Building-like spaces

By

Andreas BALSER and Alexander LYTCHAK

Abstract

We study convex subsets of buildings, discuss some structural features and derive several characterizations of buildings.

1. Introduction

The main purpose of this paper is to investigate the structure of convex subsets of spherical buildings. Such a convex subset X inherits from the ambient spherical building G the following fundamental property (see Subsection 4.1):

($*$) For each $x \in X$ there is some (conicality radius) $r_x > 0$, such that for all $y, z \in X$ with $d(x, y) < r_x$ the triangle xyz is spherical.

In order to study the geometry of X it seems natural to forget about the ambient space G and to work directly inside X. Moreover it is reasonable to investigate the slightly bigger (synthetically defined) class of all (complete) CAT(1) spaces that satisfy the above property ($*$) and have finite geometric dimension. We call such spaces *building-like*.

Remark 1.1. The property ($*$) can be regarded as a variant of a constant curvature 1 condition lying between sphericality (i.e. between convex subsets of Hilbert spheres) and local conicality (i.e. spherical complexes, see Subsection 3.2). Observe that local conicality has almost no implications on the topology (compare [Ber83]), in contrast to the very special topology of building-like spaces, see Theorem 1.1.

The first result describes the global topology of building-like spaces, characterizes buildings among them and provides a synthetic approach to buildings.

Theorem 1.1. *For a building-like space X of dimension n, the following are equivalent:*

1. *X is a building;*
2. *X is geodesically complete;*
3. *Each point has an antipode;*
4. *X contains an n-dimensional Euclidean sphere;*

5. X *is not contractible*;
6. *The radius of X is equal to π.*

Moreover we show that if an n-dimensional building-like space X contains an $(n-1)$-dimensional Euclidean sphere then X is either a building or it has radius $\frac{\pi}{2}$.

Now we turn to the local geometry of building-like spaces and prove the following structural results (see Section 6 for more details). Each building-like space X has the same local dimension at all points, and a (topologically and metrically) naturally defined (thick) decomposition in cells. The most important feature of the building-like space X is the boundary ∂X, that can be described (in analogy to convex subsets of Riemannian manifolds) by the property that it is the largest subset of X whose complement is convex and everywhere dense. The boundary is empty if and only if X is a building, moreover the boundary has the following local description that is a differential analog of Theorem 1.1:

Theorem 1.2. *Let X be a building-like space, $x \in X$ a point. The following are equivalent*:

1. $x \in \partial X$;
2. *Some geodesic terminates in x*;
3. *There are small neighborhoods U of x, such that $U \setminus \{x\}$ is contractible*;
4. *The link S_x is not a building*;
5. *The link S_x is contractible.*

Remark 1.2. In fact if X is a convex subset of a building G and not an abstract building-like space, then all links are building-like (compare Subsection 4.3) and one can deduce Theorem 1.2 from Theorem 1.1. In this case $x \in \partial X$ if and only if $\partial(S_x)$ is not empty.

We finish the investigation of the local structure of building-like spaces by showing that spherical subsets of X can be extended up to the boundary ∂X. The corresponding result for spherical buildings is shown in [KL97, Prop. 3.9.1], for Euclidean buildings it is [BT72, Prop. 2.8.1]. However, our argument (unlike the proof in [KL97]) gives a unified approach and remains valid in the Euclidean and the hyperbolic situation (see Remark 1.4).

Proposition 1.3. *Let X be a building-like space of dimension n. Let $C \subset X$ be a convex spherical subset. Then C is contained in an n-dimensional spherical subset $\tilde{C}$, whose boundary $\partial \tilde{C}$ (as a spherical set) is contained in ∂X.*

Now we discuss the local-global equivalence of our notion. The main result of [CL01] shows that a CAT(1) space that is locally isometric to a building is itself a building. This result has the following natural extension in the setting of building-like spaces:

Theorem 1.4. *Let X be a connected $CAT(1)$ space of dimension at least 2. If each point has a convex neighborhood which is (isometric to) a building-like space, then X is building-like.*

The proof of [CL01, Thm. 4.1] provides the following extension of the well known theorem of Cartan, saying that the universal covering of a complete manifold with sectional curvature 1 is a Euclidean sphere.

Corollary 1.5. *Let X be a complete geodesic space, such that each point $x \in X$ has a building-like neighborhood of dimension ≥ 3. Then the universal covering $\tilde{X}$ of X is $CAT(1)$, and therefore building-like.*

In the proof of Theorem 1.2 we use a result (in which we leave the universe of building-like and even that of locally conical spaces), that we consider to be of independent interest.

Theorem 1.6. *Let X be an n-dimensional $CAT(1)$ space that has at least one pair of antipodes. Assume that each pair of points $x, y \in X$ with $d(x, y) \geq \pi$ is contained in an n-dimensional sphere S^n. If X contains an open relatively compact subset U then X is a building.*

Remark 1.3. In dimension $n = 1$ the above result coincides with Theorem 1.1. of [Nag04].

From the very special case of Theorem 1.6 for compact spaces we derive an unpublished result of Kleiner:

Corollary 1.7. *Let X be a locally compact $CAT(0)$ space of dimension n. If each pair of points is contained in a flat $\mathbb{R}^n$, then X is a building.*

Observe that local compactness is necessary: Consider a (complete, geodesic) $CAT(1)$ space Y of dimension 1 and diameter π. If Y is not a building, then neither is the Euclidean cone CY, although it satisfies the assumptions of Corollary 1.7, except for local compactness. Suitable spaces Y exist, e.g. the spaces from [Nag04, Prop. 1.2].

