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On discontinuous Sturm-Liouville problems with transmission conditions

By

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Abstract

We consider a discontinuous Sturm-Liouville equation together with eigenparameter dependent boundary conditions and two supplementary transmission conditions at the point of discontinuity. By modifying some techniques of [2], [11] and [14] we extend and generalize some approach and results of classic regular Sturm-Liouville problems to the similar problems with discontinuities. In particular, we introduce a special Hilbert space formulation such a way that the considered problem can be interpreted as an eigenvalue problem of suitable self-adjoint operator, then we construct the Green function and resolvent operator and derive an asymptotic formulas for eigenvalues and normalized eigenfunctions.

1. Introduction

The Sturmian theory is an important aid in solving many problems of mathematical physics. Usually, the eigenvalue parameter appear linearly only in the differential equation of the classic Sturm-Liouville problems. However, in mathematical physics are encountered such problems, where eigenvalue parameter appear in both differential equation and boundary conditions (various physical applications can be found in [2]). There is a substantial literature on this type of problems (see, for example, [1], [2], [3], [8], [9], [14] and more recently [15], [16], [17] and corresponding references cited therein). In these works, only continuous problems have been investigated. The purpose of this paper is to extend some classic results of Sturmian theory to the discontinuous case, in which two supplementary transmission conditions added to the boundary conditions. In fact, we investigate both continuous and discontinuous cases (the cases $\delta = 1$ and $\delta \neq 1$ in below, respectively) in this study.

Let us consider the Sturm-Liouville equation

$$\tau u := -u'' + q(x)u = \lambda u \quad \text{for} \quad x \in [-1, 0) \cup (0, 1] \tag{1.1}$$

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(i.e. on $[-1,1]$ except one inner point $x=0$), where $q(x)$ is a real-valued, continuous in both $[-1,0)$ and $(0,1]$ and has finite limites $q(\pm 0)=\lim_{x\to\pm 0} q(x)$, together with standard boundary condition at $x=-1$

$$L_1 u := \alpha_1 u(-1) + \alpha_2 u'(-1) = 0, \tag{1.2}$$

transmission conditions at the point of discontinuity $x=0$

$$L_2 u := u(-0) - \delta u(+0) = 0, \tag{1.3}$$

$$L_3 u := u'(-0) - \delta u'(+0) = 0, \tag{1.4}$$

and eigenparameter dependent boundary condition at $x=1$

$$L_4(\lambda) u := \lambda(\beta_1' u(1) - \beta_2' u'(1)) + (\beta_1 u(1) - \beta_2 u'(1)) = 0, \tag{1.5}$$

where $\lambda \in C$ is a complex spectral parameter and all coefficients of the boundary and transmission conditions are real constants. Naturally, we assume that $|\alpha_1|+|\alpha_2|\neq 0$, $\delta\neq 0$, $|\beta_1'|+|\beta_2'|\neq 0$ and $|\beta_1|+|\beta_2|\neq 0$. Moreover, we shall assume that $\rho := \beta_1'\beta_2 - \beta_1\beta_2' > 0$.

Some special cases of this problem arises after an application of the method of separation of variables to the varied assortment of physical problems, such as, in heat and mass transfer problems (see, for example, [10]), in vibrating string problems when the string loaded additionally with point masses (see, for example, [10]), in thermal conduction problem for a thin laminated plate (see, for example, [12]).

Note that such properties as isomorphism, coerciveness with respect to the spectral parameter, completeness of root functions, distributions of eigenvalues of some discontinuous boundary value problems with transmission conditions and its applications to the corresponding initial-boundary value problems for parabolic equations have been investigated in [5], [6], [7], [12].

2. Operator-theoretic formulation in suitable Hilbert space

In this section, we introduce the special inner product in the Hilbert space $(L_2(-1,0)\oplus L_2(0,1))\oplus C$ and define a linear operator A in it such a way that the considered problem (1.1)–(1.5) can be interpreted as the eigenvalue problem of A. So, we define a new Hilbert space inner product on $H := (L_2(-1,0)\oplus L_2(0,1))\oplus C$ by

$$\langle F,G\rangle_H = \frac{1}{|\delta|}\int_{-1}^{0} f(x)\overline{g(x)}dx + |\delta|\int_{0}^{1} f(x)\overline{g(x)}dx + \frac{|\delta|}{\rho} f_1\overline{g_1}$$

for $F=\binom{f(x)}{f_1}$, $G=\binom{g(x)}{g_1}\in H$. For convenience we shall use the notations

$$R_1(u) := \beta_1 u(1) - \beta_2 u'(1),$$
$$R_1'(u) := \beta_1' u(1) - \beta_2' u'(1).$$

In this Hilbert space, we construct the operator $A : H \to H$ with domain

$$D(A) = \left\{ \begin{array}{c} F = \binom{f(x)}{f_1} \mid f(x), f'(x) \text{are absolutely continuous in} \\ [-1,0) \cup (0,1] \text{ and have finite one-hand sided limits} \\ f(\mp 0), f'(\mp 0), \text{respectively; } \tau f \in L_2(-1,0) \oplus L_2(0,1) \\ L_1 f = L_2 f = L_3 f = 0; f_1 = R_1'(f) \end{array} \right\} \tag{2.1}$$

and action law

$$AF = \begin{pmatrix} \tau f \\ -R_1(f) \end{pmatrix} \text{ with } F = \begin{pmatrix} f(x) \\ R_1'(f) \end{pmatrix} \in D(A). \tag{2.2}$$

Thus, we can pose the boundary value-transmission problem (1.1)–(1.5) as

$$AU = \lambda U, \quad U := \begin{pmatrix} u(x) \\ R_1'(u) \end{pmatrix} \in D(A) \tag{2.3}$$

in the Hilbert space H.

It is readily verified that the eigenvalues of the operator A coincide with those of the problem (1.1)–(1.5).

