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Bessel-like processes and SDE

By

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1. Introduction

The Bessel process (X_t, P_x) with the fractional dimension $\gamma > 0$ is a diffusion process on $[0, \infty)$ determined by the local generator

$$L = \frac{1}{2}\left(\frac{d^2}{dx^2} + \frac{\gamma - 1}{x}\frac{d}{dx}\right)$$

with the point 0 as

$$\begin{aligned} &\text{a reflecting boundary if } 0 < \gamma < 2, \\ &\text{an entrance boundary if } \gamma \geq 2 \end{aligned}$$

(cf. [4]). When the dimension γ is a positive integer n, this process is nothing but the radial part of the n-dimensional Brownian motion. If we consider the squared process $\{Y_t := X_t^2\}$, then Y_t is represented as a pathwise unique solution of the following stochastic differential equation (SDE):

$$dY_t = 2\sqrt{Y_t}\, dB_t + \gamma dt, \quad Y_t \geq 0 \quad (t \geq 0), \tag{1.1}$$

where $\{B_t\}$ is an $\mathcal{F}_t$-Brownian motion defined on a standard probability space $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t \geq 0})$ with a filtration.

Generalizing the SDE (1.1), we first consider the following SDE:

$$dY_t = 2\sqrt{Y_t}\, dB_t + b(Y_t)dt, \quad Y_t \geq 0 \quad (t \geq 0), \tag{1.2}$$

where we assume that b is a continuous function on $[0, \infty)$ satisfying that

$$b(0) = \gamma > 0, \quad |b(y)| \leq C(1 + y) \text{ for some constant } C.$$

Then, applying Yamada-Watanabe's pathwise uniqueness theorem ([6]) and Yamada's comparison theorem ([5]), we can see that for any $Y_0 = y \geq 0$, the SDE (1.2) has a pathwise unique solution Y_t, which defines a diffusion process on $[0, \infty)$. For this process Y_t of (1.2), we define its square root $\{X_t := \sqrt{Y_t}\}$,

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which is also a diffusion process on $[0,\infty)$. We call the diffusion process (X_t, P_x) the γ-dimensional Bessel-like process.

In this paper we discuss the possibility of describing the process X_t by an SDE. If we apply the Itô formula formally to $u(y) = \sqrt{y}$, we have the following equation in terms of stochastic differentials:

$$X_t = X_0 + B_t + \int_0^t \frac{b(X_s^2) - 1}{2X_s} ds, \tag{1.3}$$

whenever $X_t > 0$. However, the behavior of X_t when it takes value 0 is delicate so that the SDE (1.3) does not hold globally in time, generally. It even happens that X_t is not a semimartingale, in general. Indeed, the second term in the left-hand side of (1.3) cannot be a global stochastic differential in the case $0 < \gamma < 1$, as we shall see. So our basic problem should be to ask when X_t is a semimartingale.

Let us define $\mathcal{N}_t$ by

$$X_t - X_0 = \sqrt{Y_t} - \sqrt{Y_0} = B_t + \mathcal{N}_t. \tag{1.4}$$

We first confirm that (1.4) coincides with the Fukushima decomposition for additive functionals of the process (Y_t, P_y) so that $\mathcal{N}_t$ is an additive functional locally of zero energy.

We next obtain a precise condition on $b(x)$ so that $\mathcal{N}_t$ is a process of bounded variation. Finally we give a new representation of $\mathcal{N}_t$ in terms of the local time in general situation.

2. Results

As in Introduction, let (Y_t, P_y) be the diffusion process on $[0,\infty)$ governed by the SDE (1.2) and (X_t, P_x) be the γ-dimensional Bessel-like process defined by setting $X_t = \sqrt{Y_t}$, where $\gamma = b(0) > 0$. The local generator of (Y_t, P_y) is given by

$$A = 2y\frac{d^2}{dy^2} + b(y)\frac{d}{dy}. \tag{2.1}$$

Let σ_0 be the hitting time to 0 of (Y_t, P_y). It is easy to see that $P_y(\sigma_0 < \infty) > 0$ for every $y \geq 0$ if and only if