Remark 1.4. We would like to emphasize that most of the results discussed above can be word by word transferred from the $CAT(1)$ to the $CAT(0)$ and to the $CAT(-1)$ setting. For the corresponding notion of Euclidean (resp. hyperbolic) building-like spaces, that include convex subsets of locally conical Euclidean (resp. hyperbolic) buildings (in contrast to spherical building-like spaces discussed above) all the local results mentioned above hold true (except Theorem 1.1). The boundary of such a (Euclidean or hyperbolic) building-like space is empty if and only if the space is geodesically complete. In the Euclidean case (but certainly not in the hyperbolic) this is enough to deduce that such a space is a Euclidean building. The corresponding statements of Theorem 1.4 resp. Corollary 1.5 are valid in all dimensions in the Euclidean and hyperbolic cases.

Our investigations were mainly motivated by the question if a group that operates on a building by isometries and fixes some non-trivial convex subset must have a fixed point. We refer to [BL05] for an answer in small dimensions and to [KL06] for a complete answer to the closely related question about groups operating by isometries on symmetric spaces or Euclidean buildings.

Now we describe the structure of the paper: In Section 3 basic properties of spherical subsets of CAT(1) spaces are discussed that may also be of independent use. These results are used in Section 6, the heart of this paper, where the local structure of building-like spaces is discussed in detail and Theorem 1.2 is shown. In Section 5 we present the proof of Theorem 1.4. Section 4 and Section 7 are independent of the rest and contain the proofs of Theorem 1.1 and Theorem 1.6. The reader only interested in these results may skip the rest.

We would like to thank Juan Souto for useful comments on a previous version.

2. Preliminaries

2.1. Notations

By $\mathbb{R}^n$ resp. S^n we denote the Euclidean space resp. the Euclidean sphere of dimension n. By d we denote distances in metric spaces, by $B_r(x)$ we will denote the closed ball of radius r around x. Geodesics are always parametrized by arclength. For a point x in a metric space X we set $\operatorname{rad}_x(X) := \sup_{z \in X} d(x, z)$. The radius of X is defined by $\operatorname{rad}(X) := \inf_{x \in X} \operatorname{rad}_x(X)$.

By $X * Z$ we denote the spherical join of X and Z ([BH99, pp. 63]).

2.2. CAT(1) spaces

A complete metric space will be called CAT(1) if each pair of points with distance $< \pi$ is connected by a geodesic and all triangles of perimeter less than 2π are not thicker than in S^2. We refer to [BH99, ch. II]. A CAT(1) space is geodesically complete if each geodesic can be prolonged to an infinite local geodesic. A subset C of a CAT(1) space is convex (more precisely π-convex) if all points in C with distance $< \pi$ are joined by a geodesic in C.

In a CAT(1) space X we will denote by $S_x = S_x X$ the link at the point x. For each point $x \in X$ there is a natural 1-Lipschitz (logarithmic) map $p_x : X \to S_x * S^0$, where x is sent to a pole of S^0 and the distances to x are preserved. We refer to [Lyt05].

By $\dim(X)$ we denote the geometric dimension of X studied in [Kle99]. The easy proof by induction of the following lemma can be found in [BL05]:

Lemma 2.1. *Let X be an n-dimensional CAT(1) space, and let $S \subset X$ be an embedded S^n. Then for each $x \in X$ there is an antipode $y \in S$, i.e. a point satisfying $d(x, y) \geq \pi$. Therefore we have $\operatorname{rad}(X) \geq \pi$.*

This result implies that if in an n-dimensional CAT(1) space X each point is contained in some S^n, then X is geodesically complete.

2.3. Buildings

We refer to [KL97, sect. 3] for an account on spherical buildings. In the proofs below we will use some characterizations of buildings among finite dimensional geodesically complete CAT(1) spaces derived in [Lyt05] and in [CL01].

3. Spherical parts of CAT(1) spaces

3.1. Spherical subsets

We call a subset T of a CAT(1) space X *spherical* if T admits an isometric embedding into a Hilbert sphere. A convex subset T of a CAT(1) space X is spherical if and only if for every choice of three points $x_1, x_2, x_3 \in T$, the triangle $x_1 x_2 x_3$ is spherical.

Lemma 3.1. *Let X be a CAT(1) space. Assume that T_1, T_2 are convex spherical subsets of X. If for all $z_1, z_2, z_3 \in T_1 \cup T_2$ the triangle $z_1 z_2 z_3$ is spherical, then the convex hull of $T_1 \cup T_2$ is spherical too.*

Proof. The last observation implies that it is enough to prove the result in the case where T_1 and T_2 are of dimension ≤ 2. One sees easily that there is an isometric embedding $I : T_1 \cup T_2 \to S^5$. Set $Y_i = I(T_i)$. Due to [LS97, Thm. A] the inverse $f = I^{-1} : (Y_1 \cup Y_2) \to X$ has a 1-Lipschitz extension to the convex hull Y of $Y_1 \cup Y_2$. Remark that f maps geodesics connecting points in $Y_1 \cup Y_2$ isometrically onto their images and we only have to show that $f : Y \to f(Y)$ is an isometry. This reduces the statement to the case where T_1 and T_2 are 1-dimensional. In this case the proof can be finished in the same way as in [Lyt05, L. 4.1], where the special case $d(x_1, x_2) = \frac{\pi}{2}$ for all $x_i \in T_i$ is covered. $\square$

3.2. Locally conical spaces

The following definition from [CL01] is a generalization of the concept of a simplicial complex, which was discussed e.g. in [Bal90]:

Definition 3.1. Let X be a CAT(1) space. We call X *locally conical* if for each $x \in X$ there is an $r_x > 0$, such that for all $y, z \in B_{r_x}(x)$ the triangle xyz is spherical. The maximal $r_x > 0$ with this property is called the *conicality radius* at x.

