taking account of this estimate, we set

$$\bar{f}_{\boldsymbol{n}} = \frac{1}{\sqrt{\boldsymbol{n}!}}(\boldsymbol{n}\vee 1)^{-1/4}\int_{s_1\vee\cdots\vee s_n}^{T}\int_0^{s_1\wedge\cdots\wedge s_n}\left(\frac{1}{\sqrt{t-s}}\right)^{n+N}\mathbf{1}_{[0,T]}(s_1\vee\cdots\vee s_n)\,dsdt,$$

where $\boldsymbol{n}\vee 1 = (n_1\vee 1,\ldots,n_N\vee 1)$ and $\boldsymbol{n}^a = n_1^a\times\cdots\times n_N^a$, ($\boldsymbol{n} = (n_1,\ldots,n_N)$).

Since $|f_{\boldsymbol{n}}^{\varepsilon}| \le C_1\bar{f}_{\boldsymbol{n}}$ for all $\boldsymbol{n}$, it is enough to show that

$$\sum_{n=2}^{\infty}(1+n)^{\alpha}\sum_{|\boldsymbol{n}|=n}\boldsymbol{n}!\|\bar{f}_{\boldsymbol{n}}\|_{(\delta)}^2 < \infty. \tag{3.5}$$

By a slight computation we have

$$\begin{aligned}
&\int\cdots\int_{[0,T]^n}\left(\int_{s_1\vee\cdots\vee s_n}^{T}\int_0^{s_1\wedge\cdots\wedge s_n}\left(\frac{1}{\sqrt{t-s}}\right)^{n+N}\mathbf{1}_{[0,T]}(s_1\vee\cdots\vee s_n)\,dsdt\right)^2\\
&\qquad\qquad\times(s_1\vee\cdots\vee s_n - s_1\wedge\cdots\wedge s_n)^{-\delta}ds_1\cdots ds_n\\
&= \frac{2n(n-1)}{(n-\delta)(n-1-\delta)}\left\{\int_0^T\int_0^{t_2}\int_0^{t_1}\int_0^{s_2}(t_1-s_2)^{n-\delta}(t_1-s_1)^{-(n+N)/2}\right.\\
&\qquad\qquad\times(t_2-s_2)^{-(n+N)/2}ds_1ds_2dt_1dt_2\\
&\quad\left.+\int_0^T\int_0^{t_1}\int_0^{t_2}\int_0^{s_2}(t_2-s_2)^{n/2-N/2-\delta}(t_1-s_1)^{-(n+N)/2}ds_1ds_2dt_2dt_1\right\}\\
&= \frac{2n(n-1)}{(n-\delta)(n-1-\delta)}(I_1+I_2).
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
I_1 = \left(\frac{2}{n+N-2}\right)^2\int_0^T\int_0^{t_1}&\{(t_1-s_2)^{2-N-\delta}\\
&+(t_1-s_2)^{n-\delta}(T-s_2)^{1-n/2-N/2}t_1^{1-n/2-N/2}\\
&-(T-s_2)^{1-n/2-N/2}(t_1-s_2)^{1+n/2-N/2-\delta}\\
&-t_1^{1-n/2-N/2}(t_1-s_2)^{1+n/2-N/2-\delta}\}ds_2dt_1.
\end{aligned}$$

If $\delta < 3-N$, then we can easily know that all of four integrals above are uniformly bounded from above. The second term I_2 is computed explicitly;

$$I_2 = \frac{1}{n/2-N/2-\delta+1}\times\frac{1}{n/2-N/2-\delta+2}\times\frac{1}{3-N-\delta}\times\frac{1}{4-N-\delta}.$$

Therefore we have

$$\|\bar{f}_{\boldsymbol{n}}\|_{(\delta)}^2 \le C_2\frac{1}{\boldsymbol{n}!}(\boldsymbol{n}\vee 1)^{-1/2}\frac{1}{n^2},$$

C_2 being a constant. Noting that there exists a constant C_3 satisfying

$$\sum_{|\boldsymbol{n}|=n} (\boldsymbol{n} \vee 1)^{-1/2} \leq C_3 n^{N/2-1}$$

(see, for instance, Imkeller, Perez-Abreu and Vives [5, Proposition 6]),

$$\sum_{n=2}^{\infty}(1+n)^{\alpha} \sum_{|\boldsymbol{n}|=n} \boldsymbol{n}! \|\bar{f}_{\boldsymbol{n}}\|_{(\delta)}^2 \leq C_4 \sum_{n=2}^{\infty}(1+n)^{\alpha+N/2-3}$$

holds with a constant C_4, and the right hand side of the inequality above is finite if $\alpha < 2-N/2$. Therefore (3.5) is satisfied if $\delta < 3-N$ and $\alpha < 2-2/N$, which completes the proof. □

Remark 3. Letting $\varepsilon \to 0$ in (3.2) and (3.3) we obtain the chaos representation of the unrenormalized intersection local time $\gamma(T)$ of Brownian motion;

$$\gamma(T) = \sum_{|\boldsymbol{n}|\geq 1} I_{\boldsymbol{n}}(\gamma_{\boldsymbol{n}}),$$

$$\begin{aligned}\gamma_{\boldsymbol{n}} &= \left(\frac{1}{\sqrt{2\pi}}\right)^N \frac{1}{\boldsymbol{n}!} H_{\boldsymbol{n}}(0) \int_{s_1\vee\cdots\vee s_n}^{T} \int_0^{s_1\wedge\cdots\wedge s_n} \left(\frac{1}{\sqrt{t-s}}\right)^{n+N} dsdt \\ &= \begin{cases} \left(\frac{1}{\sqrt{2\pi}}\right)^N (-1)^{n/2} \frac{(\boldsymbol{n}-1)!!}{\boldsymbol{n}!} \int_{s_1\vee\cdots\vee s_n}^{T} \int_0^{s_1\wedge\cdots\wedge s_n} \left(\frac{1}{\sqrt{t-s}}\right)^{n+N} dsdt, \\ \qquad\qquad \boldsymbol{n} \in (2\mathbb{Z}_+)^N, \\ 0, \qquad\qquad \boldsymbol{n} \notin (2\mathbb{Z}_+)^N, \end{cases}\end{aligned}$$

where $\boldsymbol{n}-1 = (n_1-1, \ldots, n_N-1)$ and $\boldsymbol{n}!! = n_1!! \times \cdots \times n_N!!$, $(\boldsymbol{n} = (n_1, \ldots, n_N))$. We set $(-1)!! = 1$.