$$\int_0^1 \exp\left(\int_y^1 \frac{b(z)}{2z} dz\right) dy < \infty. \tag{2.2}$$

Throughout this paper, we assume this condition (2.2), otherwise, (1.3) is easily verified. Then the scale function $s(y)$ and the speed measure $m(dy)$ of the process (Y_t, P_y) are given by

$$s(x) = \int_0^x s'(y)dy, \quad s'(y) = \exp\left(-\int_1^y \frac{b(z)}{2z} dz\right), \tag{2.3}$$

and

$$m(dy) = m'(y)dy = (2ys'(y))^{-1}dy. \tag{2.4}$$

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ associated with (Y_t, P_y) is defined by

$$\mathcal{E}(u,v) = \int_0^t u'(x)v'(x)(s'(x))^{-1}dx,$$
$$\mathcal{F} = \{u : \mathcal{E}(u,u) < \infty\} \cap L^2([0,\infty); m).$$

Theorem 2.1. (i) *$\mathcal{N}_t$ of (1.4) is a continuous additive functional locally of zero energy in the sense of Dirichlet form theory.*
(ii) *If $0 \le t_1 < t_2$ satisfy that $X_{t_1} = X_{t_2} = 0$ and $X_s > 0$ for all $s \in (t_1, t_2)$,*

$$\mathcal{N}_{t_2} - \mathcal{N}_{t_1} = \int_{t_1+}^{t_2-} \frac{b(X_s^2) - 1}{2X_s} ds, \tag{2.5}$$

where the integral should be read as an improper integral.

Proposition 2.1. *The following limits (2.6) (when $x > 0$) and (2.7) exist and we call $L_t^X(x)$ and ℓ_t the local time at x and the local time at 0 of the process (X_t, P_x), respectively.*

$$L_t^X(x) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t \mathrm{I}_{[x-\epsilon, x+\epsilon]}(X_s)dx, \tag{2.6}$$

and

$$\ell_t = \lim_{\epsilon \to 0} \frac{2\gamma}{\epsilon^\gamma K(\epsilon)} \int_0^t \mathrm{I}_{[0,\epsilon]}(X_s)dx. \tag{2.7}$$

Here $K(x)$ is a slowly varying function defined by

$$K(x) = \exp \int_1^x \frac{b(y^2) - \gamma}{y} dy. \tag{2.8}$$

$L_t(x)$ is continuous in $(t,x) \in [0,\infty) \times (0,\infty)$ P_x-a.s. and satisfies that

$$\int_0^t f(X_s)ds = \int_0^\infty f(x)L_t^X(x)dx, \tag{2.9}$$

for any bounded Borel function $f(x)$ on $[0,\infty)$. Furthermore, it holds that for any $0 < \alpha < \gamma/2$

$$L_t^X(x) - \frac{1}{2}x^{\gamma-1}K(x)\,\ell_t = o(x^\alpha) \quad (x \searrow 0). \tag{2.10}$$

Theorem 2.2. *Suppose that*

$$\int_0^1 \frac{|b(y^2) - 1|}{y} \exp\left(\int_1^y \frac{b(z^2) - 1}{z} dz\right) dy < \infty. \tag{2.11}$$

Then $\mathcal{N}_t$ is of bounded variation locally in $t \geq 0$ P_x-almost surely.

Moreover, it holds that $\int_0^t \frac{|b(X_s^2)-1|}{X_s} ds < \infty$ P_x-a.s. and

$$X_t = x + B_t + \int_0^t \frac{b(X_s^2) - 1}{2X_s} ds + c\,\ell_t, \tag{2.12}$$

where c is given by the following limit which is shown to exist:

$$c = \lim_{\epsilon \searrow 0} \frac{\epsilon^{\gamma-1} K(\epsilon)}{4} \in [0, \infty). \tag{2.13}$$

We remark that if $\gamma > 1$, the condition (2.11) is satisfied and $c = 0$. In the case $\gamma = 1$, under the condition (2.11), $c > 0$ holds if and only if $\int_{0+}^1 \frac{b(y^2)-1}{y} dy$ exists.

Theorem 2.3. *Suppose that*

$$\int_0^1 \frac{|b(y^2) - 1|}{y} \exp\left(\int_1^y \frac{b(z^2) - 1}{z} dz\right) dy = \infty. \tag{2.14}$$

Then $\mathcal{N}_t$ is of unbounded variation in each bounded interval $[0, t]$ P_0-almost surely.