Remark 3.1. A simplicial space in the sense of [Bal90] is locally conical. It is possible to prove the converse if X is geodesically complete.

The above condition is equivalent to the requirement that $B_{r_x}(x)$ is canonically (via p_x) isometric to a convex subset of $\{x\} * S_x$. Observe that a closed convex subset of a locally conical space is locally conical.

3.3. Very spherical subsets

The following property was studied in [Lyt05] under the additional assumption of geodesic completeness:

Definition 3.2. We say that points x, y in a CAT(1) space X are *close* if for each $z \in X$ the triangle xyz is spherical. We say that a convex subset $T \subset X$ is *very spherical* in X if each pair of points of T are close.

Remark that a convex subset $T \subset X$ is very spherical if and only if for all $z \in X$ it is mapped isometrically under the logarithmic map $p_z : X \to S_z X * S^0$. A very spherical subset is spherical. For very spherical subsets T_1, T_2 of X, the convex hull of T_1 and T_2 in X is a spherical subset by Lemma 3.1. The closure of a very spherical subset is very spherical and the union of a chain of very spherical subsets is very spherical too. Hence every very spherical subset is contained in a maximal one.

Lemma 3.2. *Let T be a finite-dimensional very spherical subset of X. Let m be an inner point of the spherical convex set T. If $q \in X$ is close to m, then the convex hull of q and T is very spherical.*

Proof. The convex hull C of q and T is spherical by Lemma 3.1. Let z be arbitrary. Under the map p_z the subset T is mapped (isometrically) onto a spherical subset $\bar{T} \subset S_z * S^0$. Set $\bar{m} := p_z(m)$ and $\bar{q} := p_z(q)$. We have $d(q, m) = d(\bar{q}, \bar{m})$ since q and m are close. Let $x_1 \in T$ be arbitrary. Choose $x_2 \in T$ such that m is an inner point of $x_1 x_2$. Set $\bar{x}_i := p_z(x_i)$. Now in the triangle $\bar{x}_1 \bar{x}_2 \bar{q}$ we have $d(\bar{x}_i, \bar{q}) \leq d(x, q)$, $d(\bar{x}_1, \bar{x}_2) = d(x_1, x_2)$ and $d(\bar{q}, \bar{m}) = d(q, m)$. Since the triangle $x_1 x_2 q$ is spherical, we obtain from the CAT(1) property in $S_z * S^0$, that $d(\bar{x}_i, \bar{q}) = d(x_i, q)$ (and the triangle $\bar{x}_1 \bar{x}_2 \bar{q}$ must be spherical). Since C is spherical and x_1 was arbitrary, this shows that C is mapped isometrically onto its image by p_z. Since z was arbitrary, this shows that C is very spherical. □

The proof of the following lemma is provided by the fact that a triangle xyz is spherical if and only if for some m on xy the triangles xmz and ymz are spherical and $\angle_m(x, z) + \angle_m(y, z) = \pi$.

Lemma 3.3. *Let xmy be a geodesic in X. Assume that m is close to x and to y. Then x is close to y if and only if $S_m X$ splits as $S^0 * Y$, where the sphere S^0 consists of the starting directions of mx and my.*

4. Building-like spaces

4.1. Basics

We recall the basic definition from the introduction:

Definition 4.1. Let X be a CAT(1) space. We will say that X is *building-like* if it has finite dimension and each point x has a neighborhood $B_{r_x}(x)$ consisting of points close to x, i.e. for all $y \in B_{r_x}(x)$ and each $z \in X$ the triangle xyz is spherical.

First of all we observe that spherical buildings are building-like. Namely for each point x in a building G there is some $r_x > 0$ such that for each $y \in B_{r_x}(x)$ the points x and y are contained in some Weyl chamber of the building. Now for each other point $z \in G$ this chamber and z are contained in some apartment, hence the triangle xyz is spherical.

Observe now that the class of building-like spaces is stable with respect to spherical joins. Much more important is that a closed convex subset of a building-like space is building-like, in particular convex subsets of buildings are building-like.

The definition implies that a building-like space is locally conical. From Lemma 3.3 we see that in a building-like space X the maximal r_x satisfying the condition of Definition 4.1 coincides with the conicality radius at x from Definition 3.1 (see also Lemma 4.1 below).

Example 4.1. Directly from the definition we see that a 0-dimensional CAT(1) space is always building-like. It is a building if and only if it has at least two points.

Example 4.2. Using Theorem 1.1 one easily derives the following characterization of 1-dimensional building-like spaces: Let X be a CAT(1) space of dimension 1. Then X is building-like if and only if X is a building or X is a locally conical metric tree of diameter $\leq \pi$. In the latter case we have $\mathrm{rad}(X) = \frac{\mathrm{diam}(X)}{2} \leq \frac{\pi}{2}$.

Lemma 3.3 directly implies the next

Lemma 4.1. *Let* $\gamma = xy$ *be a geodesic in a building-like space* X, *such that for each inner point* m *on* γ *the link* S_m *splits as* $S_m = S^0 * Z_m$, *with* $S^0 = \{\gamma_m^+, \gamma_m^-\}$. *Then* x *is close to* y.

We finish the basics with some remarks about antipodes. Namely from Definition 4.1 we deduce that if x and z in a building-like space X are antipodes (i.e. satisfy $d(x, z) \geq \pi$), then $d(x, z) = \pi$ and for each $y \in B_{r_x}(x)$ the broken line xyz is in fact a geodesic. In particular the set $Ant(x)$ of all antipodes of x is discrete and if it is not empty, then X contains an isometric copy of $S^0 * S_x$.