Theorem 2.1. *The operator A is symmetric.*

Proof. Let

$$F = \begin{pmatrix} f(x) \\ R_1'(f) \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} g(x) \\ R_1'(g) \end{pmatrix}$$

are arbitrary element of $D(A)$. By two partial integration we get

(2.4)
$$\begin{aligned} \langle AF, G\rangle_H - \langle F, AG\rangle_H = \frac{1}{|\delta|} W(f,\overline{g};-0) - \frac{1}{|\delta|} W(f,\overline{g};-1) + |\delta| W(f,\overline{g};1) \\ - |\delta| W(f,\overline{g};+0) + \frac{|\delta|}{\rho}(R_1'(f)R_1(\overline{g}) - R_1(f)R_1'(\overline{g})), \end{aligned}$$

where, as usual, $W(f,g;x)$ denotes the Wronskians of the functions f and g, i.e.

$$W(f,g;x) := f(x)g'(x) - f'(x)g(x).$$

Since $F, G \in D(A)$, the first components of these elements, i.e. f and g satisfy the boundary condition (1.2). From this fact, we easily have that

$$W(f,\overline{g};-1) = 0, \tag{2.5}$$

since α_1 and α_2 are real. Further, as f and g also satisfy both transmission conditions we get

$$W(f,\overline{g};-0) = \delta^2 W(f,\overline{g};+0). \tag{2.6}$$

Moreover, the direct calculations gives

$$R_1'(f)R_1(\overline{g}) - R_1(f)R_1'(\overline{g}) = -\rho W(f, \overline{g}; 1). \tag{2.7}$$

Now, substituting (2.5)–(2.7) in (2.4) gives

$$\langle AF, G\rangle_H = \langle F, AG\rangle_H \quad (F, G \in D(A)),$$

so A is symmetric. The proof is complete. □

Recalling that the eigenvalues of the problem (1.1)–(1.5) are coincide with the eigenvalues of A we have the next corollary.

Corollary 2.2. *All eigenvalues of the problem* (1.1)–(1.5) *are real.*

As all eigenvalues are real it is enough to investigate only the real-valued eigenfunctions. Taking this into account, we can now assume that all eigenfunctions of the problem (1.1)–(1.5) are real-valued.

3. Asymptotic representations of the basic solutions

Let us define two '*basic*' solutions

$$\phi(x, \lambda) = \begin{cases} \phi_1(x, \lambda), \ x \in [-1, 0) \\ \phi_2(x, \lambda), \ x \in (0, 1] \end{cases} \quad \text{and } \chi(x, \lambda) = \begin{cases} \chi_1(x, \lambda), \ x \in [-1, 0) \\ \chi_2(x, \lambda), \ x \in (0, 1] \end{cases}$$

of equation (1.1) by the following procedure.

At first consider the next initial-value problem:

$$-u'' + q(x)u = \lambda u, \ x \in [-1, 0], \tag{3.1}$$

$$u(-1) = \alpha_2, \tag{3.2}$$

$$u'(-1) = -\alpha_1. \tag{3.3}$$

By virtue of [11, Theorem1.5] this problem has a unique solution $u = \phi_1(x, \lambda)$, which is an entire function of $\lambda \in C$ for each fixed $x \in [-1, 0]$. Slightly modifying the method of [11, Theorem1.5] we can prove that the initial-value problem

$$-u'' + q(x)u = \lambda u, \ x \in [0, 1], \tag{3.4}$$

$$u(1) = \beta_2'\lambda + \beta_2, \tag{3.5}$$

$$u'(1) = \beta_1'\lambda + \beta_1 \tag{3.6}$$

has a unique solution $u = \chi_2(x, \lambda)$, which is an entire function of parameter λ for each fixed $x \in [0, 1]$. The other functions $\phi_2(x, \lambda)$ and $\chi_1(x, \lambda)$ can be defined in terms of $\phi_1(x, \lambda)$ and $\chi_2(x, \lambda)$, respectively. Applying the method used in the proof of [13, Theorem 2] we can prove that the initial-value problem

$$-u'' + q(x)u = \lambda u, \ x \in [0, 1], \tag{3.7}$$

$$u(0) = \frac{1}{\delta}\phi_1(0, \lambda), \tag{3.8}$$

$$u'(0) = \frac{1}{\delta}\phi_1'(0, \lambda) \tag{3.9}$$

has a unique solution $u = \phi_2(x, \lambda)$, which is an entire function of λ for each fixed $x \in [0, 1]$. Similarly, the initial-value problem

$$-u'' + q(x)u = \lambda u,\ x \in [-1, 0], \tag{3.10}$$

$$u(0) = \delta\chi_2(0, \lambda), \tag{3.11}$$

$$u'(0) = \delta\chi_2'(0, \lambda) \tag{3.12}$$

also has a unique solution $u = \chi_1(x, \lambda)$, which is an entire function of λ for each fixed $x \in [-1, 0]$.

By virtue of (3.2) and (3.3) the solution $\phi(x, \lambda)$ satisfies the first boundary condition (1.2). Moreover, by virtue of (3.8) and (3.9), $\phi(x, \lambda)$ also satisfies both transmission conditions (1.3) and (1.4). Similarly, by virtue of (3.5), (3.6), (3.11) and (3.12) the other solution $\chi(x, \lambda)$ satisfies the second boundary condition (1.5) and both transmission conditions (1.3) and (1.4).

It is well-known, from the ordinary linear differential equations theory, that each of the Wronskians $\omega_1(\lambda) = W(\phi_1(x, \lambda), \chi_1(x, \lambda))$ and

$$\omega_2(\lambda) = W(\phi_2(x, \lambda), \chi_2(x, \lambda))$$

are independent on x in $[-1, 0]$ and $[0, 1]$, respectively.

Lemma 3.1. *The equality $\omega_1(\lambda) = \delta^2\omega_2(\lambda)$ holds for each $\lambda \in C$.*

Proof. Since the above Wronskians are independent on x, then using (3.8), (3.9), (3.11) and (3.12) we have

$$\begin{aligned} \omega_1(\lambda) &= \phi_1(0, \lambda)\chi_1'(0, \lambda) - \phi_1'(0, \lambda)\chi_1(0, \lambda) \\ &= (\delta\phi_2(0, \lambda)) \cdot (\delta\chi_2'(0, \lambda)) - (\delta\phi_2'(0, \lambda)) \cdot (\delta\chi_2(0, \lambda)) \\ &= \delta^2\omega_2(\lambda). \end{aligned} \tag{3.13}$$

□

Corollary 3.2. *The zeros of $\omega_1(\lambda)$ and $\omega_2(\lambda)$ are coincide.*

Taking the Lemma 3.1 into account we denote both $\omega_1(\lambda)$ and $\delta^2\omega_2(\lambda)$ by $\omega(\lambda)$.

Recalling the definitions of $\phi_i(x, \lambda)$ and $\chi_i(x, \lambda)$ we conclude the next corollary.