Next we would like to describe the Bessel-like process (X_t, P_x) by another kind of stochastic equation involving the local times $L_t^X(x)$ and ℓ_t. In the situation of Theorem 2.3, it follows from (2.9) that the SDE (2.12) can be rewritten as

$$X_t = X_0 + B_t + \int_0^{M_t} \frac{b(x^2) - 1}{2x} L_t^X(x) dx + c\,\ell_t, \tag{2.15}$$

where

$$M_t = \sup_{0 \leq s \leq t} X_s. \tag{2.16}$$

We introduce a renormalized local time $\tilde{L}_t^X(x)$ defined by

$$\tilde{L}_t^X(x) = L_t^X(x) - \frac{1}{2}\gamma x^{\gamma-1} K(x) \ell_t. \tag{2.17}$$

Then the equation (2.15) can be written as

$$X_t = X_0 + B_t + \int_0^{M_t} \frac{b(x^2) - 1}{2x} \tilde{L}_t^X(x) dx + F(M_t)\ell_t, \tag{2.18}$$

where

$$F(x) = \frac{1}{4} \int_0^x (b(y^2) - 1)\gamma y^{\gamma-2} K(y) dy + c. \tag{2.19}$$

It is easy to see by integration by parts that

$$F(x) = \frac{1}{4} x^{\gamma-1} K(x) \quad (x > 0). \tag{2.20}$$

We have thus rewritten the SDE (2.12) in the form (2.18) with F given by (2.20) when the condition (2.11) holds, that is, when the process $\mathcal{N}_t$ is of bounded variation. If we note (2.10), however, the integral in the right-hand side of (2.18) is convergent, so that the right-hand side of (2.18) is meaningful, even in the case that the condition (2.14) holds. And, indeed, we have the following theorem which holds for all cases of Bessel-like processes.

Theorem 2.4. *X_t satisfies the following equation*:

$$X_t = X_0 + B_t + \int_0^{M_t} \frac{b(x^2) - 1}{2x} \tilde{L}_t^X(x) dx + F(M_t)\ell_t, \tag{2.21}$$

where $F(x)$ is given by (2.20).

3. Proofs

Let (Y_t, P_y) be the diffusion process associated with the SDE (1.4), of which local generator is (2.1).

Lemma 3.1.

$$\int_0^t \mathrm{I}_{\{0\}}(Y_s) ds = 0. \tag{3.1}$$

Proof. Define a sequence of functions $\{f_n\}$ by

$$f_n'(y) = ((-ny + 1) \vee 0) \wedge 1, \quad f_n(y) = \int_0^y f_n'(z) dz.$$

Since $Af_n(y) \geq 2y \left(-n\mathrm{I}_{[0,\frac{1}{n}]}(y) \right) + b(0)\mathrm{I}_{\{0\}}(y)$, for sufficiently large n, we have

$$\begin{aligned} E_y[f_n(Y_n) - f_n(Y_0)] &= \int_0^t E_y[Af_n(Y_s)] ds \\ &\geq \int_0^t E_y \left[-2nY_s \mathrm{I}_{[0,\frac{1}{n}]}(Y_s) \right] ds + b(0) \int_0^t E_y[\mathrm{I}_{\{0\}}(Y_s)] ds, \end{aligned}$$

which implies (3.1) with $n \to \infty$. □

For the scale function $s(y)$ of (2.3) we denote

$$a(y) = 4ys'(y)^2 \quad (y \geq 0).$$

Lemma 3.2. *Let Y_t be the solution of* (1.2) *with $Y_0 = y \geq 0$. Then there exists a reflected Brownian motion W_t starting at $s(y)$ such that*

$$Y_t = s^{-1}(W_{A_t}) \quad (t \geq 0), \tag{3.2}$$

where

$$A_t = \int_0^t a(Y_s)ds.$$

Proof. First we claim that

$$s(Y_t) - s(y) = \int_0^t s'(Y_s)2\sqrt{Y_s}dB_s + \varphi_t, \tag{3.3}$$

where φ_t increases only at time t with $Y_t = 0$.

