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On the parametrization of the simple modules for Ariki-Koike algebras at roots of unity

By

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Abstract

Following the ideas of M.Geck and R.Rouquier in [19], we show that there exists a “canonical basic set” of Specht modules in bijection with the set of simple modules of Ariki-Koike algebras at roots of unity. Moreover, we determine the parametrization of this set and we give the consequences of these results on the representation theory of Ariki-Koike algebras.

1. Introduction

Let H be an Iwahori-Hecke algebra of a finite Weyl group W (or of an extended Weyl group) defined over $A := \mathbb{Z}[v, v^{-1}]$ where v is an indeterminate. For any ring homomorphism $\theta : A \to k$ into a field k, we have the corresponding specialized algebra $H_k := k \otimes_A H$. Assume that K is the field of fractions of A. Then, the algebra H_K is split semi-simple and isomorphic to the group algebra $K[W]$.

One of the major problems is the determination of the simple H_k-modules when H_k is not semi-simple. This problem may be attacked by using the corresponding decomposition matrix, which relates the simple H_k-modules to the simple H_K-modules, by a process of modular reduction.

In [19], using an ordering of simple H_K-modules by Lusztig a-function, Geck and Rouquier showed that there exists a canonical set $\mathcal{B} \subset \mathrm{Irr}(H_K)$ in bijection with $\mathrm{Irr}(H_k)$ (see also [15] and [16]). As a consequence, this set gives a natural way for labeling the simple H_k-modules. The Lusztig a-function has several interpretations: one in terms of Kazhdan-Lusztig basis of H and one in terms of Schur elements (the coincidence is shown in [25]).

In this paper, we consider the case of Ariki-Koike algebras. Let $d \in \mathbb{N}_{>0}$ and $\mathcal{H}_n := \mathcal{H}_n(v; u_0, \dots, u_{d-1})$ be the Ariki-Koike algebra of type $G(d, 1, n)$ where v, u_0, $u_1, \dots, u_{d-1}$ are $d+1$ parameters. This algebra appeared independently in [5] and in [8] and can be seen as an analogue of the Iwahori-Hecke algebra for the complex reflection group $(\mathbb{Z}/d\mathbb{Z}) \wr \mathfrak{S}_n$.

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As the Iwahori-Hecke algebras of type A_{n-1} and B_n are special cases of Ariki-Koike algebras, it is a natural question to ask whether the set $\mathcal{B}$ is well-defined for the Ariki-Koike algebras. We don't have Kazhdan-Lusztig type basis for Ariki-Koike algebras but we do have Schur elements. Hence, we can also define an a-function for this type of algebras.

When the parameters are generic, the representation theory of $\mathcal{H}_n$ over a field has been studied in [5]. It was shown that this algebra is semi-simple and that the simple modules $S^{\underline{\lambda}}$ are parametrized by the d-tuples of partitions $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(d-1)})$ of rank n.

Now, we consider the following choice of parameters:

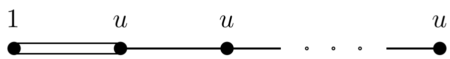
$$v = \eta_e, \quad u_i = \eta_e^{v_j}, \quad j = 0, \dots, d-1,$$

where $\eta_e := \exp(\frac{2i\pi}{e}) \in \mathbb{C}$ and where $0 \leq v_0 \leq \cdots \leq v_{d-1} < e$.

In this case, $\mathcal{H}_n$ is not semi-simple in general (it may be semi-simple if n is small). We can also attach to each d-partition $\underline{\lambda}$ an $\mathcal{H}_n$-module $S^{\underline{\lambda}}$ but it is reducible in general. In [20], Graham and Lehrer constructed a bilinear form on each $S^{\underline{\lambda}}$ and proved that the simple modules of $\mathcal{H}_n$ are given by the non zero $D^{\underline{\lambda}} := S^{\underline{\lambda}}/\mathrm{rad}(S^{\underline{\lambda}})$. Then, Ariki and Mathas (see [2] and [6]) described the non zero $D^{\underline{\lambda}}$ and showed that they are indexed by some "Kleshchev" multipartitions. Unfortunately, we only know a recursive description of this kind of multipartitions.

The aim of this paper is to prove that there exists a set $\mathcal{B}$ labeled by simple modules of a semi-simple Ariki-Koike algebra which satisfy the same property as the canonical set of [19] for Iwahori-Hecke algebras. Moreover, we prove that the parametrization of this set coincides with the parameterization of the set of simple $\mathcal{H}_n$-modules found by Foda, Leclerc, Okado, Thibon and Welsh (see [14]). The proof requires results of Ariki and Foda et al. about Ariki-Koike algebras, some properties of a-function and some combinatorial objects such as symbols.

These results have several consequences. First, it extends some results proved in [15] where the set $\mathcal{B}$ was determined for Hecke algebras of type A_{n-1} and in [22] where $\mathcal{B}$ was determined for Hecke algebras of type D_n and B_n with the following diagram:



In particular, it yields the determination of the set $\mathcal{B}$ for Hecke algebras of type B_n and, hence, completes the classification of the canonical basic set for Hecke algebras of finite Weyl groups with one parameter.

Moreover, using the results of Ariki and the ideas developed by Lascoux, Leclerc and Thibon in [24] for Hecke algebras of type A_{n-1}, we obtain a "triangular" algorithm for computing the decomposition matrix for Ariki-Koike algebras when the parameters are roots of unity.

The paper is divided in six parts. First, we briefly summarize the results of Ariki and Foda et al. about simple modules for Ariki-Koike algebras and links with quantum groups. Then, in the second part, following [7], we introduce Schur elements and a-function associated to the simple modules of a semi-simple Ariki-Koike algebra. In the third part, we prove some combinatorial properties about the multipartitions of Foda et al. and the a-function. The fourth part contains the main theorem of the paper: we give an interpretation of the multipartitions of Foda et al. in terms of the a-function. Finally, in the fifth part, we study a particular case of Ariki-Koike algebras : the case of Hecke algebras of type B_n with equal parameters.

2. Preliminaries

Let R be a commutative associative ring with unit and let v, $u_0, \dots, u_{d-1}$ be $d+1$ invertible elements in R. Let $n \in \mathbb{N}$. We define the Ariki-Koike algebra $\mathcal{H}_{R,n} := \mathcal{H}_{R,n}(v; u_0, \dots, u_{d-1})$ over R to be the unital associative R-algebra generated by $T_0, T_1, \dots, T_{n-1}$ subject to the relations:

$$
\begin{array}{ll}
(T_i - v)(T_i + 1) = 0 & \text{for } 1 \leq i \leq n-1, \\
(T_0 - u_0)(T_0 - u_1) \cdots (T_0 - u_{d-1}) = 0, & \\
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} & \text{for } 1 \leq i \leq n-2, \\
T_i T_j = T_j T_i & \text{for } |i-j| > 1, \\
T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0. &
\end{array}
$$

These relations are obtained by deforming the relations of the wreath product $(\mathbb{Z}/d\mathbb{Z}) \wr \mathfrak{S}_n$. The last three ones are known as type B braid relations.

It is known that the simple modules of $(\mathbb{Z}/d\mathbb{Z}) \wr \mathfrak{S}_n$ are indexed by the d-tuples of partitions. We will see that the same is true for the semi-simple Ariki-Koike algebras defined over a field. In this paper, we say that $\underline{\lambda}$ is a d-partition of rank n if:

- $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(d-1)})$ where, for $i = 0, \dots, d-1$, $\lambda^{(i)} = (\lambda^{(i)}_1, \dots, \lambda^{(i)}_{r_i})$ is a partition of rank $|\lambda^{(i)}|$ such that $\lambda^{(i)}_1 \geq \cdots \geq \lambda^{(i)}_{r_i} > 0$,
- $\displaystyle\sum_{k=0}^{d-1} |\lambda^{(k)}| = n$.

We denote by Π^d_n the set of d-partitions of rank n.

For each d-partition $\underline{\lambda}$ of rank n, we can associate a $\mathcal{H}_{R,n}$-module $S^{\underline{\lambda}}$ which is free over R. This is called a Specht module[*1].

Assume that R is a field. Then, for each d-partition of rank n, there is a natural bilinear form which is defined over each $S^{\underline{\lambda}}$. We denote by rad the radical associated to this bilinear form. The non zero $D^{\underline{\lambda}} := S^{\underline{\lambda}}/\mathrm{rad}(S^{\underline{\lambda}})$ form a complete set of non-isomorphic simple $\mathcal{H}_{R,n}$-modules (see for example [4, chapter 13]). In particular, if $\mathcal{H}_{R,n}$ is semi-simple, we have $\mathrm{rad}(S^{\underline{\lambda}}) = 0$ for all $\underline{\lambda} \in \Pi^d_n$ and the set of simple modules are given by the $S^{\underline{\lambda}}$.

[*1] Here, we use the definition of the classical Specht modules. Note that the results in [11] are given in terms of dual Specht modules. The passage from classical Specht modules to their duals is provided by the map $(\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(d-1)}) \mapsto (\lambda^{(d-1)'}, \lambda^{(d-2)'}, \dots, \lambda^{(0)'})$ where, for $i = 0, \dots, d-1$, $\lambda^{(i)'}$ denotes the conjugate partition.

Now, we explain in more details the representation theory of Ariki-Koike algebras. We summarize the results of Ariki and Foda et al. following [6] and [14].

2.1. Simple modules of Ariki-Koike algebras and decomposition numbers

Here, we assume that R is a field and we consider the Ariki-Koike algebra $\mathcal{H}_{R,n}$ defined over R with parameters v, $u_0, \dots, u_{d-1}$. First, we have a criterion of semi-simplicity:

Theorem 2.1 (Ariki [1]). *$\mathcal{H}_{R,n}$ is split semi-simple if and only if we have*:

- *for all $i \neq j$ and for all $d \in \mathbb{Z}$ such that $|d| < n$, we have*:

$$v^d u_i \neq u_j,$$

- $\displaystyle\prod_{i=1}^{n}(1 + v + \cdots + v^{i-1}) \neq 0.$

In this case, as noted at the beginning of this section, Ariki proved that the simple modules are labeled by the d-partitions $\underline{\lambda}$ of rank n.

Using the results of Dipper and Mathas ([13]), the case where $\mathcal{H}_{R,n}$ is not semi-simple can be reduced to the case where all the u_i are powers of v.

Here, we assume that R is a field of characteristic 0 and that v is a primitive e^{th}-root of unity with $e \geq 2$:

$$\begin{aligned} u_j &= \eta_e^{v_j} \text{ for } j = 0, \dots, d-1, \\ v &= \eta_e, \end{aligned}$$

where $\eta_e := \exp(\frac{2i\pi}{e})$ and where $0 \leq v_0 \leq \cdots \leq v_{d-1} < e$. The problem is to find the non zero $D^{\underline{\lambda}}$. To solve it, it is convenient to use the language of decomposition matrices.

We define $R_0(\mathcal{H}_{R,n})$ to be the Grothendieck group of finitely generated $\mathcal{H}_{R,n}$-modules, which is generated by the simple modules of $\mathcal{H}_{R,n}$. For a finitely generated $\mathcal{H}_{R,n}$-module M, we denote by $[M]$ its equivalence class in $R_0(\mathcal{H}_{R,n})$.

Now, let $R_1(\mathcal{H}_{R,n})$ be the Grothendieck group of finitely generated projective $\mathcal{H}_{R,n}$-modules. This is generated by the indecomposable projective $\mathcal{H}_{R,n}$-modules. For a finitely generated projective $\mathcal{H}_{R,n}$-module P, we denote by $[P]_p$ its equivalent class in $R_1(\mathcal{H}_{R,n})$. We have an injective homomorphism

$$c : R_1(\mathcal{H}_{R,n}) \to R_0(\mathcal{H}_{R,n})$$

given by $c([P]_p) = [P]$ where P is a finitely generated projective $\mathcal{H}_{R,n}$-module.

Following [6], we denote by $\mathcal{F}_n$ the free abelian group with $\mathbb{C}$-basis the set $\{[\![S^{\underline{\lambda}}]\!] \mid \underline{\lambda} \in \Pi_n^d\}$. $\mathcal{F}_n$ can be seen as the Grothendieck group of a semi-simple Ariki-Koike algebra.

Let $\Phi_n^d = \{\underline{\mu} \in \Pi_n^d |\ D^{\underline{\mu}} \neq 0\}$. $R_0(\mathcal{H}_{R,n})$ is generated by the set $\{[D^{\underline{\mu}}] |\ \underline{\mu} \in \Phi_n^d\}$. Hence, for all $\underline{\lambda} \in \Pi_n^d$, there exist numbers $d_{\underline{\lambda},\underline{\mu}}$ with $\underline{\mu} \in \Phi_n^d$ such that :

$$[S^{\underline{\lambda}}] = \sum_{\underline{\mu} \in \Phi_n^d} d_{\underline{\lambda},\underline{\mu}} [D^{\underline{\mu}}].$$

The matrix $(d_{\underline{\lambda},\underline{\mu}})_{\underline{\lambda} \in \Pi_n^d, \underline{\mu} \in \Phi_n^d}$ is called the decomposition matrix of $\mathcal{H}_{R,n}$.

Hence, we have a homomorphism:

$$d : \mathcal{F}_n \to R_0(\mathcal{H}_{R,n}),$$

defined by $d(\llbracket S^{\underline{\lambda}} \rrbracket) = [S^{\underline{\lambda}}] = \sum_{\underline{\mu} \in \Phi_n^d} d_{\underline{\lambda},\underline{\mu}} [D^{\underline{\mu}}]$. d is called the decomposition map. Note that if $\mathcal{H}_{R,n}$ is semi-simple, the decomposition matrix is just the identity.

Now, by Brauer reciprocity, we can see that the indecomposable projective $\mathcal{H}_{R,n}$-modules are labeled by Φ_n^d and that there exists an injective homomorphism:

$$e : R_1(\mathcal{H}_{R,n}) \to \mathcal{F}_n,$$

such that for $\underline{\mu} \in \Phi_n^d$, if $P^{\underline{\mu}}$ is the indecomposable projective $\mathcal{H}_{R,n}$-module which is the projective cover of $D^{\underline{\mu}}$, we have:

$$e([P^{\underline{\mu}}]_p) = \sum_{\underline{\lambda} \in \Pi_n^d} d_{\underline{\lambda},\underline{\mu}} \llbracket S^{\underline{\lambda}} \rrbracket.$$

By [20, Theorem 3.7], we obtain the following commutative diagram:

$$\begin{tikzcd} & R_1(\mathcal{H}_{R,n}) \arrow[dl, "e"'] \arrow[dr, "c"] & \\ \mathcal{F}_n \arrow[rr, "d"] & & R_0(\mathcal{H}_{R,n}) \end{tikzcd}$$

Now, we turn to the problem of determining which $D^{\underline{\lambda}}$ are non zero. Ariki and Mathas solve it by using deep results about quantum groups.

2.2. Links with quantum groups and Kleshchev d-partitions

We keep the notations of the first paragraph of this section. First, we introduce some notations and we define Kleshchev d-partitions following [6].

Let $\underline{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(d-1)})$ be a d-partition of rank n. The diagram of $\underline{\lambda}$ is the following set:

$$[\underline{\lambda}] = \left\{ (a, b, c) \mid 0 \leq c \leq d-1,\ 1 \leq b \leq \lambda_a^{(c)} \right\}.$$

The elements of this diagram are called the nodes of $\underline{\lambda}$. Let $\gamma = (a, b, c)$ be a node of $\underline{\lambda}$. The residue of γ associated to the set $\{e; v_0, \ldots, v_{d-1}\}$ is the element of $\mathbb{Z}/e\mathbb{Z}$ defined by:

$$\mathrm{res}(\gamma) = (b - a + v_c)(\mathrm{mod}\ e).$$

If γ is a node with residue i, we say that γ is an i-node. Let $\underline{\lambda}$ and $\underline{\mu}$ be two d-partitions of rank n and $n+1$ such that $[\underline{\lambda}] \subset [\underline{\mu}]$. There exists a node γ such that $[\underline{\mu}] = [\underline{\lambda}] \cup \{\gamma\}$. Then, we denote $[\underline{\mu}]/[\underline{\lambda}] = \gamma$. If $\text{res}(\gamma) = i$, we say that γ is an addable i-node for $\underline{\lambda}$ and a removable i-node for $\underline{\mu}$.