4.2. Characterization of buildings

We turn to Theorem 1.1:

Proof of Theorem 1.1. The step $(4) \to (3)$ is given by Lemma 2.1.

Assume (3). Let $\gamma : [-t, 0] \to X$ be a geodesic with $\gamma(0) = x$. Set $y = \gamma(-\varepsilon)$ for some $\varepsilon < r_x$. Choose an antipode z of y. There must be a geodesic from y to z starting in the direction of γ and since γ cannot branch between y and x, we see that yxz is a geodesic. Therefore γ does not terminate in x and X is geodesically complete.

The implication $(2) \to (1)$ follows from [Lyt05, Prop. 4.5], which says that a finite dimensional geodesically complete space X must be a building if the set of antipodes of each point $x \in X$ is discrete.

$(1) \to (4)$ is clear.

The implication $(1) \to (5)$ is well-known, since the homology of an n-dimensional spherical building is non-trivial in dimension n. On the other hand if a point $x \in X$ has no antipodes, then the contraction along geodesics starting at x shows that X is contractible, hence (5) implies (3).

Finally $(3) \rightarrow (6)$ is clear and $(6) \rightarrow (3)$ is shown in Corollary 6.3.   □

4.3.   Type bounds

It seems to be difficult to distinguish between properties of abstract building-like spaces and convex subsets of buildings. The only advantages of the existence of an ambient building we found is stability under ultralimits and building-likeness of the links. We make this more precise:

Definition 4.2.   Let W be a finite Coxeter group. We call a space X building-like of type bounded by W if X admits an isometric embedding onto a convex subset of a building G of type W.

If the type of X is bounded by W then so is the type of each convex subset and of each link S_x of X. Moreover if X_i is a sequence of building-like spaces of type bounded by W, then so is the ultralimit $\lim_\omega X_i$.

5.   Local-Global Equivalence

Before we are going to study the local structure of building-like spaces in detail we prove Theorem 1.4, showing that the global property of being building-like is in fact a local one:

Proof of Theorem 1.4.   By local conicality and Lemma 4.1 it is enough to show that for arbitrary points $x, z \in X$ and a geodesic η starting at z the triangle xyz is spherical if $y = \eta(\varepsilon)$ and $\varepsilon > 0$ is small enough.

First, we assume that $d(x, z) < \pi$. The case $d(x, z) \geq \pi$ can be easily deduced afterwards using the fact that the links are connected due to the assumption $\dim(X) \geq 2$ and Theorem 1.2.

We can cover the geodesic $\gamma = xz$ by finitely many convex open building-like subsets. Choosing ε small enough, we may assume that the geodesic $\bar{\gamma} = xy$ is contained in the union of these subsets. More precisely we can find points $x = x_0, x_1, \ldots, x_n = z$ on xz and $x = y_0, y_1, \ldots, y_n = y$ on xy such that for all i the points $x_i, x_{i+1}, y_i, y_{i+1}$ are contained in some open convex building-like subset U_i.

Moving x_i along γ, we may assume that for each x_i the directions γ^+ and γ^- define a factor of S_{x_i} for $0 < i < n$ (by local conicality, the points where this is not true are discrete in γ). Making ε smaller we may assume that y_i is close to x_i inside of U_i. Moving y_i on $\bar{\gamma}$ we may assume that $\bar{\gamma}^+$ and $\bar{\gamma}^-$ define a factor of S_{y_i} for $0 < i < n$.

The triangle xx_1y_1 is spherical since x_1, y_1 are close. Assume by induction that xx_iy_i is spherical. Our assumption on the link at y_i implies that we can glue the spherical triangle $x_iy_iy_{i+1}$ to obtain a spherical triangle xx_iy_{i+1}. Similarly, we can glue the spherical triangle $x_ix_{i+1}y_{i+1}$, so the triangle $xx_{i+1}y_{i+1}$ is spherical. For $i + 1 = n$ we get the result.   □

6. Local structure

In this section let X be a fixed building-like space of dimension n.

6.1. Simple remarks on the links

Let $x \in X$ be a point, $\gamma : [0,r] \to X$ a geodesic starting at x in the direction $v \in S_x$. From the local conicality of X at x we derive that for all small t (for $t < \min\{r, r_x\}$) the link $S_{\gamma(t)}X$ has the form $S_{\gamma(t)}X = S^0 * Z$, where $S^0 = \{\gamma^+, \gamma^-\}$ and Z is canonically isometric to $S_v(S_x)$.

Assume now that γ can be prolonged beyond x to a geodesic $\gamma : [-r, r] \to X$ and let $w \in S_x$ be the incoming direction of γ. Then S_x contains an isometrically embedded $S_v(S_x) * \{v, w\}$. Moreover $S_v(S_x) = S_w(S_x)$. To see this observe that v and w are antipodes in S_x. Each direction in $S_v(S_x)$ corresponds to a direction in $S_{\gamma(t)}X$, orthogonal to γ. Since X is building-like at $\gamma(t)$, this direction gives a germ of spherical triangle with one side $\gamma(-t)\gamma(t)$. This triangle defines a geodesic in S_x from v to w starting at the given direction of $S_v(S_x)$. Now the conclusion follows from the lune lemma ([BB99, L. 2.5]).