In order to see this, let us define $s_n(y)$ by

$$s_n(y) = \int_0^y s'\left(z \vee \frac{1}{n}\right) dz.$$

Since $As_n(y) = s'(\frac{1}{n})\mathrm{I}_{[0,\frac{1}{n}]}(y)b(y)$, by the Itô formula

$$s_n(Y_t) - s_n(y) = \int_0^t s'\left(Y_s \vee \frac{1}{n}\right) 2\sqrt{Y_s}dB_s + s'\left(\frac{1}{n}\right) \int_0^t \mathrm{I}_{[0,\frac{1}{n}]}(Y_s)b(Y_s)ds.$$

Noting that the left-hand side converges to $s(Y_t) - s(y)$, the first term of the right-hand side is represented using some Brownian motion $\tilde{B}_t$ as

$$\int_0^t s'\left(Y_s \vee \frac{1}{n}\right) 2\sqrt{Y_s}dB_s = \tilde{B}_{\int_0^t s'(Y_s \vee \frac{1}{n})^2 4Y_s ds},$$

and the last term is nonnegative and non-increasing in n, we see that

$$\int_0^t a(Y_s)ds < \infty,$$

and (3.3) is valid. Next, we set

$$A_t = \int_0^t a(Y_s)ds.$$

Since A_t is strictly increasing by Lemma 3.1, for $0 \leq t < A_\infty$

$$\overline{B}_t = \int_0^{A_t^{-1}} s'(Y_s)2\sqrt{Y_s}dB_s$$

is a Brownian motion up to A_∞. Thus $(W_t = s(Y_{A_t^{-1}}), \bar{\ell}_t = \varphi_{A_t^{-1}})$ solves the Skorohod equation for the reflected Brownian motion up to A_∞;

$$W_t = s(y) + \overline{B}_t + \bar{\ell}_t \quad (0 \leq t \leq A_\infty).$$

Hence W_t is a reflected Brownian motion on $[0,\infty)$ starting at $s(y)$, which yields the conclusion. $\square$

3.1. Proof of Theorem 2.1

For $u \in \mathcal{F}_{loc}$ the additive functional $A^{[u]}$ is defined by

$$A_t^{[u]} = u(Y_t) - u(Y_0).$$

By the Fukushima decomposition ([2]), we have the following: For any quasi-continuous function $u \in \mathcal{F}_{loc}$, $A^{[u]}$ can be decomposed uniquely as

$$A_t^{[u]} = M_t^{[u]} + N_t^{[u]}, \tag{3.4}$$

where $M^{[u]}$ is a martingale additive functional locally of finite energy and $N^{[u]}$ is an additive functional locally of zero energy.

Let $u(y) = \sqrt{y}$. Since $u \in \mathcal{F}_{loc}$, we have the decomposition (3.4) with this u, so that it suffices to show that for each $R > 0$

$$M_{t\wedge\tau_R}^{[u]} = B_{t\wedge\tau_R}, \tag{3.5}$$

where τ_R stands for the hitting time to $R > 0$ of (Y_t, P_y).

For $n \geq 1$ define $u_n(y)$ by

$$u_n(0) = 0, \quad u_n'(y) = u'\left(\frac{1}{n} \vee y\right) = \frac{1}{2\sqrt{\frac{1}{n} \vee y}}.$$

Applying Itô formula, we obtain

$$M_t^{[u_n]} = \int_0^t \frac{\sqrt{Y_s}}{\sqrt{\frac{1}{n} \vee Y_s}} dB_s, \tag{3.6}$$

and

$$N_t^{[u_n]} = \int_0^t \mathrm{I}_{[\frac{1}{n},\infty)}(Y_s) \frac{b(Y_s) - 1}{2\sqrt{Y_s}} ds + \int_0^t \mathrm{I}_{[0,\frac{1}{n}]}(Y_s) \frac{\sqrt{n}}{2} b(Y_s) ds. \tag{3.7}$$

We choose a $\rho_R \in C_0^\infty([0,\infty))$ satisfying

$$\rho_R(y) = 1 \quad (0 \leq y \leq 1).$$

Then it holds that $u^R = u \cdot \rho_R$, $u_n^R = u_n \cdot \rho_R \in \mathcal{F}$ and

$$\lim_{n\to\infty} \mathcal{E}(u_n^R - u^R, u_n^R - u^R) = 0,$$

which implies

$$\lim_{n\to\infty} E_m(|M_{t\wedge\tau_R}^{[u_n^R]} - M_{t\wedge\tau_R}^{[u^R]}|^2) = 0.$$

On the other hand, it follows from (3.6) that

$$\lim_{n\to\infty} E_m(|M^{[u^R_n]}_{t\wedge\tau_R} - B_{t\wedge\tau_R}|^2) = 0.$$

Thus we obtain

$$M^{[u\cdot\rho_R]}_{t\wedge\tau_R} = B_{t\wedge\tau_R},$$

yielding (3.5).