Now, we consider the following order on the set of removable and addable nodes of a d-partition: we say that $\gamma = (a, b, c)$ is below $\gamma' = (a', b', c')$ if $c < c'$ or if $c = c'$ and $a < a'$.

This order will be called the AM-order and the notion of normal nodes and good nodes below are linked with this order (in the next paragraph, we will give another order on the set of nodes which is distinct from this one).

Let $\underline{\lambda}$ be a d-partition and let γ be an i-node, we say that γ is a normal i-node of $\underline{\lambda}$ if, whenever η is an i-node of $\underline{\lambda}$ below γ, there are more removable i-nodes between η and γ than addable i-nodes between η and γ. If γ is the highest normal i-node of $\underline{\lambda}$, we say that γ is a good i-node.

We can now define the notion of Kleshchev d-partitions associated to the set $\{e; v_0, \ldots, v_{d-1}\}$:

Definition 2.1. The Kleshchev d-partitions are defined recursively as follows.

- The empty partition $\underline{\emptyset} := (\emptyset, \emptyset, \ldots, \emptyset)$ is Kleshchev.
- If $\underline{\lambda}$ is Kleshchev, there exist $i \in \{0, \ldots, e-1\}$ and a good i-node γ such that if we remove γ from $\underline{\lambda}$, the resulting d-partition is Kleshchev.

We denote by $\Lambda^0_{\{e;v_0,\ldots,v_{l-1}\}}$ the set of Kleshchev d-partitions associated to the set $\{e; v_0, \ldots, v_{d-1}\}$. If there is no ambiguity concerning $\{e; v_0, \ldots, v_{d-1}\}$, we denote it by Λ^0.

Now, let $\mathfrak{h}$ be a free $\mathbb{Z}$-module with basis $\{h_i, \mathfrak{d} \mid 0 \leq i < e\}$ and let $\{\Lambda_i, \delta \mid 0 \leq i < e\}$ be the dual basis with respect to the pairing:

$$\langle\ ,\ \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{Z}$$

such that $\langle \Lambda_i, h_j \rangle = \delta_{ij}$, $\langle \delta, \mathfrak{d} \rangle = 1$ and $\langle \Lambda_i, \mathfrak{d} \rangle = \langle \delta, h_j \rangle = 0$ for $0 \leq i, j < e$. For $0 \leq i < e$, we define the simple roots of $\mathfrak{h}^*$ by:

$$\alpha_i = \begin{cases} 2\Lambda_0 - \Lambda_{e-1} - \Lambda_1 + \delta & \text{if } i = 0, \\ 2\Lambda_i - \Lambda_{i-1} - \Lambda_{i+1} & \text{if } i > 0, \end{cases}$$

where $\Lambda_e := \Lambda_0$. The Λ_i are called the fundamental weights.

Now, let q be an indeterminate and let $\mathcal{U}_q$ be the quantum group of type $A^{(1)}_{e-1}$. This is a unital associative algebra over $\mathbb{C}(q)$ which is generated by elements $\{e_i, f_i \mid i \in \{0, \ldots, e-1\}\}$ and $\{k_h \mid h \in \mathfrak{h}\}$ (see for example [28, chapter 6] for the relations).

For $j \in \mathbb{N}$ and $l \in \mathbb{N}$, we define:

- $[j]_q := \dfrac{q^j - q^{-j}}{q - q^{-1}}$,
- $[j]^!_q := [1]_q[2]_q \cdots [j]_q$,
- $\begin{bmatrix} l \\ j \end{bmatrix}_q = \dfrac{[l]^!_q}{[j]^!_q [l-j]^!_q}$.

We now introduce the Ariki's theorem. For details, we refer to [4]. Let $\mathcal{A}=\mathbb{Z}[q,q^{-1}]$. We consider the Kostant-Lusztig $\mathcal{A}$-form of $\mathcal{U}_q$ which is denoted by $\mathcal{U}_{\mathcal{A}}$: this is a $\mathcal{A}$-subalgebra of $\mathcal{U}_q$ generated by the divided powers $e_i^{(l)}:=\frac{e_i^l}{[l]_q^!}$, $f_j^{(l)}:=\frac{f_j^l}{[l]_q^!}$ for $0\leq i,j<e$ and $l\in\mathbb{N}$ and by k_{h_i}, $k_{\mathfrak{d}}$, $k_{h_i}^{-1}$, $k_{\mathfrak{d}}^{-1}$ for $0\leq i<e$ (see [26, §3.1]). Now, if S is a ring and u an invertible element in S, we can form the specialized algebra $\mathcal{U}_{S,u}:=S\otimes_{\mathcal{A}}\mathcal{U}_{\mathcal{A}}$ by specializing the indeterminate q to $u\in S$.

Let $\underline{\lambda}$ and $\underline{\mu}$ be two d-partitions of rank n and $n+1$ such that there exists an i-node γ such that $[\underline{\mu}]=[\underline{\lambda}]\cup\{\gamma\}$. We define:

$$\begin{aligned}
N_i^a(\underline{\lambda},\underline{\mu}) =&\sharp\{\text{addable }\ i-\text{nodes of }\underline{\lambda}\ \text{ above }\gamma\}\\
&-\sharp\{\text{removable }\ i-\text{nodes of }\underline{\mu}\ \text{ above }\gamma\},\\
N_i^b(\underline{\lambda},\underline{\mu}) =&\sharp\{\text{addable }i-\text{nodes of }\underline{\lambda}\ \text{ below }\gamma\}\\
&-\sharp\{\text{removable }i-\text{nodes of }\underline{\mu}\ \text{ below }\gamma\},\\
N_i(\underline{\lambda}) =&\sharp\{\text{addable }i-\text{nodes of }\underline{\lambda}\}\\
&-\sharp\{\text{removable }i-\text{nodes of }\underline{\lambda}\},\\
N_{\mathfrak{d}}(\underline{\lambda}) =&\sharp\{0-\text{nodes of }\underline{\lambda}\}.
\end{aligned}$$

For $n\in\mathbb{N}$, let $\mathcal{F}_n$ be the associated space which is defined in the previous paragraph. Let $\mathcal{F}:=\bigoplus_{n\in\mathbb{N}}\mathcal{F}_n$. $\mathcal{F}$ is called the Fock space. For each d-partition $\underline{\lambda}$, we identify the Specht module $S^{\underline{\lambda}}$ with $\underline{\lambda}$ so that a basis of $\mathcal{F}_n$ is given by the set Π_n^d. $\mathcal{F}$ becomes a $\mathcal{U}_q$- module with the following action:

$$e_i\underline{\lambda}=\sum_{\operatorname{res}([\underline{\lambda}]/[\underline{\mu}])=i}q^{-N_i^a(\underline{\mu},\underline{\lambda})}\underline{\mu},\qquad f_i\underline{\lambda}=\sum_{\operatorname{res}([\underline{\mu}]/[\underline{\lambda}])=i}q^{N_i^b(\underline{\lambda},\underline{\mu})}\underline{\mu},$$
$$k_{h_i}\underline{\lambda}=q^{N_i(\underline{\lambda})}\underline{\lambda},\qquad k_{\mathfrak{d}}\underline{\lambda}=q^{-N_{\mathfrak{d}}(\underline{\lambda})}\underline{\lambda},$$

where $i=0,\dots,e-1$. This action was discovered by Hayashi in [21].

Let $\mathcal{M}$ be the $\mathcal{U}_q$-submodule of $\mathcal{F}$ generated by the empty d-partition. It is isomorphic to an integrable highest weight module (see [9]). Now, this result allows us to apply the canonical basis theory and the crystal graph theory to $\mathcal{M}$.

In particular, the crystal graph gives a way for labeling the Kashiwara-Lusztig's canonical basis of $\mathcal{M}$ (see [4] for more details). Based on Misra and Miwa's result, Ariki and Mathas observed that this graph is given by:

- vertices: the Kleshchev d-partitions,
- edges: $\underline{\lambda}\xrightarrow{i}\underline{\mu}$ if and only if $[\underline{\mu}]/[\underline{\lambda}]$ is a good i-node.

Thus, the canonical basis $\mathfrak{B}$ of $\mathcal{M}$ is labeled by the Kleshchev d-partitions:

$$\mathfrak{B}=\{G(\underline{\lambda})\mid\underline{\lambda}\in\Lambda^0_{\{e;v_0,\dots,v_{d-1}\}}\}.$$

This set is a basis of the $\mathcal{U}_{\mathcal{A}}$-module $\mathcal{M}_{\mathcal{A}}$ generated by the empty d-partition and for any specialization of q into an invertible element u of a field R, we obtain a basis of the specialized module $\mathcal{M}_{R,u}$ by specializing the set $\mathfrak{B}$.

Now, we have the following theorem of Ariki which shows that the problem of computing the decomposition numbers of $\mathcal{H}_{R,n}$ can be translated to that of computing the canonical basis of $\mathcal{M}$. This theorem was first conjectured by Lascoux, Leclerc and Thibon ([24]) in the case of Hecke algebras of type A_{n-1}.

Theorem 2.2 (Ariki [2]). *We have $\Phi_n^d = \Lambda^0_{\{e;v_0,\ldots,v_{d-1}\}}$. Moreover, assume that R is a field of characteristic 0. Then, for each $\underline{\lambda} \in \Phi_n^d$, there exist polynomials $d_{\underline{\mu},\underline{\lambda}}(q) \in \mathbb{Z}[q]$ and a unique element $G(\underline{\lambda})$ of the canonical basis such that*:

$$G(\underline{\lambda}) = \sum_{\underline{\mu}\in\Pi_n^d} d_{\underline{\mu},\underline{\lambda}}(q)\underline{\mu} \qquad \textit{and} \qquad G(\underline{\lambda}) = \underline{\lambda} \ (\textit{mod } q).$$

Finally, for all $\underline{\mu} \in \Pi_n^d$ and $\underline{\lambda} \in \Phi_n^d$, we have $d_{\underline{\mu},\underline{\lambda}}(1) = d_{\underline{\mu},\underline{\lambda}}$.

Then, if R is a field of characteristic 0 and if we identify the d-partitions $\underline{\lambda}$ with the modules $S^{\underline{\lambda}}$, we see that the canonical basis elements specialized at $q = 1$ corresponds to the indecomposable projective $\mathcal{H}_{R,n}$-modules.

As noted in the introduction, the problem of this parametrization of the simple $\mathcal{H}_{R,n}$-modules is that we only know a recursive description of the Kleshchev d-partitions. We now deal with another parametrization of this set found by Foda et al. which uses almost the same objects as Ariki and Mathas.

2.3. Parametrization of the simple modules by Foda et al.

The principal idea of Foda et al. ([14]) is to use another structure of $\mathcal{U}_q$-module over $\mathcal{F}$ by choosing another order on the set of the nodes of the d-partitions.

Here, we say that $\gamma = (a, b, c)$ is above $\gamma' = (a', b', c')$ if:

$$b - a + v_c < b' - a' + v_{c'} \text{ or if } b - a + v_c = b' - a' + v_{c'} \text{ and } c > c'.$$

This order will be called the FLOTW order.

This order allows us to define functions $\overline{N}_i^a(\underline{\lambda}, \underline{\mu})$ and $\overline{N}_i^b(\underline{\lambda}, \underline{\mu})$ given by the same way as $N_i^a(\underline{\lambda}, \underline{\mu})$ et $N_i^b(\underline{\lambda}, \underline{\mu})$ for the AM order.

Now, we have the following result:

Theorem 2.3 (Jimbo, Misra, Miwa, Okado [23]). *$\mathcal{F}$ is a $\mathcal{U}_q$-module with action*:

$$e_i\underline{\lambda} = \sum_{\mathrm{res}([\underline{\lambda}]/[\underline{\mu}])=i} q^{-\overline{N}_i^a(\underline{\mu},\underline{\lambda})}\underline{\mu}, \qquad f_i\underline{\lambda} = \sum_{\mathrm{res}([\underline{\mu}]/[\underline{\lambda}])=i} q^{\overline{N}_i^b(\underline{\lambda},\underline{\mu})}\underline{\mu},$$

$$k_{h_i}\lambda = v^{N_i(\underline{\lambda})}\underline{\lambda}, \qquad k_{\mathfrak{d}}\underline{\lambda} = q^{-N_{\mathfrak{d}}(\underline{\lambda})}\underline{\lambda},$$

where $0 \le i \le n-1$. This action will be called the JMMO action.

We denote by $\overline{\mathcal{M}}$ the $\mathcal{U}_q$-module generated by the empty d-partition with the above action. This is a highest weight module and the d-partitions of the crystal graph are obtained recursively by adding good nodes to d-partitions of the crystal graph.

Foda et al. showed that the analogue of the notion of Kleshchev d-partitions for this action is as follows:

Definition 2.2. We say that $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(d-1)})$ is a FLOTW d-partition associated to the set $\{e; v_0, \dots, v_{d-1}\}$ if and only if:

1. for all $0 \leq j \leq d-2$ and $i = 1, 2, \dots$, we have:

$$\lambda_i^{(j)} \geq \lambda_{i+v_{j+1}-v_j}^{(j+1)},$$
$$\lambda_i^{(d-1)} \geq \lambda_{i+e+v_0-v_{d-1}}^{(0)};$$

2. for all $k > 0$, among the residues appearing at the right ends of the length k rows of $\underline{\lambda}$, at least one element of $\{0, 1, \dots, e-1\}$ does not occur.

We denote by $\Lambda^1_{\{e;v_0,\dots,v_{d-1}\}}$ the set of FLOTW d-partitions associated to the set $\{e; v_0, \dots, v_{d-1}\}$. If there is no ambiguity concerning $\{e; v_0, \dots, v_{d-1}\}$, we denote it by Λ^1.

Hence, the crystal graph of $\overline{\mathcal{M}}$ is given by:

- vertices: the FLOTW d-partitions,
- edges: $\underline{\lambda} \xrightarrow{i} \underline{\mu}$ if and only if $[\underline{\mu}]/[\underline{\lambda}]$ is good i-node with respect to the FLOTW order.

So, the canonical basis elements are labeled by the FLOTW d-partitions and if we specialize these elements to $q = 1$, we obtain the same elements as in Theorem 2.2. Hence, there is a bijection between the set of Kleshchev d-partitions and the set of FLOTW d-partitions.