Corollary 3.3. *The function $\omega(\lambda)$ is an entire function.*

Theorem 3.4. *The eigenvalues of the problem* (1.1)–(1.5) *are coincide with the zeros of the function $\omega(\lambda)$.*

Proof. Let $\omega(\lambda_0) = 0$. Then $W(\phi_1(x, \lambda_0), \chi_1(x, \lambda_0)) = 0$ for all $x \in [-1, 0]$. Consequently, the functions $\phi_1(x, \lambda_0)$ and $\chi_1(x, \lambda_0)$ are linearly dependent, i.e.

$$\chi_1(x, \lambda_0) = k_1\phi_1(x, \lambda_0),\ x \in [-1, 0]$$

for some $k_1 \neq 0$. By using (3.2) and (3.3), from this equality we have

$$\begin{aligned}\alpha_1\chi(-1,\lambda_0)+\alpha_2\chi'(-1,\lambda_0) &= \alpha_1\chi_1(-1,\lambda_0)+\alpha_2\chi_1'(-1,\lambda_0)\\ &= k_1(\alpha_1\phi_1(-1,\lambda_0)+\alpha_2\phi_1'(-1,\lambda_0))\\ &= k_1(\alpha_1\alpha_2+\alpha_2(-\alpha_1))=0,\end{aligned}$$

so $\chi(x,\lambda_0)$ satisfies the first boundary condition (1.2). Recalling that the solution $\chi(x,\lambda_0)$ satisfies also the other boundary condition (1.5) and both transmission conditions (1.3) and (1.4), we conclude that $\chi(x,\lambda_0)$ is an eigenfunction of the problem (1.1)–(1.5), i.e. λ_0 is an eigenvalue. Thus, each zero of $\omega(\lambda)$ is an eigenvalue.

Now let λ_0 be an eigenvalue and $u_0(x)$ be an any eigenfunction corresponding to this eigenvalue. Suppose, if possible, that $\omega(\lambda_0) \neq 0$. Whence $W(\phi_1(x,\lambda_0), \chi_1(x,\lambda_0)) \neq 0$ and $W(\phi_2(x,\lambda_0), \chi_2(x,\lambda_0)) \neq 0$. So, by virtue of well-known properties of Wronskians, it follows that each of the pairs $\phi_1(x,\lambda_0)$, $\chi_1(x,\lambda_0)$ and $\phi_2(x,\lambda_0)$, $\chi_2(x,\lambda_0)$ are linearly independent. Therefore the solution $u_0(x)$ of Equation (1.1) may be represented in the form

$$u_0(x)=\begin{cases}c_1\phi_1(x,\lambda_0)+c_2\chi_1(x,\lambda_0), & x\in[-1,0),\\ c_3\phi_2(x,\lambda_0)+c_4\chi_2(x,\lambda_0), & x\in(0,1],\end{cases}$$

where at least one of the constants c_1, c_2, c_3 and c_4 is not zero. Considering the true equalities

$$\begin{aligned}L_\nu(u_0(x)) &= c_1L_\nu(\phi_1(x,\lambda_0))+c_2L_\nu(\chi_1(x,\lambda_0))\\ &\quad+c_3L_\nu(\phi_2(x,\lambda_0))+c_4L_\nu(\chi_2(x,\lambda_0))=0,\ \nu=1,2,3,4\end{aligned} \tag{3.14}$$

as the homogeneous system of linear equations of the variables c_1, c_2, c_3 and c_4, and taking (3.8), (3.9), (3.11) and (3.12) into account it follows that the determinant of this system is equal to

$$\begin{aligned}\begin{vmatrix}0 & \omega_1(\lambda_0) & 0 & 0\\ \phi_1(0,\lambda_0) & \chi_1(0,\lambda_0) & -\delta\phi_2(0,\lambda_0) & -\delta\chi_2(0,\lambda_0)\\ \phi_1'(0,\lambda_0) & \chi_1'(0,\lambda_0) & -\delta\phi_2'(0,\lambda_0) & -\delta\chi_2'(0,\lambda_0)\\ 0 & 0 & \omega_2(\lambda_0) & 0\end{vmatrix} &= -\delta^2\omega_1(\lambda_0)\omega_2^2(\lambda_0)\\ &= -\frac{1}{\delta^2}\omega^3(\lambda_0)\end{aligned}$$

and therefore it is not equal to zero by assumption. Consequently, this homogeneous system of linear equations has the only trivial solution $(c_1, c_2, c_3, c_4) = (0, 0, 0, 0)$. Thus we get contradiction, which completes the proof. □

Theorem 3.5. *Let* $\lambda = s^2$, $\operatorname{Im} s = t$. *Then, the following asymptotic equalities hold as* $|\lambda| \to \infty$:

(1) *In the case* $\alpha_2 \neq 0$

$$\phi_1^{(k)}(x,\lambda)=\alpha_2\frac{d^k}{dx^k}\cos[s(x+1)]+O\left(\frac{1}{|s|^{1-k}}e^{|t|(x+1)}\right), \tag{3.15}$$

$$\phi_2^{(k)}(x,\lambda)=\frac{\alpha_2}{\delta}\frac{d^k}{dx^k}\cos[s(x+1)]+O\left(\frac{1}{|s|^{1-k}}e^{|t|(x+1)}\right), \tag{3.16}$$

for $k = 0$ *and* $k = 1$.

(2) *In the case* $\alpha_2 = 0$

$$\phi_1^{(k)}(x,\lambda) = -\frac{\alpha_1}{s}\frac{d^k}{dx^k}\sin[s(x+1)] + O\left(\frac{1}{|s|^{2-k}}e^{|t|(x+1)}\right), \tag{3.17}$$

$$\phi_2^{(k)}(x,\lambda) = -\frac{\alpha_1}{\delta s}\frac{d^k}{dx^k}\sin[s(x+1)] + O\left(\frac{1}{|s|^{2-k}}e^{|t|(x+1)}\right) \tag{3.18}$$

for $k = 0$ *and* $k = 1$.

Moreover, each of asymptotic equalities hold uniformly for x.

Proof. The above asymptotic formulas for $\phi_1(x,\lambda)$ have been found in [11, Lemma 1.7]. But the similar formulas for the solution $\phi_2(x,\lambda)$ need individual considerations, since this solution is defined by the initial conditions having special non-standard forms.

The initial-value problem (3.7), (3.8), (3.9) can be transformed into an equivalent integral equation

$$\begin{aligned} u(x) &= \delta^{-1}\phi_1(0,\lambda)\cos\sqrt{\lambda}x + \frac{\delta^{-1}}{\sqrt{\lambda}}\phi_1'(0,\lambda)\sin\sqrt{\lambda}x \\ &+ \frac{1}{\sqrt{\lambda}}\int_0^x \sin[\sqrt{\lambda}(x-y)]q(y)u(y)dy. \end{aligned} \tag{3.19}$$

Let $\alpha_2 \neq 0$. Substituting (3.15) in (3.19) we have

$$\begin{aligned} \phi_2(x,\lambda) &= \frac{\alpha_2}{\delta}\cos\sqrt{\lambda}(x+1) + \frac{1}{\sqrt{\lambda}}\int_0^x \sin[\sqrt{\lambda}(x-y)]q(y)\phi_2(y,\lambda)dy \\ &+ O\left(\frac{1}{\sqrt{\lambda}}e^{|t|(x+1)}\right). \end{aligned} \tag{3.20}$$