6.2. Regular points

We will call a point x in X *regular* if $S_x = S^{n-1}$. The set of all regular points will be denoted by $R := R(X)$. If $x \in R$ is arbitrary, then $B_{r_x}(x)$ is isometric to a convex subset C of the sphere S^n and we have $S_xC = S^{n-1}$. In particular C contains a neighborhood of x in X. Hence the subset $R(X)$ is open in X and locally isometric to S^n. Due to Lemma 4.1 each convex subset of R is a very spherical subset of X.

We are going to prove that $R(X)$ is dense in X. The next lemma is in fact true in arbitrary locally conical spaces.

Lemma 6.1. *Let X be a building-like space with $dim(X) = n$. Let $C \subset X$ be a convex subset with $dim(C) = dim(X)$. Then there is a point $x \in C$ with $S_xC = S_xX = S^{n-1}$.*

Proof. In dimension $n = 1$ the statement follows directly from local conicality. Let $n > 1$ be arbitrary. Choose $q \in C$ with $\dim(S_qC) = n - 1$. Due to [Kle99, Thm. B.3] there must be a point $z \neq q$ with $z \in C \cap B_{r_q}(q)$ and $\dim(S_zC) = n-1$. Then for each inner point y of the geodesic $\gamma = qz$ we deduce $\dim(S_yC) = n - 1$, since there is a natural isometric embedding $S_zC \to S_yC$. However S_yC and S_yX split as $S^0 * Z$ resp. $S^0 * \tilde{Z}$, ($S^0 = \{\gamma^+, \gamma^-\}$). Thus in a small neighborhood of y the set $\tilde{X}$ (resp. $\tilde{C}$) of points $\bar{y} \in X$ (resp. $\bar{y} \in C$) with $\angle_y(\bar{y}, q) = \frac{\pi}{2}$ is a convex subset of X of dimension $n-1$. Arguing by induction, we find a point $x \in \tilde{C}$ arbitrarily close to y with $S_x\tilde{X} = S_x\tilde{C} = S^{n-2}$. The local structure of X near y implies $S_xC = S_xX = S^{n-1}$. $\square$

We deduce

Proposition 6.2. *Let X be a building-like space of dimension n. Then for each $x \in X$ holds $dim(S_x) = n - 1$. Moreover $R(X)$ is dense in X.*

Proof. R is open and non-empty by Lemma 6.1. Let T be an open convex subset of regular points and $x \in X$. Due to Lemma 3.1 the convex hull C of x and T is spherical. We get $\dim(S_x) \geq \dim(S_x C) \geq \dim(S) - 1 = n - 1$. Applying Lemma 6.1 to an arbitrarily small neighborhood of x we obtain the second statement. $\qquad\square$

6.3. Easy applications

Let X be a building-like space of dimension n again. The remark at the end of Subsection 4.1 and the implication $(4) \to (1)$ of Theorem 1.1 show that, if X is not a building, then no regular point $x \in R(X)$ can have an antipode in X. The next result finishes the proof of Theorem 1.1:

Corollary 6.3.   *If X is not a building, then $rad(X) < \pi$.*

Proof.   Choose a regular point $x \in X$. For small $\varepsilon > 0$ the ball $B_\varepsilon(x)$ consists of regular points. If for some $z \in X$ we had $d(z, x) > \pi - \varepsilon$, then we could prolong the geodesic zx inside the ball $B_\varepsilon(x)$ (using Lemma 2.1) and obtain an antipode $\bar{z}$ of z in $B_\varepsilon(x)$. Since $\bar{z}$ is regular, we would deduce that X is a building.

Thus if X is not a building we obtain $rad(X) \leq \pi - \varepsilon$. $\qquad\square$

The following result is mentioned in the introduction:

Lemma 6.4.   *Let X be as above. Assume that X contains a Euclidean sphere S of dimension $n - 1$. Then either X is a building or the radius of X is $\frac{\pi}{2}$.*

Proof.   Let $x \in S$ be a point. Since x has an antipode y in S, we see that X contains the isometrically embedded subset $S_x * \{x, y\}$. Therefore S_x is an $(n-1)$-dimensional building-like space that contains an $(n-2)$-dimensional sphere $S_x S$. Hence, an inductive argument shows that X contains an n-dimensional spherical hemi-sphere H whose boundary sphere is S. Let m be the midpoint of this hemisphere. Assume that $rad_m(X) > \frac{\pi}{2}$. Since the set of regular points is dense in X we find a regular point x with $d(x, m) > \frac{\pi}{2}$. Consider the geodesic xm. By Lemma 2.1 we find a direction $v \in S^{n-1} = S_m H$ that is antipodal to the starting direction of mx. Therefore we can prolong the geodesic xm inside H and find an antipode of x in H. This implies that X is a building. $\qquad\square$

6.4. Regular directions

Before we embark on the proof of Theorem 1.2 we will make some additional remarks about regular points. The picture will be completed in Subsection 6.6.

Definition 6.1.   We call the starting direction $v \in S_x$ of a geodesic γ *regular* if for all sufficiently small ε the point $\gamma(\varepsilon)$ is regular.

Observe that the direction $v = \gamma^+ \in S_x$ is regular if and only if $S_v(S_x) = S^{n-2}$. Due to Proposition 6.2 the set of regular directions R_x in S_x is open

and dense. Moreover R_x is locally isometric to S^{n-1}, in particular it is locally compact. Using the second observation in Subsection 6.1 we see that if x is an inner point of the geodesic γ, then $\gamma^+ \in S_x$ is regular if and only if the opposite direction γ^- is regular.

6.5. Boundary

In this subsection we are going to prove Theorem 1.2. We start with a (technical) definition of the boundary:

Definition 6.2. A point x in a building-like space X will be called an *inner point* of X, if it is an inner point of a geodesic connecting regular points. The set of all inner points will be denoted by X_0, its complement by ∂X.