3. Schur elements and a-functions

The aim of this section is to introduce Schur elements and a-functions associated to simple modules of Ariki-Koike algebras. It is convenient to express them in terms of symbols. Most of the results explained here are given in [7]. First, we give some notations and some definitions:

3.1. Symbols

Let $n \in \mathbb{N}$ and $d \in \mathbb{N}_{>0}$. The notion of symbols are usually associated to d-partitions. Here, we generalize it to d-compositions. This generalization is useful in our argument in the next section. A d-composition $\underline{\lambda}$ of rank n is a d-tuple $(\lambda^{(0)}, \dots, \lambda^{(d-1)})$ where:

- for all $i = 0, \dots, d-1$, we have $\lambda^{(i)} = (\lambda_1^{(i)}, \dots, \lambda_{h^{(i)}}^{(i)})$ for $h^{(i)} \in \mathbb{N}$ and $\lambda_j^{(i)} \in \mathbb{N}_{>0}$ $(j = 1, \dots, h^{(i)})$, $h^{(i)}$ is called the height of $\lambda^{(i)}$,
- $\displaystyle\sum_{i=0}^{d-1} \sum_{j=1}^{h^{(i)}} \lambda_j^{(i)} = n.$

Let $\underline{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(d-1)})$ be a d-composition and let $h^{(i)}$ be the heights of the compositions $\lambda^{(i)}$. Then the height of $\underline{\lambda}$ is the following positive integer:

$$h_{\underline{\lambda}} = \max\{h^{(0)}, \ldots, h^{(d-1)}\}$$

Let k be a positive integer. The ordinary symbol associated to $\underline{\lambda}$ and k is the following set:

$$\mathbf{B} := (B^{(0)}, \ldots, B^{(d-1)}),$$

where $B^{(i)}$, for $i = 0, \ldots, d-1$, is given by:

$$B^{(i)} := (B_1^{(i)}, \ldots, B_{h_{\underline{\lambda}}+k}^{(i)}),$$

in which:

$$B_j^{(i)} := \lambda_j^{(i)} - j + h_{\underline{\lambda}} + k \quad (1 \leq j \leq h_{\underline{\lambda}} + k),$$

and $\lambda_j^{(i)} := 0$ if $j > h^{(i)}$. Note that the entries of each $B^{(i)}$ $(i = 0, \ldots, d-1)$ are given in descending order, contrary to the usual convention.

We denote by $h_{\mathbf{B}} := h_{\underline{\lambda}} + k$ the height of the symbol $\mathbf{B}$. Now, we fix a sequence of rational positive numbers:

$$m = (m^{(0)}, \ldots, m^{(d-1)}).$$

Let k be a positive integer. We define a set $\mathbf{B}[m]'$ associated to m and k by adding on each part $B_j^{(i)}$ of the ordinary symbol the number $m^{(i)}$. So, we have:

$$\mathbf{B}[m]' = (B'^{(0)}, \ldots, B'^{(d-1)}),$$

where $B'^{(i)}$, for $i = 0, \ldots, d-1$, is given by:

$$B'^{(i)} := (B_1'^{(i)}, \ldots, B_{h_{\underline{\lambda}}+k}'^{(i)}),$$

in which:

$$B_j'^{(i)} := \lambda_j^{(i)} - j + h_{\underline{\lambda}} + k + m^{(i)} \quad (1 \leq j \leq h_{\underline{\lambda}} + k).$$

Note that this set is defined in [7] when the numbers $m^{(i)}$ are integers using the notion of m-translated symbols.

Example 3.1. Let $d = 3$ and $m = \left(1, \frac{1}{2}, 2\right)$. We consider the following 3-partition:

$$\underline{\lambda} = ((4,2), (0), (5,2,1)).$$

Let $k = 0$. The ordinary symbol associated to $\underline{\lambda}$ is given by:

$$\mathbf{B} = \left\{ \begin{array}{lcccc} B^{(0)} & = & 6 & 3 & 0 \\ B^{(1)} & = & 2 & 1 & 0 \\ B^{(2)} & = & 7 & 3 & 1 \end{array} \right.$$

The set $\mathbf{B}[m]'$ associated to $\underline{\lambda}$ is given by:

$$\mathbf{B}[m]' = \begin{cases} B'^{(0)} & = & 7 & 4 & 1 \\ B'^{(1)} & = & 5/2 & 3/2 & 1/2 \\ B'^{(2)} & = & 9 & 5 & 3 \end{cases}$$

3.2. Schur elements

Let $\mathcal{H}_{R,n}$ be a semi-simple Ariki-Koike algebra over a field R with parameters $\{v; u_0, \dots, u_{d-1}\}$. $\mathcal{H}_{R,n}$ is a symmetric algebra and thus, it has a symmetrizing normalized trace τ (see [7] and [18, chapter 7] for the representation theory of symmetric algebras). The Schur elements are defined to be the non zero elements $s_{\underline{\lambda}}(v, u_0, \dots, u_{d-1}) \in R$ such that:

$$\tau = \sum_{\underline{\lambda} \in \Pi_n^d} \frac{1}{s_{\underline{\lambda}}(v, u_0, \dots, u_{d-1})} \chi_{\underline{\lambda}},$$

where, for $\underline{\lambda} \in \Pi_n^d$, $\chi_{\underline{\lambda}}$ is the irreducible character associated to the simple $\mathcal{H}_{R,n}$-module $S^{\underline{\lambda}}$.

Now, we give the expressions of these Schur elements. They were conjectured in [27], calculated in [17] and the following expression is given in [7].

First, let's fix some notations: let $\mathbf{B} = (B^{(0)}, \dots, B^{(d-1)})$ be an ordinary symbol of height h associated to a d-partition $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(d-1)})$ of rank n. Following [7], we denote:

$$\delta_{\mathbf{B}}(v, u_0, \dots, u_{d-1}) = \prod_{\substack{0 \leq i \leq j < d \\ (\alpha,\beta) \in B^{(i)} \times B^{(j)} \\ \alpha > \beta \text{ if } i=j}} (v^{\alpha} u_i - v^{\beta} u_j),$$

$$\theta_{\mathbf{B}}(v, u_0, \dots, u_{d-1}) = \prod_{\substack{0 \leq i,j < d \\ \alpha \in B^{(i)} \\ 1 \leq k \leq \alpha}} (v^k u_i - u_j),$$

$$\sigma_{\mathbf{B}} = \binom{d}{2}\binom{h}{2} + n(d-1),$$

$$\tau_{\mathbf{B}} = \binom{d(h-1)+1}{2} + \binom{d(h-2)+1}{2} + \dots + \binom{2}{2},$$

$$|\mathbf{B}| = \sum_{\substack{0 \leq i < d \\ \alpha \in B^{(i)}}} \alpha,$$

$$\nu_{\mathbf{B}}(v, u_0, \dots, u_{d-1}) = \prod_{0 \leq i < j < d} (u_i - u_j)^h \theta_{\mathbf{B}}(v, u_0, \dots, u_{d-1}).$$

Now, we have the following proposition:

Proposition 3.1 (Geck, Iancu, Malle, see [7]). *Let $S^{\underline{\lambda}}$ be the simple $\mathcal{H}_{R,n}$-module associated to the d-partition $\underline{\lambda}$ and let $\boldsymbol{B}$ be an ordinary symbol* *associated to $\underline{\lambda}$, then the Schur element of $S^{\underline{\lambda}}$ is given by*:

$$s_{\underline{\lambda}}(v, u_0, \ldots, u_{d-1}) = \left((v-1)\prod_{0\le i<d} u_i\right)^{-n} (-1)^{\sigma_B} v^{\tau_B - |\boldsymbol{B}|+n} \frac{\nu_{\boldsymbol{B}}(v, u_0, \ldots, u_{d-1})}{\delta_{\boldsymbol{B}}(v, u_0, \ldots, u_{d-1})}.$$

3.3. a-function for Ariki-Koike algebras

In this paragraph, we deal with the a-value of the simple modules in the case of Ariki-Koike algebras. Let d and e be two positive integers and define complex numbers by:

$$\eta_d = \exp\left(\frac{2i\pi}{d}\right) \qquad \text{and} \qquad \eta_e = \exp\left(\frac{2i\pi}{e}\right).$$

We fix a sequence of rational numbers $m = (m^{(0)}, \ldots, m^{(d-1)})$ such that we have $dm^{(j)} \in \mathbb{N}$ for all $j = 0, \ldots, d-1$. Let y be an indeterminate and let $L := \mathbb{Q}[\eta_d](y)$. We consider the Ariki-Koike algebra $\mathcal{H}_{L,n}$ defined over the field L with the following choice of parameters:

$$u_j = y^{dm^{(j)}} \eta_d^j \quad \text{for} \quad j = 0, \ldots, d-1,$$
$$v = y^d.$$

By Theorem 2.1, $\mathcal{H}_{L,n}$ is a split semi-simple Ariki-Koike algebra. Thus, the simple modules are labeled by the d-partitions of rank n.

We can now define the a-value of the simple $\mathcal{H}_{L,n}$-modules. Let $\underline{\lambda}$ be a d-partition of rank n and let $S^{\underline{\lambda}}$ be the simple module associated to this d-partition. Let $s_{\underline{\lambda}}$ be the Schur element of $S^{\underline{\lambda}}$. This is a Laurent polynomial in y. Let $\mathrm{val}_y(s_{\underline{\lambda}})$ be the valuation of $s_{\underline{\lambda}}$ in y that is to say:

$$\mathrm{val}_y(s_{\underline{\lambda}}) := -\min\{l \in \mathbb{Z} \mid y^l s_{\underline{\lambda}} \in \mathbb{Z}[\eta_d][y]\}.$$

Then, the a-value of $S^{\underline{\lambda}}$ is defined by:

$$a(S^{\underline{\lambda}}) := -\frac{\mathrm{val}_y(s_{\underline{\lambda}})}{d}.$$

For all $\underline{\lambda} \in \Pi_n^d$, $a(S^{\underline{\lambda}})$ is a rational number which depends on $(m^{(0)}, \ldots, m^{(d-1)})$. To simplify, we will denote $a(\underline{\lambda}) := a(S^{\underline{\lambda}})$.

Proposition 3.2. *Let $\underline{\lambda}$ be a d-partition of rank n and let $S^{\underline{\lambda}}$ be the simple $\mathcal{H}_{L,n}$-module.*

Let k be a positive integer and $\boldsymbol{B}[m]'$ be the set associated to m and k as in paragraph 3.1. *Then, if h is the height of $\boldsymbol{B}[m]'$, we have*:

$$a(\underline{\lambda}) = f(n,h,m) + \sum_{\substack{0\le i\le j<d \\ (a,b)\in B'^{(i)}\times B'^{(j)} \\ a>b \ \text{if} \ i=j}} \min\{a,b\} - \sum_{\substack{0\le i,j<d \\ a\in B'^{(i)} \\ 1\le k\le a}} \min\{k, m^{(j)}\},$$

where:

$$f(n,h,m)= n\sum_{j=0}^{d-1} m^{(j)} - \tau_{\boldsymbol{B}} + |\boldsymbol{B}| - n - h \sum_{0\le i<j<d} \min\{m^{(i)}, m^{(j)}\} + \sum_{\substack{0\le i,j<d \\ \alpha\in B^{(i)} \\ 1\le k\le m^{(i)}}} \min\{k, m^{(j)}\}.$$

Proof. We use Proposition 3.1:

$$a(\underline{\lambda}) = n\sum_{j=0}^{d-1} m^{(j)} - \tau_{\mathbf{B}} + |\mathbf{B}| - n - h \sum_{0\le i<j<d} \min\{m^{(i)}, m^{(j)}\} + \sum_{\substack{0\le i\le j<d \\ (\alpha,\beta)\in B^{(i)}\times B^{(j)} \\ \alpha>\beta \text{ if } i=j}} \min\{\alpha+m^{(i)}, \beta+m^{(j)}\} - \sum_{\substack{0\le i,j<d \\ \alpha\in B^{(i)} \\ 1\le k\le \alpha}} \min\{k+m^{(i)}, m^{(j)}\}.$$

We have:

$$\sum_{\substack{0\le i,j<d \\ \alpha\in B^{(i)} \\ 1\le k\le \alpha}} \min\{k+m^{(i)}, m^{(j)}\} = \sum_{\substack{0\le i,j<d \\ \alpha\in B^{(i)} \\ 1\le k\le \alpha+m^{(i)}}} \min\{k, m^{(j)}\} - \sum_{\substack{0\le i,j<d \\ \alpha\in B^{(i)} \\ 1\le k\le m^{(i)}}} \min\{k, m^{(j)}\}.$$

Using the set $\mathbf{B}[m]'$ of paragraph 3.1, we conclude that:

$$a(\underline{\lambda}) = f(n,h,m) + \sum_{\substack{0\le i\le j<d \\ (a,b)\in B'^{(i)}\times B'^{(j)} \\ a>b \text{ if } i=j}} \min\{a,b\} - \sum_{\substack{0\le i,j<d \\ a\in B'^{(i)} \\ 1\le k\le a}} \min\{k, m^{(j)}\}.$$

□

Remark 3.1. The formula of the a-function does not depend on the choice of the ordinary symbol.

The next section gives an interpretation of the parametrization of the simple modules by Foda et al. in terms of this a-function when the Ariki-Koike algebra is not semi-simple.

4. Foda et al. d-partitions and a-functions

First, we introduce some definitions:

4.1. Notations and hypothesis

Here, we keep the notations of the previous section. As in the second section, we assume that we have nonnegative integers:

$$0 \leq v_0 \leq v_1 \leq \cdots \leq v_{d-1} < e.$$

Then, for $j = 0, \ldots, d-1$, we define rational numbers:

$$m^{(j)} = v_j - \frac{je}{d} + se,$$

where s is a positive integer such that $m^{(j)} \geq 0$ for $j = 0, \ldots, d-1$. We have $dm^{(j)} \in \mathbb{N}$ for all $j = 0, \ldots, d-1$.

Let $\mathcal{H}_{L,n}$ be the Ariki-Koike algebra over $L = \mathbb{Q}[\eta_d](y)$ with the following parameters:

$$\begin{aligned} u_j &= y^{dm^{(j)}} \eta_d^j \quad \text{for} \quad j = 0, \ldots, d-1, \\ v &= y^d. \end{aligned}$$

If we specialize the parameter y into $\eta_{de} := \exp(\frac{2i\pi}{de})$, we obtain an Ariki-Koike algebra $\mathcal{H}_{R,n}$ over $R = \mathbb{Q}(\eta_{de})$ with the following parameters:

$$\begin{aligned} u_j &= \eta_{de}^{dm^{(j)}} \eta_d^j = \eta_e^{v_j} \quad \text{for} \quad j = 0, \ldots, d-1, \\ v &= \eta_e. \end{aligned}$$

This algebra is split but not semi-simple and we will discuss about its representation theory.

Here, we keep the notations used at the previous section and we add the following ones. First, we extend the combinatorial definitions of diagrams and residues introduced in paragraph 2.2 to the d-compositions of rank n in an obvious way.

Let $\underline{\lambda}$ be a d-composition of rank n. The nodes at the right ends of the parts of $\underline{\lambda}$ will be called the nodes of the border of $\underline{\lambda}$.

Example 4.1. Assume that $d = 2$, $v_0 = 0$, $v_1 = 2$ and that $e = 4$. Let $\underline{\lambda} = (4.2.3, 3.5)$, then the diagram of $\underline{\lambda}$ is the following one:

$$\underline{\lambda} = \left(\begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & \mathbf{3} \\ \hline 3 & \mathbf{0} & & \\ \hline 2 & 3 & \mathbf{0} & \\ \hline \end{array} \; , \; \begin{array}{|c|c|c|c|c|} \hline 2 & 3 & \mathbf{0} & & \\ \hline 1 & 2 & 3 & 0 & \mathbf{1} \\ \hline \end{array} \right).$$

The nodes in bold-faced type are the nodes of the border of $\underline{\lambda}$.

We will also use the following notations: Let $\underline{\lambda}$ and $\underline{\mu}$ be d-compositions of rank n and $n+1$ and let $k \in \{0, \ldots, e-1\}$ be a residue.

- We write $\underline{\lambda} \overset{k}{\underset{(j,p)}{\mapsto}} \underline{\mu}$ if $[\underline{\mu}]/[\underline{\lambda}]$ is a k-node on the part $\mu_j^{(p)}$.
- We write $\underline{\lambda} \overset{k}{\mapsto} \underline{\mu}$ if $[\underline{\mu}]/[\underline{\lambda}]$ is a k-node.

The next paragraph gives combinatorial properties concerning FLOTW d-partitions.