Multiplying by $e^{-|t|(x+1)}$ and letting $F(x,\lambda) = e^{-|t|(x+1)}\phi_2(x,\lambda)$, we have the next 'asymptotic integral equation'

$$\begin{aligned} F(x,\lambda) &= \frac{\alpha_2 e^{-|t|(x+1)}}{\delta}\cos\sqrt{\lambda}(x+1) \\ &+ \frac{1}{\sqrt{\lambda}}\int_0^x \sin[\sqrt{\lambda}(x-y)]q(y)e^{-|t|(x-y)}F(y,\lambda)dy + O\left(\frac{1}{\sqrt{\lambda}}\right). \end{aligned}$$

Letting $M(\lambda) = \max_{x\in[0,1]}|F(x,\lambda)|$ from the last equation we derive that

$$M(\lambda) \leq M_0\left(|\alpha_2\delta^{-1}| + \frac{1}{\sqrt{|\lambda|}}\right)$$

for some $M_0 > 0$. Consequently, $M(\lambda) = O(1)$ as $|\lambda| \to \infty$, so

$$\phi_2(x, \lambda) = O(e^{|t|(x+1)}) \ as \ |\lambda| \to \infty.$$

Substituting this asymptotic equality in the integral term of the (3.20) gives (3.16) for the case $k = 0$. The proof of (3.16) for the case $k = 1$ can be obtained at once by differentiating (3.19) and then following the same procedure as in the case $k = 0$. The proof of (3.18) is similar to that of (3.16) and hence omitted. □

Theorem 3.6. *Let $\lambda = s^2$, $\operatorname{Im} s = t$. Then, the following asymptotic formulas hold for the eigenvalues of the boundary-value-transmission problem* (1.1)–(1.5):

Case 1: $\beta_2' \neq 0, \quad \alpha_2 \neq 0$

$$s_n = \frac{1}{2}\pi(n-1) + O\left(\frac{1}{n}\right), \tag{3.21}$$

Case 2: $\beta_2' \neq 0, \quad \alpha_2 = 0$

$$s_n = \frac{1}{2}\pi\left(n - \frac{1}{2}\right) + O\left(\frac{1}{n}\right), \tag{3.22}$$

Case 3: $\beta_2' = 0, \quad \alpha_2 \neq 0$

$$s_n = \frac{1}{2}\pi\left(n - \frac{1}{2}\right) + O\left(\frac{1}{n}\right), \tag{3.23}$$

Case 4: $\beta_2' = 0, \quad \alpha_2 = 0$

$$s_n = \frac{1}{2}\pi n + O\left(\frac{1}{n}\right). \tag{3.24}$$

Proof. Let us consider Case 1 only. Writing

$$\omega_2(\lambda) = \phi_2(x, \lambda)\chi_2'(x, \lambda) - \phi_2'(x, \lambda)\chi_2(x, \lambda)$$

for $x = 1$ and then using $\chi_2(1, \lambda) = \beta_2'\lambda + \beta_2$, $\chi_2'(1, \lambda) = \beta_1'\lambda + \beta_1$ as given by (3.5) and (3.6), respectively, we have the following representation for $\omega_2(\lambda)$:

$$\omega_2(\lambda) = (\beta_1'\lambda + \beta_1)\phi_2(1, \lambda) - (\beta_2'\lambda + \beta_2)\phi_2'(1, \lambda). \tag{3.25}$$

Now writing $x = 1$ in (3.16) and then substituting in (3.25) we derive that

$$\omega_2(\lambda) = \delta^{-1}\beta_2'\alpha_2 s^3 \sin(2\sqrt{\lambda}) + O(|s|^2 e^{2|t|}). \tag{3.26}$$

By applying well-known Rouche's Theorem (which assert that if $f(z)$ and $g(z)$ are analytic inside and on a closed contour Γ, and $|g(z)| < |f(z)|$ on Γ, then $f(z)$ and $f(z) + g(z)$ have the same number zeros inside Γ, provided that each zeros are counted according to their multiplicity) on a sufficiently large contour, it follows that $\omega_2(\lambda)$ has the same number of zeros inside the contour as the leading term in (3.26). Hence, if $\lambda_0 < \lambda_1 < \lambda_2 < \dots$, are the zeros of $\omega_2(\lambda)$ and $s_n^2 = \lambda_n$, we have

$$s_n = \frac{\pi}{2}(n-1) + \delta_n, \tag{3.27}$$

where $|\delta_n| < \frac{\pi}{4}$, for sufficiently large n. By substituting (3.27) into (3.26) we have

$$\delta_n = O\left(\frac{1}{n}\right),$$

so the proof completes for Case 1. The proofs for the other cases are similar. $\square$

Theorem 3.7. *The following asymptotic formulas hold for the eigenfunctions*

$$\phi_{\lambda_n}(x) = \begin{cases} \phi_1(x,\lambda_n), & x \in [-1,0), \\ \phi_2(x,\lambda_n), & x \in (0,1] \end{cases}$$

of the problem (1.1)–(1.5):

Case 1: $\beta_2' \neq 0, \quad \alpha_2 \neq 0$

$$\phi_{\lambda_n}(x) = \begin{cases} \alpha_2 \cos\left(\frac{1}{2}\pi(n-1)(x+1)\right), & x \in [-1,0) \\ \alpha_2 \frac{1}{\delta} \cos\left(\frac{1}{2}\pi(n-1)(x+1)\right), & x \in (0,1] \end{cases} + O\left(\frac{1}{n}\right), \tag{3.28}$$

Case 2: $\beta_2' \neq 0, \quad \alpha_2 = 0$

$$\begin{aligned} &\phi_{\lambda_n}(x) \\ &= \begin{cases} -2\alpha_1 \frac{1}{\pi(n-1/2)} \sin\left(\frac{1}{2}\pi\left(n-\frac{1}{2}\right)(x+1)\right), & x \in [-1,0) \\ -2\alpha_1 \frac{1}{\delta}\frac{1}{\pi(n-1/2)} \sin\left(\frac{1}{2}\pi\left(n-\frac{1}{2}\right)(x+1)\right), & x \in (0,1] \end{cases} + O\left(\frac{1}{n^2}\right), \end{aligned} \tag{3.29}$$

Case 3: $\beta_2' = 0, \quad \alpha_2 \neq 0$

$$\phi_{\lambda_n}(x) = \begin{cases} \alpha_2 \cos\left(\frac{1}{2}\pi\left(n-\frac{1}{2}\right)(x+1)\right), & x \in [-1,0) \\ \alpha_2 \frac{1}{\delta} \cos\left(\frac{1}{2}\pi\left(n-\frac{1}{2}\right)(x+1)\right), & x \in (0,1] \end{cases} + O\left(\frac{1}{n}\right), \tag{3.30}$$

Case 4: $\beta_2' = 0, \quad \alpha_2 = 0$

$$\phi_{\lambda_n}(x) = \begin{cases} -2\alpha_1 \frac{1}{\pi n} \sin\left(\frac{1}{2}\pi n(x+1)\right), & x \in [-1,0) \\ -2\alpha_1 \frac{1}{\delta}\frac{1}{\pi n} \sin\left(\frac{1}{2}\pi n(x+1)\right), & x \in (0,1] \end{cases} + O\left(\frac{1}{n^2}\right). \tag{3.31}$$

All this asymptotic formulas hold uniformly for x.