For (ii), let any t_1', t_2' with $t_1 < t_1' < t_2' < t_2$ be fixed. Since $\min_{s\in[t_1',t_2']} Y_s > 0$, a simple use of Itô formula to (1.2) with $u(y) = \sqrt{y}$ gives

$$X_{t_2'} - X_{t_1'} = B_{t_2'} - B_{t_1'} + \int_{t_1'}^{t_2'} \frac{b(X_s^2) - 1}{X_s} ds,$$

so that it holds

$$\mathcal{N}_{t_2'} - \mathcal{N}_{t_1'} = \int_{t_1'}^{t_2'} \frac{b(X_s^2) - 1}{X_s} ds.$$

Since $\mathcal{N}_t$ is continuous, letting $t_1 \searrow t_1$, $t_2 \nearrow t_2$, we obtain (2.5).

3.2. Proof of Proposition 2.2

Let $L^W_t(y)$ be the local time of the reflected Brownian motion W_t on $[0,\infty)$ starting at $s(y)$, that is jointly continuous in $(t,x) \in [0,\infty)\times[0,\infty)$ and satisfies that

$$\int_0^t f(W_s)ds = \int_0^\infty f(x)L^W_t(x)dx \tag{3.8}$$

for every bounded Borel function $f(x)$ on $[0,\infty)$. Since $X_t = \sqrt{Y_t}$, by Lemma 3.2 and (3.8)

$$\begin{aligned}\int_0^t f(X_s)ds &= \int_0^t f\left(\sqrt{s^{-1}(W_{A_s})}\right) ds \\ &= \int_0^{A_t} f(\sqrt{s^{-1}(W_r)})\frac{dr}{a(s^{-1}(W_r))} \\ &= \int_0^\infty f(\sqrt{s^{-1}(z)})\frac{1}{a(s^{-1}(z))}L^W_{A_t}(z)dz \\ &= \int_0^\infty f(x)L^W_{A_t}(s(x^2))\frac{2xs'(x^2)}{a(x^2)}dx.\end{aligned}$$

Note that by (2.8)

$$\frac{2xs'(x^2)}{a(x^2)} = \frac{x^{\gamma-1}}{2}K(x),$$

so that, setting

$$L^X_t(x) = L^W_{A_t}(s(x^2))\frac{x^{\gamma-1}}{2}K(x), \quad \ell_t = L^W_{A_t}(0), \tag{3.9}$$

we see that (2.9) is valid. Furthermore, observe that

$$L_t^X(x) - \frac{x^{\gamma-1}}{2}K(x)\ell_t = (L_{A_t}^W(s(x^2)) - L_{A_t}^W(0))\frac{x^{\gamma-1}}{2}K(x)$$

and use a fact on the Brownian local time that for any $0 < \eta < 1/2$,

$$L_{A_t}^W(y) - \ell_t = o(y^\eta) \quad (y \to 0).$$

Then we obtain (2.10).

Finally (2.6) and (2.7) follow from (2.9) and (3.9).

3.3. Proof of Theorem 2.3

Recalling (3.7), we set

$$\mathcal{N}_t^{n,1} = \int_0^t \mathrm{I}_{[\frac{1}{\sqrt{n}},\infty)}(X_s)\frac{b(X_s^2)-1}{2X_s}ds, \tag{3.10}$$

and

$$\mathcal{N}_t^{n,2} = \int_0^t \mathrm{I}_{[0,\frac{1}{\sqrt{n}}]}(X_s)\frac{\sqrt{n}}{2}b(X_s^2)ds. \tag{3.11}$$

