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Centralization of positive definite functions, weak containment of representations and Thoma characters for the infinite symmetric group*

By

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Introduction

In this paper, we study positive definite functions on the infinite symmetric group $\mathfrak{S}_\infty$ and their relations to its unitary representations. Our principal method is taking the limits of the centralizations of such functions. This method can be applied to study positive definite invariant functions and especially characters of factor representations of type II_1 (cf. [Di]) for other discrete groups such as the infinite Weyl groups of type B_∞/C_∞ or of type D_∞.

Let us explain this a little more in detail. Let G be a countably infinite discrete group and K be a finite group acting on G in such a way that, for every $k \in K$, $G \ni g \mapsto k(g) \in G$ is an automorphism. Then, for a function f on G, we put

$$f^K(g) := \frac{1}{|K|} \sum_{k \in K} f(k(g)) \qquad (g \in G)$$

and call it a centralization of f with respect to K. We treat mainly the case where K is a subgroup of G and its action is through the inner automorphism.

Take an increasing sequence of finite subgroups $G_n \nearrow G$ $(n = 1, 2, \ldots)$. For a positive definite function f on G we consider a series of centralized functions $f_n = f^{G_n}$ on G. If this series converges pointwise to a function on G, then $\lim_{n\to\infty} f_n$ is a positive definite invariant function (or class function). Relations of positive definite invariant functions to factor representations of G are given in [Tho1].

Our problems treated here for the group $G = \mathfrak{S}_\infty$ are the following.

(1) For special interesting positive definite functions f given in [Bo], [BS], determine $\lim_{n\to\infty} f_n$.

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(2) For irreducible unitary representations (= IURs) of $\mathfrak{S}_\infty$ given in [Hi2] as induced representations from subgroups of wreath product type, and also for non-irreducible induced representations of similar kind, take their matrix elements f, and calculate the limits $\lim_{n\to\infty} f_n$, which heavily depend on the choice of increasing sequences of finite subgroups $G_n \nearrow G$.

(3) Translate the results in (1) and (2) into certain results in the weak containment topology of the space of unitary representations.

(4) Analyse relations of the results in (2) to the problem of determining Thoma characters in [Tho2], and also to the problem of irreducible decompositions of factor representations in [Ob2].

The contents of this paper are organized as follows.

Sections 1 to 5 are devoted to Problem (1).

In Section 6, we assert in Theorem 7 that for any infinite-dimensional IUR ρ of $G = \mathfrak{S}_\infty$ given in [Hi2], the closure of one point set $\{\,\rho\,\}$ in $\mathrm{Rep}(G)$ contains at least one of the trivial representation $\mathbf{1}_G$, the sign representation sgn_G and the regular representation λ_G. After this theorem, we propose as conjectures that a similar phenomenon is true in general. This relates in part to Problems (2) and (3). After a preparation in Section 6 about matrix elements of induced representations, Theorem 7 is proved in Sections 7–10.

In Section 11, Thoma's results in [Tho2] are reviewed about extremal positive definite invariant functions on $\mathfrak{S}_\infty$. In Section 12, explicit construction of IURs in [Hi2] is reviewed. After these preparations, we give in Section 13 a rather complete result about the limits of centralizations of matrix elements of IURs, in the form of Theorem 13. From this result, we see that there exists a certain kind of IUR ρ for which the limits of centralizations of its matrix elements give all of Thoma characters and so all the characters of factor representation of type II_1, or in the contrary, a certain IUR ρ for which the limits contain only the delta function δ_e supported on the identity element $e \in G$, the character of the regular representation λ_G.

A proof of Theorem 13 is given in Section 14. In Section 15, we treat the case of non-irreducible induced representations of a similar kind.

The last section, Section 16, is devoted to Problem (4).

1. Centralizations of positive definite functions

The infinite symmetric group consists of all finite permutations on the set of natural numbers $\boldsymbol{N}$, and is denoted by $\mathfrak{S}_\infty$. The symmetric group $\mathfrak{S}_N$ is imbedded in it as the permutation group of the set $\boldsymbol{I}_N := \{\,1, 2, \ldots, N\,\} \subset \boldsymbol{N}$.

A function $F(g)$ on $G = \mathfrak{S}_\infty$ is called *central* if $F(\sigma g \sigma^{-1}) = F(g)$ $(g, \sigma \in G)$. For a function f on G and a finite subgroup $G' \subset G$, we define a *centralization* of f on G' as

$$f^{G'}(g) := \frac{1}{|G'|} \sum_{\sigma \in G'} f(\sigma g \sigma^{-1}). \tag{1.1}$$

Taking an increasing sequence of finite subgroups $G_N \nearrow G$, we consider a series f^{G_N} of centralizations of f on G_N and study its pointwise convergence limit.

In particular, when we take a series $\mathfrak{S}_N \nearrow \mathfrak{S}_\infty = G$, we put

$$f_N(g) := f^{\mathfrak{S}_N}(g) = \frac{1}{|\mathfrak{S}_N|} \sum_{\sigma \in \mathfrak{S}_N} f(\sigma g \sigma^{-1}). \tag{1.2}$$

Note that for $N' > N$, we have $f_{N'} = (f_N)_{N'}$, but usually

$$f_{N'}|_{\mathfrak{S}_N} \neq f_N|_{\mathfrak{S}_N}.$$

Consider special kinds of positive definite functions on $G = \mathfrak{S}_\infty$ given as

$$f(g) := r^{|g|} \qquad (-1 \le r \le 1,\ g \in G), \tag{1.3}$$

$$f'(g) := q^{\|g\|} \qquad (0 \le q \le 1,\ g \in G), \tag{1.4}$$

$$f''(g) := \mathrm{sgn}(g) \cdot q^{\|g\|} \qquad (0 \le q \le 1,\ g \in G), \tag{1.5}$$

where $|g|$ denotes the usual length of a permutation of g, and $\|g\|$ denotes the block length of g, which is by definition the number of different simple permutations appearing in a reduced expression of g (cf. [Bo] for (1.3), and [BS] for (1.4)).

Problem (M. Bożejko). *Let π_f, $\pi_{f'}$ and $\pi_{f''}$ be cyclic unitary representations of $G = \mathfrak{S}_\infty$ corresponding to the positive definite functions in* (1.3), (1.4), *and* (1.5) *by GNS construction. Then, are π_f, $\pi_{f'}$ and $\pi_{f''}$ irreducible ? If not, give irreducible decompositions of them.*

We give here a partial answer to this question as follows.

Theorem 1. *Let $|r| < 1$. Then for the positive definite function f in* (1.3) *its centralization f_N converges pointwise to the delta function δ_e on $G = \mathfrak{S}_\infty$ as N tends to ∞:*

$$f_N(e) = 1; \quad f_N(g) \to 0 \ \text{for}\ g \neq e \quad (N \to \infty), \tag{1.6}$$

where e denotes the neutral element of G.

Theorem 2. *Let $0 < q < 1$. Then, for the positive definite function f' in* (1.4) *and f'' in* (1.5), *their centralizations f'_N and f''_N converge pointwise to the delta function δ_e on $G = \mathfrak{S}_\infty$ as N tends to ∞: for $F = f'$ or f'',*

$$F_N(e) = 1; \quad F_N(g) \to 0 \ \text{for}\ g \neq e \quad (N \to \infty). \tag{1.7}$$

The delta function δ_e is a positive definite function associated to the regular representation λ_G of G which corresponds to a cyclic vector $v_0 = \delta_e \in L_2(G)$: $\delta_e(g) = \langle \lambda_G(g)v_0, v_0\rangle$, and also is the character of this representation which is known to be a factor representation of type II_1.

Concerning to the definition of weak containment of unitary representations, we refer to [Di, §18]. Then, we get the following theorem as a direct consequence of Theorems 1 and 2.

Theorem 3. *Each of the representations π_f, $\pi_{f'}$ and $\pi_{f''}$ contains weakly the regular representation λ_G of G.*

2. Lengths of permutations, sums of power series

Take $g \neq e$ from G, and decompose it into a product of mutually disjoint cycles (= cyclic permutations) as

$$g = g_1 g_2 \cdots g_m, \quad g_j = (i_{j1}\ \ i_{j2}\ \ \ldots\ \ i_{j\ell_j}). \tag{2.1}$$

We call ℓ_j the *length* of the cycle g_j, and put $n_\ell(g) = |\{\, j\,;\, \ell_j = \ell\}|$ the number of cycles g_j with length ℓ. For $\sigma \in G$, put $h = \sigma g \sigma^{-1}$, then

$$h = \sigma g \sigma^{-1} = h_1 h_2 \cdots h_m, \quad h_j = (\sigma(i_{j1})\ \ \sigma(i_{j2})\ \ \ldots\ \ \sigma(i_{j\ell_j})\,). \tag{2.2}$$

Thus we should evaluate the length $|h|$ from below to get an evaluation of $r^{|h|}$ from above.

To do so, let us introduce some notations. Take an element $h \in G, h \neq e$, and express it in a product of mutually disjoint cycles as

$$h = h_1 h_2 \cdots h_m. \tag{2.3}$$

Let us denote by $\mathrm{supp}(h)$ the set of numbers i for which $h(i) \neq i$, then $\mathrm{supp}(h) = \sqcup_{j=1}^m \mathrm{supp}(h_j)$. Assume a cycle h_j is given as $h_j = (a_{j1}\ \ a_{j2}\ \ldots\ a_{j\ell_j})$. Then, $\mathrm{supp}(h_j) = \{\, a_{j1}, a_{j2}, \ldots, a_{j\ell_j}\,\}$. Put

$$a_j^- := \min_{1 \le k \le \ell_j} a_{jk}, \qquad a_j^+ := \max_{1 \le k \le \ell_j} a_{jk}, \tag{2.4}$$

and define an interval $[h_j] \subset \boldsymbol{I}_N$ as $[h_j] := [a_j^-, a_j^+]$ and denote by $|[h_j]|$ its width $a_j^+ - a_j^-$, which is different from $|\mathrm{supp}(h_j)| = \ell_j$, the order of the set $\mathrm{supp}(h_j)$. Note that the number of different possible cycles h_j with the same $\mathrm{supp}(h_j)$ is equal to $(\ell_j - 1)!$.

Lemma 4. (i) *For an element $h \in G = \mathfrak{S}_\infty, h \neq e$, let $h = h_1 h_2 \cdots h_m$ in* (2.3) *be its decomposition into disjoint cycles. Then,*

$$|h| \ \ge \sum_{1 \le j \le m} 2\,|[h_j]| - (2m - 1/2)\,|\mathrm{supp}(h)|. \tag{2.5}$$

(ii) *For $g \in G, g \neq e$, let $g = g_1 g_2 \cdots g_m$ in* (2.1) *be its decomposition into disjoint cycles. Then, for $\sigma \in G$, we have*

$$|\sigma g \sigma^{-1}| \ge \sum_{1 \le j \le m} 2\,|[\sigma g_j \sigma^{-1}]| - (2m - 1/2)\,|\mathrm{supp}(g)|. \tag{2.6}$$

Proof. The assertion (ii) is deduced from (i) immediately through (2.2) if we take into account that, for $h = \sigma g \sigma^{-1}$, $\mathrm{supp}(h) = \sigma \cdot \mathrm{supp}(g)$ and so $|\mathrm{supp}(h)| = |\mathrm{supp}(g)|$.

Let us prove (i). The length $|h|$ is equal to the number of inversions under h, that is, the number of pairs $i < i'$ in $\boldsymbol{I}_N$ such that $h(i) > h(i')$.

Consider the cycle h_j. Take an integer i from $[h_j] \setminus \mathrm{supp}(h)$. Then, there exists at least one $i' \in \mathrm{supp}(h_j)$ such that $i' < i$ and $h_j(i') = h(i') > h(i) = i$. If not, the set of integers $\{\, i' \in \mathrm{supp}(h_j)\,;\, i' < i \,\} \neq \emptyset$ remains invariant under h_j and accordingly under h. This is impossible since $\{\, i' \in \mathrm{supp}(h_j)\,;\, i' > i \,\}$ is not empty and $\mathrm{supp}(h_j)$ contains no non-trivial invariant set. Similarly, there exists at least one $i'' \in \mathrm{supp}(h_j)$ such that $i'' > i$ and $h(i'') < h(i) = i$. Thus, every $i \in [h_j] \setminus \mathrm{supp}(h)$ contributes at least 2 for the numbers of inversions, and so in total the contribution to $|h|$ is

$$\geq 2\,|[h_j] \setminus \mathrm{supp}(h)| \geq 2\,|[h_j]| - 2\,|\mathrm{supp}(h)|. \tag{2.7}$$

Other inversions are between pairs of integers in $\mathrm{supp}(h)$. Their total number is evaluated from below by $1/2 \cdot |\mathrm{supp}(h)|$.