4. Substitution for parameters of generalized Wiener functionals

We discussed in [15] how to substitute Wiener functionals for parameters of a generalized Wiener functional. In this section we introduce the substitution discussed in [15] in a little more mild situation. Our idea is naive. Let $\{\Phi(x); x \in \mathbb{R}^N\}$ be a generalized Wiener functional parametrized by $x \in \mathbb{R}^N$ and F a non-degenerate Wiener functional in Malliavin's sense. Then, formally, it holds that

$$\Phi(F) = \int_{\mathbb{R}^N} \Phi(x)\delta_x(F)dx.$$

We define the product $\Phi(x)\delta_x(F)$ through the Wiener multiplication, as is a natural extension of usual multiplication, and we apply Bochner integral to the integration above.

To begin with, we define the Wiener product in our situation. For this sake we introduce the contraction of functions; Let $\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{r} \in \mathbb{Z}_+^N$. Suppose $\boldsymbol{r} \leq \boldsymbol{n} \wedge \boldsymbol{m}$. (For $\boldsymbol{n} = (n_1, \ldots, n_N), \boldsymbol{m} = (m_1, \ldots, m_N) \in \mathbb{Z}_+^N$, $\boldsymbol{n} \leq \boldsymbol{m}$ means $n_i \leq m_i$ for all $i = 1, \ldots, N$ and $\boldsymbol{n} \wedge \boldsymbol{m} = (n_1 \wedge m_1, \ldots, n_N \wedge m_N)$.) For

$f_{\boldsymbol{n}} \in L^2_{\boldsymbol{n}}$ and $g_{\boldsymbol{m}} \in L^2_{\boldsymbol{m}}$, the contraction $f_{\boldsymbol{n}} \otimes_{\boldsymbol{r}} g_{\boldsymbol{m}}$ of $\boldsymbol{r}$ indices of $f_{\boldsymbol{n}}$ and $g_{\boldsymbol{m}}$ is defined by

$$f_{\boldsymbol{n}} \otimes_{\boldsymbol{r}} g_{\boldsymbol{m}} = \int \cdots \int f_{\boldsymbol{n}}(*, t_1^{(1)}, \ldots, t_{r_1}^{(1)}; \cdots ; *, t_1^{(N)}, \ldots, t_{r_N}^{(N)}) \times g_{\boldsymbol{m}}(*, t_1^{(1)}, \ldots, t_{r_1}^{(1)}; \cdots ; *, t_1^{(N)}, \ldots, t_{r_N}^{(N)}) dt_1^{(1)} \cdots dt_{r_N}^{(N)}.$$

If $\boldsymbol{r} = \boldsymbol{0}$, $f_{\boldsymbol{n}} \otimes_{\boldsymbol{0}} g_{\boldsymbol{m}}$ means the tensor product $f_{\boldsymbol{n}} \otimes g_{\boldsymbol{m}}$. We denote the symmetrization of $f_{\boldsymbol{n}} \otimes_{\boldsymbol{r}} g_{\boldsymbol{m}}$ by $f_{\boldsymbol{n}} \tilde{\otimes}_{\boldsymbol{r}} g_{\boldsymbol{m}}$. We define the Wiener product as follows:

Definition 4.1. Let $F = \sum I_{\boldsymbol{n}}(f_{\boldsymbol{n}})$ and $G = \sum I_{\boldsymbol{n}}(g_{\boldsymbol{n}})$ belong to $\boldsymbol{D}^{ser}$. Let $\mathcal{W} = \{w_{\boldsymbol{n}}; \boldsymbol{n} \in \mathbb{Z}_+^N\}$ be a set of symmetric positive functions each of whose essential infimum is positive or each of which is bounded. Suppose

$$h_{\boldsymbol{n}} = \sum_{\boldsymbol{p}+\boldsymbol{q}-2\boldsymbol{r}=\boldsymbol{n}} \boldsymbol{r}! \binom{\boldsymbol{p}}{\boldsymbol{r}} \binom{\boldsymbol{q}}{\boldsymbol{r}} g_{\boldsymbol{p}} \tilde{\otimes}_{\boldsymbol{r}} f_{\boldsymbol{q}} \tag{4.1}$$

converges in $L^2_{\boldsymbol{n}}(w_{\boldsymbol{n}} d\boldsymbol{t})$, where, for $\boldsymbol{n} = (n_1, \ldots, n_N)$ and $\boldsymbol{k} = (k_1, \ldots, k_N)$,

$$\binom{\boldsymbol{n}}{\boldsymbol{k}} = \prod \binom{n_i}{k_i}.$$

Then the Wiener product $F \diamond_1 G \in \mathcal{D}^{ser}_{\mathcal{W}}$ of F and G is defined by

$$F \diamond_1 G = \sum I_{\boldsymbol{n}}(h_{\boldsymbol{n}}).$$

Remark 4. (4.1) is derived from the Wiener product formula:

$$I_{\boldsymbol{n}}(f_{\boldsymbol{n}}) I_{\boldsymbol{m}}(g_{\boldsymbol{m}}) = \sum_{\boldsymbol{r}} \boldsymbol{r}! \binom{\boldsymbol{n}}{\boldsymbol{r}} \binom{\boldsymbol{m}}{\boldsymbol{r}} I_{\boldsymbol{n}+\boldsymbol{m}-2\boldsymbol{r}}(f_{\boldsymbol{n}} \tilde{\otimes}_{\boldsymbol{r}} g_{\boldsymbol{m}}). \tag{4.2}$$

Summing up the adequate kernels on the right hand side of (4.2), we obtain (4.1). We should mention that Wiener product is also investigated in the framework of white noise analysis. Refer, for instance, Obata [11], Yan [16], Chung and Chung [1].