4.2. Combinatorial properties

First, we give some lemmas:

Lemma 4.1. *Let $\underline{\lambda}$ be a FLOTW d-partition associated to $\{e; v_0, \dots, v_{d-1}\}$. Let ξ be a removable node on a part $\lambda_{j_1}^{(i_1)}$, let $\lambda_{j_2}^{(i_2)}$ be a part of $\underline{\lambda}$ and assume that*:

$$\lambda_{j_2}^{(i_2)} - j_2 + v_{i_2} \equiv \lambda_{j_1}^{(i_1)} - j_1 + v_{i_1} - 1 \ (mod\ e).$$

Then, we have:

$$\lambda_{j_2}^{(i_2)} \geq \lambda_{j_1}^{(i_1)} \iff \lambda_{j_2}^{(i_2)} - j_2 + m^{(i_2)} + 1 \geq \lambda_{j_1}^{(i_1)} - j_1 + m^{(i_1)}.$$

Proof. Using the hypothesis, there exists some $t \in \mathbb{Z}$ such that:

$$\lambda_{j_2}^{(i_2)} - j_2 + v_{i_2} = \lambda_{j_1}^{(i_1)} - j_1 + v_{i_1} - 1 + te.$$

a) We assume that:

$$\lambda_{j_2}^{(i_2)} - j_2 + m^{(i_2)} + 1 \geq \lambda_{j_1}^{(i_1)} - j_1 + m^{(i_1)}.$$

We want to show:

$$\lambda_{j_2}^{(i_2)} \geq \lambda_{j_1}^{(i_1)}.$$

Assume to the contrary that:

$$\lambda_{j_2}^{(i_2)} < \lambda_{j_1}^{(i_1)}.$$

We have:

$$\lambda_{j_2}^{(i_2)} - j_2 + v_{i_2} - i_2\frac{e}{d} + 1 \geq \lambda_{j_1}^{(i_1)} - j_1 + v_{i_1} - i_1\frac{e}{d},$$

then:

$$-1 + te - i_2\frac{e}{d} + 1 \geq -i_1\frac{e}{d}.$$

So, we have:

$$te \geq (i_2 - i_1)\frac{e}{d}.$$

We now distinguish two cases.
<u>If $i_1 \geq i_2$:</u>
Then, we have $t \geq 0$ because $i_2 - i_1 \geq 1 - d$, thus:

$$\lambda_{j_2}^{(i_2)} - j_2 + v_{i_2} \geq \lambda_{j_1}^{(i_1)} - j_1 + v_{i_1} - 1,$$

so:

$$j_1 - j_2 \geq v_{i_1} - v_{i_2}.$$

Now, we use the characterization of the FLOTW d-partitions:

$$\lambda_{j_2}^{(i_2)} \geq \lambda_{j_2+v_{i_1}-v_{i_2}}^{(i_1)}.$$

We obtain:

$$\lambda_{j_2}^{(i_2)} \geq \lambda_{j_1}^{(i_1)},$$

a contradiction.
<u>If $i_1 < i_2$:</u>
Then, we have $t > 0$, thus:

$$\lambda_{j_2}^{(i_2)} - j_2 + v_{i_2} \geq \lambda_{j_1}^{(i_1)} - j_1 + v_{i_1} - 1 + e,$$

so:

$$j_1 - j_2 \geq v_{i_1} - v_{i_2} + e.$$

We use the characterization of the FLOTW d-partitions:

$$\lambda_{j_2}^{(i_2)} \geq \lambda_{j_2+v_{i_1}-v_{i_2}+e}^{(i_1)}.$$

We obtain:

$$\lambda_{j_2}^{(i_2)} \geq \lambda_{j_1}^{(i_1)},$$

a contradiction again.
b) We assume that:

$$\lambda_{j_2}^{(i_2)} - j_2 + m^{(i_2)} + 1 < \lambda_{j_1}^{(i_1)} - j_1 + m^{(i_1)}.$$

We want to show:

$$\lambda_{j_2}^{(i_2)} < \lambda_{j_1}^{(i_1)}.$$

Assume to the contrary that:

$$\lambda_{j_2}^{(i_2)} \geq \lambda_{j_1}^{(i_1)}.$$

As above, we have:

$$te < (i_2 - i_1)\frac{e}{d}.$$

Again, there are two cases to consider.
<u>If $i_1 > i_2$:</u>
Then, we have $t < 0$, and so:

$$\lambda_{j_2}^{(i_2)} - j_2 + v_{i_2} \leq \lambda_{j_1}^{(i_1)} - j_1 + v_{i_1} - 1 - e,$$

thus:

$$j_2 - (j_1 + 1) \geq v_{i_2} - v_{i_1} + e.$$

We use the characterization of the FLOTW d-partitions:

$$\lambda^{(i_1)}_{j_1+1} \geq \lambda^{(i_2)}_{j_1+1+e+v_{i_2}-v_{i_1}},$$

so:

$$\lambda^{(i_1)}_{j_1+1} \geq \lambda^{(i_2)}_{j_2}.$$

Now, ξ is a removable node of $\underline{\lambda}$. Thus, we obtain:

$$\lambda^{(i_1)}_{j_1} > \lambda^{(i_1)}_{j_1+1},$$

a contradiction.
<u>If $i_1 \leq i_2$:</u>
Then, we have $t \leq 0$ because $i_1 - i_2 \geq 1 - d$, thus:

$$\lambda^{(i_2)}_{j_2} - j_2 + v_{i_2} \leq \lambda^{(i_1)}_{j_1} - j_1 + v_{i_1} - 1,$$

so:

$$j_2 - (j_1 + 1) \geq v_{i_2} - v_{i_1}.$$

We use the characterization of the FLOTW d-partitions:

$$\lambda^{(i_1)}_{j_1+1} \geq \lambda^{(i_2)}_{j_1+1+v_{i_2}-v_{i_1}},$$

so:

$$\lambda^{(i_1)}_{j_1+1} \geq \lambda^{(i_2)}_{j_2}.$$

Now, ξ is a removable node of $\underline{\lambda}$, so:

$$\lambda^{(i_1)}_{j_1} > \lambda^{(i_1)}_{j_1+1}.$$

Again, this is a contradiction. □

Lemma 4.2. *Let $\underline{\lambda}$ be a FLOTW d-partition associated to $\{e; v_0, \dots, v_{d-1}\}$. We define*:

$$l_{max} := \max\{\lambda^{(0)}_1, \lambda^{(1)}_1, \dots, \lambda^{(l-1)}_1\}$$

Then, there exists a removable k-node ξ_1, for some k, on a part $\lambda^{(i_1)}_{j_1} = l_{max}$ which satisfies the following property: *if ξ_2 is a $k-1$-node on the border of a part $\lambda^{(i_2)}_{j_2}$, then*:

$$\lambda^{(i_1)}_{j_1} > \lambda^{(i_2)}_{j_2}.$$

Proof. Let $\lambda^{(l_1)}$, …, $\lambda^{(l_r)}$ be the partitions of $\underline{\lambda}$ such that $\lambda^{(l_1)}_1 = \cdots = \lambda^{(l_r)}_1 = l_{\max}$ are the parts of maximal length. Let $k_1, \dots, k_r$ be the residues of the removable nodes $\xi_1, \dots, \xi_r$ on the parts of length $l_{\max}$.

We want to show that there exists $1 \leq i \leq r$ such that there is no node with residue $k_i - 1$ on the border of a part with length $l_{\max}$.

Assume that, for each $1 \leq i \leq r$, there exists a node on the border of a part of length $l_{\max}$ with residue $k_i - 1$. Then, there exists a partition $\lambda^{(l_{s_1})}$, for some $1 \leq s_1 \leq r$, with a $k_1 - 1$-node on the border of a part of length $l_{\max}$:

$$\lambda^{(l_{s_1})} = \underbrace{\begin{array}{|c|c|} \hline \cdots & \cdots \\ \cdots & \cdots \\ \hline & k_1 - 1 \\ \hline \cdots & \cdots \\ \cdots & \cdots \\ \hline & k_{s_1} \\ \hline \cdots & \\ \cdots & \end{array}}_{l_{\max}}$$

We have $s_1 \neq 1$, otherwise the nodes on the border of the parts with length $l_{\max}$ on $\lambda^{(l_{s_1})}$ would describe the set $\{0, \ldots, e-1\}$. This violates the second condition to be a FLOTW d-partition.

We use the same idea for the residue k_{s_1}, there exists $\lambda^{(l_{s_2})}$, for some $1 \leq s_2 \leq r$, as below:

$$\lambda^{(l_{s_2})} = \underbrace{\begin{array}{|c|c|} \hline \cdots & \cdots \\ \cdots & \cdots \\ \hline & k_{s_1} - 1 \\ \hline \cdots & \cdots \\ \cdots & \cdots \\ \hline & k_{s_2} \\ \hline \cdots & \\ \cdots & \end{array}}_{l_{\max}}$$

We have $s_2 \neq s_1$ (for the same reasons as above) and $s_2 \neq 1$, otherwise the nodes on the border of the parts with length $l_{\max}$ on $\lambda^{(l_{s_2})}$ and on $\lambda^{(l_{s_1})}$ would describe all the set $\{0, \ldots, e-1\}$. Continuing in this way, we finally obtain that there exists $1 \leq s_r \leq r$ such that:

$$s_r \notin \{1, s_1, s_2, \ldots, s_{r-1}\}.$$

This is impossible since $s_r \in \{1, \ldots, r\}$ and for all $i \neq j$, we have $s_i \neq s_j$.

So, there exists $0 \leq i \leq r$ such that there is no $k_i - 1$-node on the border of the parts with maximal length. $\square$

Lemma 4.3. *Let $\underline{\lambda}$ be a FLOTW d-partition associated to $\{e; v_0, \ldots, v_{d-1}\}$. Let k and ξ_1 be as in the previous lemma.*

Let $\xi_1, \xi_2, \ldots, \xi_s$ be the removable k-nodes of $\underline{\lambda}$. We assume that they are on parts $\lambda_{j_1}^{(i_1)} \geq \lambda_{j_2}^{(i_2)} \geq \cdots \geq \lambda_{j_s}^{(i_s)}$.

Let $\gamma_1, \gamma_2, \ldots, \gamma_r$ be the $k-1$-nodes on the border of $\underline{\lambda}$. We assume that they are on parts $\lambda_{p_1}^{(l_1)} \geq \lambda_{p_2}^{(l_2)} \geq \ldots \geq \lambda_{p_r}^{(l_r)}$.

We remove the nodes ξ_u such that $\lambda_{j_u}^{(i_u)} > \lambda_{p_1}^{(l_1)}$ from $\underline{\lambda}$. Let $\underline{\lambda}'$ be the resulting d-partition. Then, $\underline{\lambda}'$ is a FLOTW d-partition associated to $\{e; v_0, \ldots, v_{d-1}\}$ and the rank of $\underline{\lambda}'$ is strictly smaller than the rank of $\underline{\lambda}$.

Proof. The rank of $\underline{\lambda}'$ is strictly smaller than the rank of $\underline{\lambda}$ by the previous lemma. We verify the two conditions of the FLOTW d-partitions for $\underline{\lambda}'$.
First Condition:
First, we have to verify that if $\lambda_j^{(i)} > \lambda_{p_1}^{(l_1)}$, $\lambda_j^{(i)} = \lambda_{j+v_{i+1}-v_i}^{(i+1)}$ and if we remove a node from $\lambda_j^{(i)}$, then, we also remove a node from $\lambda_{j+v_{i+1}-v_i}^{(i+1)}$. Observe that:

$$\lambda_{j+v_{i+1}-v_i}^{(i+1)} - (j + v_{i+1} - v_i) + v_{i+1} = \lambda_j^{(i)} - j + v_i.$$

Thus, the residue on the border of $\lambda_{j+v_{i+1}-v_i}^{(i+1)}$ is k. Moreover, the associated node is a removable one, otherwise we would have:

$$\lambda_{j+v_{i+1}-v_i+1}^{(i+1)} > \lambda_{j+1}^{(i)},$$

contradicting to our assumption that $\underline{\lambda}$ is a FLOTW d-partition.

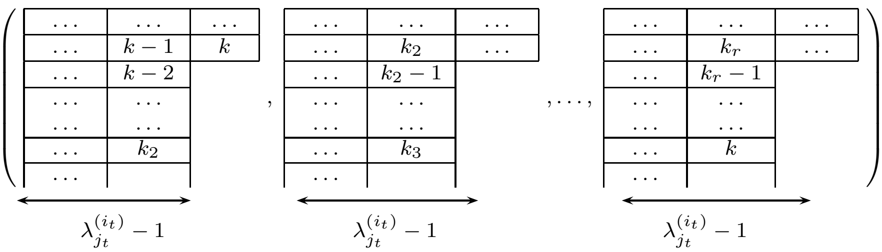
Since $\lambda_j^{(i)} = \lambda_{j+v_{i+1}-v_i}^{(i+1)}$ and $\lambda_{j+v_{i+1}-v_i}^{(i+1)} > \lambda_{p_1}^{(l_1)}$, the k-node on the border of $\lambda_{j+v_{i+1}-v_i}^{(i+1)}$ must be removed. So, the first condition of the FLOTW d-partitions holds for $\underline{\lambda}'$.
Second Condition :
The only problem may arrive when there exists $t \in \{1, \ldots, s\}$ such that:

- if we delete the nodes $\xi_1, \xi_2, \ldots, \xi_{t-1}$, the resulting d-partition $\underline{\lambda} \setminus \{\xi_1, \ldots, \xi_{t-1}\}$ satisfies the second condition.
- $\underline{\lambda} \setminus \{\xi_1, \ldots, \xi_t\}$ doesn't satisfy the second condition.

This implies that the set of residues of the nodes on the border of the parts of $\underline{\lambda}$ with length $\lambda_{j_t}^{(i_t)} - 1$ is equal to the following set:

$$\{0, \ldots, e-1\} \setminus \{k-1\}.$$

Note that the second condition is satisfied for all the other lengths than $\lambda_{j_t}^{(i_t)} - 1$. For example, this problem occurs when $\underline{\lambda}$ is as follows:



We want to show that among the residues on the border of the parts of $\underline{\lambda}'$ with length $\lambda_{j_t}^{(i_t)} - 1$, k does not occur.

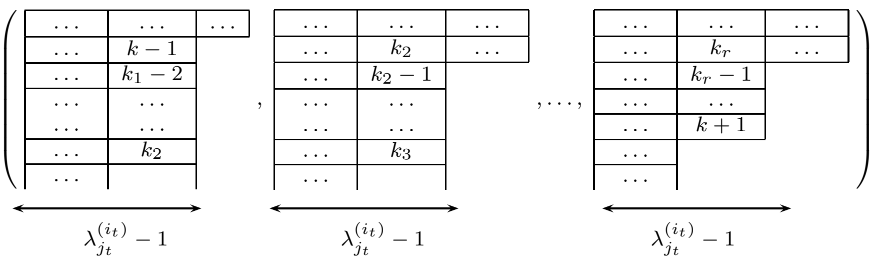
There exists at least one k-node on the border of a part of $\underline{\lambda}$ with length $\lambda_{j_t}^{(i_t)} - 1$. Such a k-node must be a removable one. If otherwise, we would have a $k-1$-node on the border of a part of $\underline{\lambda}$ with length $\lambda_{j_t}^{(i_t)} - 1$ contradicting to our assumption.

We have $\lambda_{j_t}^{(i_t)} - 1 > \lambda_{p_1}^{(l_1)}$ because there is no $k-1$-node on parts with length $\lambda_{j_t}^{(i_t)} - 1$. So, all the k-nodes on the border of parts with length $\lambda_{j_1}^{(i_1)} - 1$ must be removed. Thus, we do remove all of them.