Summing up all these evaluations, we get the desired inequality (2.5). □

Lemma 5. *Let ρ be a real number such that $0 < \rho < 1$. Then, for a fixed non-negative integer $s \geq 0$,*

$$\sum_{s \leq p < \infty} \binom{p}{s} \rho^p = \frac{\rho^s}{(1-\rho)^{s+1}}. \tag{2.8}$$

Proof. For $s = 0$, we have a formula for infinite series as

$$\sum_{0 \leq p < \infty} \rho^p = \frac{1}{1-\rho}. \tag{2.9}$$

Then, defferentiating s times the both sides of this equality, we get the following one from which we get the desired one by multiplying both side by $\rho^s/s!$:

$$\sum_{s \leq p < \infty} p(p-1)\cdots(p-s+1)\rho^{p-s} = s! \cdot \frac{1}{(1-\rho)^{s+1}}.$$

□

3. Proof of Theorem 1

It is enough to consider $\hat{f}(g) = |f(g)| = |r|^{|g|}$. Put $\rho = |r|^2$, then,

$$\begin{aligned}\hat{f}_N(g) &= \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \hat{f}(\sigma g \sigma^{-1}) = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} |r|^{|\sigma g \sigma^{-1}|} \\ &\leq \frac{|r|^{-(2m-1/2)|\mathrm{supp}(g)|}}{N!} \sum_{\sigma \in \mathfrak{S}_N} \prod_{1 \leq j \leq m} \rho^{|[\sigma g_j \sigma^{-1}]|} \quad \text{(by Lemma 4).}\end{aligned}$$

Fix two numbers $1 \leq b_j^- < b_j^+ \leq N$, and consider possible cycles h_j of length ℓ_j for which

$$[h_j] = B_j, \qquad B_j := [b_j^-, b_j^+] \subset \boldsymbol{I}_N. \tag{3.1}$$

Then, the number of such cycles is equal to $(\ell_j - 1)! \times$ {the number of different choices of $(\ell_j - 2)$ integers from the interval (b_j^-, b_j^+)}:

$$(\ell_j - 1)! \times \binom{b_j^+ - b_j^- - 1}{\ell_j - 2}. \tag{3.2}$$

Let $S(\,(g_j, B_j)_{1 \le j \le m})$ be the subset of $\mathfrak{S}_N$ of all such σ that satisfies

$$[h_j] = B_j \quad \text{for} \quad h_j = \sigma g_j \sigma^{-1} \qquad (1 \le j \le m), \tag{3.3}$$

and put $s(\,(g_j, B_j)_{1 \le j \le m}) = |S(\,(g_j, B_j)_{1 \le j \le m})|$. Then,

$$\frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \prod_{1 \le j \le m} \rho^{|[\sigma g_j \sigma^{-1}]|} = \frac{1}{N!} \sum s(\,(g_j, B_j)_{1 \le j \le m}) \prod_{1 \le j \le m} \rho^{|B_j|}, \tag{3.4}$$

where the summation runs over all systems of m intervals $\{\, B_j\,; 1 \le j \le m \,\}$ in $\boldsymbol{I}_N$. Since the family of m subsets $\mathrm{supp}(\sigma g_j \sigma^{-1})$ of $\boldsymbol{I}_N$ are mutually disjoint, a possible system $\{\, B_j \,\}$ should satisfy certain conditions, for example, their extremities are all different. For any non possible one, we put $s(\,(g_j, B_j)_{1 \le j \le m}) = 0$.

We want to evaluate from above the number $s(\,(g_j, B_j)_{1 \le j \le m})$. We note the following fact. Assume N sufficiently large so that $A := \mathrm{supp}(g) \subset \boldsymbol{I}_N$. Let $\mathfrak{S}_A$ be the full permutation group acting on A, and consider the commutant

$$C_A(g) := \{\, s \in \mathfrak{S}_A \,;\, sgs^{-1} = g \,\}.$$

Let $n_l(g), \ell \ge 2$, be the number of cycles g_j such that $\ell_j = |\mathrm{supp}(g_j)| = \ell$. Then, the order $|C_A(g)|$ is equal to $\prod_{\ell \ge 2} n_\ell(g)! \cdot \ell^{n_\ell(g)}$. However, since we consider independently for each j the cycle $\sigma g_j \sigma^{-1}$, the first factor $\prod_{\ell \ge 2} n_\ell(g)!$ does not appear in the next discussion.

Let $g_j = (i_{j1}, i_{j2}, \ldots, i_{j\ell_j})$, then $h_j = \sigma g_j \sigma^{-1}$ is given by (2.2). This means that the cycle h_j determines the integers $\sigma(i_{j1}), \sigma(i_{j2}), \ldots, \sigma(i_{j\ell_j})$ modulo cyclic permutations. On the other hand, for integers $p \in \boldsymbol{I}_N \setminus \mathrm{supp}(g)$, $\sigma(p)$'s can be given arbitrariry from $\boldsymbol{I}_N \setminus \sigma \cdot \mathrm{supp}(g)$. Thus, taking into account the evaluation (3.2) and $\prod_{\ell \ge 2} \ell^{n_\ell(g)} = \prod_{1 \le j \le m} \ell_j$, we get

$$s(\,(g_j, B_j)_{1 \le j \le m}) \le \prod_{1 \le j \le m} \ell_j! \cdot \binom{|B_j| - 2}{\ell_j - 2} \times (N - |\mathrm{supp}(g)|)!.$$

This evaluation is necessarily from above because the evaluation (3.2) is given not counting any restriction coming from other $\sigma g_{j'} \sigma^{-1}$ for $j' \ne j$.

Fix the width $k_j = |B_j| \ge \ell_j$. Then, the number of such intervals in $\boldsymbol{I}_N$ is $(N - k_j + 1) < N$. Therefore the left hand side of (3.4) is evaluated from above by

$$\begin{aligned} &C \cdot \frac{N^m \cdot (N - |\mathrm{supp}(g)|)!}{N!} \cdot \prod_{1 \le j \le m} \sum_{\ell_j \le k_j \le N} \binom{k_j - 2}{\ell_j - 2} \rho^{k_j} \\ &= C \cdot \frac{N^m \cdot (N - |\mathrm{supp}(g)|)!}{N!} \cdot \frac{\rho^{|\mathrm{supp}(g)|}}{(1 - \rho)^{|\mathrm{supp}(g)| - m}} \qquad \text{(by Lemma 5)}, \end{aligned}$$

where C denotes a constant independent of N and k_j's.

The above last term tends to 0 as $N \to \infty$. This proves that, for the positive definite function f in the theorem, its centralization f_N tends to the delta function δ_e pointwise on $\mathfrak{S}_\infty$. This proves Theorem 1.

4. Evaluation of block length of a permutation

To prove Theorem 2, we need an evaluation of the block length $\|h\|$ from below for $h \in \mathfrak{S}_N$, similar to (2.5) for the length $|h|$ but a little finer.

Let $h = h_1 h_2 \cdots h_m$ be as in §2 a cycle decomposition of $h \in \mathfrak{S}_N$. Consider intervals $[h_j], 1 \le j \le m$, as before. If $[h_j]$ and $[h_{j'}]$ have a non-empty intersection, we join them to get a bigger interval. In this way, we devide the union $\cup_{1 \le j \le m}[h_j]$ into *connected components*. Let M be the number of such connected components. Then we have a partition of the index set $\boldsymbol{I}_m = \{1, 2, \ldots, m\}$ into M subsets $J_p, 1 \le p \le M$, such that $C_p := \cup_{j \in J_p}[h_j]$ are these connected components.

Lemma 6. *For an element $h \in \mathfrak{S}_N$, let the notations be as above. Let the connected components $C_p = \cup_{j \in J_p}[h_j]$ be $[c_p^-, c_p^+]$ for $1 \le p \le M$. Then the block length of h is given as*

$$\|h\| = \sum_{1 \le p \le M} (|C_p| - 1) = \sum_{1 \le p \le M} (c_p^+ - c_p^-) - M. \tag{4.1}$$

Proof. Any cycle h_j can be expressed as a product of simple reflections $s_i = (i \;\; i+1)$ with $\mathrm{supp}(s_i) = \{i, i+1\} \subset [h_j]$ and conversely all such simple reflections are necessary. Thus, to express all $h_j, j \in J_p$, we should use all simple reflections $s_i, c_p^- \le i < c_p^+$, since $C_p = \cup_{j \in J_p}[h_j]$. □

5. Proof of Theorem 2

Since $|f''(g)| = f'(g)$ $(g \in \mathfrak{S}_\infty)$, it is sufficient to prove the theorem for the positive definite function f'. Let us consider its centralization

$$f'_N(g) = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} f'(\sigma g \sigma^{-1}) = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} q^{\|\sigma g \sigma^{-1}\|}. \tag{5.1}$$

Let $g = g_1 g_2 \cdots g_m$ be as before a cycle decomposition of $g \in \mathfrak{S}_N$. Then, for $\sigma \in \mathfrak{S}_N$, a cycle decomposition of $h = \sigma g \sigma^{-1}$ is $h = h_1 h_2 \cdots h_m, h_j = \sigma g_j \sigma^{-1}$. In the sum in (5.1), we devide $\sigma \in \mathfrak{S}_N$ into subsets according to the mutual relation among intervals $[h_j] = [\sigma g_j \sigma^{-1}]$.

Denote by $\mathcal{P}(\boldsymbol{I}_m)$ the set of all partitions of the index set $\boldsymbol{I}_m$, and by $\mathcal{D}_M(\boldsymbol{I}_N)$ the set of all families of *mutually disjoint* non-empty intervals of $\boldsymbol{I}_N$ of number $M \le m$. For a partition $\Delta = \{J_p\,;\, 1 \le p \le M\} \in \mathcal{P}(\boldsymbol{I}_m)$ with non-empty J_p's, we put $|\Delta| = M$, and for a $\Gamma = \{C_p\,;\, 1 \le p \le M = |\Delta|\} \in \mathcal{D}_{|\Delta|}(\boldsymbol{I}_N)$, denote by $S_{\Delta,\Gamma}$ the subset of $\mathfrak{S}_N$ consisting of all $\sigma \in \mathfrak{S}_N$ for which the connected components of $[h_j] = [\sigma g_j \sigma^{-1}], 1 \le j \le m$, are given by Δ and Γ as

$$C_p = \bigcup_{j \in J_p} [h_j] \quad \text{for} \quad 1 \le p \le M = |\Delta|, \quad h_j = \sigma g_j \sigma^{-1}. \tag{5.2}$$

Then, $S_{\Delta,\Gamma}$ with $\Delta \in \mathcal{P}(\boldsymbol{I}_m), \Gamma \in \mathcal{D}_{|\Delta|}(\boldsymbol{I}_N)$, gives a partition of the group $\mathfrak{S}_N$, and so the sum in (5.1) is expressed by the help of Lemma 6 as follows:

$$f'_N(g) = \sum_{\Delta \in \mathcal{P}(\boldsymbol{I}_m)} q^{-|\Delta|} \cdot F_{\Delta,N}(g) \tag{5.3}$$

with
$$F_{\Delta,N}(g) := \frac{1}{N!} \sum_{\Gamma \in \mathcal{D}_{|\Delta|}(\boldsymbol{I}_N)} \sum_{\sigma \in S_{\Delta,\Gamma}} \prod_{1 \le p \le M} q^{|C_p|}$$

$$= \frac{1}{N!} \sum_{\Gamma \in \mathcal{D}_{|\Delta|}(\boldsymbol{I}_N)} |S_{\Delta,\Gamma}| \prod_{1 \le p \le M} q^{c_p^+ - c_p^-}, \tag{5.4}$$

where $C_p = [c_p^+, c_p^-]$ and $|C_p| = c_p^+ - c_p^- \ge L_p := \sum_{j \in J_p} \ell_j$ with $\ell_j =$ the length of cycles g_j and h_j.

It is enough for us to prove $F_{\Delta,N} \to 0$ $(N \to \infty)$ for any fixed $\Delta \in \mathcal{P}(\boldsymbol{I}_m)$. To do so, we should evaluate the order $|S_{\Delta,\Gamma}|$ from above.

Let $S'_{\Delta,\Gamma}$ be the subset of $\mathfrak{S}_N$ consisting of all such σ that satisfies the following condition weaker than (5.2):

$$C_p \supset \bigcup_{j \in J_p} [h_j] \quad \text{for} \quad 1 \le p \le M = |\Delta|, \quad h_j = \sigma g_j \sigma^{-1}. \tag{5.5}$$

Then naturally $S'_{\Delta,\Gamma} \supset S_{\Delta,\Gamma}$. We evaluate the order $|S'_{\Delta,\Gamma}|$.

For an interval C_p, denote by $\overline{C_p}$ its underlying set $\{ c_p^-, c_p^- + 1, \ldots, c_p^+ \} \subset \boldsymbol{I}_N$. Then, the condition (5.5) is equivalent to

$$\overline{C_p} \supset \bigcup_{j \in J_p} \sigma \cdot \mathrm{supp}(g_j) \quad \text{for} \quad 1 \le p \le M = |\Delta|. \tag{5.6}$$

Therefore the number of different ways of chices of $\sigma \cdot \mathrm{supp}(g_j), j \in J_p$, is

$$\frac{|C_p|!}{\prod_{j \in J_p} \ell_j! \cdot (|C_p| - L_p)!} = \frac{L_p!}{\prod_{j \in J_p} \ell_j!} \times \binom{|C_p|}{L_p}, \tag{5.7}$$

where $L_p = \sum_{j \in J_p} \ell_j$. Then $t(g; \Delta, \Gamma) := |S'_{\Delta,\Gamma}|$ is given as

$$t(g; \Delta, \Gamma) = \prod_{1 \le p \le M} L_p! \times \prod_{1 \le p \le M} \binom{|C_p|}{L_p} \times (N - |\mathrm{supp}(g)|)!, \tag{5.8}$$

where $M = |\Delta|$, and $|\mathrm{supp}(g)| = \sum_{1 \le p \le M} L_p$.

From this we obtain a majorized power series in q for the expression (5.4) of $F_{\Delta,N}(g)$ as follows:

$$\begin{aligned} F_{\Delta,N}(g) &= \frac{1}{N!}\sum_{\Gamma} t(g;\Delta,\Gamma)\prod_{p=1}^{M} q^{|C_p|} \\ &\leq \frac{C(g)\cdot(N-|\mathrm{supp}(g)|)!}{N!}\prod_{1\leq p\leq M}\ \sum_{c_p^-+L_p\leq c_p^+\leq N}\binom{c_p^+-c_p^-}{L_p} q^{c_p^+-c_p^-}, \end{aligned}$$

with a constant $C(g)$ depending only on g. Here in the last summation, we admit, for $1\leq p\leq M$, any pairs of positive integers $c_p^- < c_p^+ \leq N$ satisfying $L_p \leq c_p^+ - c_p^-$, forgetting the original meaning of $C_p = [c_p^-, c_p^+], 1\leq p\leq M$, and keeping only the inequality coming from $L_p \leq |C_p|$. (This corresponds roughly that, for a set of intervals $\Gamma = \{\, C_p\,;\ 1\leq p\leq M\,\}$, we discard the condition of mutual-disjointness of C_p's.)