Remark 5. In [15] we assumed that the right hand side of (4.1) converges absolutely. In this paper we modify to assume that the right hand side of (4.1) converges in $L^2_{\boldsymbol{n}}(w_{\boldsymbol{n}} d\boldsymbol{t})$. From this modification we have that

$$I_{\boldsymbol{n}}\left(\sum_{\text{finite}} \boldsymbol{r}! \binom{\boldsymbol{p}}{\boldsymbol{r}} \binom{\boldsymbol{q}}{\boldsymbol{r}} g_{\boldsymbol{p}} \tilde{\otimes}_{\boldsymbol{r}} f_{\boldsymbol{q}}\right) \to I_{\boldsymbol{n}}(h_{\boldsymbol{n}}) \quad \text{in} \quad \mathcal{D}^{ser}_{\mathcal{W}}.$$

The definition of the substitution for parameters of a generalized Wiener functional is as follows;

Definition 4.2. Let $\Phi(x) \in \boldsymbol{D}_2^s$ for dx-almost all $x \in \mathbb{R}^N$. Let F be a non-degenerate smooth Wiener functional in Malliavin's sense. Let $\mathcal{W} = \{w_{\boldsymbol{n}}; \boldsymbol{n} \in \mathbb{Z}_+^N\}$ be a set of symmetric positive functions each of whose essential infimum is positive or each of which is bounded. Let $\mathcal{A} = \{a_n; n \in \mathbb{Z}_+\}$ be a sequence of non-negative numbers. Suppose that there exists $\Phi(x) \diamond_1 \delta_x(F)$ in $\mathcal{D}_{\mathcal{W}}^{\mathcal{A}}$, and moreover that it is Bochner integrable. Then we define the substitution $\Phi(F) \in \mathcal{D}_{\mathcal{W}}^{\mathcal{A}}$ as follows:

$$\Phi(F) = \int \Phi(x) \diamond_1 \delta_x(F) dx.$$

Remark 6. If we restrict ourselves to the case where $w_{\boldsymbol{n}} = 1$ in Definitions 4.1 and 4.2, then we can also define the Wiener product and the substitution in the framework of an abstract Wiener space in the same manner; Let (B, H, μ) be an abstract Wiener space. Let $\mathcal{L}_n^2(H)$ be the totality of real valued symmetric n-ple continuous linear functionals on $H^{\otimes n}$ of Hilbert-Schmidt class. We denote the n-ple Wiener integral of $f_n \in \mathcal{L}_n^2(H)$ by $I_n(f_n)$. Set $\boldsymbol{D}^{ser}$, $\boldsymbol{D}_2^s$, $\mathcal{D}_{\mathcal{W}}^{ser}$ and $\mathcal{D}_{\mathcal{W}}^{\mathcal{A}}$ as those in (2.1), (2.2), (2.3) and (2.4), respectively, in the case where $N = 1$. (We replace $L_{\boldsymbol{n}}^2$ by $\mathcal{L}_n^2(H)$ in (2.1) and apply the Hilbert-Schmidt norm for $\|\cdot\|$ in (2.2). Since we assume that $w_{\boldsymbol{n}} = 1$, we also replace $L_{\boldsymbol{n}}^2(w_{\boldsymbol{n}} d\boldsymbol{t})$ by $\mathcal{L}_n^2(H)$ in (2.3) and apply the Hilbert-Schmidt norm for $\|\cdot\|_{w_{\boldsymbol{n}}}$ in (2.4).) Then the Wiener product and the substitution are defined in the same way as those in Definitions 4.1 and 4.2. (We understand $g_p \otimes_r f_q$ as follows;

$$g_p \otimes_r f_q = \sum_{n_1, \ldots, n_r = 1}^{\infty} g_p(*, h_{n_1}, \ldots, h_{n_r}) f_q(*, h_{n_1}, \ldots, h_{n_r}),$$

where $\{h_i\}_{i=1}^{\infty}$ denotes a complete orthonormal system of H.)

5. Local time representation of intersection local time

We study a local time representation of intersection local time $\gamma(T)$ of Brownian motion. In introduction we lead the relationship (1.2) through the formal argument. In this section we decompose the right hand side of (1.2) and then justify each component arisen from this decomposition. In the process of justification we regret to make some modifications to (1.2). Finally we show a local time representation. Our assertions are as follows;

Proposition 5.1. *Assume $\tau < t$ and $x \neq 0$. Let $L(\tau, x)$ be the N-dimensional Brownian local time. Let $\mathcal{A}_1 = \{a_n; n \in \mathbb{Z}_+\}$ be a sequence of non-negative numbers satisfying*

$$\sum a_n (4n \vee 1)^n (n \vee 1)^{3N/2} \log(t/\tau)^{-n} < \infty.$$

Then $L(\tau, x) \diamond_1 \delta_x(B_t)$ exists in $\mathcal{D}_{(0)}^{\mathcal{A}_1}$, and moreover is Bochner integrable (Therefore $L(\tau, B_t)$ can be read as an element of $\mathcal{D}_{(0)}^{\mathcal{A}_1}$).

Proposition 5.2. *Assume $\delta < 2 - N$ and $\alpha < 1 - N/2$. Then $L(\tau, B_t)$ exists in $\mathcal{D}^{\alpha}_{(\delta)}$. Moreover $L^{(1)}(\tau, B_t)$ is continuous in $\mathcal{D}^{\alpha}_{(\delta)}$ with respect to τ, and*

$$L^{(1)}(t-, B_t) = \lim_{\tau \nearrow t} L^{(1)}(\tau, B_t)$$

is Bochner integrable in $\mathcal{D}^{\alpha}_{(\delta)}$ with respect to t.