Then, the set of residues of the nodes on the border of parts of $\underline{\lambda}'$ with length $\lambda_{j_t}^{(i_t)} - 1$ is equal to:

$$\{0, \dots, e-1\} \setminus \{k\}.$$

Keeping the above example, $\underline{\lambda}'$ is given by:



Now, we can delete the remaining removable k-nodes of length $\lambda_{j_t}^{(i_t)}$ without violating the second condition. Repeating the same argument for those ξ_u with $\lambda_{j_u}^{(i_u)} > \lambda_{p_1}^{(l_1)}$, we conclude that the second condition holds.
Thus, $\underline{\lambda}'$ is a FLOTW d-partition. $\square$

Thanks to this lemma, we can now associate to each FLOTW d-partition a residue sequence which have "good" properties according to the a-function.

Let $\underline{\lambda}$ be a FLOTW d-partition associated to $\{e; v_0, \dots, v_{d-1}\}$. By using Lemma 4.2, there exists a removable node ξ_1 with residue k on a part $\lambda_{j_1}^{(i_1)}$ with maximal length, such that there doesn't exist a $k-1$-node on the border of a part with the same length as the length of the part of ξ_1.

Let $\gamma_1,\ \gamma_2, \dots, \gamma_r$ be the $k-1$-nodes on the border of $\underline{\lambda}$, on parts $\lambda_{p_1}^{(l_1)} \geq \lambda_{p_2}^{(l_2)} \geq \dots \geq \lambda_{p_r}^{(l_r)}$. Then, Lemma 4.1 implies that:

$$\lambda_{j_1}^{(i_1)} - j_1 + m^{(i_1)} - 1 > \lambda_{p_k}^{(l_k)} - p_k + m^{(l_k)} \qquad k = 1, \dots, r.$$

As in Lemma 4.3, let $\xi_1,\ \xi_2, \dots, \xi_u$ be the removable k-nodes on the border of $\underline{\lambda}$ on parts $\lambda_{j_1}^{(i_1)} \geq \lambda_{j_2}^{(i_2)} \geq \dots \geq \lambda_{j_u}^{(i_u)}$ such that:

$$\lambda_{j_u}^{(i_u)} > \lambda_{p_1}^{(l_1)}.$$

By Lemma 4.1, we have:

$$\lambda_{j_t}^{(i_t)} - j_t + m^{(i_t)} - 1 > \lambda_{p_k}^{(l_k)} - p_k + m^{(l_k)} \qquad t = 1, \dots, u \qquad k = 1, \dots, r.$$

We remove the nodes $\xi_1, \xi_2, \dots, \xi_u$ from $\underline{\lambda}$. Let $\underline{\lambda}'$ be the resulting d-partition. $\underline{\lambda}'$ is a FLOTW d-partition by Lemma 4.3.

Now the $a-$sequence of residues of $\underline{\lambda}$ is defined recursively as follows:

Definition 4.1. $a - \text{sequence}(\underline{\lambda}) = a - \text{sequence}(\underline{\lambda}'), \underbrace{k, \dots, k}_{u}.$

Note that if we have $a - \text{sequence}(\underline{\lambda}') = a - \text{sequence}(\underline{\lambda}''), \underbrace{k', \dots, k'}_{u'}$ for some $k' \in \{0, 1, \dots, e-1\}$ and $u' \in \mathbb{N}$, then we have $k \neq k'$. Indeed, let ξ'_1 be a removable k'-node on a part ${\lambda'}_{a_1}^{(b_1)}$ with maximal length, such that there doesn't exist a $k'-1$-node on the border of a part of length ${\lambda'}_{a_1}^{(b_1)}$. Assume that $k = k'$. We can't have ${\lambda'}_{a_1}^{(b_1)} = \lambda_{j_1}^{(i_1)}$, otherwise $\xi'_1 = \xi_t$ for some $t \in \{1, 2, \dots, u\}$, a contradiction. Thus, we have ${\lambda'}_{a_1}^{(b_1)} < \lambda_{j_1}^{(i_1)}$ and as $\xi'_1 \neq \xi_t$ for all $t \in \{1, 2, \dots, u\}$, there exists a $k'-1$-node on the border of a part of length ${\lambda'}_{a_1}^{(b_1)}$ in $\underline{\lambda}'$, a contradiction again.

Example 4.2. We assume that $d = 2$, $v_0 = 0$, $v_1 = 1$ and $e = 4$. The FLOTW 2-partitions are the 2-partitions $(\lambda^{(0)}, \lambda^{(1)})$ which satisfy:

- for all $i \in \mathbb{N}_{>0}$:

$$\lambda_i^{(0)} \geq \lambda_{i+1}^{(1)},$$
$$\lambda_i^{(1)} \geq \lambda_{i+3}^{(0)};$$

- for all $k > 0$, among the residues appearing at the right ends of the length k rows of $\underline{\lambda}$, at least one element of $\{0, 1, 2, 3\}$ does not occur.

We consider the 2-partition $\underline{\lambda} = (2.2, 2.2.1)$ with the following diagram:

$$\left(\begin{array}{|c|c|} \hline 0 & 1 \\ \hline 3 & 0 \\ \hline \end{array} \; , \; \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 0 & 1 \\ \hline 3 \\ \cline{1-1} \end{array} \right)$$

$\underline{\lambda}$ is a FLOTW 2-partition.

We search the $a-$sequence of $\underline{\lambda}$: we have to find $k \in \{0, 1, 2, 3\}$, $u \in \mathbb{N}_{>0}$ and a 2-partition $\underline{\lambda}'$ such that:

$$a - \text{sequence}(\underline{\lambda}) = a - \text{sequence}(\underline{\lambda}'), \underbrace{k, \dots, k}_{u}.$$

The parts with maximal length are the parts with length 2 and the residues of the removable nodes on these parts are 1 and 0.

For $k = 1$, we have a part of length 2 with a node on the border which have $k - 1 = 0 \pmod{e}$ as a residue.

So, we have to take $k=0$ and we can remove this node as 3-node(s) on the border have maximal length 1. There is no other removable 0-node in $\underline{\lambda}$, so:

$$a-\text{sequence}(\underline{\lambda}) = a-\text{sequence}(2.1, 2.2.1), 0.$$

We can verify that the 2-partition $(2.1, 2.2.1)$ is a FLOTW 2-partition.

Now, the removable nodes on the part with maximal length have 1 as a residue and there is no 0-node, so:

$$a-\text{sequence}(\underline{\lambda}) = a-\text{sequence}(1.1, 2.1.1), 1, 1, 0.$$

Repeating the same procedure, we obtain:

$$a-\text{sequence}(\underline{\lambda}) = 1, 0, 0, 3, 3, 2, 1, 1, 0.$$

Let $\underline{\lambda}$ be a d-composition of rank n and let k be a residue. Following the notations of paragraphe 4.1, let $\underline{\mu}$ be a d-composition of rank $n+1$ such that:

$$\underline{\lambda} \underset{(j,p)}{\overset{k}{\mapsto}} \underline{\mu}.$$

Then, we write:

$$\underline{\lambda} \underset{(j,p)}{\overset{k\text{-opt}}{\mapsto}} \underline{\mu},$$

if we have:

$$\lambda_{j'}^{(p')} - j' + m^{(p')} \leq \lambda_j^{(p)} - j + m^{(p)},$$

for all d-composition $\underline{\mu'}$ of rank $n+1$ such that $\underline{\lambda} \underset{(j',p')}{\overset{k}{\mapsto}} \underline{\mu'}$.

Remark 4.1. Let $k \in \{0, 1, \dots, e-1\}$. Assume that $\underline{\lambda}$ is a d-partition and that $\underline{\mu}$ and $\underline{\mu'}$ are d-compositions such that:

$$\underline{\lambda} \underset{(j,p)}{\overset{k\text{-opt}}{\mapsto}} \underline{\mu} \qquad \text{and} \qquad \underline{\lambda} \underset{(j',p')}{\overset{k\text{-opt}}{\mapsto}} \underline{\mu'}$$

This implies that there exists $t \in \mathbb{Z}$ such that:

$$\lambda_{j'}^{(p')} - j' + v_{p'} = \lambda_j^{(p)} - j + v_p + te \equiv k \pmod e.$$

Moreover, we have:

$$\lambda_{j'}^{(p')} - j' + m^{(p')} = \lambda_j^{(p)} - j + m^{(p)}.$$

We obtain:

$$te = (p' - p)\frac{e}{d}.$$

Hence, we have $p = p'$ and since $\underline{\lambda}$ is a d-partition, we have $(j,p) = (j',p')$ and $\underline{\mu'} = \underline{\mu}$.

Proposition 4.1. *Let $\underline{\lambda}$ be a FLOTW d-partition of rank n and let*:

$$a-\text{sequence}(\underline{\lambda})=s_1,s_2,\dots,s_n.$$

Then, there exists a sequence of FLOTW d-partitions $\underline{\lambda}[0]=\underline{\emptyset}$, $\underline{\lambda}[1],\dots,\underline{\lambda}[n]=\underline{\lambda}$ such that for all $l\in\{0,\dots,n-1\}$, we have:

$$\underline{\lambda}[l]\underset{(j_l,p_l)}{\overset{s_l\text{-}opt}{\mapsto}}\underline{\lambda}[l+1].$$

Proof. Let $\underline{\lambda}$ be a FLOTW d-partition. Assume that:

$$a-\text{sequence}(\underline{\lambda})=a-\text{sequence}(\underline{\lambda}')\,\underbrace{i_s,\dots,i_s}_{a_s},$$

where $\underline{\lambda}'$ is the FLOTW d-partition as in Definition 4.1. Define $\underline{\lambda}[n-a_s]:=\underline{\lambda}'$. Then, there exist d-compositions $\underline{\lambda}[i]$, $i=n-a_s+1,\dots,n$ such that

$$\underline{\lambda}[n-a_s]\overset{i_s\text{-opt}}{\mapsto}\underline{\lambda}[n-a_s+1]\overset{i_s\text{-opt}}{\mapsto}\cdots\overset{i_s\text{-opt}}{\mapsto}\underline{\lambda}[n-1]\overset{i_s\text{-opt}}{\mapsto}\underline{\lambda}[n].$$

(we omit the indices of the nodes here to simplify the notations.)

We claim that $\underline{\lambda}[n]=\underline{\lambda}$. Indeed, by the discussion above Definition 4.1, the nodes of $\underline{\lambda}[n]/\underline{\lambda}[n-a_s]$ are precisely the nodes of $\underline{\lambda}/\underline{\lambda}'$. Note also that in the above graph, the nodes are succesively added on the greatest part where we have an addable i_s-node. Moreover, as $\underline{\lambda}$ is a d-partition, it follows that for all $i\in\{n-a_s,n-a_s+1,\dots,n\}$, $\underline{\lambda}[i]$ is a d-partition.

Continuing in this way, there exist d-partitions $\underline{\lambda}[l]$ $(l=1,\dots,n)$ such that:

$$\underline{\emptyset}\overset{i_1\text{-opt}}{\mapsto}\underline{\lambda}[1]\overset{i_1\text{-opt}}{\mapsto}\cdots\overset{i_1\text{-opt}}{\mapsto}\underline{\lambda}[a_1]\overset{i_2\text{-opt}}{\mapsto}\cdots\overset{i_2\text{-opt}}{\mapsto}\underline{\lambda}[a_1+a_2]\overset{i_3-\text{opt}}{\mapsto}\cdots\overset{i_s\text{-opt}}{\mapsto}\underline{\lambda}[n]. \tag{1}$$

By Lemma 4.3 and by definition of the $a-$sequence, $\underline{\lambda}[a_1]$, $\underline{\lambda}[a_1+a_2],\dots$, $\underline{\lambda}[n]$ are FLOTW d-partitions. Now, assume that there exists a d-partition $\underline{\mu}$ which appears in (1) and which is not a FLOTW d-partition. Then, there exists $r\in\{1,\dots,s\}$ such that:

$$\underline{\lambda}[a_1+\dots+a_{r-1}]\overset{i_r\text{-opt}}{\mapsto}\cdots\overset{i_r\text{-opt}}{\mapsto}\underline{\mu}\overset{i_r\text{-opt}}{\mapsto}\cdots\overset{i_r\text{-opt}}{\mapsto}\underline{\lambda}[a_1+\dots+a_r] \tag{2}$$

We denote $\underline{\nu}:=\underline{\lambda}[a_1+\dots+a_{r-1}]$. Then, we have two cases to consider.

- If $\underline{\mu}$ violates the first condition to be a FLOTW d-partition, then there exist $j\in\mathbb{N}_{>0}$ and $i\in\{0,1,\dots,d-2\}$ such that $\mu_j^{(i)}<\mu_{j+v_{i+1}-v_i}^{(i+1)}$ (the case $\mu_j^{(d-1)}<\mu_{j+e+v_0-v_{d-1}}^{(0)}$ is similar to this one). It implies that we have added an i_r-node on $\nu_{j+v_{i+1}-v_i}^{(i+1)}$ and that $\nu_j^{(i)}=\nu_{j+v_{i+1}-v_i}^{(i+1)}$. We obtain:

$$\nu_j^{(i)}-j+m^{(i)}>\nu_{j+v_{i+1}-v_i}^{(i+1)}-(j+v_{i+1}-v_i)+m^{(i+1)}.$$

It implies that we can't add any i_r-node on $\mu_j^{(i)}$ in (2) because we add k-nodes by decreasing order with respect to $\nu_{j'}^{(i')}-j'+m^{(i')}$. Thus, $\underline{\lambda}[a_1+\dots+a_r]$ is not a FLOTW d-partition. This is a contradiction.

• Assume that $\underline{\mu}$ is the first place where the second condition to be a FLOTW d-partition is violated. Thus, there exist $a_1 + \cdots + a_{r-1} < j < a_1 + \cdots + a_r$ such that $\underline{\mu} = \underline{\mu}[j]$ and $\underline{\mu}[j-1]$ is a FLOTW d-partition. In (2), we have added an i_r-node on a part $\nu_j^{(i)}$ and the set of residues appearing at the right ends of parts of length $\nu_j^{(i)} + 1$ of $\underline{\mu}$ is equal to $\{0, \ldots, e-1\}$. If there is a $i_r - 1$-node on the border of a part of length greater than $\nu_j^{(i)}$ which will be occupied before reaching $\underline{\lambda}[a_1 + \cdots + a_r]$, Lemma 4.1 implies that this node had to be occupied before the i_r-node on the part $\nu_j^{(i)}$ is added. Thus, the set of residues appearing at the right ends of parts of length $\nu_j^{(i)} + 1$ in $\underline{\lambda}[a_1 + \cdots + a_r]$ is equal to $\{0, \ldots, e-1\}$. This is a contradiction as $\underline{\lambda}[a_1 + \cdots + a_r]$ is a FLOTW d-partition.

□

Now, we associate a graph with a FLOTW d-partition as follows:

Definition 4.2. Let $\underline{\lambda}$ be a FLOTW d-partition of rank n, let $s_1, s_2, \ldots, s_n$ be its $a-$ sequence of residues. Then by Remark 4.1 and Proposition 4.1, there exist unique d-partitions $\underline{\lambda}[l]$ $(l = 1, \ldots, n)$ such that:

$$\underline{\emptyset} \underset{(j_1,p_1)}{\overset{s_1\text{-opt}}{\mapsto}} \underline{\lambda}[1] \underset{(j_2,p_2)}{\overset{s_2\text{-opt}}{\mapsto}} \underline{\lambda}[2] \cdots \underset{(j_n,p_n)}{\overset{s_n\text{-opt}}{\mapsto}} \underline{\lambda}[n] = \underline{\lambda},$$

where $\underline{\emptyset}$ is the empty d-partition. We call this the a-graph of $\underline{\lambda}$.