Now, in the last term, first we sum up with respect to $d_p := c_p^+ - c_p^-$ over $L_p \leq d_p \leq \infty$, and then get by Lemma 5, $q^{L_p}/(1-q)^{L_p+1}$. The integer c_p^- runs over $[1, N-L_p+1]$, and so we get at most N times of the above. Multiplying these evaluations, we get finally

$$\begin{aligned} F_{\Delta,N}(g) &\leq \frac{C(g)\cdot N^M (N-|\mathrm{supp}(g)|)!}{N!}\prod_{p=1}^{M}\frac{q^{L_p}}{(1-q)^{L_p+1}} \\ &= \frac{C(g)\cdot N^M (N-|\mathrm{supp}(g)|)!}{N!}\cdot\frac{q^{|\mathrm{supp}(g)|}}{(1-q)^{|\mathrm{supp}(g)|+M}}. \end{aligned} \tag{5.9}$$

The right hand side of (5.9) tends to 0 as $N\to\infty$. This implies, as is explained at the beginning of the proof, that $f'_N(g)\to 0$ for $g\neq e$ as is desired. So Theorem 2 is now completely proved.

6. Closures in $\mathrm{Rep}(\mathfrak{S}_\infty)$ of unitary representations

In this section, we state a rather astonishing property of unitary representations of the infinite symmetric group $\mathfrak{S}_\infty$.

For a locally compact group G, a topology is introduced in the set $\mathrm{Rep}(G)$ of its unitary representations by means of 'weak containment', for which we refer to [Di, §18]. In consequence, a topology is introduced in the dual $\widehat{G}$ of G.

For the infinite symmetric group $G = \mathfrak{S}_\infty$, any irreducible unitary representation (= IUR) known until now can be realized as an induced representation $\mathrm{Ind}_H^G\pi$ from a wreath product type subgroup H and its irreducible unitary representation π, as is proved in [Hi2].

Theorem 7. *For any irreducible unitary representation of the infinite symmetric group $G=\mathfrak{S}_\infty$ given in* [Hi2]*, its closure in* $\mathrm{Rep}(G)$*, with respect to the topology of weak containment, contains at least one of the trivial representation $\mathbf{1}_G$, the sign representation sgn_G and the regular representation λ_G.*

Method of Proof. Take an IUR ρ given as an induced representation $\mathrm{Ind}_H^G \pi$. Take a positive definite function f_π associated to π which is given as its matrix element. Then, a positive definite function F associated to ρ is given as an inducing up of f_π: $F = \mathrm{Ind}_H^G f_\pi$, which is defined as an extension of f_π to G by putting 0 outside of H (see the next section).

Using explicit form of a wreath product subgroup H, we can work as in the previous sections. In more detail, choosing an appropriate increasing sequence of subgroups $G_N \nearrow \mathfrak{S}_\infty$ $(N \to \infty)$, $G_N = \mathfrak{S}_{J_N}$ with $J_N \nearrow \boldsymbol{N}$, we calculate the centralization

$$F^{G_N}(g) := \frac{1}{|G_N|} \sum_{\sigma \in G_N} F(\sigma g \sigma^{-1}) \qquad (g \in G = \mathfrak{S}_\infty) \tag{6.1}$$

on G_N of F, and prove that $F^{G_N}(g)$ converges respectively to the constant function 1, the sign $\mathrm{sgn}(g)$ or the delta function $\delta(g)$ pointwise, as $N \to \infty$.

The key points are

(i) a kind of reduction from F to f_π, and

(ii) an estimation of the order of $\{ \sigma \in G_N \,;\ \sigma h \sigma^{-1} \in H \}$ for an element $h \in H, \neq e$.

According to the result in Theorem 7, we can propose certain conjectures.

Conjecture 1 (a weaker form). *For the infinite symmetric group $G = \mathfrak{S}_\infty$, every infinite-dimensional IUR is not closed in the dual space $\widehat{G}$ as a one point set, with respect to the weak containment topology.*

Recall that this topology can be defined in two different ways. The one is by means of the so-called hull-kernel topology according to the containment relation among kernels of representations, and the other is by means of the convergence of positive definite functions associated with representations, cf. for instance, [Di, §3, §18].

Recall further the following fact [Di, §4, §9, §18]. Let G' be a locally compact, unimodular and separable group. Assume that G' is of type I. Then, for an IUR π of G', the one point set $\{ [\pi] \}$ in $\widehat{G'}$ is closed if and only if the representation π is CCR, or equivalently, $\pi(L^1(G')) \subset \mathcal{C}(\mathcal{H}_\pi)$ (cf. [Di, §13]). Here, $\mathcal{C}(\mathcal{H}_\pi)$ denotes the algebra of all compact operators on the representation space $\mathcal{H}_\pi$ of π.

In our present case, the group $G = \mathfrak{S}_\infty$ is not of type I. Here again, if an IUR π is CCR, then the one point set $\{[\pi]\}$ is closed. However the converse is not known to be true. Furthermore, since G is discrete, an IUR π of G is CCR if and only if $\pi(g)$ is compact for any $g \in G$, and so $\dim \pi$ is finite.

Thus the above Conjecture 1 makes sense, and we propose further the following more exact one.

Conjecture 2. *For the infinite symmetric group $G = \mathfrak{S}_\infty$, every infinite-dimensional IUR contains in its closure in* $\mathrm{Rep}(G)$ *at least one of the trivial representation $\mathbf{1}_G$, the sign representation* sgn_G *and the regular representation λ_G.*

7. Inducing up of positive definite functions

In a general setting, let G be a discrete group, and H its subgroup. Take a unitary representation π of H on a Hilbert space $\mathcal{V}_\pi$, and consider an induced representation $\rho = \mathrm{Ind}_H^G \pi$.

The representation space $\mathcal{H}_\rho$ of ρ is given as follows. For a vector $v \in \mathcal{V}_\pi$, and a representative g_0 of a right coset $Hg_0 \in H\backslash G$, put

$$E_{v,g_0}(g) = \begin{cases} \pi(h)v & (g = hg_0, h \in H), \\ 0 & (g \notin Hg_0). \end{cases} \tag{7.1}$$

Let $\mathcal{H}$ be a linear span of these $\mathcal{V}_\pi$-valued functions on G, and define an inner product on it as

$$\langle E_{v,g_0}, E_{v',g_0'} \rangle = \begin{cases} \langle \pi(h)v, v' \rangle & \text{if } hg_0 = g_0' \ (\exists h \in H), \\ 0 & \text{if } Hg_0 \neq Hg_0'. \end{cases} \tag{7.2}$$

The space $\mathcal{H}_\rho$ is nothing but the completion of $\mathcal{H}$.

The representation ρ is given as

$$\rho(g_1)E(g) = E(gg_1) \quad (g_1, g \in G, E \in \mathcal{H}_\rho). \tag{7.3}$$

Now take a non-zero vector $v \in \mathcal{V}_\pi$ and put $E = E_{v,e} \in \mathcal{H}_\rho$. Consider a positive definite function on H associated to π as

$$f_\pi(h) = \langle \pi(h)v, v \rangle \qquad (h \in H), \tag{7.4}$$

and also such a one on G associated to ρ as

$$F(g) = \langle \rho(g)E, E \rangle \qquad (g \in G). \tag{7.5}$$

Then, we can easily prove the following lemma.

Lemma 8. *The positive definite function F on G associated to $\rho = \mathrm{Ind}_H^G \pi$ is equal to the inducing up of the positive definite function f_π on H associated to π: $F = \mathrm{Ind}_H^G f_\pi$, which is, by definition, equal to f_π on H and to zero outside of H.*

8. Case of characters $\mathbf{1}_G$ and sgn_G

Firstly we treat the case where the closure of an induced representation $\rho = \mathrm{Ind}_H^G \pi$ contains characters $\mathbf{1}_G$ or sgn_G.

Let H be a subgroup of $G = \mathfrak{S}_\infty$ of the product form $H = H_1 H_2$, where $H_1 = \mathfrak{S}_I$ and $H_2 \subset \mathfrak{S}_J$ with an infinite subset $I \subset \boldsymbol{N}$ and $J = \boldsymbol{N} \setminus I$. Denote by χ_1 a character $\mathbf{1}_{\mathfrak{S}_I}$ or $\mathrm{sgn}_{\mathfrak{S}_I}$ of the group $\mathfrak{S}_I \cong \mathfrak{S}_\infty$, and by π_2 a unitary representation (= UR) of H_2. Take a UR $\pi = \chi_1 \otimes \pi_2$ of $H_1 H_2$ and induce it up to G to get $\rho = \mathrm{Ind}_H^G \pi$.

Theorem 9. *Let a unitary representation $\pi = \chi_1 \otimes \pi_2$ of $H = H_1 H_2$ be as above. Then the closure of its induced representation $\rho = \mathrm{Ind}_H^G \pi$ of $G = \mathfrak{S}_\infty$ contains the character $\chi_G = \mathbf{1}_G$ or sgn_G corresponding to $\chi_1 = \mathbf{1}_{\mathfrak{S}_I}$ or $\mathrm{sgn}_{\mathfrak{S}_I}$.*

Proof. Let $J_N \subset \boldsymbol{N}$ be a series of increasing subsets such that $|J_N| = N$, $J_N \nearrow \boldsymbol{N}$, and that the ratio $|I \cap J_N|/|J_N| \to 1$ as $N \to \infty$, so that $|J \cap J_N|/N \to 0$. Then, $G_N := \mathfrak{S}_{J_N} \nearrow G = \mathfrak{S}_\infty$ and we consider the centralizations of a positive definite function F associated to ρ along the series of increasing subgroups G_N: for $g \in G$,

$$F^{G_N}(g) := \frac{1}{|G_N|} \sum_{\sigma \in G_N} F(\sigma g \sigma^{-1}) = \frac{1}{N!} \sum_{\sigma \in G_N} F(\sigma g \sigma^{-1}). \tag{8.1}$$

Take a unit vector v from the representation space $\mathcal{H}_{\pi_2}$ and put a positive definite function f_π associated to π as

$$f_\pi(h_1 h_2) = \chi_1(h_1) \cdot \langle \pi_2(h_2)v, v\rangle \qquad (h_1 \in H_1, h_2 \in H_2). \tag{8.2}$$

Then $F = \mathrm{Ind}_H^G f_\pi$ is such a one associated to $\rho = \mathrm{Ind}_H^G \pi$, by Lemma 8.

Now take an arbitrary $g \in G$. Since $J_N \nearrow \boldsymbol{N}$, if N is sufficiently large, there exists a $\sigma_0 \in G_N$ such that $g' = \sigma_0 g \sigma_0^{-1} \in H_1 \cap G_N = \mathfrak{S}_{I \cap J_N}$ or

$$S' := \mathrm{supp}(g') \subset I \cap J_N. \tag{8.3}$$

Then we have $F^{G_N}(g) = F^{G_N}(g')$.

Fix $g' \in \mathfrak{S}_I$, and consider the asymptotic behavior of the value $F^{G_N}(g')$ as $N \to \infty$. In the formula (8.1) for g', instead of g, we devide the sum over $\sigma \in G_N = \mathfrak{S}_{J_N}$ into three parts as follows.

Case 1: σ such that $\sigma g' \sigma^{-1} \in \mathfrak{S}_I \cap G_N$ or equivalently $\sigma S' \subset I \cap J_N$;

Case 2: σ such that $\sigma g' \sigma^{-1} \in H = H_1 H_2$, but not in Case 1;

Case 3: σ such that $\sigma g' \sigma^{-1} \notin H$.

In Case 1, $F(\sigma g' \sigma^{-1}) = f_\pi(\sigma g' \sigma^{-1}) = \chi_G(g') = \chi_G(g)$. The number of such $\sigma \in G_N = \mathfrak{S}_{J_N}$ is equal to

$$\frac{|I \cap J_N|!}{(|I \cap J_N| - |S'|)!} \times |J_N \setminus S'|! = \frac{|I \cap J_N|!}{(|I \cap J_N| - k)!} \times (N-k)! \tag{8.4}$$

with $k = |S'| = |\mathrm{supp}(g)|$. Therefore, since $|I \cap J_N|/N \to 1$, the partial sum for Case 1 in (8.1) is evaluated as follows when N tends to ∞:

$$\frac{1}{|G_N|} \sum_{\sigma \in G_N:\,\mathrm{Case}\ 1} F(\sigma g' \sigma^{-1}) = C_N \cdot \chi_G(g), \tag{8.5}$$

$$C_N = \frac{1}{N!} \cdot \frac{|I \cap J_N|!}{(|I \cap J_N| - k)!} \cdot (N-k)! = \prod_{p=0}^{k-1} \frac{|I \cap J_N| - p}{N - p} \longrightarrow 1. \tag{8.6}$$

In Case 2, we have $|F(\sigma g' \sigma^{-1})| \le 1$ and the evaluation in Case 1 shows us that the partial sum for this case tends to zero as $N \to \infty$. (This follows directly from $\lim_{N\to\infty} C_N = 1$.) In Case 3, we have $F(\sigma g' \sigma^{-1}) = 0$ and there is no contribution to the sum in (8.1).

Altogether we get finally $F^{G_N}(g) \to \chi_G(g)$ $(g \in G)$. This proves our assertion. □

9. A reduction to a subgroup $\mathfrak{S}_I \cong \mathfrak{S}_\infty$, $I \subset \boldsymbol{N}$

To treat the case where the closure of $\rho = \mathrm{Ind}_H^G \pi$ contains the regular representation λ_G, it is better to prepare a preliminary step.

We take a subgroup $H \subset G = \mathfrak{S}_\infty$ of the product form $H = H_1 H_2$, where $H_1 \subset \mathfrak{S}_I \cong \mathfrak{S}_\infty$ and $H_2 \subset \mathfrak{S}_J$ with an infinite subset $I \subset \boldsymbol{N}$ and $J = \boldsymbol{N} \setminus I$. Take also an infinite-dimensional UR π_1 of H_1 and a UR π_2 of H_2. Then we take a UR $\pi = \pi_1 \otimes \pi_2$ of $H = H_1 H_2$ and its induced one $\rho = \mathrm{Ind}_H^G \pi$ of G.