Theorem 5.1. *It holds that*

$$\gamma(T) = \int_0^T L^{(1)}(t-, B_t)dt. \tag{5.1}$$

To prove Proposition 5.1 we prepare some lemmas. For the uniform estimate of Hermite polynomials, Imkeller, Perez-Abreu and Vives [5] obtained the following lemma. Refer also Szegö [12]:

Lemma 5.1 ([5], [12])**.** *Let $1/4 \le \delta \le 1/2$ and $n \in \mathbb{N}$. Then there exists a constant C which is independent of δ such that*

$$\sup_x \left| H_n(x) \mathrm{e}^{-\delta x^2} \right| \le C\sqrt{n!}(n \vee 1)^{-(8\delta - 1)/12}.$$

We note the following estimate concerning Stirling's formula:

Lemma 5.2. *There exist positive constants C_1 and C_2 satisfying that*

$$C_1\sqrt{2\pi}(n+1)^{n+1/2}\mathrm{e}^{-n-1} \le n! \le C_2\sqrt{2\pi}(n+1)^{n+1/2}\mathrm{e}^{-n-1}$$

for all $n = 0, 1, 2, \ldots$.

The lemma above is easily obtained from the following estimate (cf. Lebedev [7, §1.4]):

Lemma 5.3. *Let $\Gamma(x)$ be the gamma function. Set*

$$\Gamma(x) = \sqrt{2\pi} x^{x-1/2} \mathrm{e}^{-x}\{1 + r(x)\}.$$

Then, for $x > 0$, it holds that

$$|r(x)| \le \mathrm{e}^{1/12x} - 1.$$

Proof of Proposition 5.1. We first show that $L(\tau, x) \diamond_1 \delta_x(B_t) \in \mathcal{D}^{ser}_{(0)}$.

Note that $L(\tau, x)$ and $\delta_x(B_t)$ admit the following Itô-Wiener chaos expansions (cf. Uemura [13], [14], Imkeller and Weisz [6]);

$$L(\tau, x) = \sum I_{\boldsymbol{n}}(f_{\boldsymbol{n}}(\tau, x)),$$

$$f_{\boldsymbol{n}}(\tau, x) = \frac{1}{\boldsymbol{n}!}\int_0^\tau \left(\frac{1}{\sqrt{s}}\right)^n H_{\boldsymbol{n}}\left(\frac{x}{\sqrt{s}}\right) p_N(s, x) \mathbf{1}_{[0,s]}(s_1) \times \cdots \times \mathbf{1}_{[0,s]}(s_n)\, ds,$$

and

$$\delta_x(B_t) = \sum I_{\boldsymbol{n}}(g_{\boldsymbol{n}}(t,x)),$$
$$g_{\boldsymbol{n}}(t,x) = \frac{1}{\boldsymbol{n}!}\left(\frac{1}{\sqrt{t}}\right)^n H_{\boldsymbol{n}}\left(\frac{x}{\sqrt{t}}\right) p_N(t,x)\mathbf{1}_{[0,t]}(s_1)\times\cdots\times\mathbf{1}_{[0,t]}(s_n),$$

where $n = |\boldsymbol{n}|$. Then we set (formally)

(5.2)

$$\begin{aligned} h_{\boldsymbol{n}}(\tau,t,x) &= \sum_{\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{n}}\sum_{\boldsymbol{r}} \boldsymbol{r}!\binom{\boldsymbol{p}+\boldsymbol{r}}{\boldsymbol{r}}\binom{\boldsymbol{q}+\boldsymbol{r}}{\boldsymbol{r}} g_{\boldsymbol{p}+\boldsymbol{r}}(t,x)\otimes_{\boldsymbol{r}} f_{\boldsymbol{q}+\boldsymbol{r}}(\tau,x) \\ &= \sum_{\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{n}}\sum_{\boldsymbol{r}} \frac{1}{\boldsymbol{p}!\boldsymbol{q}!\boldsymbol{r}!}\int_0^\tau \left(\frac{1}{\sqrt{s}}\right)^{q+r} s^r H_{\boldsymbol{q}+\boldsymbol{r}}\left(\frac{x}{\sqrt{s}}\right) p_N(s,x) \\ &\qquad\qquad \times \mathbf{1}_{[0,s]}(s_1)\times\cdots\times\mathbf{1}_{[0,s]}(s_q)\,ds \\ &\quad \times\left(\frac{1}{\sqrt{t}}\right)^{p+r} H_{\boldsymbol{p}+\boldsymbol{r}}\left(\frac{x}{\sqrt{t}}\right) p_N(t,x)\mathbf{1}_{[0,t]}(s_{q+1})\times\cdots\times\mathbf{1}_{[0,t]}(s_n), \end{aligned}$$

where $p = |\boldsymbol{p}|$, $q = |\boldsymbol{q}|$ and $r = |\boldsymbol{r}|$. We prove that the right hand side of (5.2) converges in $L^2_{\boldsymbol{n}}(d\boldsymbol{s})$. To this end it is enough to show that

$$\begin{aligned} \sum_{\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{n}}\sum_{\boldsymbol{r}} \frac{1}{\boldsymbol{p}!\boldsymbol{q}!\boldsymbol{r}!}\Bigg\| \int_0^\tau \left(\frac{1}{\sqrt{s}}\right)^{q+r} s^r H_{\boldsymbol{q}+\boldsymbol{r}}\left(\frac{x}{\sqrt{s}}\right) p_N(s,x) \\ \times\mathbf{1}_{[0,s]}(s_1)\times\cdots\times\mathbf{1}_{[0,s]}(s_q)\,ds \\ \times\left(\frac{1}{\sqrt{t}}\right)^{p+r} H_{\boldsymbol{p}+\boldsymbol{r}}\left(\frac{x}{\sqrt{t}}\right) p_N(t,x)\mathbf{1}_{[0,t]}(s_{q+1})\times\cdots\times\mathbf{1}_{[0,t]}(s_n)\Bigg\| < \infty. \end{aligned}$$