Proof. Let us consider Case 1 only. Substituting (3.16) into the integral term of (3.20), it is easy to see that

$$\int_0^x \sin[\sqrt{\lambda}(x-y)]q(y)\phi_2(y,\lambda)dy = O(e^{|t|(x+1)}). \tag{3.32}$$

Substituting into (3.20) we have

$$\phi_2(x,\lambda) = \frac{\alpha_2}{\delta}\cos\sqrt{\lambda}(x+1) + O\left(\frac{1}{\sqrt{\lambda}}e^{|t|(x+1)}\right). \tag{3.33}$$

We already know that all eigenvalues are real. Further, putting $\lambda = -R$, $R > 0$ in (3.26) it follows that $\omega(-R) \to \infty$ as $R \to +\infty$, so $\omega(-R) \neq 0$ for sufficiently large $R > 0$. Consequently, the set of eigenvalues is bounded below. Now, writing $\sqrt{\lambda} = s_n$ in (3.33) we obtain

$$\phi_2(x,\lambda_n) = \frac{\alpha_2}{\delta}\cos[s_n(x+1)] + O\left(\frac{1}{s_n}\right),$$

since $t_n = \operatorname{Im} s_n = 0$ for sufficiently large n. After some routine calculations we easily obtain that

$$\cos[s_n(x+1)] = \cos\left(\frac{1}{2}\pi(n-1)(x+1)\right) + O\left(\frac{1}{n}\right).$$

Consequently,

$$\phi_2(x,\lambda_n) = \frac{\alpha_2}{\delta}\cos\left(\frac{1}{2}\pi(n-1)(x+1)\right) + O\left(\frac{1}{n}\right).$$

Similarly we can find that

$$\phi_1(x,\lambda_n) = \alpha_2\cos\left(\frac{1}{2}\pi(n-1)(x+1)\right) + O\left(\frac{1}{n}\right).$$

Since

$$\phi_{\lambda_n}(x) = \begin{cases}\phi_1(x,\lambda_n), & x \in [-1,0),\\ \phi_2(x,\lambda_n), & x \in (0,1],\end{cases}$$

the proof for Case 1 is completed. The proofs for the other cases are similar. □

4. Asymptotic formulas for normalized eigenfunctions

It is evident that the two-component vectors

$$\Phi_n := \begin{pmatrix}\phi_{\lambda_n}(x)\\ R_1'(\phi_{\lambda_n})\end{pmatrix},\ n = 0,1,2,\ldots \tag{4.1}$$

are the eigenelements of the operator A corresponding to the eigenvalue λ_n. For $n \neq m$,

$$\langle \Phi_n, \Phi_m \rangle_H = 0, \; n, m = 0, 1, 2, \ldots, \tag{4.2}$$

since A is symmetric. Denoting

$$\psi_n := \frac{\phi_{\lambda_n}(x)}{\|\Phi_n\|_H}, \tag{4.3}$$

it is easily seen that the eigenelements

$$\Psi_n := \begin{pmatrix} \psi_{\lambda_n}(x) \\ R_1'(\psi_{\lambda_n}) \end{pmatrix}, \; n, m = 0, 1, 2, \ldots \tag{4.4}$$

are orthonormal. That is,

$$x = 0 \quad \text{and} \quad \langle \Psi_n, \Psi_m \rangle_H = \delta_{nm},$$

where δ_{nm} is the kronecker delta.

Lemma 4.1. *The following asymptotic equalities hold*:
(1) *in case* $\alpha_2 \neq 0$

$$R_1'(\phi_{\lambda_n}) = O\left(\frac{1}{n}\right), \tag{4.5}$$

(2) *in case* $x = 1$

$$R_1'(\phi_{\lambda_n}) = O\left(\frac{1}{n^2}\right). \tag{4.6}$$

Proof. It follows from the equality $\omega_2(\lambda_n) = 0$ that

$$\lambda_n R_1'(\phi_{2\lambda_n}) + R_1(\phi_{2\lambda_n}) = 0. \tag{4.7}$$

(1) Let $\alpha_2 \neq 0$. Then from the formula (3.16) we get

$$R_1(\phi_{2\lambda_n}) = \beta_1 \phi_{2\lambda_n}(1) - \beta_2 \phi_{2\lambda_n}'(1) = \beta_1 O(1) - \beta_2 O(|s_n|).$$

Now applying Theorem 3.6 we have

$$R_1(\phi_{2\lambda_n}) = O(n). \tag{4.8}$$

Substituting (4.8) into (4.7), and taking Theorem 3.6 into account, we get

$$R_1'(\phi_{2\lambda_n}) = -\frac{1}{\lambda_n} R_1(\phi_{2\lambda_n}) = O\left(\frac{1}{n}\right).$$

Now let $\alpha_2 = 0$. By using Theorem 3.6 we obtain

$$\begin{aligned} R_1(\phi_{2\lambda_n}) &= \beta_1 \phi_{2\lambda_n}(1) - \beta_2 \phi'_{2\lambda_n}(1) \\ &= \beta_1 O(|s_n|^{-1}) - \beta_2 O(1) = \beta_1 O\left(\frac{1}{n}\right) - \beta_2 O(1) \\ &= O(1). \end{aligned}$$

Taking into account that $\lambda_n \sim \left(\frac{\pi}{2}n\right)^2$ and using (4.7) we have

$$R'_1(\phi_{2\lambda_n}) = -\frac{1}{\lambda_n} R_1(\phi_{2\lambda_n}) = O\left(\frac{1}{n^2}\right).$$

The proof is complete. □

Theorem 4.2. *Let Φ_n be defined as in* (4.1). *Then the following asymptotic formulas hold for the norms $\|\Phi_n\|_H$ of the eigenelements Φ_n:*

Case 1: *If $\beta'_2 \neq 0$ and $\alpha_2 \neq 0$, then*

$$\|\Phi_n\|_H = \frac{|\alpha_2|}{\sqrt{|\delta|}} + O\left(\frac{1}{n}\right), \tag{4.9}$$

Case 2: *If $\beta'_2 \neq 0$ and $\alpha_2 = 0$, then*

$$\|\Phi_n\|_H = \frac{2|\alpha_1|}{\sqrt{|\delta|}} \frac{1}{\pi(n - 1/2)} + O\left(\frac{1}{n^2}\right), \tag{4.10}$$