Example 4.3. Assume that $d = 2$, $v_0 = 0$, $v_1 = 1$ and $e = 4$. Let $\underline{\lambda} = (2.2, 2.2.1)$.

$$a - \text{sequence}(\underline{\lambda}) = 1, 0, 0, 3, 3, 2, 1, 1, 0.$$

Then, the a-graph associated to $\underline{\lambda}$ is the following one:

$$(\emptyset, \emptyset) \underset{(1,1)}{\overset{1\text{-opt}}{\mapsto}} (\emptyset, 1) \underset{(1,0)}{\overset{0\text{-opt}}{\mapsto}} (1, 1) \underset{(2,1)}{\overset{0\text{-opt}}{\mapsto}} (1, 1.1) \underset{(2,0)}{\overset{3\text{-opt}}{\mapsto}} (1.1, 1.1) \underset{(3,1)}{\overset{3\text{-opt}}{\mapsto}} (1.1, 1.1.1) \underset{(1,1)}{\overset{2\text{-opt}}{\mapsto}}$$
$$\underset{(1,1)}{\overset{2\text{-opt}}{\mapsto}} (1.1, 2.1.1) \underset{(1,0)}{\overset{1\text{-opt}}{\mapsto}} (2.1, 2.1.1) \underset{(2,1)}{\overset{1\text{-opt}}{\mapsto}} (2.1, 2.2.1) \underset{(2,0)}{\overset{0\text{-opt}}{\mapsto}} (2.2, 2.2.1)$$

Now, we deal with a fundamental property concerning the a-graph o f a FLOTW d-partition. First, we have to show some properties about a-functions.

Definition 4.3. Let $\underline{\mu}$ and $\underline{\nu}$ be d-compositions of rank n. Let $\mathbf{B}_{\underline{\mu}}$ and $\mathbf{B}_{\underline{\nu}}$ be two ordinary symbols of $\underline{\mu}$ et $\underline{\nu}$ with the same height. Let $\mathbf{B}_{\underline{\mu}}[m]'$, $\mathbf{B}_{\underline{\nu}}[m]'$ be as in paragraph 3.1. Then we write:

$$\underline{\mu} \prec \underline{\nu},$$

if:

$$\sum_{\substack{0\leq i\leq j<d\\ (a,b)\in B_{\underline{\mu}}'^{(i)}\times B_{\underline{\mu}}'^{(j)}\\ a>b \text{ if } i=j}} \min\{a,b\} - \sum_{\substack{0\leq i,j<d\\ a\in B_{\underline{\mu}}'^{(i)}\\ 1\leq k\leq a}} \min\{k,m^{(j)}\} < \sum_{\substack{0\leq i\leq j<d\\ (a,b)\in B_{\underline{\nu}}'^{(i)}\times B_{\underline{\nu}}'^{(j)}\\ a>b \text{ if } i=j}} \min\{a,b\} - \sum_{\substack{0\leq i,j<d\\ a\in B_{\underline{\nu}}'^{(i)}\\ 1\leq k\leq a}} \min\{k,m^{(j)}\}.$$

In particular, if $\underline{\mu}$ and $\underline{\nu}$ are some d-partitions, we have:

$$\underline{\mu}\prec\underline{\nu} \iff a(\underline{\mu})<a(\underline{\nu}).$$

The following results give the consequences of the properties showed in this section in terms of a-function:

Lemma 4.4. *Let $\underline{\lambda}$ be a d-composition of rank n, let $\boldsymbol{B}:=(B^{(1)},\ldots,B^{(l-1)})$ be an ordinary symbol of $\underline{\lambda}$, let β_1 and β_2 be two elements of $\boldsymbol{B}[m]'$, we assume that*:

$$\beta_1<\beta_2.$$

Let $l\in\mathbb{N}_{>0}$. We add l nodes to $\underline{\lambda}$ on the part associated to β_1. Let $\underline{\mu}$ be the resulting d-composition of rank n+l. We add l nodes to $\underline{\lambda}$ on the part associated to β_2. Let $\underline{\nu}$ be the resulting d-composition of rank $n+l$. Let $\boldsymbol{B}_{\underline{\mu}}$ and $\boldsymbol{B}_{\underline{\nu}}$ be the two ordinary symbols of $\underline{\mu}$ and $\underline{\nu}$ with the same height as $\boldsymbol{B}$ and let $\boldsymbol{B}_{\underline{\mu}}[m]'$, $\boldsymbol{B}_{\underline{\nu}}[m]'$ be as in the paragraph 3.1. *Then, we have*:

$$\underline{\nu}\prec\underline{\mu}.$$

In particular, if $\underline{\mu}$ and $\underline{\nu}$ are some d-partitions, then:

$$a(\underline{\mu})>a(\underline{\nu}).$$

Proof. We assume that β_1 is on $B^{(i_1)}$ and that β_2 is on $B^{(i_2)}$. Then, we have:

$$\begin{aligned}&\sum_{\substack{0\leq i\leq j<d\\ (a,b)\in B_{\underline{\mu}}'^{(i)}\times B_{\underline{\mu}}'^{(j)}\\ a>b \text{ if } i=j}} \min\{a,b\} - \sum_{\substack{0\leq i,j<d\\ a\in B_{\underline{\mu}}'^{(i)}\\ 1\leq t\leq a}} \min\{t,m^{(j)}\}\\ &-\left(\sum_{\substack{0\leq i\leq j<d\\ (a,b)\in B'^{(i)}\times B'^{(j)}\\ a>b \text{ if } i=j}} \min\{a,b\} - \sum_{\substack{0\leq i,j<d\\ a\in B'^{(i)}\\ 1\leq t\leq a}} \min\{t,m^{(j)}\}\right)\\ &=\sum_{\substack{a\in B^{(i)}\\ 0\leq i\leq d-1\\ a\neq\beta_1 \text{ if } i=i_1}} (\min\{a,\beta_1+l\}-\min\{a,\beta_1\}) - \sum_{\substack{0\leq j\leq d-1\\ \beta_1<t\leq\beta_1+l}} \min\{t,m^{(j)}\}.\end{aligned}$$

and we have an analogous formula for $\underline{\nu}$.
Now, for $a \in B^{(j)}$, $j = 0, \dots, d-1$, we have:

$$\min\{a, \beta_1 + l\} - \min\{a, \beta_1\} = \begin{cases} 0 & \text{if } a < \beta_1, \\ a - \beta_1 & \text{if } \beta_1 \leq a < \beta_1 + l, \\ l & \text{if } a \geq \beta_1 + l. \end{cases}$$

As $\beta_1 < \beta_2$, we have:

$$\sum_{\substack{a \in B^{(i)} \\ 0 \leq i \leq d-1 \\ a \neq \beta_1 \text{ if } i = i_1}} (\min\{a, \beta_1 + l\} - \min\{a, \beta_1\}) > \sum_{\substack{a \in B^{(i)} \\ 0 \leq i \leq d-1 \\ a \neq \beta_2 \text{ if } i = i_2}} (\min\{a, \beta_2 + l\} - \min\{a, \beta_2\}).$$

Moreover, $\min\{t, m^{(j)}\} \leq \min\{t + \beta_2 - \beta_1, m^{(j)}\}$ implies that:

$$\sum_{\substack{0 \leq j \leq d-1 \\ \beta_1 < t \leq \beta_1 + l}} \min\{t, m^{(j)}\} \leq \sum_{\substack{0 \leq j \leq d-1 \\ \beta_2 < t \leq \beta_2 + l}} \min\{t, m^{(j)}\}.$$

□

Proposition 4.2. *Let $\underline{\lambda}[n]$ be a FLOTW d-partition of rank n and $s_1, s_2, \dots, s_n$ be its* $a-$sequence *of residues. We consider the a-graph of $\underline{\lambda}[n]$:*

$$(\emptyset, \emptyset) \underset{(j_1,p_1)}{\overset{s_1\text{-}opt}{\mapsto}} \underline{\lambda}[1] \underset{(j_2,p_2)}{\overset{s_2\text{-}opt}{\mapsto}} \underline{\lambda}[2] \cdots \underset{(j_n,p_n)}{\overset{s_n\text{-}opt}{\mapsto}} \underline{\lambda}[n],$$

where all the d-partitions appearing in this graph are FLOTW d-partitions. Then, if we have another graph of d-compositions:

$$(\emptyset, \emptyset) \underset{(j'_1,p'_1)}{\overset{s_1}{\mapsto}} \underline{\mu}[1] \underset{(j'_2,p'_2)}{\overset{s_2}{\mapsto}} \underline{\mu}[2] \cdots \underset{(j'_n,p'_n)}{\overset{s_n}{\mapsto}} \underline{\mu}[n],$$

we have $\underline{\lambda}[n] \prec \underline{\mu}[n]$ if $\underline{\lambda}[n] \neq \underline{\mu}[n]$.

Proof. We will argue by induction on $n - r \in \mathbb{N}$. Assume that $\underline{\lambda}[r-1] = \underline{\mu}[r-1]$. We want to show that if we have the following graphs:

$$(1) \qquad \underline{\lambda}[r-1] \underset{(j_r,p_r)}{\overset{s_r\text{-opt}}{\mapsto}} \underline{\lambda}[r] \underset{(j_{r+1},p_{r+1})}{\overset{s_{r+1}\text{-opt}}{\mapsto}} \cdots \underset{(j_n,p_n)}{\overset{s_n\text{-opt}}{\mapsto}} \underline{\lambda}[n]$$

$$(2) \qquad \underline{\mu}[r-1] \underset{(j'_r,p'_r)}{\overset{s_r}{\mapsto}} \underline{\mu}[r] \underset{(j'_{r+1},p'_{r+1})}{\overset{s_{r+1}}{\mapsto}} \cdots \underset{(j'_n,p'_n)}{\overset{s_n}{\mapsto}} \underline{\mu}[n]$$

then $\underline{\lambda}[n] \prec \underline{\mu}[n]$ if $\underline{\lambda}[n] \neq \underline{\mu}[n]$.
<u>Assume that $n - r = 0$</u>:
Assume that $\underline{\lambda}[n] \neq \underline{\mu}[n]$. To simplify the notations, we write $\underline{\lambda} := \underline{\lambda}[n-1] = \underline{\mu}[n-1]$. We have:

$$\underline{\lambda} \underset{(j_n,p_n)}{\overset{s_n\text{-opt}}{\mapsto}} \underline{\lambda}[n] \qquad \text{and} \qquad \underline{\lambda} \underset{(j'_n,p'_n)}{\overset{s_n}{\mapsto}} \underline{\mu}[n]$$

As $\underline{\lambda}$ is a d-partition, by Remark 4.1, we have

$$\lambda_j^{(p)} - j + m^{(p)} > \lambda_{j'}^{(p')} - j' + m^{(p')}$$

Let $\mathbf{B}_{\underline{\lambda}}$ be an ordinary symbol associated to $\underline{\lambda}$. Then, $\underline{\lambda}[n]$ and $\underline{\mu}[n]$ are obtained from $\underline{\lambda}$ by adding a node on parts associated to two elements β_1 and β_2 of $\mathbf{B}_{\underline{\lambda}}[m]'$. We have:

$$\beta_1 > \beta_2.$$

We are in the setting of Lemma 4.4, hence:

$$\underline{\lambda}[n] \prec \underline{\mu}[n].$$

Assume that $n - r > 0$:
Let t be such that $r - 1 \leq t < n$. If the residues of the right ends of the parts (j_r, p_r) and (j'_r, p'_r) of $\underline{\mu}[t]$ are the same, we say that t is admissible. Let $t < t'$ be the two first consecutive admissible indices. Suppose that the lengths of the part (j_r, p_r) (resp. (j'_r, p'_r)) increases by N (resp. N') between $\underline{\mu}[t]$ and $\underline{\mu}[t']$. If the first node added to $\underline{\mu}[t]$ is on the (j_r, p_r)-part, we do nothing. Otherwise, we add N nodes to the (j'_r, p'_r)-part and N' nodes to the (j_r, p_r)-part instead of adding N nodes to the (j_r, p_r)-part and N' nodes to the (j'_r, p'_r)-part.

Next we consider the consecutive admissible indices $t' < t''$ and repeat the same procedure until we reach the final consecutive admissible indices. Then we get a new graph as follows:

$$\underline{\lambda}[r-1] \underset{(j_r, p_r)}{\overset{s_r\text{-opt}}{\mapsto}} \underline{\lambda}[r] \underset{(j''_{r+1}, p''_{r+1})}{\overset{s_{r+1}}{\mapsto}} \cdots \underset{(j''_n, p''_n)}{\overset{s_n}{\mapsto}} \underline{\nu}[n], \tag{3}$$

In $\underline{\nu}[n]$ and $\underline{\mu}[n]$, only two parts ((j_r, p_r)-part and (j'_r, p'_r)-part) may differ.

- If $(j_r, p_r) = (j'_r, p'_r)$, we conclude by induction.
- If $(j_r, p_r) \neq (j'_r, p'_r)$, to simplify, we write $\underline{\lambda} := \underline{\lambda}[n]$, $\underline{\mu} := \underline{\mu}[n]$ and $\underline{\nu} := \underline{\nu}[n]$. Assume that $\underline{\nu} \neq \underline{\mu}$, then we are in the setting of Lemma 4.4: let $\underline{\gamma} = (\gamma^{(0)}, \ldots, \gamma^{(l-1)})$ be the d-composition defined by:

$$\begin{aligned} \gamma_j^{(p)} &:= \mu_j^{(p)} && \text{if } (j,p) \neq (j'_r, p'_r), \\ \gamma_{j'_r}^{(p'_r)} &:= \nu_{j'_r}^{(p'_r)}. \end{aligned}$$

Then, $\underline{\mu}$ is obtained from $\underline{\gamma}$ by adding nodes on $\gamma_{j'_r}^{(p'_r)}$ where as $\underline{\nu}$ is obtained from $\underline{\gamma}$ by adding the same number of nodes on $\gamma_{j_r}^{(p_r)}$. Now, we know that:

$$\lambda_{j_r}^{(p_r)} - j_r + m^{(p_r)} > \lambda_{j'_r}^{(p'_r)} - j'_r + m^{(p'_r)}.$$

This implies:

$$\gamma_{j_r}^{(p_r)} - j_r + m^{(p_r)} > \gamma_{j'_r}^{(p'_r)} - j'_r + m^{(p'_r)}.$$

This follows from the fact that the residue associated to (j_r, p_r) and (j'_r, p'_r) in $\underline{\lambda}$ are the same.

Thus, $\underline{\nu}$ and $\underline{\mu}$ are obtained from $\underline{\gamma}$ by adding the same number of nodes on parts associated to two elements β_1 and β_2 of $\mathbf{B}_\gamma[m]'$ where $\mathbf{B}_\gamma$ is an ordinary symbol associated to γ. The above discussion shows that:

$$\beta_1 > \beta_2.$$

So, by Lemma 4.4, we have:

$$\underline{\nu} \prec \underline{\mu}.$$

By induction hypothesis, we have :

$$\underline{\lambda} \prec \underline{\nu} \qquad \text{or} \qquad \underline{\lambda} = \underline{\nu}.$$

We conclude that:

$$\underline{\lambda} \prec \underline{\mu}.$$

□

The following proposition is now clear.

Proposition 4.3. *Let $\underline{\lambda}[n]$ be a FLOTW d-partition of rank n and let $s_1, s_2, \ldots, s_n$ be its $a-$sequence of residues. We consider the a-graph of $\underline{\lambda}[n]$:*

$$(\emptyset,\emptyset) \underset{(j_1,p_1)}{\overset{s_1\text{-}opt}{\mapsto}} \underline{\lambda}[1] \underset{(j_2,p_2)}{\overset{s_2\text{-}opt}{\mapsto}} \underline{\lambda}[2] \cdots \underset{(j_n,p_n)}{\overset{s_n\text{-}opt}{\mapsto}} \underline{\lambda}[n]$$

then, if we have another graph with d-partitions:

$$(\emptyset,\emptyset) \underset{(j'_1,p'_1)}{\overset{s_1}{\mapsto}} \underline{\mu}[1] \underset{(j'_2,p'_2)}{\overset{s_2}{\mapsto}} \underline{\mu}[2] \cdots \underset{(j'_n,p'_n)}{\overset{s_n}{\mapsto}} \underline{\mu}[n]$$

we have $a(\underline{\lambda}[n]) < a(\underline{\mu}[n])$ if $\underline{\lambda}[n] \neq \underline{\mu}[n]$.