For $j = 1, 2$, take a unit vector v_j from the representation space $\mathcal{H}_{\pi_j}$ and put a positive definite function f_π associated to π as

$$f_\pi(h_1 h_2) = f_{\pi_1}(h_1) \cdot f_{\pi_2}(h_2), \quad f_{\pi_j}(h_j) = \langle \pi_j(h_j) v_j, v_j \rangle \quad (h_j \in H_j).$$

Then $F = \mathrm{Ind}_H^G f_\pi$ is a positive definite function associated to $\rho = \mathrm{Ind}_H^G \pi$.

Let $J_N \subset \boldsymbol{N}$ be a series of increasing subsets with the same property as in the proof of Theorem 9, so that putting $J'_N = I \cap J_N$, we have

$$J'_N \nearrow I \quad \text{and} \quad |J'_N|/|J_N| = |J'_N|/N \to 1 \ (N \to \infty).$$

For our later use, we put $G' := \mathfrak{S}_I \supset H_1$, which is naturally isomorphic to $\mathfrak{S}_\infty$, and put $F' := \mathrm{Ind}_{H_1}^{G'} f_{\pi_1}$. Then, F' is a positive definite function on G' associated to $\mathrm{Ind}_{H_1}^{G'} \pi_1$.

We have $G_N := \mathfrak{S}_{J_N} \nearrow G = \mathfrak{S}_\infty$ and $G'_N := \mathfrak{S}_{J'_N} = G' \cap G_N \nearrow G'$. We compare centralizations F^{G_N} in (8.1) of a positive definite function $F = \mathrm{Ind}_H^G f_\pi$ with those $(F')^{G'_N}$ of $F' = \mathrm{Ind}_{H_1}^{G'} f_{\pi_1}$, concerning their limits as $N \to \infty$.

Take an arbitrary $g \in G$. Then, if N is sufficiently large, there exists a $\sigma_0 \in G_N$ such that $g' = \sigma_0 g \sigma_0^{-1} \in \mathfrak{S}_I \cap G_N = \mathfrak{S}_{J'_N}$ with $J'_N = I \cap J_N$ (in another notation, $g' \in G'_N \subset G'$), or equivalently $S' := \mathrm{supp}(g') \subset J'_N$. Then, $F^{G_N}(g) = F^{G_N}(g')$.

Fix $g' \in \mathfrak{S}_I = G'$, and devide the sum over $\sigma \in G_N = \mathfrak{S}_{J_N}$ in (8.1) for $F^{G_N}(g')$ into three parts according to Cases 1, 2 and 3 for σ as in the proof of Theorem 9.

Case 1: In Case 1, since $g' \in G'_N \subset G'$, and $\sigma g' \sigma^{-1} \in G'_N$, there exists a $\sigma' \in G'_N$ such that $\sigma g' \sigma^{-1} = \sigma' g' \sigma'^{-1}$. Since $G' \cap H = H_1$, we have $F(\sigma g' \sigma^{-1}) = F(\sigma' g' \sigma'^{-1}) = F'(\sigma' g' \sigma'^{-1})$.

Note that $(\sigma g' \sigma^{-1})(i) = i$ for $i \notin \sigma(S') := \{\, \sigma(j)\, ;\, j \in S' \,\}$, then we see that the restriction $\sigma|S'$ of σ determines the element $\sigma g' \sigma^{-1}$ completely. So we count the number of $\sigma \in G_N = \mathfrak{S}_{J_N}$ (resp. $G_N \cap \mathfrak{S}_I = \mathfrak{S}_{J'_N} = G'_N$) in Case 1 that have the same restriction $\sigma|S'$ on $S' \subset I$. They are equal to $|J_N \setminus S'|! = (N-k)!$ and $(|J'_N| - k)!$ respectively, with $k = |S'| = |\mathrm{supp}(g)|$. Hence,

$$\frac{1}{N!} \sum_{\sigma \in G_N:\ \text{Case 1}} F(\sigma g' \sigma^{-1}) = C_N \times \frac{1}{|J'_N|!} \sum_{\sigma \in G_N \cap \mathfrak{S}_I = \mathfrak{S}_{J'_N}} F'(\sigma g' \sigma^{-1})$$

with
$$C_N = \frac{|J'_N|!}{N!} \cdot \frac{(N-k)!}{(|J'_N| - k)!} \longrightarrow 1 \quad (N \to \infty).$$

Since $G_N \cap \mathfrak{S}_I = \mathfrak{S}_{J'_N} = G'_N$, the right hand side of the above equality, except the constant factor C_N, is nothing but the centralization, with respect to G'_N of positive definite function F' on G':

$$(F')^{G'_N}(g') := \frac{1}{|G'_N|} \sum_{\sigma \in G'_N} F'(\sigma g' \sigma^{-1}). \tag{9.1}$$

Cases 2 and 3: In Case 2, the partial sum over $\sigma \in G_N$ in this case tends to zero as $N \to \infty$ similarly as in the proof of Theorem 9. In Case 3, we have no contribution to the sum in (8.1).

Altogether we get the following lemma.

Lemma 10. *Let the notations be as above, in particular,* $H = H_1 H_2, H_1 \subset \mathfrak{S}_I, H_2 \subset \mathfrak{S}_J$ *with* $|I| = \infty, J = \boldsymbol{N} \setminus I$, *and* $\pi = \pi_1 \otimes \pi_2$ *with a UR* π_j *of* H_j, *and take* $f_\pi(h_1 h_2) = f_{\pi_1}(h_1) f_{\pi_2}(h_2)$ $(h_j \in H_j)$. *Put* $F = \mathrm{Ind}_H^G f_\pi$ *for* $G = \mathfrak{S}_\infty$, *and* $F' = \mathrm{Ind}_{H_1}^{G'} f_{\pi_1}$ *for* $G' = \mathfrak{S}_I \cong \mathfrak{S}_\infty$.

For an increasing sequence of subsets $J_N \nearrow \boldsymbol{N}$, *put* $G_N = \mathfrak{S}_{J_N}, G'_N = G' \cap G_N = \mathfrak{S}_{J'_N}$ *with* $J'_N = I \cap J_N$. *For any* $g \in G = \mathfrak{S}_\infty$, *there exists a* $g' \in G'$ *conjugate to* g *in* G. *If the sequence* J_N *satisfies* $|J'_N|/|J_N| \to 1 (N \to \infty)$, *then,*

$$\lim_{N \to \infty} F^{G_N}(g) = \lim_{N \to \infty} (F')^{G'_N}(g'). \tag{9.2}$$

10. Case of the regular representation λ_G

We follow the notations in the previous section. For a subgroup $H = H_1 H_2 \subset G = \mathfrak{S}_\infty$, we take as H_1 a so-called wreath product type subgroup imbedded into $G' = \mathfrak{S}_I \cong \mathfrak{S}_\infty$ in a saturated way, and $H_2 \subset \mathfrak{S}_J, J = \boldsymbol{N} \setminus I$. Let us explain for H_1 in more detail.

Take any finite group T and a countable infinite index set Y. Put $T_\eta = T$ for any $\eta \in Y$, and take a restricted direct product $D_Y(T) := \prod'_{\eta \in Y} T_\eta$. Denote by $\mathfrak{S}_Y$ the group of all finite permutations on Y, then it acts naturally on $D_Y(T)$ by permuting components of

$$d = (t_\eta)_{\eta \in Y} \in D_Y(T).$$

The semidirect product group $D_Y(T) \rtimes \mathfrak{S}_Y$ is called a wreath product of T with $\mathfrak{S}_Y$ and is denoted by $\mathfrak{S}_Y(T)$, where, for $\sigma \in \mathfrak{S}_Y$ and $d \in D_Y(T)$, $\sigma \cdot d \cdot \sigma^{-1} = (t'_\eta)$ with $t'_\eta = t_{\sigma^{-1}(\eta)}$ $(\eta \in Y)$.

We imbed $\mathfrak{S}_Y(T)$ into $\mathfrak{S}_I$ as follows. Take a faithful permutation representation of T into a finite symmetric group $\mathfrak{S}_n$, and identify T with its image in $\mathfrak{S}_n$. On the other hand, an ordered set $\mathcal{J} = (p_1, p_2, \ldots, p_n)$ of different n integers $p_j \in \boldsymbol{N}$ is called an *ordered n-set* and denote by $\overline{\mathcal{J}} := \{ p_1, p_2, \ldots, p_n \}$ its underlying subset of $\boldsymbol{N}$. We decompose I into infinite number of ordered n-sets $\mathcal{J}_\eta, \eta \in Y$: $I = \sqcup_{\eta \in Y} \overline{\mathcal{J}_\eta}$. For each η, denote by ι_η the order-preserving correspondence $p_j \mapsto j$ $(1 \leq j \leq n)$ from $\mathcal{J}_\eta = (p_1, p_2, \ldots, p_n)$ onto $\boldsymbol{I}_n = \{ 1, 2, \ldots, n \}$. Then ι_η gives us an imbedding

$$\varphi_\eta : T_\eta = T \subset \mathfrak{S}_n \ni \sigma \mapsto \iota_\eta^{-1} \cdot \sigma \cdot \iota_\eta \in \mathfrak{S}_{\overline{\mathcal{J}_\eta}} \subset \mathfrak{S}_I. \tag{10.1}$$

This fixes imbeddings of $D_Y(T)$ and $\mathfrak{S}_Y$, and the one Φ of $\mathfrak{S}_Y(T)$ into $\mathfrak{S}_I$, which depends on a partition $\mathcal{I} = \{ \mathcal{J}_\eta \}_{\eta \in Y}$ of I into ordered n-sets.

We take $H_1 = \Phi(\mathfrak{S}_Y(T)) \subset \mathfrak{S}_I$, which is denoted also by $H(\mathcal{I}, T)$. In case T is trivial and imbedded into $\mathfrak{S}_1 = \{ e \}, n = 1$, we have $H(\mathcal{I}, T) = \mathfrak{S}_I$. Except this trivial case, we call such a subgroup as $H(\mathcal{I}, T)$ *properly of wreath product type*.

We take URs π_j of H_j for $j = 1, 2$, and then a tensor product representation $\pi = \pi_1 \otimes \pi_2$ of $H = H_1 H_2$, and induced it up to G: $\rho = \mathrm{Ind}_H^G \pi$. To get an irreducible UR of G by this method, we should choose as π_1 an IUR coming from an infinite tensor product (with respect to a reference vector) of a fixed irreducible finite-dimensional representation of T, and of course similar kinds of restrictions are necessary for H_2 and π_2. Further details are given in [Hi1] and [Hi2], and are summarized in Section 12 below. For our later use, we define for $\mathcal{I} = (\mathcal{J}_\eta)_{\eta \in Y}$ and $T \subset \mathfrak{S}_n$ the folowing

$$\mathrm{supp}(H(\mathcal{I}, T)) = \mathrm{supp}(\mathcal{I}) := \sqcup_{\eta \in Y} \overline{\mathcal{J}_\eta} \subset \boldsymbol{N}.$$

Theorem 11. *Let a subgroup $H \subset G = \mathfrak{S}_\infty$ be given as $H = H_1 H_2$, with a proper wreath product type subgroup $H_1 = H(\mathcal{I}, T)$ of $G' = \mathfrak{S}_I \cong \mathfrak{S}_\infty$, and $H_2 \subset \mathfrak{S}_J, J = \boldsymbol{N} \setminus I$. Let π_1 be an infinite-dimensional UR of H_1 and π_2 a UR of H_2. Take a tensor product representation $\pi = \pi_1 \otimes \pi_2$ of $H = H_1 H_2$. Then the closure of its induced representation $\rho = \mathrm{Ind}_H^G \pi$ of G contains the regular representation λ_G.*

Proof. By Lemma 10, we may and do assume that $H = H_1 = H(\mathcal{I}, T)$, that is, $I = \boldsymbol{N}$. The finite group T is contained in $\mathfrak{S}_n$ with $n \geq 2$. For $\pi = \pi_1$ and $f_\pi(h) = \langle \pi(h) v, v \rangle, v \in \mathcal{H}_\pi, \|v\| = 1$, we have $|F(h)| \leq 1$ for $F = \mathrm{Ind}_H^G f_\pi$. Therefore, taking $G_N = \mathfrak{S}_{J_N}, J_N \nearrow \boldsymbol{N}$, we have the following evaluation for $g \in G$

$$(10.2) \qquad |F^{G_N}(g)| \leq \frac{1}{|G_N|} \sum_{\sigma \in G_N} |F(\sigma g \sigma^{-1})| \leq \frac{D_N(g; H)}{|G_N|} = \frac{D_N(g; H)}{|J_N|!}$$

with $D_N(g; H) := |\{\sigma \in G_N \,;\, \sigma g \sigma^{-1} \in H\}|$.

We evaluate the number $D_N(g; H)$ from above. Replacing $T \subset \mathfrak{S}_n$ by $\mathfrak{S}_n$, we consider a bigger subgroup $\widetilde{H} \supset H = H(\mathcal{I}, T) = \Phi(\mathfrak{S}_Y(T))$, that is,

$$\widetilde{H} = H(\mathcal{I}, \mathfrak{S}_n) = \Phi(\mathfrak{S}_Y(\mathfrak{S}_n)).$$

Then, naturally $D_N(g; H) \leq D_N(g; \widetilde{H})$, and thus we evaluate the latter.

Recall that these subgroups are defined by means of a partition of $I = \boldsymbol{N}$ into ordered n-sets as $I = \sqcup_{\eta \in Y} \overline{\mathcal{J}_\eta}$. We introduce a linear order $\eta_1, \eta_2, \ldots$ in Y, and put $J_N := \sqcup_{1 \leq i \leq N} \overline{\mathcal{J}_{\eta_i}}$. Then, $|J_N| = nN$ and $J_N \nearrow \boldsymbol{N}$.