Case 3: *If $\beta'_2 = 0$ and $\alpha_2 \neq 0$, then*

$$\|\Phi_n\|_H = \frac{|\alpha_2|}{\sqrt{|\delta|}} + O\left(\frac{1}{n}\right), \tag{4.11}$$

Case 4: *If $\beta'_2 = 0$ and $\alpha_2 = 0$, then*

$$\|\Phi_n\|_H = \frac{2|\alpha_1|}{\sqrt{|\delta|}} \frac{1}{\pi n} + O\left(\frac{1}{n^2}\right). \tag{4.12}$$

Proof. Let $\beta'_2 \neq 0$ and $\alpha_2 \neq 0$. In this case, using (3.28) we have

$$\begin{aligned} \int_{-1}^{0} (\phi_{\lambda_n}(x))^2 dx &= \alpha_2^2 \int_{-1}^{0} \left[\cos\left(\frac{1}{2}\pi(n-1)(x+1)\right) + O\left(\frac{1}{n}\right)\right]^2 dx \\ &= \alpha_2^2 \int_{-1}^{0} \cos^2\left(\frac{1}{2}\pi(n-1)(x+1)\right) dx + O\left(\frac{1}{n}\right) \\ &= \frac{\alpha_2^2}{2} + O\left(\frac{1}{n}\right). \end{aligned} \tag{4.13}$$

Similarly, we have

$$\int_0^1 (\phi_{\lambda_n}(x))^2 dx = \frac{\alpha_2^2}{2\delta^2} + O\left(\frac{1}{n}\right). \tag{4.14}$$

Using (4.5), (4.13) and (4.14) we get

$$\begin{aligned} \|\Phi_n\|_H^2 &= \frac{1}{|\delta|}\int_{-1}^0 (\phi_{\lambda_n}(x))^2 dx + |\delta| \int_0^1 (\phi_{\lambda_n}(x))^2 dx + \frac{|\delta|}{\rho}(R_1'(\phi_{\lambda_n}))^2 \\ &= \left(\frac{\alpha_2^2}{2|\delta|} + O\left(\frac{1}{n}\right)\right) + \left(\frac{\alpha_2^2}{2|\delta|} + O\left(\frac{1}{n}\right)\right) + \frac{|\delta|}{\rho} O\left(\frac{1}{n^2}\right) \\ &= \frac{\alpha_2^2}{|\delta|} + O\left(\frac{1}{n}\right). \end{aligned} \tag{4.15}$$

Consequently,

$$\|\Phi_n\|_H = \sqrt{\frac{\alpha_2^2}{|\delta|} + O\left(\frac{1}{n}\right)} = \frac{|\alpha_2|}{\sqrt{|\delta|}} + O\left(\frac{1}{n}\right),$$

which proves the formula (4.9).

Now let $\beta_2' \neq 0$ and $\alpha_2 = 0$. In this case from (3.29) we get

$$\begin{aligned} \|\Phi_n\|_H^2 &= \frac{1}{|\delta|}\int_{-1}^0 (\phi_{\lambda_n}(x))^2 dx + |\delta| \int_0^1 (\phi_{\lambda_n}(x))^2 dx + \frac{|\delta|}{\rho}(R_1'(\phi_{\lambda_n}))^2 \\ &= \frac{1}{|\delta|}\left\{\left(-2\alpha_1 \frac{1}{\pi(n-1/2)}\right)^2 \cdot \frac{1}{2} + O\left(\frac{1}{n^3}\right)\right\} \\ &\quad + |\delta| \left\{\left(-2\alpha_1 \frac{1}{\delta}\frac{1}{\pi(n-1/2)}\right)^2 \cdot \frac{1}{2} + O\left(\frac{1}{n^3}\right)\right\} + O\left(\frac{1}{n^4}\right) \\ &= \frac{4\alpha_1^2}{|\delta|}\frac{1}{(\pi(n-1/2))^2} + O\left(\frac{1}{n^3}\right). \end{aligned} \tag{4.16}$$

From this it follows that

$$\|\Phi_n\|_H = \frac{2|\alpha_1|}{\sqrt{|\delta|}}\frac{1}{\pi(n-1/2)} + O\left(\frac{1}{n}\right),$$

which proves the formula (4.10).

The proofs for the other cases are similar. □

Theorem 4.3. *The first components of the normalized eigenelements (4.4) have the following asymptotic representation as $n \to \infty$:*

Case 1: *If $\beta_2' \neq 0$ and $\alpha_2 \neq 0$, then*

(4.17)
$$\psi_n(x) = \begin{cases} \operatorname{sgn}(\alpha_2)\sqrt{|\delta|}\cos\left(\frac{1}{2}\pi(n-1)(x+1)\right) + O\left(\frac{1}{n}\right) & \text{for } x \in [-1,0), \\ \operatorname{sgn}\left(\frac{\alpha_2}{\delta}\right)\frac{1}{\sqrt{|\delta|}}\cos\left(\frac{1}{2}\pi(n-1)(x+1)\right) + O\left(\frac{1}{n}\right) & \text{for } x \in (0,1], \end{cases}$$

Case 2: *If* $\beta_2' \neq 0$ *and* $\alpha_2 = 0$, *then*

(4.18)
$$\psi_n(x) = \begin{cases} \operatorname{sgn}(-\alpha_1)\sqrt{|\delta|}\sin\left(\frac{1}{2}\pi\left(n-\frac{1}{2}\right)(x+1)\right) + O\left(\frac{1}{n}\right) & \text{for } x \in [-1,0), \\ \operatorname{sgn}\left(\frac{-\alpha_1}{\delta}\right)\frac{1}{\sqrt{|\delta|}}\sin\left(\frac{1}{2}\pi(n-\frac{1}{2})(x+1)\right) + O\left(\frac{1}{n}\right) & \text{for } x \in (0,1], \end{cases}$$

Case 3: *If* $\beta_2' = 0$ *and* $\alpha_2 \neq 0$, *then*

$$\frac{1}{|\delta|}\int_{-1}^{0} f(x)\overline{g(x)}dx + |\delta|\int_{0}^{1} f(x)\overline{g(x)}dx + \frac{|\delta|}{\rho}R_1'(f)\overline{g_1} = 0, \tag{4.19}$$

Case 4: *If* $\beta_2' = 0$ *and* $\alpha_2 \neq 0$, *then*

(4.20)
$$\psi_n(x) = \begin{cases} \operatorname{sgn}(-\alpha_1)\sqrt{|\delta|}\sin\left(\frac{1}{2}\pi n(x+1)\right) + O\left(\frac{1}{n}\right) & \text{for } \quad x \in [-1,0), \\ \operatorname{sgn}\left(-\frac{\alpha_1}{\delta}\right)\frac{1}{\sqrt{|\delta|}}\sin\left(\frac{1}{2}\pi n(x+1)\right) + O\left(\frac{1}{n}\right) & \text{for } \quad x \in (0,1]. \end{cases}$$