Appealing to Lemma 5.1, it is easy to see that

$$\begin{aligned} &\Bigg\|\int_0^\tau \left(\frac{1}{\sqrt{s}}\right)^{q+r} s^r H_{\boldsymbol{q}+\boldsymbol{r}}\left(\frac{x}{\sqrt{s}}\right) p_N(s,x)\mathbf{1}_{[0,s]}(s_1)\times\cdots\times\mathbf{1}_{[0,s]}(s_q)\,ds \\ &\qquad \times\left(\frac{1}{\sqrt{t}}\right)^{p+r} H_{\boldsymbol{p}+\boldsymbol{r}}\left(\frac{x}{\sqrt{t}}\right) p_N(t,x)\mathbf{1}_{[0,t]}(s_{q+1})\times\cdots\times\mathbf{1}_{[0,t]}(s_n)\Bigg\| \\ &\quad \le C_1\sqrt{(\boldsymbol{q}+\boldsymbol{r})!}\sqrt{(\boldsymbol{p}+\boldsymbol{r})!}\,((\boldsymbol{q}+\boldsymbol{r})\vee 1)^{-(8\alpha-1)/12}((\boldsymbol{p}+\boldsymbol{r})\vee 1)^{-(8\alpha-1)/12} \\ &\quad\quad \times\Bigg\|\int_0^\tau s^{r/2-q/2-N/2}\mathrm{e}^{-\beta|x|^2/s}\mathbf{1}_{[0,s]}(s_1)\times\cdots\times\mathbf{1}_{[0,s]}(s_q)\,ds \\ &\qquad\qquad \times t^{-r/2-p/2-N/2}\mathrm{e}^{-\beta|x|^2/t}\mathbf{1}_{[0,t]}(s_{q+1})\times\cdots\times\mathbf{1}_{[0,t]}(s_n)\Bigg\| \\ &\quad \le C_2\sqrt{(\boldsymbol{q}+\boldsymbol{r})!}\sqrt{(\boldsymbol{p}+\boldsymbol{r})!}\left(\frac{\tau}{t}\right)^{r/2}\left(\frac{1}{\beta|x|^2}\right)^{\rho} t^{-p/2-N/2}\mathrm{e}^{-\beta|x|^2/t} \\ &\quad\quad \times\Bigg\|\int_0^\tau s^{-q/2-N/2+\rho}\mathbf{1}_{[0,s]}(s_1)\times\cdots\times\mathbf{1}_{[0,s]}(s_q)\,ds \end{aligned}$$

$$\times \mathbf{1}_{[0,t]}(s_{q+1}) \times \cdots \times \mathbf{1}_{[0,t]}(s_n)\Big\|,$$

where $\alpha \in [1/4, 1/2)$, $\beta = 1/2 - \alpha$ and $\rho > 0$. It is easy to see that

(5.3)
$$\int_0^T \cdots \int_0^T \left(\int_0^\tau s^{-q/2-N/2+\rho}\mathbf{1}_{[0,s]}(s_1)\cdots\mathbf{1}_{[0,s]}(s_q)\,ds\mathbf{1}_{[0,t]}(s_{q+1})\cdots\mathbf{1}_{[0,t]}(s_n)\right)^2 ds_1\cdots ds_n$$
$$\leq \frac{C}{\rho+q/2}t^p$$

if $\rho > N/2 - 1/2$, C being a positive constant. From (5.2) and (5.3) we have

$$\sum_{\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{n}}\sum_{\boldsymbol{r}}\boldsymbol{r}!\binom{\boldsymbol{p}+\boldsymbol{r}}{\boldsymbol{r}}\binom{\boldsymbol{q}+\boldsymbol{r}}{\boldsymbol{r}}\|g_{\boldsymbol{p}+\boldsymbol{r}}(t,x)\otimes_{\boldsymbol{r}} f_{\boldsymbol{q}+\boldsymbol{r}}(\tau,x)\|$$
$$\leq C\left\{\left(\frac{1}{\beta|x|^2}\right)^\rho t^{-N/2}\mathrm{e}^{-\beta|x|^2/t}\sqrt{\frac{1}{\rho}}\right.$$
$$\left.\times\sum_{\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{n}}\frac{1}{\boldsymbol{p}!\boldsymbol{q}!}\sum_{\boldsymbol{r}}\frac{1}{\boldsymbol{r}!}\sqrt{(\boldsymbol{q}+\boldsymbol{r})!}\sqrt{(\boldsymbol{p}+\boldsymbol{r})!}\left(\frac{\tau}{t}\right)^{r/2}\right\}.$$

Applying Lemma 5.2 we have

$$\frac{1}{r!}\sqrt{(q+r)!}\sqrt{(p+r)!}\left(\frac{\tau}{t}\right)^{r/2}$$
$$\leq C_1(q+r+1)^{q/2+r/2+1/4}(p+r+1)^{p/2+r/2+1/4}$$
$$\times(r+1)^{-r-1/2}\mathrm{e}^{-p/2-q/2}\left(\frac{\tau}{t}\right)^{r/2}$$
$$\leq C_2(q+r+1)^{q/2}(p+r+1)^{p/2}\left(\frac{\tau}{t}\right)^{r/2} \leq C_2(n+r+1)^{n/2}\left(\frac{\tau}{t}\right)^{r/2}$$
$$\leq C_2(n+1)^{n/2}(r+1)^{n/2}\left(\frac{\tau}{t}\right)^{r/2}$$
$$\leq C_3\sqrt{n!}(n+1)^{-1/4}\mathrm{e}^{n/2}(r+1)^{n/2}\left(\frac{\tau}{t}\right)^{r/2},$$

where $n = p + q$. Put $c = \sqrt{t/\tau} > 1$. A slight computation gives

$$\sum(r+1)^{n/2}c^{-r} \leq \int_0^\infty (1+x)^{n/2}c^{-x}dx + c\left(\frac{n\vee 1}{2\log c}\right)^{n/2}\mathrm{e}^{-n/2}$$

and

$$\int_0^\infty (1+x)^{n/2}c^{-x}dx \leq \frac{c}{(\log c)^{n/2+1}}\Gamma(n/2+1)$$
$$\leq C_4\frac{c}{(\log c)^{n/2+1}}(n/2+1)^{n/2+1}\mathrm{e}^{-n/2}.$$