Take an arbitrary $g \in G, \neq e$, and decompose it into disjoint cycles as in (2.1):

$$(10.3) \qquad g = g_1 g_2 \cdots g_m, \quad g_j = (i_{j1} \;\; i_{j2} \;\; \ldots \;\; i_{j\ell_j}),$$

then, $\mathrm{supp}(g) = \sqcup_{1\le j\le m}\mathrm{supp}(g_j)$, with $\mathrm{supp}(g_j) = \{\, i_{j1}, i_{j2}, \ldots, i_{j\ell_j}\,\}$. For $\sigma \in G$, put $h = \sigma g \sigma^{-1}$ and $h_j = \sigma g_j \sigma^{-1}$, then,

$$h = \sigma g \sigma^{-1} = h_1 h_2 \cdots h_m, \quad h_j = (\sigma(i_{j1})\ \ \sigma(i_{j2})\ \ \ldots\ \ \sigma(i_{j\ell_j})\,). \tag{10.4}$$

We treat the case where $D_N(g; H) > 0$ for sufficiently large N. Take a $\sigma \in G_N$ such that $h = \sigma g \sigma^{-1} \in \widetilde{H}$. Then, we have the following two cases:

Case I: For a certain $j, 1 \le j \le m$, $\mathrm{supp}(h_j) = \sigma\,\mathrm{supp}(g_j) \subset \overline{\mathcal{J}_{\eta_i}}$ for some $1 \le i \le N$.

Case II: For any $j, 1 \le j \le m$, $\mathrm{supp}(h_j) = \sigma\,\mathrm{supp}(g_j) \not\subset \overline{\mathcal{J}_{\eta_i}}$ for any $1 \le i \le N$.

Denote by $D_N^I(g; \widetilde{H})$ (resp. $D_N^{II}(g; \widetilde{H})$) the number of $\sigma \in G_N$ with $h = \sigma g \sigma^{-1} \in \widetilde{H}$ which is in Case I (resp. Case II). Then we have the following evaluations from above.

Lemma 12.

$$D_N^I(g; \widetilde{H}) \le m \cdot N \cdot n(n-1) \cdot (N'-2)!, \qquad N' = nN,$$

$$D_N^{II}(g; \widetilde{H}) \le \left(\sum_{j=1}^{m} \frac{\ell_j(\ell_j - 1)}{2} + |\mathrm{supp}(g)| \right) \cdot N \cdot n(n-1) \cdot (N'-2)!\,.$$

Assume this lemma be granted, then

$$\frac{D_N(g; H)}{|G_N|} \le \frac{D_N(g; \widetilde{H})}{|J_N|!} \le \frac{D_N^I(g; \widetilde{H}) + D_N^{II}(g; \widetilde{H})}{N'!} \longrightarrow 0. \tag{10.5}$$

This has to be proved for Theorem 11. □

Proof of Lemma 12.

Case I. Let $\sigma \in G_N$ be in Case I. For a certain pair (j, i) with $1 \le j \le m$ and $1 \le i \le N$, we have $\sigma\,\mathrm{supp}(g_j) \subset \overline{\mathcal{J}_{\eta_i}}$. The number of pairs (j, i) is $m \cdot N$.

We pick up i_{j1}, i_{j2} in the expression (10.3) of g_j, and watch their images $k_1 = \sigma(i_{j1}), k_2 = \sigma(i_{j2})$ under σ. They are chosen from $\overline{\mathcal{J}_{\eta_i}}$ of n elements, and so the number of choices is $n(n-1)$.

After choosing k_1, k_2, other $\sigma(p)$ for $p \in J_N \setminus \{i_{j1}, i_{j2}\}$ can be chosen from $J_N \setminus \{k_1, k_2\}$ for which the number of elements is $nN - 2 = N' - 2$.

Case II. We divide $\sigma \in G_N$ in Case II, into two subsets Σ_1, Σ_2 according to that for a certain pair $(j, i), 1 \le j \le m, 1 \le i \le N$, the intersection $\sigma\,\mathrm{supp}(g_j) \cap \overline{\mathcal{J}_{\eta_i}}$ contains at least two elements, or for any pair (j, i), the intersection $\sigma\,\mathrm{supp}(g_j) \cap \overline{\mathcal{J}_{\eta_i}}$ contains at most one elements.

For $\sigma \in \Sigma_1$, we have a pair (j, i) such that for some $p_1, p_2 \in \mathrm{supp}(g_j)$, both $\sigma(p_1), \sigma(p_2)$ belong to $\overline{\mathcal{J}_{\eta_i}}$. The number of ways to choose $\{p_1, p_2\}$ is equal to $\ell_j(\ell_j - 1)/2$ and so the number $|\Sigma_1|$ corresponds to the first term in the right hand side of the second inequality.

For $\sigma \in \Sigma_2$, put

$$Y^j = \{\, \eta_i\ ;\ \sigma\,\mathrm{supp}(g_j) \cap \overline{\mathcal{J}_{\eta_i}} \ne \emptyset, 1 \le i \le N\,\}.$$

Then, the union $\sqcup_{\eta_i \in Y^1} \overline{\mathcal{J}_{\eta_i}}$ is the disjoint union of $\sigma\,\mathrm{supp}(g_j)$ over such j that $Y^j = Y^1$. Therefore, for any such j, there exists at least one $p \in \mathrm{supp}(g_j)$ such that $\sigma(p)$ and $\sigma(i_{11})$ for $i_{11} \in \mathrm{supp}(g_1)$ both belong to one of $\overline{\mathcal{J}_{\eta_i}}, \eta_i \in Y^1$. The number of p is limited by $|\mathrm{supp}(g)|$, and so we get an evaluation of the number $|\Sigma_2|$ as is given in the second term in the right hand side of the second inequality.

This completes the proof of Lemma 12. □

11. Indecomposable positive definite class functions

For the infinite symmetric group $G = \mathfrak{S}_\infty$, all the indecomposable (or extremal) positive definite class functions, which we call Thoma characters, are classified and are given explicitly in [Tho2].

After Satz 3 in [Tho2], they are written as follows. Let $\alpha = (\alpha_1, \alpha_2, \dots), \beta = (\beta_1, \beta_2, \dots)$ be decreasing sequences of non-negative real numbers such that

$$\sum_{1 \le k < \infty} \alpha_k + \sum_{1 \le k < \infty} \beta_k \le 1, \tag{11.1}$$

and put $\gamma_0 = 1 - (\|\alpha\| + \|\beta\|) \ge 0$, with $\|\alpha\| := \sum_{1 \le k < \infty} \alpha_k$, $\|\beta\| := \sum_{1 \le k < \infty} \beta_k$, so that $\|\alpha\| + \|\beta\| + \gamma_0 = 1$.

Take a $g \in G$ and let $g = g_1 g_2 \cdots g_m$ be a cycle decomposition in (10.3), where the length of cycle g_j is denoted by ℓ_j. For $\nu \ge 2$, let $n_\nu(g) = |\{\, j \,;\, \ell_j = \nu \,\}|$ the number of g_j with length ν. Then the character $f_{\alpha,\beta}$ determined by the parameter (α, β) is given by

$$f_{\alpha,\beta}(g) = \prod_{2 \le \nu < \infty} \left(\sum_{1 \le k < \infty} \alpha_k^\nu + (-1)^{\nu+1} \sum_{1 \le k < \infty} \beta_k^\nu \right)^{n_\nu(g)}. \tag{11.2}$$

The case where $\alpha_1 = 1$ (resp. $\beta_1 = 1$ and $\gamma_0 = 1$) corresponds to the identity representation $\mathbf{1}_G$ (resp. the sign representation sgn_G, and the regular representation λ_G). Except the cases of 1-dimensional representations $\mathbf{1}_G$ and sgn_G, the indecomposable positive definite invariant function $f = f_{\alpha,\beta}$ defines a character in the sense of [Di, §6] of a II_1 type factor representation π_f of G. Here π_f denotes a cyclic representation associated to f by GNS construction. Recall that, for the $*$-algebra $C_c(G)$ of functions on G with compact supports, $f = f_{\alpha,\beta}$ defines a $*$-homomorphism (i.e., linear map with $f(\varphi^*) = \overline{f(\varphi)}$)

$$C_c(G) \ni \varphi \longmapsto f(\varphi) := \sum_{g \in G} f(g)\varphi(g) \in \boldsymbol{C},$$

which induces a character $\pi_f(\varphi) \to f(\varphi)$ of the von Neumann algebra generated by $\{\, \pi_f(\varphi) \,;\, \varphi \in C_c(G) \,\}$ (cf. [Tho1]). These factor representations can be decomposed into irreducible representations, but explicit decompositions are known only in the case where $\gamma_0 = 0$, in [Ob2].

Now let us rewrite the formula (11.2) in another form. Put

$$\chi_G^{(k)} = \mathbf{1}_G, \quad \chi_G^{(-k)} = \mathrm{sgn}_G, \qquad \alpha_{-k} = \beta_k$$

for $k=1,2,\ldots$. Then, when $\ell_j=\nu$, we have $(-1)^{\nu+1}=(-1)^{\ell_j+1}=\mathrm{sgn}_G(g_j)=\chi_G^{(-k)}(g_j)$. Therefore the formula (11.2) is written as

$$(11.3)\qquad f_{\alpha,\beta}(g)=\prod_{1\le j\le m}\left(\sum_{1\le k<\infty}\chi_G^{(k)}(g_j)\alpha_k^{\ell_j}+\sum_{1\le k<\infty}\chi_G^{(-k)}(g_j)(\alpha_{-k})^{\ell_j}\right)$$
$$=\prod_{1\le j\le m}\left(\sum_{k\in\boldsymbol{Z}^*}\chi_G^{(k)}(g_j)\alpha_k^{\ell_j}\right)\quad\text{with}\quad \boldsymbol{Z}^*=\boldsymbol{Z}\setminus\{0\}.$$

We expand this product into a sum of monomial products as follows. Let $K_+=\max\{k\ ;\ \alpha_k>0\}, K_-=\min\{k\ ;\ \alpha_k>0\}$, and let $\boldsymbol{Z}_{\alpha,\beta}$ be the intersection of the interval $[K_-,K_+]\subset\boldsymbol{Z}$ with $\boldsymbol{Z}^*$. Then the sum over $k\in\boldsymbol{Z}^*$ in (11.3) is actually over $k\in\boldsymbol{Z}_{\alpha,\beta}$. Thus we get

$$(11.4)\qquad f_{\alpha,\beta}(g)=\sum_{(k_1,k_2,\ldots,k_m)\in(\boldsymbol{Z}_{\alpha,\beta})^m}\ \prod_{1\le j\le m}\chi_G^{(k_j)}(g_j)(\alpha_{k_j})^{\ell_j},$$

where $g=g_1g_2\cdots g_m$ is a cycle decomposition and ℓ_j is the length of cycle g_j.

As is shown later, this expression of $f_{\alpha,\beta}$ has its own intrinsic meaning in relation to the centralization of matrix elements of certain induced representations of G which contain all irreducible unitary representations (= IURs) constructed in [Hi2].

12. IURs of $G=\mathfrak{S}_\infty$ as induced representations

Take a subgroup H of G of the form

$$(12.1)\qquad H=H_0H_PH_Q,\quad H_P=\textstyle\prod'_{p\in P}H_p,\quad H_Q=\prod'_{q\in Q}H_q,$$

where $H_0=\mathfrak{S}_B$ with a finite subset $B\subset\boldsymbol{N}$, $H_p=\mathfrak{S}_{I_p}$ with an infinite subset $I_p\subset\boldsymbol{N}$, and $H_q=H(\mathcal{I}_q,T_q)$ properly of wreath product type subgroup with $T_q\subset\mathfrak{S}_{n_q}, n_q>1$, and an infinite partition $\mathcal{I}_q=(\mathcal{J}_{\eta_q})_{\eta_q\in Y_q}$ of $I_q:=\mathrm{supp}(H(\mathcal{I}_q,T_q))$ into ordered n_q-sets $\mathcal{J}_{\eta_q}$. Thus H is determined by the data

$$\mathfrak{c}:=(B,(I_p)_{p\in P},(\mathcal{I}_q,T_q)_{q\in Q})$$

and is denoted also by $H^{\mathfrak{c}}$. We assume that H is "saturated" in G in the sense that

$$(12.2)\qquad \boldsymbol{N}=B\sqcup(\sqcup_{p\in P}I_p)\sqcup(\sqcup_{q\in Q}I_q)$$

is a partition of $\boldsymbol{N}$. We admit the cases where some of B,P and Q are empty.

As an IUR of H, we take

$$(12.3)\qquad \pi=\pi_0\otimes(\otimes_{p\in P}\chi_p)\otimes(\otimes^b_{q\in Q}\pi_q),$$

where π_0 is an IUR of $H_0=\mathfrak{S}_B$, χ_p is a character of $H_p=\mathfrak{S}_{I_p}$ (and so trivial one or sign), and π_q is an IUR of $H_q=H(\mathcal{I}_q,T_q)$, and the tensor product $\otimes^b_{q\in Q}\pi_q$ is taken with respect to a reference vector $b = (b_q)_{q\in Q}, b_q \in V(\pi_q), \|b_q\| = 1$, if $\dim \pi_q > 1$ for infinitely many $q \in Q$. Here $V(\pi_q)$ denotes the representation space of π_q.