In the following paragraph, we see consequences of this result: we give an interpretation of the parametrization of the simple modules by Foda et al. for Ariki-Koike algebras in terms of a-functions.

4.3. Principal result

First, recall the FLOTW order given in the paragraph 2.3, we have the following proposition.

Proposition 4.4. *Let $\underline{\lambda}$ be a FLOTW d-partition and let $\gamma = (a, b, c)$, $\gamma' = (a', b', c')$ be two nodes of $\underline{\lambda}$ with the same residue. Then:*

$$b - a + m^{(c)} > b' - a' + m^{(c')} \iff \gamma \textit{ is below } \gamma' \textit{ with respect to the FLOTW order.}$$

Proof. As γ and γ' have the same residue, there exists $t \in \mathbb{Z}$ such that:

$$b - a + v_c = b' - a' + v_{c'} + te.$$

Then, we have:

$$b - a + m^{(c)} > b' - a' + m^{(c')} \iff (c - c')\frac{e}{d} < te.$$

Then $t \geq 0$ and if $t = 0$, we have $c' > c$. So, γ is below γ'. □

Now, we consider the quantum group $\mathcal{U}_q$ of type $A^{(1)}_{e-1}$ as in the paragraph 2.2. Let $\overline{\mathcal{M}}$ be the $\mathcal{U}_q$-module generated by the empty d-partition with respect to the JMMO action (see paragraph 2.3).

First, we have the following result which gives the action of the divided powers on the multipartitions with respect to the JMMO action. The demonstration is analogous to the case of the AM action (see for example [28, Lemma 6.15] for $d = 1$), we give the proof for the convenience of the reader.

Proposition 4.5. *Let $\underline{\lambda}$ be a d-partition, $i \in \{0, \dots, e-1\}$ and j a positive integer. Then*:

$$f_i^{(j)}\underline{\lambda} = \sum q^{\overline{N}_i^b(\underline{\lambda},\underline{\mu})}\underline{\mu},$$

where the sum is taken over all the $\underline{\mu}$ which satisfy:

$$\underline{\lambda}\underbrace{\overset{i}{\mapsto}\cdots\overset{i}{\mapsto}}_{j}\underline{\mu},$$

and where:

$$\overline{N}_i^b(\underline{\lambda},\underline{\mu}) := \sum_{\gamma \in [\underline{\mu}]/[\underline{\lambda}]} \left(\sharp \left\{ \begin{array}{c} \textit{addable } i-\textit{nodes} \\ \textit{of } \underline{\mu} \textit{ below } \gamma \end{array} \right\} - \sharp \left\{ \begin{array}{c} \textit{removable } i-\textit{nodes} \\ \textit{of } \underline{\lambda} \textit{ below } \gamma \end{array} \right\} \right)$$

Proof. The proof is by induction on $j \in \mathbb{N}_{>0}$:

- If $j = 1$, this is the definition of the JMMO action (see paragraph 2.3).
- If $j > 1$, let $\underline{\mu}$ be a d-partition such that:

$$\underline{\lambda}\underbrace{\overset{i}{\underset{(r_1,p_1)}{\mapsto}}\cdots\overset{i}{\underset{(r_j,p_j)}{\mapsto}}}_{j}\underline{\mu}.$$

We assume that the nodes γ_l associated to (r_l, p_l), $l = 1, \dots, j$, are such that for all $l = 1, \dots, j-1$, γ_l is below γ_{l+1}. Let $\underline{\mu}_s$, $s = 1, \dots, j$, be the d-partitions such that $[\underline{\mu}_s]/[\underline{\mu}_{s-1}] = \gamma_s$.

By induction, the coefficient of $\underline{\mu}_s$ in $f_i^{(j-1)}\underline{\lambda}$ is $q^{\overline{N}_i^b(\underline{\lambda},\underline{\mu}_s)}$. Hence, the coefficient of $\underline{\mu}$ in $f_i f_i^{(j-1)}\underline{\lambda}$ is $\sum_{s=1}^{j} q^{\overline{N}_i^b(\underline{\lambda},\underline{\mu}_s)+\overline{N}_i^b(\underline{\mu}_s,\underline{\mu})}$.

Now, we define the following numbers:

$$\overline{N}_i^b(\underline{\lambda},\underline{\mu},\gamma_s) := \left(\sharp\left\{\begin{array}{c}\text{addable } i-\text{nodes}\\ \text{of } \underline{\mu} \text{ below } \gamma_s\end{array}\right\} - \sharp\left\{\begin{array}{c}\text{removable } i-\text{nodes}\\ \text{of } \underline{\lambda} \text{ below } \gamma_s\end{array}\right\}\right),$$

where $s = 1,\dots,j$.
We have:

$$\begin{aligned}\overline{N}_i^b(\underline{\lambda},\underline{\mu}_s) &= \sum_{t=1}^{s-1}\overline{N}_i^b(\underline{\lambda},\underline{\mu}_s,\gamma_t) + \sum_{t=s+1}^{j}\overline{N}_i^b(\underline{\lambda},\underline{\mu}_s,\gamma_t)\\ &= \sum_{t=1}^{s-1}\overline{N}_i^b(\underline{\lambda},\underline{\mu},\gamma_t) + \sum_{t=s+1}^{j}(\overline{N}_i^b(\underline{\lambda},\underline{\mu},\gamma_t)+1)\\ &= \overline{N}_i^b(\underline{\lambda},\underline{\mu}) - \overline{N}_i^b(\underline{\lambda},\underline{\mu}_s,\gamma_s) + j - s.\end{aligned}$$

And:

$$\overline{N}_i^b(\underline{\mu}_s,\underline{\mu}) = \overline{N}_i^b(\underline{\lambda},\underline{\mu}_s,\gamma_s) - s + 1.$$

Hence, the coefficient of $\underline{\mu}$ in $f_i f_i^{(j-1)}\underline{\lambda}$ is $q^{\overline{N}_i^b(\underline{\lambda},\underline{\mu})}\sum_{s=1}^{j} q^{j+1-2s}$.

Now, we have $f_i f_i^{(j-1)}\underline{\lambda} = [j]_q f_i^{(j)}$. Thus, the coefficient of $\underline{\mu}$ in $f_i^{(j)}\underline{\lambda}$ is $q^{\overline{N}_i^b(\underline{\lambda},\underline{\mu})}$. □

Now, we have the following result:

Proposition 4.6. *Let $\underline{\lambda}$ be a FLOTW d-partition and let* $a-$sequence$(\underline{\lambda})$ $= \underbrace{i_1,\dots,i_1}_{a_1},\underbrace{i_2,\dots,i_2}_{a_2},\dots,\underbrace{i_s,\dots,i_s}_{a_s}$ *be its* $a-$sequence *of residues where for all $j = 1,\dots,s-1$, we have $i_j \neq i_{j+1}$. Then, we have*:

$$f_{i_s}^{(a_s)} f_{i_{s-1}}^{(a_{s-1})}\cdots f_{i_1}^{(a_1)}\underline{\emptyset} = \underline{\lambda} + \sum_{a(\underline{\mu})>a(\underline{\lambda})} c_{\underline{\lambda},\underline{\mu}}(q)\underline{\mu},$$

where $c_{\underline{\lambda},\underline{\mu}}(q) \in \mathbb{Z}[q,q^{-1}]$.

Proof. By Proposition 4.3, we have:

$$f_{i_s}^{(a_s)} f_{i_{s-1}}^{(a_{s-1})}\dots f_{i_1}^{(a_1)}\underline{\emptyset} = c_{\underline{\lambda},\underline{\lambda}}(q)\underline{\lambda} + \sum_{a(\underline{\mu})>a(\underline{\lambda})} c_{\underline{\lambda},\underline{\mu}}(q)\underline{\mu},$$

where $c_{\underline{\lambda},\underline{\mu}}(q) \in \mathbb{Z}[q,q^{-1}]$. So, we have to show that:

$$c_{\underline{\lambda},\underline{\lambda}}(q) = 1.$$

Assume that the last part of the a-graph of $\underline{\lambda}$ is given by:

$$\underbrace{\underline{\nu} \underset{(r_1,p_1)}{\overset{i_s}{\mapsto}} \cdots \underset{(r_{a_s},p_{a_s})}{\overset{i_s}{\mapsto}}}_{a_s} \underline{\lambda}.$$

Let $\underline{\mu}$ be a d-partition obtained by removing a_s i_s-nodes from $\underline{\lambda}$ and assume that $\underline{\mu} \neq \underline{\nu}$. Then, by construction of the $a-$ sequence of residues and Proposition 4.4, the nodes γ_l associated to (r_l, p_l), $l = 1, \ldots, a_s$, are the lowest i_s-nodes of $\underline{\lambda}$ (with respect to the FLOTW order). If $\underline{\mu} \neq \underline{\nu}$ then at least one of the lowest i_s-node is a node of $\underline{\mu}$. When this node is added, the i_s-node can be added to two parts, one to obtain $\underline{\mu}[t]$, the other to obtain $\underline{\nu}[t]$ for some t. As we choose the higher node, this contradicts to the optimality of $\underline{\nu}[t]$. Hence, we have to show that:

$$\overline{N}_i^b(\underline{\nu}, \underline{\lambda}) = 0.$$

There is no addable node of $\underline{\lambda}$ below the γ_l and there is no removable node of $\underline{\nu}$ below the γ_l. Hence this is clear. $\square$

Remark 4.2. This proposition shows that $\underline{\lambda}$ is a FLOTW d-partition if and only if there exists a sequence of residues $\underbrace{i_1, \ldots, i_1}_{a_1}, \underbrace{i_2, \ldots, i_2}_{a_2}, \ldots, \underbrace{i_s, \ldots, i_s}_{a_s}$ such that $\underline{\lambda}$ is the minimal d-partition with respect to a-function which appears in the expression $f_{i_s}^{(a_s)} f_{i_{s-1}}^{(a_{s-1})} \cdots f_{i_1}^{(a_1)} \underline{\emptyset}$.

This result is similar to a conjecture of Dipper, James and Murphy ([12]): in this paper, it was conjectured that $\underline{\lambda}$ is a Kleshchev d-partition if and only if there exists a sequence of residues $\underbrace{i_1, \ldots, i_1}_{a_1}, \underbrace{i_2, \ldots, i_2}_{a_2}, \ldots, \underbrace{i_s, \ldots, i_s}_{a_s}$ such that:

$$f_{i_s}^{(a_s)} f_{i_{s-1}}^{(a_{s-1})} \cdots f_{i_1}^{(a_1)} \underline{\emptyset} = n_{\underline{\lambda}} \underline{\lambda} + \sum_{\underline{\mu} \not\trianglerighteq \underline{\lambda}} n_{\underline{\mu}} \underline{\mu},$$

where $n_{\underline{\lambda}} \neq 0$. The partial order $\trianglerighteq$ is called the dominance order and it is defined as follows.

$$\underline{\mu} \trianglerighteq \underline{\lambda} \iff \sum_{k=0}^{j-1} |\mu^{(k)}| + \sum_{p=1}^{i} \mu_p^{(j)} \geq \sum_{k=0}^{j-1} |\lambda^{(k)}| + \sum_{p=1}^{i} \lambda_p^{(j)} \quad (\forall j, i).$$

Now, the next proposition follows from the same argument as [24, Lemma 6.6], the proof gives an explicit algorithm for computing the canonical basis, which is an analogue of the LLT algorithm (the details of the algorithm will be published elsewhere).

Proposition 4.7. *Following the notations of paragraph* 2.2 *, the canonical basis elements of* $\overline{\mathcal{M}}$ *are of the following form*:

$$\underline{\lambda} + \sum_{a(\underline{\mu}) > a(\underline{\lambda})} b_{\underline{\lambda},\underline{\mu}}(q) \underline{\mu},$$

where $b_{\underline{\lambda},\underline{\mu}}(q) \in q\mathbb{Z}[q]$ *and* $\underline{\lambda}$ *is a FLOTW d-partition.*

Reciprocally, for any FLOTW d-partition $\underline{\lambda}$, *there exists a canonical basis element of the above form.*

Proof. Let $\underline{\mu} \in \Lambda^1_{\{e;v_0,\ldots,v_{d-1}\}}$ be a FLOTW d-partition and assume that:

$$a - \text{sequence}(\underline{\lambda}) = \underbrace{i_1,\ldots,i_1}_{a_1},\underbrace{i_2,\ldots,i_2}_{a_2},\ldots,\underbrace{i_s,\ldots,i_s}_{a_s}.$$

Then, we define:

$$A(\underline{\lambda}) := f_{i_s}^{(a_s)} f_{i_{s-1}}^{(a_{s-1})} \cdots f_{i_1}^{(a_1)} \underline{\emptyset}.$$

By the previous proposition, we have:

$$A(\underline{\lambda}) = \underline{\lambda} + \sum_{a(\underline{\mu})>a(\underline{\lambda})} c_{\underline{\lambda},\underline{\mu}}(q)\underline{\mu},$$

where $c_{\underline{\lambda},\underline{\mu}}(q) \in \mathbb{Z}[q,q^{-1}]$.

The $A(\underline{\lambda})$ are linearly independant and so, the following set is a basis of the $\mathcal{U}_{\mathcal{A}}$-module $\overline{\mathcal{M}}_{\mathcal{A}}$ generated by the empty d-partition with respect to the JMMO action:

$$\{A(\underline{\mu}) \mid \underline{\mu} \in \Lambda^1\}.$$

Note that the vertices of the crystal graph of $\overline{\mathcal{M}}$ are given by the FLOTW l-partitions. Let $\{B_{\underline{\mu}} \mid \underline{\mu} \in \Lambda^1\}$ be the canonical basis of $\overline{\mathcal{M}}$. There exist Laurent polynomials $m_{\underline{\mu},\underline{\nu}}(q)$ such that, for all $\underline{\mu} \in \Lambda^1$, we have:

$$B_{\underline{\mu}} = \sum_{\underline{\nu}\in\Lambda^1} m_{\underline{\mu},\underline{\nu}}(q) A(\underline{\nu}).$$

Now, we consider the bar involution on $\mathcal{U}_{\mathcal{A}}$, this is the $\mathbb{Z}$-linear automorphism defined by:

$$\overline{q} := q^{-1}, \qquad \overline{k_h} = k_{-h}, \qquad \overline{e_i} = e_i, \qquad \overline{f_i} = f_i.$$

It can be extended to $\overline{\mathcal{M}}_{\mathcal{A}}$ by setting, for all $u \in \mathcal{U}_{\mathcal{A}}$:

$$\overline{u\underline{\emptyset}} := \overline{u}\underline{\emptyset}.$$

The $A(\underline{\mu})$ are clearly invariant under the bar involution and so are the $B_{\underline{\mu}}$ (this is by the definition of the canonical basis, see [4]). Therefore, we have:

$$m_{\underline{\mu},\underline{\nu}}(q) = m_{\underline{\mu},\underline{\nu}}(q^{-1}),$$

for all $\underline{\mu}$ and $\underline{\nu}$ in Λ^1.

Let $\underline{\alpha}$ be one of the minimal d-partitions with respect to the a-function such that $m_{\underline{\mu},\underline{\alpha}}(q) \neq 0$. Then, the coefficient of $\underline{\alpha}$ in $B_{\underline{\mu}}$ is:

$$b_{\underline{\mu},\underline{\alpha}}(q) := \sum_{\underline{\nu}\in\Lambda^1} m_{\underline{\mu},\underline{\nu}}(q) c_{\underline{\nu},\underline{\alpha}}(q).$$

We have:

• $m_{\underline{\mu},\underline{\nu}}(q) = 0$ if $a(\underline{\nu}) < a(\underline{\alpha})$.