Noting that

$$\prod_{i=1}^{N}\Big(\frac{n_i}{2}+1\Big)^{n_i/2} \le \prod_{i=1}^{N}\Big(\frac{n}{2}+1\Big)^{n_i/2} = \Big(\frac{n}{2}+1\Big)^{n/2} \qquad (n = n_1+\cdots+n_N),$$

we obtain

$$\begin{aligned} &\boldsymbol{n}!\left\|\sum_{\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{n}}\sum_{\boldsymbol{r}}\boldsymbol{r}!\binom{\boldsymbol{p}+\boldsymbol{r}}{\boldsymbol{r}}\binom{\boldsymbol{q}+\boldsymbol{r}}{\boldsymbol{r}} g_{\boldsymbol{p}+\boldsymbol{r}}(t,x)\otimes_{\boldsymbol{r}} f_{\boldsymbol{q}+\boldsymbol{r}}(\tau,x)\right\|^2 \\ &\le \boldsymbol{n}!\left(\sum_{\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{n}}\sum_{\boldsymbol{r}}\boldsymbol{r}!\binom{\boldsymbol{p}+\boldsymbol{r}}{\boldsymbol{r}}\binom{\boldsymbol{q}+\boldsymbol{r}}{\boldsymbol{r}} \|g_{\boldsymbol{p}+\boldsymbol{r}}(t,x)\otimes_{\boldsymbol{r}} f_{\boldsymbol{q}+\boldsymbol{r}}(\tau,x)\|\right)^2 \qquad (5.4)\\ &\le C_5\frac{1}{\rho}\left(\frac{1}{\beta|x|^2}\right)^{2\rho} t^{-N}\mathrm{e}^{-2\beta|x|^2/t}(4n\vee 1)^n(n\vee 1)^{3N/2}\log(t/\tau)^{-n}, \end{aligned}$$

which ensures that the right hand side of (5.2) converges in $L^2_{\boldsymbol{n}}(d\boldsymbol{s})$ and moreover that $L(\tau,x)\diamond_1\delta_x(B_t)\in\mathcal{D}^{\mathcal{A}_1}_{(0)}$.

Since $h_{\boldsymbol{n}}(\tau,t,x)$ is continuous with respect to $x\ne 0$, $L(\tau,x)\diamond_1\delta_x(B_t)$ is continuous in $\mathcal{D}^{\mathcal{A}_1}_{(0)}$ with respect to $x\ne 0$. From (5.4) we easily know that $\|L(\tau,x)\diamond_1\delta_x(B_t)\|_{(0),\mathcal{A}_1}$ is dx-integrable if $\rho<N/2$. Setting $N/2-1/2<\rho<N/2$, we find that $L(\tau,x)\diamond_1\delta_x(B_t)$ is Bochner integrable. This completes the proof. □

For the proof of Proposition 5.2 we note the following formula on Hermite polynomials. Refer, for instance, Gradshteyn and Ryzhik [3, 7.374].

Lemma 5.4 ([3]). *Let $\{H_n\}$ be Hermite polynomials as in* (3.4) *and $a\in\mathbb{R}$. Then it holds that*

$$\int_{-\infty}^{\infty} H_{2m+n}(ax)H_n(x)\mathrm{e}^{-x^2/2}\,dx = \sqrt{2\pi}2^{-m}\frac{(2m+n)!}{m!}(a^2-1)^m a^n.$$

We also note the following lemma;

Lemma 5.5. *It holds that*

$$\sum_{|\boldsymbol{n}|=n}\frac{(2\boldsymbol{n})!}{(\boldsymbol{n}!)^2} = \frac{2^n N(N+2)\cdots(N+2n-2)}{n!}.$$

We easily have the lemma above from the equation below, so we omit the proof;

$$\sum_{k\ge 0}\binom{2k}{k}x^k = (1-4x)^{-1/2}.$$

Proof of Proposition 5.2. From Proposition 5.1 $L(\tau,B_t)$ admits the Itô-Wiener chaos expansion in $\mathcal{D}^{\mathcal{A}_1}_{(0)}$;

$$L(\tau,B_t) = \sum I_{\boldsymbol{n}}(\eta_{\boldsymbol{n}}(\tau,t)),$$
$$\eta_{\boldsymbol{n}}(\tau,t) = \int_{\mathbb{R}^N} h_{\boldsymbol{n}}(\tau,t,x)\,dx,$$