As an IUR π_q of the group $H_q = H(\mathcal{I}_q, T_q) \cong \mathfrak{S}_{Y_q}(T_q) := D_{Y_q}(T_q) \rtimes \mathfrak{S}_{Y_q}$, we take the following one. Take an IUR ρ_{T_q} of the finite group T_q, and consider it as an IUR ρ_{η_q} of each component $T_{\eta_q} = T_q$ of $D_{Y_q}(T_q) = \prod'_{\eta_q\in Y_q} T_{\eta_q}$. Making their tensor product, we get an IUR π'_q of the restricted direct product $D_{Y_q}(T_q)$. Here, in case $\dim \rho_{T_q} > 1$, the tensor product is taken with respect to a reference vector

$$a_q = (a_{\eta_q})_{\eta_q\in Y_q} \quad \text{with} \quad a_{\eta_q} \in V(\rho_{\eta_q}), \quad \|a_{\eta_q}\| = 1.$$

For a $\sigma \in \mathfrak{S}_{Y_q}$, put for $\otimes_{\eta_q\in Y_q} w_{\eta_q} \in \otimes^{a_q}_{\eta_q\in Y_q} V(\rho_{\eta_q})$,

$$\pi'_q(\sigma)(\otimes_{\eta_q\in Y_q} w_{\eta_q}) := \chi_{Y_q}(\sigma)(\otimes_{\eta_q\in Y_q} w'_{\eta_q}), \quad w'_{\eta_q} = w_{\sigma^{-1}(\eta_q)},$$

where χ_{Y_q} is a character of $\mathfrak{S}_{Y_q}$. Then, $\pi'_q(d\cdot\sigma) := \pi'_q(d)\pi'_q(\sigma)$ gives an IUR of $\mathfrak{S}_{Y_q}(T_q)$. Pulling π'_q back to $H_q = H(\mathcal{I}_q, T_q)$ through an isomorphism similar to Φ in Section 10, we get an IUR π_q of H_q.

Thus the IUR π of $H = H^{\mathfrak{c}}$ is determined by the data $(\mathfrak{c}, \mathfrak{d})$ with

$$\mathfrak{d} := (\pi_0, (\chi_p)_{p\in P}, (b; (\rho_{T_q}, \chi_{Y_q}, a_q)_{q\in Q})),$$

and is denoted also by $\pi(\mathfrak{c}, \mathfrak{d})$.

We know in [Hi2] that, under the saturation condition (12.2), the induced representation

$$\rho(\mathfrak{c}, \mathfrak{d}) = \mathrm{Ind}_H^G \pi(\mathfrak{c}, \mathfrak{d})$$

is irreducible, and equivalence relations among these IURs are also clarified there. As far as I know, this big family of IURs of $G = \mathfrak{S}_\infty$ contains all IURs known until now.

13. Centralization of matrix elements of IURs

For an IUR given as $\rho(\mathfrak{c}, \mathfrak{d}) = \mathrm{Ind}_H^G \pi(\mathfrak{c}, \mathfrak{d})$, we take one of its matrix elements as a positive definite function on G and study limits of its centralizations. So, take a unit vector $v_0 \in V(\pi_0)$ and $v_Q \in \otimes^b_{q\in Q} V(\pi_q)$, and consider a matrix element f_π of $\pi = \pi(\mathfrak{c}, \mathfrak{d})$ given according to (12.3) as

$$f_\pi(h) = \langle \pi_0(h_0)v_0, v_0\rangle \cdot (\otimes_{p\in P}\chi_p)(h_P) \cdot \langle (\otimes^b_{q\in Q}\pi_q)(h_Q)v_Q, v_Q\rangle, \tag{13.1}$$

where $h = h_0h_Ph_Q \in H = H_0H_PH_Q$ is a decomposition according to (12.1). Then $F = \mathrm{Ind}_H^G f_\pi$ is a matrix element of $\rho(\mathfrak{c}, \mathfrak{d}) = \mathrm{Ind}_H^G \pi$. Let us study the centralizations F^{G_N} of F for certain increasing sequences $G_N \nearrow G$ of subgroups.

Take $G_N = \mathfrak{S}_{J_N}, J_N \nearrow \boldsymbol{N}$, as follows. We demand an asymptotic condition

$$\frac{|I_p \cap J_N|}{|J_N|} \to \lambda_p \ \ (p \in P), \quad \frac{|I_q \cap J_N|}{|J_N|} \to \mu_q \ \ (q \in Q), \tag{13.2}$$

then there holds

$$\sum_{p\in P}\lambda_p + \sum_{q\in Q}\mu_q \ \le\ 1. \tag{13.3}$$

In case $|P|+|Q|<\infty$, we have an equality here. Put for the family $\{\,H_p=\mathfrak{S}_{I_p}\ ;\ p\in P\,\}$,

$$P_+=\{\,p\in P\ ;\ \chi_p=\mathbf{1}_{H_p}\,\},\quad P_-=\{\,p\in P\ ;\ \chi_p=\mathrm{sgn}_{H_p}\,\}, \tag{13.4}$$

then we have the following inequality similar as (11.1)

$$\sum_{p\in P_+}\lambda_p+\sum_{p\in P_-}\lambda_p\le 1. \tag{13.5}$$

At this stage, first let us give our results in the following theorem and the succeeding corollaries, and then give the proof of the theorem in the next section.

From a technical reason for proving the convergence of sequences F^{G_N} as $N\to\infty$, we assume in the following an additional condition on the way of growing up of J_N's, in such a form that, for each $q\in Q$,

$$I_q\cap J_N \quad \text{is a union of subsets } \overline{\mathcal{J}_{\eta_q}},\ \eta_q\in Y_q \quad \text{for } N>>0. \tag{13.6}$$

Theorem 13. *Let $H=H_0H_PH_Q$ be a subgroup of $G=\mathfrak{S}_\infty$, and π be its irreducible unitary representation given above in* (12.1)–(12.2) *and in* (12.3) *respectively. For a positive definite function f_π given in* (13.1) *as a matrix element of π, put $F=\mathrm{Ind}_H^G f_\pi$. Then it is a positive definite function associated to the induced representation $\rho=\mathrm{Ind}_H^G\pi$.*

According to an increasing sequence $G_N=\mathfrak{S}_{J_N}\nearrow G$ of subgroups, the centralizations F^{G_N} of F converges pointwise to a Thoma character $f_{\alpha,\beta}$ if $J_N\nearrow\boldsymbol{N}$ satisfies the asymptotic condition (13.2). *Here the parameters $\alpha=(\alpha_1,\alpha_2,\ldots)$ and $\beta=(\beta_1,\beta_2,\ldots)$ are determined from $(\lambda_p)_{p\in P_+},(\lambda_p)_{p\in P_-}$, respectively by rearranging λ_p's as decreasing sequences.*

The inequality (13.5) *corresponds exactly to* (11.1), *and $\gamma_0=1-\sum_{p\in P}\lambda_p$.*

Put $p_+=|P_+|, p_-=|P_-|$. Then the lengths of α and β are limited by p_+ and p_- in such a sense that $\alpha_k=0\ (k>p_+),\beta_k=0\ (k>p_-)$.

Corollary 14. (i) *In the case where $|P|<\infty$ and $Q=\emptyset$, as limits of centralizations of $F=\mathrm{Ind}_H^G f_\pi$, there appear all $f_{\alpha,\beta}$ with $\alpha=(\alpha_1,\alpha_2,\ldots)$ limited by p_+ and $\beta=(\beta_1,\beta_2,\ldots)$ limited by p_- satisfying the equality*

$$\|\alpha\|+\|\beta\|=\sum_{1\le k<\infty}\alpha_k+\sum_{1\le k<\infty}\beta_k=1. \tag{13.7}$$

(ii) *In other cases, as limits of centralizations of $F=\mathrm{Ind}_H^G f_\pi$, there appear all $f_{\alpha,\beta}$ with $\alpha=(\alpha_1,\alpha_2,\ldots)$ limited by p_+ and $\beta=(\beta_1,\beta_2,\ldots)$ limited by p_- satisfying the inequality* (11.1): *$\|\alpha\|+\|\beta\|\le 1$, and in particular, $f_{\mathbf{0},\mathbf{0}}=\delta_e$ with $\alpha=\beta=\mathbf{0}=(0,0,\ldots)$ and $\gamma_0=1$.*

The invariant positive definite function $f_{\alpha,\beta}$ is a matrix element of a II_1 factor representation of G, associated to its cyclic vector. Therefore, in terms of the weak containment topology in the space $\mathrm{Rep}(G)$ of representations [Di, §18], we can translate the above corollary as follows.

Corollary 15. (i) *In the case where* $|P| < \infty$ *and* $Q = \emptyset$, *the closure in* $\mathrm{Rep}(G)$ *of one point set* $\{\rho\}$ *of irreducible unitary representation* $\rho = \mathrm{Ind}_H^G \pi$ *contains all* II_1 *factor representations corresponding to* $f_{\alpha,\beta}$ *with* α *limited by* p_+ *and* β *limited by* p_- *satisfying the equality* (13.7).

(ii) *In other cases, the closure in* $\mathrm{Rep}(G)$ *of one point set* $\{\rho\}$ *contains all* II_1 *factor representations corresponding to* $f_{\alpha,\beta}$ *with* α *limited by* p_+ *and* β *limited by* p_- *satisfying the inequality* (11.1), *and in particular, it contains the regular representation* λ_G.

Notation 13.1. For an IUR $\rho = \mathrm{Ind}_H^G \pi$, $\rho = \rho(\mathfrak{c}, \mathfrak{d}), \pi = \pi(\mathfrak{c}, \mathfrak{d})$, denote by $\mathcal{TC}(\rho)$ the set of all Thoma characters obtained here as limits of centralizations of the matrix element $F = \mathrm{Ind}_H^G f_\pi$. Then,

$$\mathcal{TC}(\rho) := \{ f_{\alpha,\beta}\ ;\ \alpha, \beta \text{ coming from } (\lambda_p)_{p\in P_+}, (\lambda_p)_{p\in P_-} \text{ satisfying Condition (TC)} \},$$

$$\text{Condition (TC):} \quad \begin{cases} \sum_{p\in P} \lambda_p = 1 & \text{if } |P| < \infty \text{ and } Q = \emptyset\ ; \\ \sum_{p\in P} \lambda_p \le 1 & \text{otherwise.} \end{cases}$$

Remark 13.1. In the case where $P = \emptyset$, the only positive definite class function obtained here as limits of centralizations of matrix elements of $\rho = \mathrm{Ind}_H^G \pi$ is $f_{\mathbf{0},\mathbf{0}} = \delta_e$, that is, $\mathcal{TC}(\rho) = \{\delta_e\}$.

Therefore, in this case, the closure of one point set $\{\rho\}$ of irreducible representation ρ contains the regular representation λ_G, but seemingly does not contain any other II_1 type factor representations.

14. Proof of Theorem 13

14.1. Case of $Q = \emptyset$

Let us first consider a case where $Q = \emptyset$. Take a $g \in \mathfrak{S}_\infty$ and let

$$g = g_1 g_2 \cdots g_m, \tag{14.1}$$

be its cycle decomposition. The centralization of $F = \mathrm{Ind}_H^G f_\pi$ over $G_N = \mathfrak{S}_{J_N}$ is

$$F^{G_N}(g) = \frac{1}{|G_N|} \sum_{\sigma \in G_N} F(\sigma g \sigma^{-1}) = \frac{1}{|J_N|!} \sum_{\substack{\sigma \in G_N \\ \sigma g \sigma^{-1} \in H}} f_\pi(\sigma g \sigma^{-1}). \tag{14.2}$$

Here, $H = H_0 H_P, H_P = \prod'_{p\in P} H_p$, and $f_\pi(h) = \langle \pi_0(h_0)v_0, v_0 \rangle \cdot \prod_{p\in P} \chi_p(h_p)$ for $h = h_0 \prod_{p\in P} h_p \in H_0 \prod'_{p\in P} H_p$.

Suppose N is sufficiently large so that $J_N \supset \sqcup_{1\le j\le m} K_j$ with $K_j :=$ $\mathrm{supp}(g_j)$. Recall that $H_0 = \mathfrak{S}_B, H_p = \mathfrak{S}_{I_p}$ $(p \in P)$, and $\mathrm{supp}(\sigma g_j \sigma^{-1}) =$

σK_j, then we see that the condition $\sigma g \sigma^{-1} \in H$ is equivalent to that each $\sigma K_j, 1 \le j \le m$, is contained in some of B, I_p $(p \in P)$. Put

$$(14.3) \qquad \begin{aligned} S(g) &:= \{\, \sigma \in G_N = \mathfrak{S}_{J_N} \;;\; \sigma g \sigma^{-1} \in H \,\}, \\ S_P(g) &:= \{\, \sigma \in S(g) \;;\; \sigma g \sigma^{-1} \in H_P \,\}, \\ S^B(g) &:= \{\, \sigma \in S(g) \;;\; \sigma g \sigma^{-1} \text{ has non-trivial} \\ &\qquad\qquad \text{component in } H_0 = \mathfrak{S}_B \,\}. \end{aligned}$$

Then, $S(g) = S_P(g) \sqcup S^B(g)$, and moreover $S_P(g)$ is decomposed into disjoint sum of its subsets as follows. Let $\delta = \{\, J_p \;;\; p \in P \,\}$ be a partition indexed by P of the set $\boldsymbol{I}_m = \{\, 1, 2, \ldots, m \,\}$ of indices of $g_j' s$ ($J_p = \emptyset$ except for finite number of p), and put

$$S_\delta(g) := \{\, \sigma \in S(g) \;;\; \sigma K_j \subset I_p \;\text{ or }\; \sigma g_j \sigma^{-1} \in \mathfrak{S}_{I_p} = H_p \; (j \in J_p, p \in P) \,\}.$$

Denote by $\mathcal{P}_m$ the set of all partitions $\delta = \{\, J_p \;;\; p \in P \,\}$ of $\boldsymbol{I}_m$ indexed by P, and by $\mathcal{P}_{m,N}$ its finite subset consisting of δ for which $J_p = \emptyset$ if $I_p \cap J_N = \emptyset$. Then $S_P(g) = \sqcup_{\delta \in \mathcal{P}_m} S_\delta(g)$, and $S_\delta(g) = \emptyset$ for $\delta \notin \mathcal{P}_{m,N}$. We have

$$(14.4) \qquad S(g) := S^B(g) \sqcup (\sqcup_{\delta \in \mathcal{P}_m} S_\delta(g)).$$