• $c_{\underline{\nu},\underline{\alpha}}(q) = 0$ if $a(\underline{\nu}) > a(\underline{\alpha})$ and if $a(\underline{\nu}) = a(\underline{\alpha})$ and $c_{\underline{\nu},\underline{\alpha}}(q) \neq 0$ then $\underline{\nu} = \underline{\alpha}$.

Hence, we have:

$$b_{\underline{\mu},\underline{\alpha}}(q) = m_{\underline{\mu},\underline{\alpha}}(q) \neq 0.$$

Now, as the FLOTW d-partitions are labeling the vertices of the crystal graph of $\overline{\mathcal{M}}_{\mathcal{A}}$, we have:

$$B_{\underline{\mu}} = \underline{\mu} \ (\text{mod } q).$$

Therefore, if $\underline{\mu} \neq \underline{\alpha}$, we have:

$$b_{\underline{\mu},\underline{\alpha}}(q) \in q\mathbb{Z}[q].$$

Moreover, we have:

$$m_{\underline{\mu},\underline{\alpha}}(q) = m_{\underline{\mu},\underline{\alpha}}(q^{-1}).$$

Therefore, we have $\underline{\mu} = \underline{\alpha}$ and $b_{\underline{\mu},\underline{\mu}}(q) = 1$. To summarize, we obtain:

$$B_{\underline{\mu}} = A(\underline{\mu}) + \sum_{a(\underline{\nu})>a(\underline{\mu})} m_{\underline{\mu},\underline{\nu}}(q) A(\underline{\nu}).$$

The theorem follows now from Proposition 4.6. □

Using the correspondence between the canonical basis elements and the indecomposable projective modules, we can reformulate this result as follows.

Theorem 4.1. *Let $\mathcal{H}_{R,n} := \mathcal{H}_{R,n}(v; u_0, \ldots, u_{d-1})$ be the Ariki-Koike algebra of type $G(d,1,n)$ over $R := \mathbb{Q}(\eta_{de})$ and assume that the parameters are given by*:

$$v = \eta_e, \qquad u_i = \eta_e^{v_j} \quad j = 0, \ldots, d-1,$$

where $\eta_e := \exp(\frac{2i\pi}{e})$ and $0 \leq v_0 \leq \cdots \leq v_{d-1} < e$.

Let $\{P_R^{\underline{\mu}} \mid \underline{\mu} \in \Lambda^0\}$ be the set of indecomposable projective $\mathcal{H}_{R,n}$-modules. Then, there exists a bijection between the set of Kleshchev d-partitions and the set of FLOTW d-partitions:

$$j : \Lambda^0 \to \Lambda^1,$$

such that, for all $\underline{\mu} \in \Lambda^0$:

$$e([P_R^{\underline{\mu}}]_p) = [\![S^{j(\underline{\mu})}]\!] + \sum_{\substack{\underline{\nu} \in \Pi_n^d \\ a(\underline{\nu})>a(j(\underline{\mu}))}} d_{\underline{\nu},\underline{\mu}} [\![S^{\underline{\nu}}]\!].$$

In particular, this theorem shows that the decomposition matrix of $\mathcal{H}_{R,n}$ has a lower triangular shape with 1 along the diagonal.

Remark 4.3. Keeping the notations of the above theorem, we can attach to each simple $\mathcal{H}_{R,n}$-module M an a-value in the following way:

$$\text{If } M = D_R^{\underline{\mu}} \text{ for } \underline{\mu} \in \Lambda^0 \text{ then } a_M := a_{S^{j(\underline{\mu})}} = a(j(\underline{\mu})).$$

Then, we have:

$$a_M = \min\{a_V \mid d_{V,M} \neq 0\}.$$

We remark that this property was shown in [16] for Hecke algebras where the a-function was defined in terms of Kazhdan-Lusztig basis.

We have also the following property: for all $\underline{\nu} \in \Lambda^1$:

$$[S^{\underline{\nu}}] = [D^{j^{-1}(\underline{\nu})}] + \sum_{\substack{\underline{\mu} \in \Lambda^0 \\ a(j(\underline{\mu})) < a(\underline{\nu})}} d_{\underline{\nu},\underline{\mu}} [D^{\underline{\mu}}].$$

In the following section, we see consequences of the above theorem on the representation theory of Hecke algebras of type B_n.

5. Determination of the canonical basic set for Hecke algebras of type B_n

Let W be a finite Weyl group, S be the set of simple reflections and let H be the associated Hecke algebra defined over $\mathbb{Z}[y, y^{-1}]$ where y is an indeterminate. Let $v = y^2$. Then, H is defined by:

- generators: $\{T_w \mid w \in W\}$,
- relations: for $s \in S$ and $w \in W$:

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ vT_{sw} + (v-1)T_w & \text{if } l(sw) < l(w). \end{cases}$$

where l is the usual lenght function.
Let $K = \mathbb{Q}(y)$ and $\theta : \mathbb{Z}[y, y^{-1}] \to k$ be a specialization such that k is the field of fractions of $\theta(\mathbb{Z}[y, y^{-1}])$. We assume that the characteristic of k is either 0 or a good prime for W. Let $H_K := K \otimes_{\mathbb{Z}[y,y^{-1}]} H$ and $H_k := k \otimes_{\mathbb{Z}[y,y^{-1}]} H$. Then, following [18, Theorem 7.4.3], we have a well-defined decomposition map d_θ between the Grothendieck groups of H_K-modules and H_k-modules. For $V \in \mathrm{Irr}(H_K)$, we have :

$$d_\theta([V]) = \sum_{M \in \mathrm{Irr}(H_k)} d_{V,M}[M],$$

where the numbers $d_{V,M} \in \mathbb{N}$ are the decomposition numbers (see [18, Chapter 7] for details).

In [19], Geck and Rouquier have defined a “canonical basic set” $\mathcal{B} \subset \mathrm{Irr}(H_K)$ which leads to a natural parametrization of the set $\mathrm{Irr}(H_k)$.

Theorem 5.1 (Geck-Rouquier [19]). *We consider the following subset of* $\mathrm{Irr}(H_K)$:

$$\mathcal{B} := \{V \in \mathrm{Irr}(H_K) \mid d_{V,M} \neq 0 \textit{ and } a(V) = a(M) \textit{ for some } M \in \mathrm{Irr}(H_k)\}.$$

Then, there exists a unique bijection $\mathcal{B} \leftrightarrow \mathrm{Irr}(H_k)$, $V \leftrightarrow \overline{V}$, *such that the following two conditions hold*:

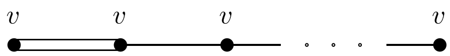
(i) *For all* $V \in \mathcal{B}$, *we have* $d_{V,\overline{V}} = 1$ *and* $a(V) = a(\overline{V})$.

(ii) *If* $V \in \mathrm{Irr}(H_K)$ *and* $M \in \mathrm{Irr}(H_k)$ *are such that* $d_{V,M} \neq 0$, *then we have* $a(M) \leq a(V)$, *with equality only for* $V \in \mathcal{B}$ *and* $M = \overline{V}$.

The set $\mathcal{B}$ *is called the canonical basic set.*

From now to the end of the paper, we assume that k is a field of characteristic 0. Then, $\mathcal{B}$ has been already determined for all specializations for type A_{n-1} in [15], for type D_n and e odd in [16] and for type D_n and e even in [22].

The aim of this section is to find $\mathcal{B}$ in the case of Hecke algebras of type B_n. Let W be a Weyl group of type B_n with the following diagram:



Let H be the correponding Hecke algebra over $\mathbb{Z}[y, y^{-1}]$. First, it is known that H_K is semi-simple unless $\theta(u)$ is a root of unity. In this case, we have $\mathcal{B} = \mathrm{Irr}(H_K)$. For $p \in \mathbb{N}_{>0}$ we put $\eta_p := \exp(\frac{2i\pi}{p})$, then we can assume that $\theta(v) = \eta_e$ and that $k := \mathbb{Q}(\eta_{2e})$.

The semi-simple algebra H_K is an Ariki-Koike algebra with parameters $u_0 = v$ and $u_1 = -1$ over K. Then, the simple H_K-modules are given by the Specht modules $S^{\underline{\lambda}}$ which are labeled by the 2-partitions $\underline{\lambda}$ of rank n. Let $\underline{\lambda} := (\lambda^{(0)}, \lambda^{(1)})$ be a 2-partition and let $h^{(0)}$ (resp. $h^{(1)}$) be the height of $\lambda^{(0)}$ (resp. $\lambda^{(1)}$). Let r be a positive integer such that $r \geq \max\{h^{(0)}, h^{(1)}\}$. Then, the a-value of the Specht module labeled by $\underline{\lambda} := (\lambda^{(0)}, \lambda^{(1)})$ is given by

$$\begin{aligned} a(\lambda^{(0)}, \lambda^{(1)}) := -\frac{1}{6}r(r-1)(2r+5) + \sum_{i=1}^{r}(i-1)(\lambda_i^{(0)} + \lambda_i^{(1)} + 1) \\ + \sum_{i,j=1}^{r} \min\{\lambda_i^{(0)} + 1 + r - i, \lambda_j^{(1)} + r - j\}, \end{aligned}$$

where we put $\lambda_j^{(0)} := 0$ (resp. $\lambda_j^{(1)} := 0$) if $h^{(0)} < j \leq r$ (resp. $h^{(1)} < j \leq r$). This formula is obtained by rewriting that in Proposition 3.2 with the above choice of parameters. Now, we have two cases to consider.

a) Assume that e is odd. In this case, we can apply results of Dipper and James: they have shown that the simple H_k-modules are given by the modules $D^{\underline{\lambda}}$ where $\underline{\lambda} = (\lambda^{(0)}, \lambda^{(1)})$ is such that $\lambda^{(0)}$ and $\lambda^{(1)}$ are e-regular. A partition $\nu = (\nu_1, \ldots, \nu_r)$ where $\nu_r > 0$ is e-regular if for all $i = 1, \ldots, r$, we can not have $\nu_i = \nu_{i+1} = \cdots = \nu_{i+e-1}$. For $\underline{\lambda}$ a e-regular 2-partition and $\underline{\mu}$ a 2-partition, we denote by $d_{\underline{\mu},\underline{\lambda}}$ the corresponding decomposition number.

Moreover, Dipper and James have shown that the decomposition numbers of H are determined by the decomposition numbers of a Hecke algebra of type A_{n-1} in the following way (see [10] and [13] for a more general case).

Let $0 \leq l \leq n$ and let $H(\mathfrak{S}_l)$ be the generic Hecke algebra of type A_{l-1}, then θ determines a decomposition map between the Grothendieck groups of $H_K(\mathfrak{S}_l)$ and $H_k(\mathfrak{S}_l)$. The simple modules of $H_K(\mathfrak{S}_l)$ are given by Specht modules parametrized by partitions of rank l. The simple modules of $H_k(\mathfrak{S}_l)$ are given by the D^λ labeled by the partitions λ of rank l which are e-regular. Let $d_{\lambda',\lambda}$ be the decomposition numbers of $H(\mathfrak{S}_l)$ where λ runs over the set of e-regular partitions of rank l and λ' over the set of partitions of rank l.

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_r)$ be two partitions of rank l. Recall that we write $\lambda \trianglerighteq \mu$ if , for all $i = 1, \ldots, r$, we have:

$$\sum_{j=1}^{i} \lambda_j \geq \sum_{j=1}^{i} \mu_j.$$

Then, if $\underline{\lambda} = (\lambda^{(0)}, \lambda^{(1)})$ is a e-regular 2-partition of rank n and $\underline{\mu} = (\mu^{(0)}, \mu^{(1)})$ a 2-partition of rank n, we have, following [10, Theorem 5.8]:

$$d_{\underline{\mu},\underline{\lambda}} = \begin{cases} d_{\mu^{(0)},\lambda^{(0)}} d_{\mu^{(1)},\lambda^{(1)}} & \text{if } |\mu^{(0)}| = |\lambda^{(0)}| \text{ and } |\mu^{(1)}| = |\lambda^{(1)}|, \\ 0 & \text{otherwise.} \end{cases}$$

Assume that $|\mu^{(0)}| = |\lambda^{(0)}|$ and $|\mu^{(1)}| = |\lambda^{(1)}|$ and that $d_{\mu^{(0)},\lambda^{(0)}} \neq 0$, $d_{\mu^{(1)},\lambda^{(1)}} \neq 0$, then following [28, Theorem 3.43] (result of Dipper and James), we have:

$$\mu^{(0)} \trianglelefteq \lambda^{(0)} \qquad \text{and} \qquad \mu^{(1)} \trianglelefteq \lambda^{(1)}.$$

Then, as in [16, Proposition 6.8], it is easy to see that:

$$a(\lambda^{(0)}, \lambda^{(1)}) \leq a(\lambda^{(0)}, \mu^{(1)}) \leq a(\mu^{(0)}, \mu^{(1)}).$$

For all e-regular 2-partition $\underline{\lambda}$, we have $d_{\underline{\lambda},\underline{\lambda}} = 1$. Hence, it proves the following proposition:

Proposition 5.1. *Keeping the above notations, assume that W is a Weyl group of type B_n and that e is odd, then the canonical basic set in bijection with* $\mathrm{Irr}(H_k)$ *is the following one*:

$$\mathcal{B} = \{S^{\underline{\lambda}} \mid \underline{\lambda} = (\lambda^{(0)}, \lambda^{(1)}) \in \Pi_n^2,\ \lambda^{(0)} \textit{ and } \lambda^{(1)} \textit{ are e-regular}\}.$$

b) Assume that e is even. Then, by using the notations of paragraph 4.1, we put $v_0 = 1$ and $v_1 = \frac{e}{2}$. Then, we have:

$$m^{(0)} = 1 \qquad \text{and} \qquad m^{(1)} = 0.$$

Then, the Ariki-Koike algebra $\mathcal{H}_{K,n}$ over $K = \mathbb{Q}(y)$ have the following parameters:

$$u_0 = y^2,$$
$$u_1 = -1,$$
$$v = y^2.$$

This is nothing but the one parameter Hecke algebra H_K of type B_n. If we specialize y to η_{2e}, we obtain the Hecke algebra H_k.

Thus, we are in the setting of Theorem 4.1. Thus, we obtain:

Proposition 5.2. *Keeping the above notations, assume that W is a Weyl group of type B_n and that e is even, then the canonical basic set in bijection with* $\mathrm{Irr}(H_k)$ *is the following one*:

$$\mathcal{B} = \{S^{\underline{\lambda}} \mid \underline{\lambda} = (\lambda^{(0)}, \lambda^{(1)}) \in \Lambda^1_{\{e;1,\frac{e}{2}\}}\}.$$

We have $(\lambda^{(0)}, \lambda^{(1)}) \in \Lambda^1_{\{e;1,\frac{e}{2}\}}$ *if and only if*:

1. *we have*:

$$\lambda_i^{(0)} \geq \lambda^{(1)}_{i-1+\frac{e}{2}},$$
$$\lambda_i^{(1)} \geq \lambda^{(0)}_{i+1+\frac{e}{2}}.$$

2. *For all $k > 0$, among the residues appearing at the right ends of the length k rows of $\underline{\lambda}$, at least one element of* $\{0, 1, \ldots, e-1\}$ *does not occur.*

Thus, we obtain the parametrization of the canonical basic set for type B_n in characteristic 0 for all specializations. Note that the canonical basic set for the exceptional types can be easily deduced from the explicit tables of decomposition numbers obtained by Geck, Lux and Müller. Hence, the above results complete the classification of the canonical basic set for all types and all specializations in characteristic 0.

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