Remark 6.1. X_0 is in general not open in X, however it is open with respect to the natural weak topology, see Lemma 6.11.

Lemma 6.5. *Let X be an n-dimensional building-like space. Then a point $x \in X$ is an inner point of X if and only if there is an n-dimensional convex spherical subset T that contains x as an inner point (of T).*

Proof. Assume that x is an inner point of the geodesic $y_1 y_2$ with regular points $y_1 y_2$. Then small neighborhoods U_i of y_i are very spherical. Hence x is an inner point of the convex hull of U_1 and U_2 that is spherical by Lemma 3.1.

On the other hand let x be an inner point of T. By Proposition 6.2 the set $R(X) \cap T$ of regular points lying in T is dense in T. Since it is also open and locally convex, one easily finds points $y_1, y_2 \in T \cap R$ such that x lies on the geodesic $y_1 y_2$. $\qquad\square$

Using Lemma 2.1 we immediately conclude:

Corollary 6.6. *A point x is an inner point if and only if no geodesic terminates in x. In particular X has no boundary if and only if it is a building.*

Now let $x \in X$ be an inner point, let $\gamma : [0, \varepsilon') \to B_{r_x}(x)$ be a geodesic starting at x. Choose T as in Lemma 6.5 and prolong γ to a geodesic $\gamma : [-r, \varepsilon')$ inside T. We see that a small neighborhood V of $y = \gamma(-r)$ in T and $\gamma(\varepsilon'')$ (for $\varepsilon'' < \varepsilon'$) span a spherical subset of X. This shows that one can find an n-dimensional spherical convex subset T' containing $\gamma[0, \varepsilon]$ (for $\varepsilon < \varepsilon'$). Repeating this argument we see that for each geodesic $\eta : (-\varepsilon', \varepsilon') \to B_{r_x}(x)$ with $\eta(0) = x$ there is an n-dimensional convex spherical subset T'' that contains $\eta[-\varepsilon, \varepsilon]$. Now we are going to prove:

Lemma 6.7. *A point x is an inner point of X if and only if S_x is a building.*

Proof. Let x be an inner point. Denote by $V \subset S_x$ the set of all directions in which a geodesic starts. By definition, V is dense in S_x; furthermore, V is convex because of the locally conical structure of X. By the last observation,

each pair of antipodes $v, w \in V$ is contained in an $(n-1)$-dimensional sphere $S^{n-1} \subset V \subset S_x$. This implies that V is geodesically complete (Lemma 2.1).

Unfortunately we cannot apply Theorem 1.6, since V may not be complete. To circumvent this difficulty, consider the ultraproduct $V^\omega = S_x^\omega$. This is a (complete) CAT(1) space of dimension $n-1$ and each pair of antipodes of V^ω are still contained in some S^{n-1} (due to the geodesic completeness of V). Now S_x contains the (non-empty) open locally compact subset R_x of regular points. Let $T \subset R_x$ be open and relatively compact. Then T is also an open and relatively compact subset of V^ω. From Theorem 1.6 (which is proved independently in Section 7), we deduce that V^ω is a building. Hence S_x is a convex subset of a building with $\mathrm{rad}(S_x) = \mathrm{rad}(S_x^\omega) = \pi$. Thus S_x itself is a building by Theorem 1.1.

Assume now that S_x is a building. Since each convex dense subset in a building is the whole building, we see that in each direction $v \in S_x$ a geodesic starts. Choosing a finite number of directions in S_x whose convex hull is a sphere of dimension $n-1$, we deduce from local conicality that x is contained in a spherical n-dimensional subset. Hence x is an inner point. $\qquad\square$

Proof of Theorem 1.2. $(1) \leftrightarrow (2)$ is Corollary 6.6, and $(1) \leftrightarrow (4)$ is the previous lemma.

Let $x \in X$ be a boundary point. Then some geodesic yx cannot be extended beyond x. Thus the contraction along geodesics starting at y gives a contraction of $U \setminus \{x\}$, where choosing y arbitrarily close to x we may choose U to be an arbitrarily small neighborhood of x.

On the other hand if x is an inner point of X, then the ball $B_{r_x}(x)$ is a convex part of the building $S_x * S^0$. Remark that $B_{r_x}(x)$ contains an n-dimensional spherical convex subset containing x. This subset defines a nontrivial element in the local $(n-1)$-dimensional homology group at x (compare [KL97, sect. 6.2]). This shows that $U \setminus \{x\}$ cannot be contractible for small neighborhoods of x.

Finally $(5) \rightarrow (4)$ is clear and (2) implies that the link S_x has radius smaller than π (as in the proof of Corollary 6.3), hence $(2) \rightarrow (5)$. $\qquad\square$

The next results show that the boundary discussed above has the description announced in the introduction. First, we show that X_0 is convex, in fact even more is true:

Lemma 6.8. *Let x be an inner point of X, $\gamma : [0, s) \to X$ a geodesic starting at x. Then γ consists of inner points of X. In particular X_0 is convex.*

Proof. The observation preceding Lemma 6.7 shows that the set I of numbers t with $\gamma(t) \in X_0$ is open in $[0, s)$. Let $t \in (0, s)$ be a boundary point of I. Then for $\varepsilon < r_{\gamma(t)}$, the point $\gamma(t - \varepsilon)$ is contained in a convex n-dimensional spherical subset T and this T together with $\gamma(t + \varepsilon)$ span an n-dimensional spherical subset containing $\gamma(t)$. Thus $t \in I$, in contradiction to the openness of I. $\qquad\square$

Since each convex dense subset of a building-like space X must contain all regular and therefore all inner points, we conclude

Corollary 6.9. ∂X *is the largest subset of X whose complement is convex and everywhere dense.*

6.6. Regular points revisited

Now we are going to investigate the combinatorial structure of X. The next lemma shows that maximal very spherical subsets of X define a decomposition of X in cells.