The right hand side of (14.5) below is a sum over $\sigma \in S(g)$, decomposed into partial sums according to the above decomposition of $S(g)$,

$$(14.5) \quad F^{G_N}(g) = \frac{1}{|J_N|!} \sum_{\sigma \in S^B(g)} f_\pi(\sigma g \sigma^{-1}) + \sum_{\delta \in \mathcal{P}_m} \frac{1}{|J_N|!} \sum_{\sigma \in S_\delta(g)} f_\pi(\sigma g \sigma^{-1}).$$

We study the second term. Put $h_j = \sigma g_j \sigma^{-1}$, then $\sigma g \sigma^{-1} = h_1 h_2 \cdots h_m$. For $\delta = \{\, J_p \;;\; p \in P \,\} \in \mathcal{P}_m$, $h_j \in H_p$ $(j \in J_p)$ and $\chi_p(h_j) = 1$ or $\chi_p(h_j) = \mathrm{sgn}(g_j) = (-1)^{\ell_j - 1}$ with $\ell_j = l(g_j)$ the length of g_j. Denote this value by $\chi_p(g_j)$, then $f_\pi(\sigma g \sigma^{-1}) = \prod_{p \in P} \prod_{j \in J_p} \chi_p(g_j)$. Hence we have

$$(14.6) \qquad \frac{1}{|J_N|!} \sum_{\sigma \in S_\delta(g)} f_\pi(\sigma g \sigma^{-1}) = \prod_{p \in P} \prod_{j \in J_p} \chi_p(g_j) \cdot \frac{|S_\delta(g)|}{|J_N|!}.$$

The number of elements $|S_\delta(g)|$ is given from the condition $\sigma K_j \subset I_p \cap J_N$ $(j \in J_p)$. Since $|K_j| = \ell_j$, we can choose for $\sqcup_{j \in J_p} \sigma K_j$ freely $\sum_{j \in J_p} \ell_j$ number of elements from $I_p \cap J_N$. Noting that $\sum_{p \in P} \sum_{j \in J_p} \ell_j = \sum_{j \in \boldsymbol{I}_m} \ell_j$, we get

$$(14.7) \quad \begin{aligned} |S_\delta(g)| = \prod_{p \in P} & |I_p \cap J_N| (|I_p \cap J_N| - 1) \cdots \Big(|I_p \cap J_N| - \textstyle\sum_{j \in J_p} \ell_j + 1 \Big) \\ & \times \Big(|J_N| - \textstyle\sum_{j \in \boldsymbol{I}_m} \ell_j \Big)!, \end{aligned}$$

and $\sum_{\delta \in \mathcal{P}_m} |S_\delta(g)| / |J_N|! \le 1$. When J_N grows up to $\boldsymbol{N}$ under the condition $|I_p \cap J_N| / |J_N| \to \lambda_p$ $(p \in P)$, we have

$$(14.8) \qquad \textstyle\sum_{p \in P} \lambda_p \;\le\; 1.$$

Furthermore, dividing the both sides of (14.7) by $|J_N|!$, and taking limits as $N \to \infty$, we obtain

$$\lim_{N\to\infty} \frac{|S_\delta(g)|}{|J_N|!} = \prod_{p\in P} \prod_{j\in J_p} \lambda_p^{\ell_j} \quad \text{with} \quad \ell_j = l(g_j).$$

Thus, we can choose the way of growing up of J_N so that the second term of (14.5) has the limit

$$\sum_{\delta\in\mathcal{P}_m} \prod_{p\in P} \prod_{j\in J_p} \chi_p(g_j)(\lambda_p)^{l(g_j)} = \prod_{j=1}^{m} \left(\sum_{p\in P} \chi_p(g_j)\lambda_p^{l(g_j)} \right). \tag{14.9}$$

On the other hand, for the first term of (14.5), noting that B is finite, we see by an evaluation similar to that of $|S_\delta(g)|$ that the limit of $|S^B(g)|/|J_N|!$ as $N \to \infty$ is equal to zero (cf. also **14.2** below).

Comparing the above formula (14.9) with the formula (11.3) or (11.4), we see that the proof of Theorem 13 in the case $Q = \emptyset$ is now complete.

14.2. Case of $Q \neq \emptyset$

Here we study the general case of $Q \neq \emptyset$. Let $S(g) = \{ \sigma \in G_N = \mathfrak{S}_{J_N} \,;\, \sigma g \sigma^{-1} \in H \}$ and $S^B(g), S_P(g)$ be as in **14.1**, and in addition put

$$S^Q(g) := \{ \sigma \in S(g) \,;\, \sigma g \sigma^{-1} \text{ has non-trivial component in } H_Q \}. \tag{14.10}$$

Then, $S(g) = (S^B(g) \cup S^Q(g)) \sqcup S_P(g)$, and accordingly the formula (14.5) is rewritten as

$$\begin{aligned} F^{G_N}(g) &= \frac{1}{|J_N|!} \sum_{\sigma\in S^B(g)\cup S^Q(g)} f_\pi(\sigma g \sigma^{-1}) \\ &\quad + \sum_{\delta\in\mathcal{P}_m} \frac{1}{|J_N|!} \sum_{\sigma\in S_\delta(g)} f_\pi(\sigma g \sigma^{-1}). \end{aligned} \tag{14.11}$$

Denote by $\Sigma_I(g;N)$ and $\Sigma_{II}(g;N)$ the first term and the second term in the right hand side. We want to prove that $\Sigma_I(g;N) \to 0$ as $N \to \infty$, under the condition

$$\frac{|I_p \cap J_N|}{|J_N|} \to \lambda_p \;\; (p \in P), \qquad \frac{|I_q \cap J_N|}{|J_N|} \to \mu_q \;\; (q \in Q). \tag{14.12}$$

If this is done, the proof of Theorem 13 will be completed, because the limit of the second term $\Sigma_{II}(g;N)$ can be obtained just as in **14.1**.

Now let $\delta' = \{ J_0, J_p \ (p \in P), J_q \ (q \in Q) \}$ be a partition of $\boldsymbol{I}_m$ for which at least one of $J_0, J_q \ (q \in Q)$ is non-empty. For $\sigma \in S(g)$, put $h = \sigma g \sigma^{-1}, h_j = \sigma g_j \sigma^{-1} \ (j \in \boldsymbol{I}_m)$, then $h = h_1 h_2 \cdots h_m$. Define

$$S_{\delta'}(g) := \{\sigma \in S(g) \,;\, h_j = \sigma g_j \sigma^{-1} \ (j \in \boldsymbol{I}_m) \text{ satisfy Condition (SQ)} \}$$

$$\text{Condition (SQ):} \quad \begin{cases} h_j \in H_0 = \mathfrak{S}_B \text{ or } \sigma K_j \subset B \ (j \in J_0), \\ h_j \in H_p = \mathfrak{S}_{I_p} \text{ or } \sigma K_j \subset I_p \ (j \in J_p, p \in P), \\ h_j \in H_q = H(\mathcal{I}_q, T_q) \ (j \in J_q, q \in Q). \end{cases}$$

Denote by $\mathcal{P}'_m$ the set of all possible such partitions δ'. Noting that $|f_\pi(\sigma g \sigma^{-1})| \leq 1$, we get the evaluation

$$|\Sigma_I(g;N)| \leq \sum_{\delta' \in \mathcal{P}'_m} \frac{|S_{\delta'}(g)|}{|J_N|!}. \tag{14.13}$$

So we should evaluate the number $|S_{\delta'}(g)|$.

For a subset $J \subset \boldsymbol{I}_m$ and a subgroup H' of H, we denote by $DF(J,H')$ the number of possible ways for choosing integers $\sigma(k) \in J_N$ $(k \in \sqcup_{j\in J} K_j)$ under Condition (SQ) for $\sigma \in S(g)$ in such a way that $\sigma(\prod_{j\in J} g_j)\sigma^{-1} = \prod_{j\in J} h_j \in H'$. ($DF$ = degree of freedom). Similarly, for $K := J_N \setminus \mathrm{supp}(g) = J_N \setminus \sqcup_{j\in \boldsymbol{I}_m} K_j$, denote by $DF'(K,H)$ the number of possible ways for choosing integers $\sigma(k) \in J_N$ $(k \in K)$ under Condition (SQ) in such a way that $\sigma g \sigma^{-1} = h \in H$ (after choosing all of $\sigma(k), k \in \mathrm{supp}(g)$). Then,

$$\begin{aligned} |S_{\delta'}(g)| = DF(J_0,H_0) \cdot \textstyle\prod_{p\in P} DF(J_p,H_p) \\ \times \textstyle\prod_{q\in Q} DF(J_q,H_q) \times DF'(J_N \setminus \sqcup_{j\in \boldsymbol{I}_m} K_j, H), \end{aligned} \tag{14.14}$$

where $K_j = \mathrm{supp}(g_j), \sqcup_{j\in \boldsymbol{I}_m} K_j = \mathrm{supp}(g)$.

In **14.1**, we calculated $DF(J_p, H_p = \mathfrak{S}_{I_p})$ as given below, noting that the condition (SQ) for this term is equivalent to $\sigma(K_j) \subset I_p$ $(j \in J_p)$ and that $|\sqcup_{j\in J_p} K_j| = \sum_{j\in J_p} \ell_j$,

$$DF(J_p,H_p) = |I_p \cap J_N|(|I_p \cap J_N| - 1) \cdots \left(|I_p \cap J_N| - \textstyle\sum_{j\in J_p} \ell_j + 1\right).$$

Similarly $DF(J_0,H_0)$ with $H_0 = \mathfrak{S}_B$ is given as follows if N is sufficiently large so that $B \subset J_N$:

$$DF(J_0,H_0) = |B|(|B|-1) \cdots \left(|B| - \textstyle\sum_{j\in J_0} \ell_j + 1\right). \tag{14.15}$$

After taking all of $\sigma(k), k \in \mathrm{supp}(g) = \sqcup_{j\in \boldsymbol{I}_m} K_j$, other elements $\sigma(i), i \in J_N \setminus \mathrm{supp}(g)$, can be chosen freely from $J_N \setminus \sigma(\mathrm{supp}(g))$, and so

$$DF'(J_N \setminus \sqcup_{j\in \boldsymbol{I}_m} K_j, H) = \left(|J_N| - \textstyle\sum_{j\in \boldsymbol{I}_m} \ell_j\right)!\,. \tag{14.16}$$

Note that, as $N \to \infty$, the factor $1/|J_N|!$ in (14.11) can be replaced by a simpler one if we note

$$\frac{1}{|J_N|!} \times \left(|J_N| - \textstyle\sum_{j\in \boldsymbol{I}_m} \ell_j\right)! \times \prod_{j\in \boldsymbol{I}_m} |J_N|^{\ell_j} \longrightarrow 1 \quad (N \to \infty).$$

Then we see that the contribution to the limit from a partial sum for δ' is majorized by

$$
\begin{aligned}
\lim_{N\to\infty}\frac{|S_{\delta'}(g)|}{|J_N|!} &= \lim_{N\to\infty}\frac{|B|}{|J_N|}\cdot\frac{|B|-1}{|J_N|}\cdot\cdots\cdot\frac{|B|-\sum_{j\in J_0}\ell_j+1}{|J_N|} \\
&\times\prod_{p\in P}\lim_{N\to\infty}\frac{|I_p\cap J_N|}{|J_N|}\cdot\frac{|I_p\cap J_N|-1}{|J_N|}\cdot\cdots\cdot\frac{|I_p\cap J_N|-\sum_{j\in J_p}\ell_j+1}{|J_N|} \\
&\times\prod_{q\in Q}\lim_{N\to\infty}\frac{DF(J_q,H_q)}{\prod_{j\in J_q}|J_N|^{\ell_j}}.
\end{aligned}
\tag{14.17}
$$

Therefore, if $J_0\neq\emptyset$ in δ', or if the first factor (containing $|B|$) actually exists in the right hand side of (14.17), then it is equal to zero and so the left hand side (contribution to the limit) is also zero.

14.3. Calculation for wreath product subgroup $H_q = H(\mathcal{I}_q, T_q)$

Now assume $J_0=\emptyset$ in δ'. Then it is enough for us to prove that the ratio

$$DF(J_q,H_q)/\textstyle\prod_{j\in J_q}|J_N|^{\ell_j} \tag{14.18}$$

tends to zero as $N\to\infty$ for $J_q\neq\emptyset$. Recall that

$$H_q = H(\mathcal{I}_q,T_q)\cong \mathfrak{S}_{Y_q}(T_q) := D_{Y_q}(T_q)\rtimes\mathfrak{S}_{Y_q},$$

with a subgroup $T_q\subset\mathfrak{S}_{n_q}, n_q>1$, and an infinite partition $\mathcal{I}_q=(\mathcal{J}_\eta)_{\eta\in Y_q}$ of $I_q := \mathrm{supp}(H(\mathcal{I}_q,T_q))$ into ordered n_q-sets $\mathcal{J}_\eta$. Replacing $T_q\subset\mathfrak{S}_{n_q}$ by $\mathfrak{S}_{n_q}$, we get a bigger group $\widetilde{H}_q = H(\mathcal{I}_q,\mathfrak{S}_{n_q})$, so that $H_q\subset\widetilde{H}_q\subset\mathfrak{S}_{I_q}$. Since $DF(J_q,H')$ is defined by the condition $\prod_{j\in J_q}h_j\in H'$ for $h_j=\sigma g_j\sigma^{-1}$, there holds

$$DF(J_q,H_q)\le DF(J_q,\widetilde{H}_q)\le DF(J_q,\mathfrak{S}_{I_q}).$$

Here the last term is given by a formula similar to that for $DF(J_p,H_p)$ by means of $\sqcup_{j\in J_q}K_j$ and I_q. Let us evaluate the middle term.

Denote by $\mathfrak{S}_{\mathcal{J}_\eta}$ the symmetric group $\mathfrak{S}_{\overline{\mathcal{J}_\eta}}$ on the underlying set of integers $\overline{\mathcal{J}_\eta}$ of ordered n_q-set $\mathcal{J}_\eta$, thus indicating the canonical isomorphisms between them for $\eta\in Y_q$. Then, $\widetilde{H}_q\cong\left(\prod'_{\eta\in Y_q}\mathfrak{S}_{\mathcal{J}_\eta}\right)\rtimes\mathfrak{S}_{Y_q}$.