where $h_{\boldsymbol{n}}(\tau,t,x)$ is as in (5.2). We find $\eta_{\boldsymbol{n}}(\tau,t)$ more explicitly. By a slight computation we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_0^\tau \left(\frac{1}{\sqrt{s}}\right)^{q+r} s^r H_{\boldsymbol{q}+\boldsymbol{r}}\left(\frac{x}{\sqrt{s}}\right) p_N(s,x)\mathbf{1}_{[0,s]}(s_1)\times\cdots\times\mathbf{1}_{[0,s]}(s_q)\,ds \\
&\qquad \times \left(\frac{1}{\sqrt{t}}\right)^{p+r} H_{\boldsymbol{p}+\boldsymbol{r}}\left(\frac{x}{\sqrt{t}}\right) p_N(t,x)\mathbf{1}_{[0,t]}(s_{q+1})\times\cdots\times\mathbf{1}_{[0,t]}(s_n)\,dx \\
&= \int_0^\tau s^r\mathbf{1}_{[0,s]}(s_1)\times\cdots\times\mathbf{1}_{[0,s]}(s_q)\times\mathbf{1}_{[0,t]}(s_{q+1})\times\cdots\times\mathbf{1}_{[0,t]}(s_n) \\
&\qquad \times \int_{\mathbb{R}^N}(-1)^{q+r}\partial_x^{\boldsymbol{q}+\boldsymbol{r}} p_N(s,x)(-1)^{p+r}\partial_x^{\boldsymbol{p}+\boldsymbol{r}} p_N(t,x)\,dx\,ds \\
&= \int_0^\tau s^r\mathbf{1}_{[0,s]}(s_1)\times\cdots\times\mathbf{1}_{[0,s]}(s_q)\times\mathbf{1}_{[0,t]}(s_{q+1})\times\cdots\times\mathbf{1}_{[0,t]}(s_n) \\
&\qquad \times \int_{\mathbb{R}^N}(-1)^{q+r}\partial_x^{\boldsymbol{n}+2\boldsymbol{r}} p_N(s,x)\cdot p_N(t,x)\,dx\,ds \\
&= \int_0^\tau\int_{\mathbb{R}^N}(-1)^{p+r}\left(\frac{1}{\sqrt{s}}\right)^{n+2r} s^r H_{\boldsymbol{n}+2\boldsymbol{r}}\left(\frac{x}{\sqrt{s}}\right)\left(\frac{1}{\sqrt{2\pi(t+s)}}\right)^N \\
&\qquad \times p_N(ts/(t+s),x)\,dx\times\mathbf{1}_{[0,s]}(s_1)\times\cdots\times\mathbf{1}_{[0,s]}(s_q) \\
&\qquad \times\mathbf{1}_{[0,t]}(s_{q+1})\times\cdots\times\mathbf{1}_{[0,t]}(s_n)\,ds \\
&= \int_0^\tau (-1)^{p+r}\left(\frac{1}{\sqrt{2\pi(t+s)}}\right)^N 2^{-n/2-r}\frac{(\boldsymbol{n}+2\boldsymbol{r})!}{(\boldsymbol{n}/2+\boldsymbol{r})!}s^r\left(\frac{-1}{t+s}\right)^{-n/2-r} \\
&\qquad \times\mathbf{1}_{[0,s]}(s_1)\times\cdots\times\mathbf{1}_{[0,s]}(s_q)\times\mathbf{1}_{[0,t]}(s_{q+1})\times\cdots\times\mathbf{1}_{[0,t]}(s_n)\,ds,
\end{aligned}$$

where $\boldsymbol{n}=\boldsymbol{p}+\boldsymbol{q}$. The last equality holds from Lemma 5.4 if $\boldsymbol{n}\in(2\mathbb{Z}_+)^N$, otherwise the above integral vanishes. $\partial_x^{\boldsymbol{p}}$ denotes $\partial^{p_1}/\partial x^{p_1}\cdots\partial^{p_N}/\partial x^{p_N}$ if $\boldsymbol{p}=(p_1,\dots,p_N)$. We then consider only $\eta_{2\boldsymbol{n}}(\tau,t)$. Noting that

$$\sum_r \frac{1}{r!}2^{-r}\frac{(2n+2r)!}{(n+r)!}\left(\frac{s}{t+s}\right)^r = \frac{(2n)!}{n!}\left(\frac{t+s}{t-s}\right)^{(2n+1)/2},$$

we obtain

$$\begin{aligned}
&\sum_{\boldsymbol{r}}\frac{1}{\boldsymbol{r}!}\int_0^\tau (-1)^{p+r}\left(\frac{1}{\sqrt{2\pi(t+s)}}\right)^N 2^{-n-r}\frac{(2\boldsymbol{n}+2\boldsymbol{r})!}{(\boldsymbol{n}+\boldsymbol{r})!}s^r\left(\frac{-1}{t+s}\right)^{-n-r} \\
&\qquad \times\mathbf{1}_{[0,s]}(s_1)\times\cdots\times\mathbf{1}_{[0,s]}(s_q)\times\mathbf{1}_{[0,t]}(s_{q+1})\times\cdots\times\mathbf{1}_{[0,t]}(s_{2n})\,ds \\
&= \int_0^\tau (-1)^{p+n}\left(\frac{1}{\sqrt{2\pi(t-s)}}\right)^N 2^{-n}\frac{(2\boldsymbol{n})!}{\boldsymbol{n}!}\left(\frac{1}{t-s}\right)^n \\
&\qquad \times\mathbf{1}_{[0,s]}(s_1)\times\cdots\times\mathbf{1}_{[0,s]}(s_q)\times\mathbf{1}_{[0,t]}(s_{q+1})\times\cdots\times\mathbf{1}_{[0,t]}(s_{2n})\,ds.
\end{aligned}$$

It is easy to see that

$$\mathcal{S} \sum_{\boldsymbol{p}+\boldsymbol{q}=2\boldsymbol{n}} \frac{(2\boldsymbol{n})!}{\boldsymbol{p}!\boldsymbol{q}!}(-1)^p \mathbf{1}_{[0,s]}(s_1) \times \cdots \times \mathbf{1}_{[0,s]}(s_q) \times \mathbf{1}_{[0,t]}(s_{q+1}) \times \cdots \times \mathbf{1}_{[0,t]}(s_{2n})$$
$$= \mathbf{1}_{(s,t]}(s_1) \times \cdots \times \mathbf{1}_{(s,t]}(s_{2n}),$$