Lemma 6.10. *Let C be a maximal very spherical subset of X. Denote by C_0 the set of inner points of C as a spherical set. Then C_0 is open in X. Moreover it is a connected component of $R(X)$.*

Proof. Let $m \in C_0$ be arbitrary. If C_0 does not contain a neighborhood of m, then we can find a point x close to m that is not in C_0. Due to Lemma 3.2 the convex hull of x and C is a very spherical subset of X in contradiction to the maximality of C. Since C_0 is open it is certainly contained in R. Therefore we only have to show that $C \setminus C_0$ does not contain regular points. Assume that x is such a point. Then a neighborhood of x is very spherical and therefore we will find points $m \in C_0$ and $x' \in X \setminus C$ that are close. Using Lemma 3.2 again we get a contradiction to the maximality of C. □

Remark 6.2. From Theorem 1.2 it is easy to deduce that a point $x \in X$ is a regular point if and only if it has a neighborhood homeomorphic to a manifold. This shows that the decomposition in maximal very spherical subsets is natural also from the topological point of view.

Lemma 6.11. *Let C be a maximal very spherical subset of X. Then the intersection $X_0 \cap C$ is open in C. For $x \in X_0 \cap C$ some neighborhood of x in C is isometric to an open subset in some Coxeter chamber.*

Proof. Let x be a point in $X_0 \cap C$. We know that S_x is a building and that in each direction $v \in S_x$ a geodesic starts. This shows that $S_x C$ is a maximal chamber of the building $S_x X$. This chamber is a convex hull of finitely many points, hence for a small number ε in each direction $v \in S_x C$ a geodesic of length at least ε starts. This and Lemma 6.8 show that $X_0 \cap C$ is open in C and that the ε-ball around x in C is isometric to the ε-ball around x in the Coxeter chamber $S_x C * \{x\}$. □

Remark 6.3. From this and well known facts about Coxeter groups in spheres it is easy to see that if a maximal very spherical subset C of X is contained in X_0, then C is a spherical join $C = C_1 * C_2 * \cdots * C_k$, where each C_i is either 1-dimensional or isometric to the Coxeter chamber of an irreducible Coxeter group.

The following lemma is a weak equivalent of the statement that the simplicial structure defined by the decomposition in maximal very spherical subsets is thick. We leave the easy proof to the reader.

Lemma 6.12. *Let C be a maximal very spherical subset of X, $x \in C$ a point. If $S_x C = \{v\} * S^{n-2}$ is a hemisphere, then either x is a boundary point of X and C is a neighborhood of x, or $S_x X = T * S^{n-2}$, where the discrete set T has more than two points. Moreover x is contained in at least 3 cells in the latter case.*

6.7. Maximal spherical subsets

This subsection is devoted to the proof of Proposition 1.3.

We start with a criterion when a spherical subset is not maximal:

Lemma 6.13. *Let C be a spherical subset of X, m a point of C, and y be a point close to m. Let v be the initial direction of my. If the convex hull H of v and $S_m C$ is spherical, then the convex hull of C and y is spherical.*

Proof. Observe that C is mapped isometrically by $p_m : X \to S_m * S^0$.

Since every triangle myq for $q \in X$ is spherical, we find that $\{y\} \cup C$ is mapped isometrically (by p_m) into the spherical set $H * S^0$. This implies that for all $x_1, x_2 \in C$, the triangle yx_1x_2 is spherical. Hence, the convex hull of C and y is spherical by Lemma 3.1. $\qquad\square$

Proof of Proposition 1.3. Since the closure and a union of a chain of spherical subsets is spherical, we may assume that the spherical subset C of X is maximal; we have to prove that it has dimension n and that $\partial C \subset \partial X$.

Let $m \in \partial C$ be an inner point of X. Then $S_m X$ is a building, hence (by induction or due to [KL97, Prop. 3.9.1]) $S_m C \subsetneqq S^{n-1} \subset S_p X$, which is a contradiction to the maximality of C by the previous lemma (since for an inner point of X a geodesic starts in all directions).

Assume now that C has dimension smaller than n (the same argument as above implies that C must be contained in ∂X in this case). Pick an inner point m of C. Similar to the proof of Lemma 6.1, one shows that one can find another inner point $\bar{m}$ of C, such that $S_{\bar{m}} C$ is a (non-trivial) spherical join factor of $S_{\bar{m}} X$. This shows the existence of a vector v as in the lemma above and therefore a contradiction to the maximality. $\qquad\square$

7. Appendix

Here, we are going to prove Theorem 1.6 and Corollary 1.7:

Proof of Theorem 1.6. By the assumptions X contains at least one isometrically embedded S^n. Due to Lemma 2.1 each point has an antipode and is therefore contained in a sphere of maximal dimension. Hence, X is geodesically complete and has diameter π.

We proceed by induction on dimension. In dimension 0 there is nothing to be done. If X is reducible, i.e. if it has a non-trivial decomposition $X = Y * Z$, then one easily sees, that the assumptions of the theorem are fulfilled for the spaces Y and Z, which have dimension smaller than X. By induction we obtain that Y, Z and therefore $X = Y * Z$ are buildings. Hence we may assume that X is irreducible.