Each of this asymptotic equalities hold uniformly for x. (Here, as usual, sgn denotes the sign function)

Proof. Let $\beta_2' \neq 0$ and $\alpha_2 \neq 0$. In this case, from (4.9) it follows that

$$\frac{1}{\|\Phi_n\|_H} = \frac{\sqrt{|\delta|}}{|\alpha_2|} + O\left(\frac{1}{n}\right). \tag{4.21}$$

Putting (3.28) and (4.21) into (4.3) we find the required asymptotic formula (4.17). Similarly, we can derive the other formulas (4.18)–(4.20). □

5. Green function, resolvent operator and self-adjointness of the problem

Let $A : H \to H$ be defined by (2.1) and (2.2), and let λ not be an eigenvalue of A. For finding the resolvent operator $R(\lambda, A) = (\lambda I - A)^{-1}$ consider the operator equation

$$(\lambda I - A)U = F \tag{5.1}$$

for $F = \binom{f(x)}{f_1} \in H$. This operator equation is equivalent to the inhomogeneous differential equation

$$u'' + (\lambda - q(x))u = f(x) \text{ for } x \in [-1,0) \cup (0,1] \tag{5.2}$$

subject to inhomogeneous boundary conditions

$$\alpha_1 u(-1) + \alpha_2 u'(-1) = 0, \tag{5.3}$$

$$\lambda(\beta_1' u(1) - \beta_2' u'(1)) + (\beta_1 u(1) - \beta_2 u'(1)) = f_1 \tag{5.4}$$

and homogeneous transmission conditions

$$u(-0) - \delta u(+0) = 0, \tag{5.5}$$

$$u'(-0) - \delta u'(+0) = 0. \tag{5.6}$$

By applying the same techniques as in our previous paper [4] we can prove that the problem (5.2)–(5.6) has a unique solution $u(x,\lambda)$, which can be represented as

(5.7)

$$u(x,\lambda) = \begin{cases} \dfrac{\chi_1(x,\lambda)}{\omega_1(\lambda)} \displaystyle\int_{-1}^{x} \phi_1(y,\lambda) f(y)dy + \dfrac{\phi_1(x,\lambda)}{\omega_1(\lambda)} \left(\displaystyle\int_{x}^{0} \chi_1(y,\lambda) f(y)dy \right. \\ \qquad \left. + \delta^2 \displaystyle\int_{0}^{1} \chi_2(y,\lambda) f(y)dy + \delta^2 f_1 \right) \text{ for } x \in [-1,0), \\ \dfrac{\chi_2(x,\lambda)}{\omega_2(\lambda)} \left(\dfrac{1}{\delta^2} \displaystyle\int_{-1}^{0} \phi_1(y,\lambda) f(y)dy + \displaystyle\int_{0}^{x} \phi_2(y,\lambda) f(y)dy \right) \\ \qquad + \dfrac{\phi_2(x,\lambda)}{\omega_2(\lambda)} \left(\displaystyle\int_{x}^{1} \chi_2(y,\lambda) f(y)dy + f_1 \right) \text{ for } x \in (0,1]. \end{cases}$$

Denoting

$$G(x,y,\lambda) = \begin{cases} |\delta| \dfrac{\chi(x,\lambda)\phi(y,\lambda)}{\omega(\lambda)} \text{ for } -1 \le y \le x \le 1, \\ |\delta| \dfrac{\phi(x,\lambda)\chi(y,\lambda)}{\omega(\lambda)} \text{ for } -1 \le x \le y \le 1, \end{cases} \tag{5.8}$$

where $x \neq 0$ and $y \neq 0$, the formula (5.7) reduced to

$$u(x,\lambda) = \frac{1}{|\delta|} \int_{-1}^{0} G(x,y,\lambda) f(y)dy + |\delta| \int_{0}^{1} G(x,y,\lambda) f(y)dy + \delta^2 f_1 \frac{\phi(x,\lambda)}{\omega(\lambda)}. \tag{5.9}$$

On the other hand, by applying the functional R_1' to the Green function with respect to the variable y and recalling that $\chi(x,\lambda)$ satisfies the initial conditions (3.5) and (3.6) we have

$$
\begin{aligned}
R_1'(G(x,\bullet;\lambda)) &= \beta_1' G(x,1;\lambda) - \beta_2' \frac{\partial G(x,1;\lambda)}{\partial y} \\
&= |\delta| \frac{\phi(x,\lambda)}{\omega(\lambda)} (\beta_1' \chi(1,\lambda) - \beta_2' \chi'(1,\lambda)) \\
&= |\delta| \frac{\phi(x,\lambda)}{\omega(\lambda)} (\beta_1' (\beta_2' \lambda + \beta_2) - \beta_2' (\beta_1' \lambda + \beta_1)) \\
&= |\delta| \rho \frac{\phi(x,\lambda)}{\omega(\lambda)}.
\end{aligned} \tag{5.10}
$$

Substituting this into (5.9) gives

$$
\begin{aligned}
u(x,\lambda) = \frac{1}{|\delta|} \int_{-1}^{0} G(x,y,\lambda) f(y) dy + |\delta| \int_{0}^{1} G(x,y,\lambda) f(y) dy \\
+ \frac{|\delta|}{\rho} R_1'(G(x,\bullet,\lambda)) f_1.
\end{aligned} \tag{5.11}
$$

Now introducing

$$
G_{x,\lambda} = \begin{pmatrix} G(x,\bullet,\lambda) \\ R_1'(G(x,\bullet,\lambda)) \end{pmatrix} \tag{5.12}
$$

which we call the Green element of the problem (5.2)–(5.6), the last formula (5.11) takes the form

$$
u(x,\lambda) = \langle G_{x,\lambda}, \overline{F} \rangle, \tag{5.13}
$$

where by $\overline{F}$ we mean

$$
\overline{F} = \begin{pmatrix} \overline{f(x)} \\ \overline{f_1} \end{pmatrix}.
$$

Now we can find the resolvent operator of A in terms of Green element $G_{x,\lambda}$.

As the function $u(x,\lambda)$ defined by (5.11) is the solution of the inhomogeneous boundary-transmission problem (5.2)–(5.6) which is equivalent to the operator equation (5.1) we have

$$
R(\lambda, A) F = \begin{pmatrix} u(x,\lambda) \\ R_1'(u(\bullet,\lambda)) \end{pmatrix} = \begin{pmatrix} \langle G_{x,\lambda}, \overline{F} \rangle \\ R_1' \langle G_{\bullet,\lambda}, \overline{F} \rangle \end{pmatrix} \tag{5.14}
$$

for arbitrary $F \in H$.