$\mathcal{S}$ denoting the symmetrization operator. Hence we have

$$\eta_{2\boldsymbol{n}}(\tau,t) = \frac{(-1)^n}{2^n \boldsymbol{n}!} \int_0^\tau \left(\frac{1}{\sqrt{2\pi(t-s)}}\right)^N \left(\frac{1}{t-s}\right)^n \mathbf{1}_{(s,t]}(s_1)\times\cdots\times\mathbf{1}_{(s,t]}(s_{2n})\, ds.$$

Set

$$\bar{\eta}_{2\boldsymbol{n}}(t) = \frac{1}{2^n \boldsymbol{n}!} \int_0^t \left(\frac{1}{\sqrt{2\pi(t-s)}}\right)^N \left(\frac{1}{t-s}\right)^n \mathbf{1}_{(s,t]}(s_1) \times \cdots \times \mathbf{1}_{(s,t]}(s_{2n})\, ds.$$

Then a slight computation gives

(5.5)
$$\|\bar{\eta}_{2\boldsymbol{n}}(t)\|_{(\delta)}^2 = \left(\frac{1}{2\pi}\right)^N \left(\frac{1}{2^n \boldsymbol{n}!}\right)^2 \frac{2n(2n-1)}{(2n-\delta)(2n-1-\delta)} \times \frac{1}{2n+2-N-\delta} \times \frac{t^{2-N-\delta}}{2-N-\delta}$$

if $n \geq 1$ and $\delta < 2-N$. Applying Lemma 5.5 we get

$$\begin{aligned} &\sum_{|\boldsymbol{n}|=n} (2\boldsymbol{n})! \|\bar{\eta}_{2\boldsymbol{n}}(t)\|_{(\delta)}^2 \\ &\leq C_1 \frac{1}{n}\left(1+\frac{N-2}{2}\right)\left(1+\frac{N-2}{4}\right)\cdots\left(1+\frac{N-2}{2n}\right) t^{2-N-\delta} \\ &\leq C_2 n^{N/2-2} t^{2-N-\delta}, \end{aligned} \tag{5.6}$$

where the last inequality above is due to the following estimate;

$$\log\left\{\left(1+\frac{N-2}{2}\right)\left(1+\frac{N-2}{4}\right)\cdots\left(1+\frac{N-2}{2n}\right)\right\}$$
$$\leq C_3 \left\{\frac{N-2}{2} + \frac{N-2}{4} + \cdots + \frac{N-2}{2n}\right\} \leq C_4 \frac{N-2}{2} \log n.$$

Since $|\eta_{2\boldsymbol{n}}(\tau,t)| \leq \bar{\eta}_{2\boldsymbol{n}}(t)$, we conclude that $L(\tau, B_t) \in \mathcal{D}^\alpha_{(\delta)}$ and is continuous with respect to τ in $\mathcal{D}^\alpha_{(\delta)}$ if $\delta < 2-N$ and $\alpha < 1-N/2$. As $\lim_{\tau\nearrow t} \eta_{\mathbf{0}}(\tau,t) = \infty$ and $\lim_{\tau\nearrow t} \eta_{2\boldsymbol{n}}(\tau,t) = (-1)^n \bar{\eta}_{2\boldsymbol{n}}(t)$, we also have

$$L^{(1)}(t-, B_t) = \lim_{\tau \nearrow t} L^{(1)}(\tau, B_t)$$

in $\mathcal{D}^\alpha_{(\delta)}$, where

$$L^{(1)}(t-, B_t) = \sum_{|\boldsymbol{n}|\geq 1} I_{2\boldsymbol{n}}((-1)^n \bar{\eta}_{2\boldsymbol{n}}(t)).$$

In order to show that $L^{(1)}(t-,B_t)$ is continuous with respect to t in $\mathcal{D}^{\alpha}_{(\delta)}$, it is sufficient to prove

$$\sup_{0<t\leq T}\|\bar{\eta}_{2\boldsymbol{n}}(t)^{1+\varepsilon}\|_{(\delta)}<\infty$$

for some $\varepsilon>0$. As $(\int_0^t|f(x)|dx)^{2(1+\varepsilon)}\leq t^{2\varepsilon}(\int_0^t|f(x)|^{1+\varepsilon}dx)^2$, it is enough to estimate $\|\bar{\eta}^{\varepsilon}_{2\boldsymbol{n}}(t)\|_{(\delta)}$, where

$$\bar{\eta}^{\varepsilon}_{2\boldsymbol{n}}(t)=\frac{1}{2^n\boldsymbol{n}!}\int_0^t\left(\frac{1}{\sqrt{2\pi(t-s)}}\right)^{N(1+\varepsilon)}\left(\frac{1}{t-s}\right)^{n(1+\varepsilon)} \times\mathbf{1}_{(s,t]}(s_1)\times\cdots\times\mathbf{1}_{(s,t]}(s_{2n})\,ds.$$

From the same computation as that in (5.5) we obtain

$$\sup_{0<t\leq T}\|\bar{\eta}^{\varepsilon}_{2\boldsymbol{n}}\|_{(\delta)}<\infty$$

if $\varepsilon<(2-N-\delta)/(2n+N)$. Therefore $L^{(1)}(t-,B_t)$ is continuous with respect to t in $\mathcal{D}^{\alpha}_{(\delta)}$. The dt-integrability of the $\mathcal{D}^{\alpha}_{(\delta)}$ norm of $L^{(1)}(t-,B_t)$ is easily obtained from (5.6). Thus we conclude that $L^{(1)}(t-,B_t)$ is Bochner integrable in $\mathcal{D}^{\alpha}_{(\delta)}$, which completes the proof. □

Proof of Theorem 5.1. Obviously it holds that

$$\gamma_{2\boldsymbol{n}}=\int_0^T(-1)^n\bar{\eta}_{2\boldsymbol{n}}(t)\,dt.$$

Therefore we easily obtain (5.1) applying Propositions 5.1 and 5.2, which completes the proof. □

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