Let U be an open relatively compact subset of X. Since X is geodesically complete, U contains an open subset $\tilde{U}$ homeomorphic to a manifold (compare for example [Ots97]). The dimension of $\tilde{U}$ is at most $\dim(X) = n$ and since each point of $\tilde{U}$ is contained in some S^n, we see that $\dim(\tilde{U}) = n$ and that $\tilde{U}$ is locally isometric to S^n.

Consider the set O of all points $x \in X$ that have a neighborhood isometric to an open subset of S^n. This set is open by definition and we have just seen that it is non-empty. If O is the whole set X then X is isometric to the sphere S^n, hence we may assume $O \neq X$.

Let $x \in O$ be arbitrary, $y \in X$ a point with $d(x, y) = \pi$. Let $S = S^n$ be a sphere of maximal dimension that contains x and y. Since $x \in O$, the sphere S contains a ball $B_r(x) \subset O \subset X$ for some small $r > 0$. We are going to show that S contains $B_r(y)$.

Let $\bar{y} \in B_r(y)$. Due to Lemma 2.1 we may continue the geodesic $\bar{y}y$ inside S and obtain an antipode $\bar{x}$ of $\bar{y}$ in S. We have $d(x, \bar{x}) = d(\bar{y}, y) \leq r$, hence $\bar{x} \in O$. So there is a spherical neighborhood U of $\bar{x}$. Since S is a sphere in X containing $\bar{x}$, we have $U \subset S$.

If we assume U to be a maximal connected spherical neighborhood of $\bar{x}$, we have $U \supset B_r(x)$. Hence, the geodesic segment $\bar{x}x$ can be extended to a geodesic $\bar{x}x\bar{y}$, implying that $x\bar{y}y$ is a geodesic too. In particular, $\bar{y} \in S$.

Thus S contains a neighborhood of y, and y is in O. This shows that the complement $T = X \setminus O$ is closed and contains all antipodes of all of its points. Since $T \neq X$ and X is irreducible we may apply the main result of [Lyt05], which says that the existence of such a subset in an irreducible geodesically complete space X implies that X is a building. $\square$

In the proof of Corollary 1.7, we use the appropriate definition of local conicality in the $CAT(0)$ setting.

Proof of Corollary 1.7.   Lemma 2.1 shows that X is geodesically complete. Since X is proper, the sequence (tX, x) converges for $t \to \infty$ to the tangent cone CS_x in the Gromov-Hausdorff topology and from Theorem 1.6 we immediately obtain that each link is a spherical building. By [CL01] it is enough to prove that X is locally conical.

Let $x \in X$ be arbitrary. Then for some $r > 0$ each n-dimensional Euclidean $\mathbb{R}^n = F \subset X$ with $d(x, F) < r$ must contain x, since otherwise we would obtain a flat $\bar{F} = \mathbb{R}^n$ in the Euclidean cone CS_x that does not contain the origin and this would contradict $\dim(CS_x) \leq n$.

Hence for all $y, z \in B_r(x)$ each maximal flat through y and z must contain x, thus the triangle xyz is flat. Therefore X is locally conical. $\square$

MATHEMATISCHES INSTITUT
UNIVERSITÄT MÜNCHEN
e-mail: balser@math.lmu.de

MATHEMATISCHES INSTITUT
RHEINISCHE FRIEDRICH-WILHELMS-UNIVERSITÄT BONN
e-mail: lytchak@math.uni-bonn.de

References

[Bal90]   W. Ballmann, *Singular spaces of nonpositive curvature*, In Sur les groupes hyperboliques d'après Mikhael Gromov (Bern, 1988), Progr. Math. **83**, p. 189–201. Birkhäuser Boston, Boston, MA, 1990.

[BB99]   W. Ballmann and M. Brin, *Diameter rigidity of spherical polyhedra*, Duke Math. J. **122** (1999), 597–609.

[Ber83]   V. N. Berestovskiĭ, *Borsuk's problem on metrization of a polyhedron*, Dokl. Akad. Nauk SSSR **268** (2) (1983), 273–277.

[BH99]   M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Springer, 1999.

[BL05]   A. Balser and A. Lytchak, *Centers of convex subsets of buildings*, Ann. Global Anal. Geom. **28** (2) (2005), 201–209.

[BT72]   F. Bruhat and J. Tits, *Groupes réductifs sur un corps local*, Inst. Hautes Études Sci. Publ. Math. **41** (1972), 5–251.

[CL01]   R. Charney and A. Lytchak, *Metric characterizations of spherical and Euclidean buildings*, Geom. Topol. **5** (2001), 521–550.

[KL97]   B. Kleiner and B. Leeb, *Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings*, IHES Publ. Math **86** (1997), 115–197.

[KL06]   ———, *Rigidity of invariant convex sets in symmetric spaces*, Invent. Math. **163** (3) (2006), 657–676.

[Kle99]   B. Kleiner, *The local structure of length spaces with curvature bounded above*, Math. Z. **231** (3) (1999), 409–456.

[LS97]   U. Lang and V. Schroeder, *Kirszbraun's theorem and metric spaces of bounded curvature*, Geom. Funct. Anal. **7** (3) (1997), 535–560.

[Lyt05]   A. Lytchak, *Rigidity of spherical buildings and joins*, Geom. Funct. Anal. **15** (3) (2005) 720–752.

[Nag04]   K. Nagano, *Diameter rigidity of CAT(1) spaces with local $\mathbb{R}$-tree structure*, preprint, 2004.

[Ots97]   Y. Otsu, *Differential geometric aspects of Alexandrov spaces*, In Comparison geometry (Berkeley, CA, 1993–94), Math. Sci. Res. Inst. Publ. **30**, p. 135–148. Cambridge Univ. Press, Cambridge, 1997.