Theorem 5.1. *The operator A is self-adjoint on the Hilbert space H.*

Proof. First, we prove that A is densely defined on H. For this suppose

$$G = \begin{pmatrix} g(x) \\ g_1 \end{pmatrix} \in H$$

is orthogonal to $D(A)$, i.e.

$$\frac{1}{|\delta|}\int_{-1}^{0} f(x)\overline{g(x)}dx + |\delta|\int_{0}^{1} f(x)\overline{g(x)}dx + \frac{|\delta|}{\rho}R_1'(f)\overline{g_1} = 0 \tag{5.15}$$

for all $F \in \binom{f(x)}{R_1'(f)} \in D(A)$. Let $C_0^\infty([-1,0)\cup(0,1])$ be a set of infinitely differentiable functions on $[-1,0)\cup(0,1]$, each element of which vanishes on some neighborhood of the points $x=-1$, $x=0$ and $x=1$. It is clear from the definition of $D(A)$ that $C_0^\infty([-1,0)\cup(0,1])\oplus\{0\}\subset D(A)$. By writing (5.15) for all $F\in C_0^\infty([-1,0)\cup(0,1])$ we can see that $g(x)$ is orthogonal to $C_0^\infty([-1,0)\cup(0,1])$ in $L_2(-1,1)$ with respect to the following inner product

$$\frac{1}{|\delta|}\int_{-1}^{0} f(x)\overline{g(x)}dx + |\delta|\int_{0}^{1} f(x)\overline{g(x)}dx = 0 \text{ for all } f\in C_0^\infty([-1,0)\cup(0,1]).$$

Consequently, $g(x)$ vanishes, since $L_2(-1,1)$ is complete with respect to the above inner product. Then, substituting $g(x)=0$ into (5.15) yields

$$R_1'(f)g_1 = 0, \tag{5.16}$$

for all $f\in L_2(-1,1)$ such that $\binom{f(x)}{R_1'(f)}\in D(A)$. Choosing

$$F_0 = \begin{pmatrix} f_0(x) \\ R_1'(f_0) \end{pmatrix} \in D(A)$$

such that $R_1'(f_0)=1$, we have from (5.16) that $g_1=0$. Consequently, $G=0$, so $D(A)$ is dense in H. Further, since A is symmetric it is enough to prove that $D(A^*)=D(A)$, where A^* is adjoint of A. Let $F\in D(A^*)$. We must show that $F\in D(A)$. By definition of A^*

$$\langle AG,F\rangle_H = \langle G,A^*F\rangle_H \text{ for all } G\in D(A). \tag{5.17}$$

From this it follows that

$$\langle (iI-A)G,F\rangle = \langle G,(-iI-A^*)F\rangle. \tag{5.18}$$

We already know that (see (5.14)) $\lambda=-i$ is regular point of A and therefore we can let

$$U = R(-i,A)(-iF-A^*F), \tag{5.19}$$

that is

$$(-iI - A)U = -iF - A^*F. \tag{5.20}$$

Substituting this into (5.18) and taking into account that A is symmetric and $U \in D(A)$ we have

$$\begin{aligned}\langle (iI - A)G, F\rangle_H &= \langle G, (-iI - A)U\rangle_H \\ &= \langle G, -iU\rangle_H - \langle G, AU\rangle_H \\ &= \langle iG, U\rangle_H - \langle AG, U\rangle_H \\ &= \langle (iI - A)G, U\rangle_H.\end{aligned}$$

Consequently,

$$\langle (iI - A)G, F - U\rangle_H = 0 \text{ for all } G \in H. \tag{5.21}$$

Since $\lambda = i$ is regular point of A we can choose

$$G = R(i, A)(F - U).$$

Substituting this into (5.21) we get

$$\|F - U\|_H = 0,$$

so $F = U$ and therefore $F \in D(A)$. The proof is complete. □

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References

[1] G. D. Birkhoff, *On the asymptotic character of the solution of the certain linear differential equations containing parameter*, Trans. Amer. Soc. **9** (1908), 219–231.

[2] C. T. Fulton, *Two-point boudary value problems with eigenvalue parameter contained in the boundary conditions*, Proc. Roy. Soc. Edinburgh **77A** (1977), 293–308.

[3] D. B. Hinton, *An expansion theorem for an eigenvalue problem with eigenvalue parameter in the boundary condition*, Quart. J. Math. Oxford **30** (1979), 33–42.

[4] M. Kadakal, F. S. Muhtarov and O. Sh. Mukhtarov, *Green function of one discontinuous boundary value problem with transmission conditions*, Bull. Pure Appl. Sci. **21E** (2) (2002), 357–369.

[5] O. Sh. Mukhtarov and H. Demir, *Coerciveness of the discontinuous initial-boundary value problem for parabolic equations*, Israel J. Math. **114** (1999), 239–252.

[6] O. Sh. Mukhtarov, M. Kandemir and N. Kuruoglu, *Distribution of eigenvalues for the discontinuous boundary value problem with functional many-point conditions*, Israel J. Math. **129** (2002), 143–156.

[7] O. Sh. Mukhtarov and S. Yakubov, *Problems for ordinary differential equations with transmission conditions*, Appl. Anal. **81** (2002), 1033–1064.

[8] A. Schneider, *A note on eigenvalue problems with eigenvalue parameter in the boundary conditions*, Math. Z. **136** (1974), 163–167.

[9] A. A. Shkalikov, *Boundary value problems for ordinary differential equations with a parameter in boundary condition*, Trudy Sem. Imeny I. G. Petrowsgo **9** (1983), 190–229.

[10] A. N. Tikhonov and A. A. Samarskii, *Equations of Mathematical Physics*, Oxford and New York, Pergamon, 1963.

[11] E. C. Titchmarsh, *Eigenfunctions Expansion Associated With Second Order Differential Equations* I, 2^{nd} edn, Oxford Univ. Press, London, 1962.

[12] I. Titeux and S. Yakubov, *Application of Abstract Differential Equations to some Mechanical Problems*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2003.

[13] E. Tunc and O. Sh. Mukhtarov, *Fundamental solutions and eigenvalues of one boundary-value problem with transmission conditions*, Applied Mathematics and Computation, 2003.

[14] J. Walter, *Regular eigenvalue problems with eigenvalue parameter in the boundary conditions*, Math. Z. **133** (1973), 301–312.

[15] S. Yakubov, *Completeness of Root Functions of Regular Differential Operators*, Longman, Scientific Technical, New York, 1994.

[16] S. Yakubov and Y. Yakubov, *Abel basis of root functions of regular boundary value problems*, Math. Nachr. **197** (1999), 157–187.

[17] ———, *Differential-Operator Equations, Ordinary and Partial Differential Equations*, Chapman and Hall/CRC, Boca Raton, 2000, p. 568.