Note that the condition (2.11) can be expressed as

$$\int_0^1 |b(x^2)-1|x^{\gamma-2}K(y)dy < \infty.$$

Using this, Lemma 3.1 and (2.10) of Proposition 2.1, we have

$$\int_0^t \frac{|b(X_s^2)-1|}{X_s}ds = \int_0^\infty L_t^X(x)\frac{|b(x^2)-1|}{x}ds < \infty,$$

and

$$\lim_{n\to\infty}\mathcal{N}_t^{n,1} = \int_0^t \frac{b(X_s^2)-1}{2X_s}ds.$$

Moreover, noting that

$$\int_0^{\frac{1}{\sqrt{n}}} b(x^2)x^{\gamma-1}K(x)dx \sim n^{-\frac{\gamma}{2}}K\left(\frac{1}{\sqrt{n}}\right) \quad (n\to\infty),$$

we obtain

$$\lim_{n\to\infty}\mathcal{N}_t^{n,2} = \lim_{n\to\infty}\frac{1}{4}n^{\frac{(1-\gamma)}{2}}K\left(\frac{1}{\sqrt{n}}\right)\ell_t = c\,\ell_t.$$

3.4. Proof of Theorem 2.4

Note that the condition (2.14) can be expressed as

$$\int_0^1 |b(x^2)-1|x^{\gamma-2}K(x)dx = \infty.$$

Denote the total variation of $\{\mathcal{N}_s\}_{0\le s\le t}$ by $V_t(\mathcal{N})$. By Theorem 2.1 we see that

$$V_t(\mathcal{N}) \ge \int_0^t \mathrm{I}_{(X_s>0)} \frac{|b(X_s^2)-1|}{2X_s} ds.$$

Thus, by (2.10)

$$\begin{aligned}\int_0^t \mathrm{I}_{(X_s>0)} \frac{|b(X_s^2)-1|}{2X_s} ds &= \int_0^\infty \frac{|b(x^2)-1|}{2x} L_t^X(x)dx \\ &= \frac{1}{4}\int_0^{M_t} |b(x^2)-1| x^{\gamma-2} K(x)dx\, \ell_t + O(1) \\ &= \infty,\end{aligned}$$

which shows $V_t(\mathcal{N}) = \infty$.

3.5. Proof of Theorem 2.5

Note that $\mathcal{N}_t^n = \mathcal{N}_t^{[u_n]}$ of (3.7) satisfies

$$\begin{aligned}\mathcal{N}_t^n = \frac{\sqrt{n}}{2}\int_0^{\frac{1}{\sqrt{n}}} b(x^2)\tilde{L}_t^X(x)dx + \frac{\sqrt{n}}{4}\int_0^{\frac{1}{\sqrt{n}}} b(x^2)x^{\gamma-1}K(x)dx\,\ell_t \\ + \int_{\frac{1}{\sqrt{n}}}^{M_t} \frac{b(x^2)-1}{2x}\tilde{L}_t^X(x)dx + \frac{1}{4}\int_{\frac{1}{\sqrt{n}}}^{M_t} (b(x^2)-1)x^{\gamma-2}K(x)dx\,\ell_t.\end{aligned}$$

Using (2.10), we see that the first term vanishes and the third term converges as $n\to\infty$ to

$$\int_0^{M_t} \frac{b(x^2)-1}{2x}\tilde{L}_t^X(x)dx.$$

For the remaining two terms, noting that

$$(y^\gamma K(y))' = b(y^2)y^{\gamma-1}K(y), \quad (y^{\gamma-1}K(y))' = y^{\gamma-2}(b(y^2)-1)K(y),$$

we have

$$\begin{aligned}&\sqrt{n}\int_0^{\frac{1}{\sqrt{n}}} b(x^2)x^{\gamma-1}K(x)dx + \int_{\frac{1}{\sqrt{n}}}^{M_t} (b(x^2)-1)x^{\gamma-2}K(x)dx \\ &= M_t^{\gamma-1}K(M_t).\end{aligned}$$

Hence

$$\lim_{n\to\infty} \mathcal{N}_t^n = \int_0^{M_t} \frac{b(x^2)-1}{2x}\tilde{L}_t^X(x)dx + \frac{1}{4}M_t^{\gamma-1}K(M_t)\,\ell_t,$$

completing the proof of Theorem 2.5.

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