By assumption (13.6), $I_q\cap J_N$ consists of subsets of the form $\overline{\mathcal{J}_\eta}$, whose number is given by $\nu(N)=\nu(q,N):=|I_q\cap J_N|/n_q$ since $|\overline{\mathcal{J}_\eta}|=n_q$. Put $\nu(N)=\nu(q,N)$ for simplicity and rename these $\mathcal{J}_\eta$'s as $\mathcal{J}_1,\mathcal{J}_2,\ldots,\mathcal{J}_{\nu(N)}$. The condition

$$\sigma(\textstyle\prod_{j\in J_q}g_j)\sigma^{-1}=\prod_{j\in J_q}h_j\in H_q\cong\left(\prod'_{\eta\in Y_q}\mathfrak{S}_{\mathcal{J}_\eta}\right)\rtimes\mathfrak{S}_{Y_q},$$

for $\sigma\in\mathfrak{S}_{J_N}$ to define $DF(J_q,H_q)$, is rewritten as

$$\sigma(\textstyle\prod_{j\in J_q}g_j)\sigma^{-1}=\prod_{j\in J_q}h_j\in\left(\prod_{i=1}^{\nu(N)}\mathfrak{S}_{\mathcal{J}_i}\right)\rtimes\mathfrak{S}_{\nu(N)}. \tag{14.19}$$

Therefore we have the following two cases for each $j \in J_q$.

CASE 1: Case where $h_j = \sigma g_j \sigma^{-1} \in \mathfrak{S}_{\mathcal{J}_i}$ or $\sigma(K_j) \subset \overline{\mathcal{J}_i}$ for some i.

CASE 2: Case where h_j is not in Case 1. So

$$h_j = ((\tau_i)_{i=1}^{\nu(N)}, a) \in \Big(\prod_{i=1}^{\nu(N)} \mathfrak{S}_{\mathcal{J}_i}\Big) \rtimes \mathfrak{S}_{\nu(N)},$$

with a cycle $a \neq \mathbf{1}$ and $\tau_i \neq \mathbf{1}$ for some i (and so the length $|a| < \ell_j$ for a).

Depending on the cases of each h_j, we evaluate the corresponding part in (14.18) by $\prod_{j\in J_p} A_j(N)$, where $A_j(N)$ is the number of possible choices of $h_j \in H_q$ irrespective of other $h_{j'}, j' \in J_p, \neq j$. In Cases 1 and 2, we have respectively

$$\begin{aligned} A_j(N) \;&\leq\; \nu(N) \times n_q!\,, \\ A_j(N) \;&\leq\; \nu(N)(\nu(N)-1)\cdots(\nu(N)-|a|+1) \times (n_q!)^{\ell_j}. \end{aligned}$$

In both cases (in Case 2, remark that $|a| < \ell_j$), we obtain $A_j(N)/|J_N|^{\ell_j} \to 0$ $(N \to \infty)$, and as a result

$$\frac{DF(J_q, H_q)}{\prod_{j\in J_q} |J_N|^{\ell_j}} \leq \text{sum of possible cases of } \prod_{j\in J_q} \frac{A_j(N)}{|J_N|^{\ell_j}} \longrightarrow 0 \qquad (N \to \infty). \tag{14.20}$$

By 14.2–14.3, the proof of Theorem 13 in the case of $Q \neq \emptyset$ is now complete.

15. Case of non-irreducible unitary representations

We keep to the notation in Section 12. Assume $Q \neq \emptyset$ in (12.1), and consider a subgroup $H' = H_0 H_P$ omitting the factor H_Q (or replacing H_Q by $H'_Q = \{\, e \,\}$), and also a subgroup $H'' = H_P$ in place of $H = H_0 H_P H_Q$. These subgroups are small and far from saturated in G. Take an IUR π' of H', and such a one π'' of H'' given as

$$\pi' = \pi_0 \otimes (\otimes_{p\in P}\, \chi_p), \qquad \pi'' = \otimes_{p\in P}\, \chi_p\,, \tag{15.1}$$

and consider induced representations of G as

$$\rho' = \mathrm{Ind}_{H'}^{G} \pi', \qquad \rho'' = \mathrm{Ind}_{H''}^{G} \pi'',$$

which are very far from to be irreducible. Let $f_{\pi'}$ and $f_{\pi''}$ be positive definite functions given as matrix elements of π' and π'' as

$$\begin{aligned} f_{\pi'}(h') &= \langle \pi_0(h'_0) v_0, v_0\rangle \cdot \Big(\prod_{p\in P} \chi_p\Big)(h'_P), \\ f_{\pi''}(h'') &= \Big(\prod_{p\in P} \chi_p\Big)(h''_P), \end{aligned}$$

for $h' = h'_0 h'_P \in H' = H_0 H_P$, and a unit vector $v_0 \in V(\pi_0)$, and $h'' = h''_P \in H'' = H_P$ respectively. Put

$$F' = \mathrm{Ind}_{H'}^{G} f_{\pi'}, \qquad F'' = \mathrm{Ind}_{H''}^{G} f_{\pi''},$$

then F' and F'' are positive definite functions on G, or matrix elements associated to the induced representations $\rho' = \mathrm{Ind}_{H'}^{G}\pi'$ and $\rho'' = \mathrm{Ind}_{H''}^{G}\pi''$ respectively.

Taking limits of centralizations of F' or F'', similarly as for $F = \mathrm{Ind}_H^G f_\pi$ in (13.1) with $H = H_0 H_P H_Q$ in (12.1) and π in (12.3), we get exactly the same family of Thoma characters $f_{\alpha,\beta}$, extremal invariant positive definite functions on G.

In more detail, repeating the discussions in Section 14 (essentially those in **14.1**), we obtain the following result, rather astonishing.

Theorem 16. *Let $G_N = \mathfrak{S}_{J_N}$ be an increasing sequence of subgroups going up to $G = \mathfrak{S}_\infty$. Assume that for every $p \in P$,*

$$|I_p \cap J_N|/|J_N| \longrightarrow \lambda_p \quad (N \to \infty).$$

Then, the centralizations of F' and F'' over G_N tend respectively to a Thoma character $f_{\alpha,\beta}$ pointwise, where the decreasing sequences of non-negative integers $\alpha = (\alpha_1, \alpha_2, \dots)$ and $\beta = (\beta_1, \beta_2, \dots)$ are reorderings of $\{ \lambda_p \ ; \ p \in P_+ \}$ and $\{ \lambda_p \ ; \ p \in P_- \}$ respectively.

These convergences are quite similar as for $F = \mathrm{Ind}_H^G f_\pi$, and are proved word for word as for the second term in (14.11) (cf. **14.1**).

16. Remarks and comments

16.1. Irreducible decompositions of factor representations

Here we treat two extreme cases of Thoma character $f_{\alpha,\beta}$, where $\gamma_0 = 0$ or $\gamma_0 = 1$, with $\|\alpha\| + \|\beta\| + \gamma_0 = 1$.

Case of $\gamma_0 = 0$ or $\|\alpha\| + \|\beta\| = 1$.

An irreducible decomposition of a factor representation π_f (of type II_1) associated to $f = f_{\alpha,\beta}$ is given in [Ob2].

His result says the following. Let i_+ and i_- be natural numbers such that

$$\alpha_{i_+} > \alpha_{i_+ + 1} = 0 \quad \text{and} \quad \beta_{i_-} > \beta_{i_- + 1} = 0.$$

Then, the factor representation π_f is decomposed as an integral of IURs $\rho(\mathfrak{c}, \mathfrak{d}) = \mathrm{Ind}_H^G \pi(\mathfrak{c}, \mathfrak{d})$ with infinite multiplicity, where $\mathfrak{c} = (I_p)_{p \in P}, \mathfrak{d} = (\chi_p)_{p \in P}$, satisfying the condition

$$p_+ := |P_+| = i_+, \quad p_- := |P_-| = i_-. \tag{16.1}$$

Here $H = \prod'_{p \in P} \mathfrak{S}_{I_p}$ a restricted direct product of $\mathfrak{S}_{I_p} \cong \mathfrak{S}_\infty$, and $\pi(\mathfrak{c}, \mathfrak{d}) = \otimes_{p \in P} \chi_p$ a character of H, and P_+, P_- are defined in (13.4).

From this result, we can define $\mathrm{supp}(\pi_f)$ in the dual space $\widehat{G}$ of G for $f = f_{\alpha,\beta}$ by

$$\begin{aligned} \mathrm{supp}(\pi_f) := \big\{ \rho(\mathfrak{c}, \mathfrak{d}) = \mathrm{Ind}_H^G(\otimes_{p \in P} \chi_p) \ ; \ &\mathfrak{c} = (I_p)_{p \in P}, \\ &\mathfrak{d} = (\chi_p)_{p \in P}, H = \textstyle\prod'_{p \in P} \mathfrak{S}_{I_p}, |P_+| = i_+, |P_-| = i_- \big\}. \end{aligned}$$

We can characterize $\mathrm{supp}(\pi_f)$ from the view point of the topology in $\mathrm{Rep}(G)$, or more exactly by means of the set $\mathcal{TC}(\rho)$ (in Notation 13.1) of Thoma characters obtained as limits of centralizations of matrix element $F = \mathrm{Ind}_H^G f_\pi$ of an IUR $\rho = \mathrm{Ind}_H^G \pi$.

Fix $f = f_{\alpha,\beta}$ and consider an IUR $\rho = \mathrm{Ind}_H^G \pi$ such that $\mathcal{TC}(\rho) \ni f_{\alpha,\beta}$. Let $\rho = \rho(\mathfrak{c}, \mathfrak{d}), \pi = \pi(\mathfrak{c}, \mathfrak{d})$, and $\mathfrak{c} = (B, (I_p)_{p\in P}, (\mathcal{I}_q, T_q)_{q\in Q})$. We say that ρ can attain $f_{\alpha,\beta}$ (or that $\mathcal{TC}(\rho)$ contains $f_{\alpha,\beta}$) *without redundancy* if $B = \emptyset$, $|P_+| = i_+, |P_-| = i_-$, and in addition $Q = \emptyset$ in case $\|\alpha\| + \|\beta\| = 1$ (or $\gamma_0 = 0$). The meaning of this terminology is that $B \neq \emptyset$ has no effect to the set $\mathcal{TC}(\rho)$, and that, if $|P_+| > i_+$ for example, we put $\lambda_p = 0$ for some $p \in P_+$ (in other words, kill the role of p) to get $f_{\alpha,\beta}$. Put

$$\mathcal{IUR}(f_{\alpha,\beta}) := \{\, \rho = \mathrm{Ind}_H^G \pi \;;\; \mathcal{TC}(\rho) \ni f_{\alpha,\beta}, \ \ \text{without redundancy}\}.$$

Proposition 17. *Assume* $\|\alpha\| + \|\beta\| = 1$. *For the indecomposable positive definite class function* $f = f_{\alpha,\beta}$, *the support* $\mathrm{supp}(\pi_f)$ *of* II_1 *factor representation* π_f *is characterized as follows*:

$$\mathrm{supp}(\pi_f) = \mathcal{IUR}(f), \qquad f = f_{\alpha,\beta}. \tag{16.2}$$

Remark 16.1. The expression given in (11.4) of $f_{\alpha,\beta}$ plays an important role for our calculation in Section 14. It has also an intimate relation to Obata's method in [Ob2] of giving irreducible decompositions of π_f, $f = f_{\alpha,\beta}$.

Case of $\gamma_0 = 1$ or $\alpha = \beta = 0$ (regular representation).

The regular representation λ_G is a factor representation associated to $f_{\mathbf{0},\mathbf{0}} = \delta_e$. When we extend the above situation in the case of the factor representation π_f associated to $f = f_{\alpha,\beta}$ with $\gamma_0 = 0$ to the case of $f_{\mathbf{0},\mathbf{0}}$ with $\gamma_0 = 1$, we can make a speculation about the support of λ_G (or the support of Plancherel measure for G). Note that, for an IUR $\rho = \mathrm{Ind}_H^G \pi$, "$\mathcal{TC}(\rho)$ contains $f_{\mathbf{0},\mathbf{0}}$ without redundancy" means that $B = P = \emptyset$ for ρ. Then, in this case, $\mathcal{TC}(\rho) = \{ f_{\mathbf{0},\mathbf{0}} \}$ as in Remark 13.1. We may take our speculation as

First working hypothesis.
The support $\mathrm{supp}(\lambda_G)$ *is equal to or is contained in the set* $\mathcal{IUR}(f_{\mathbf{0},\mathbf{0}})$.

16.2. Classification of extremal positive definite class functions

Aiming to apply our method of "*taking limits of centralizations*" of positive definite functions to other types of infinite discrete groups, we analyse relations of our present results to Thoma's results in [Tho2].

Main important points in [Tho2] can be considered as the following.

(1) Criterion for extremality of positive definite class functions;
(2) Sufficient condition for positive definiteness;
(3) Necessary condition for positive definiteness.

In that paper, after establishing a simple criterion for (1), the author studied (2) and (3) at the same time by applying a deep theory of analytic functions defined on discs.

Here in this paper, we established the second part (2) by the method of 'taking limits of centralizations', a proof quite different from that in [Tho2], and much simpler one if we take into consideration the result in Section 15.

Added in proof. The proof of Theorem 13 in Section 14 is not complete in the case where $|P| = \infty$ and $\sum_{p \in P} \lambda_p < 1$. So, to cover this case, we should replace in the statement of the theorem, "$J_N \nearrow \boldsymbol{N}$ *satisfies*" by "$J_N \nearrow \boldsymbol{N}$ *is an appropriate sequence satisfying*". Similarly in the statement of Theorem 16, "*Assume that*" should be replaced by "*Take an appropriate sequence satisfying that*".

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