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A partial horseshoe structure at an indeterminate point of birational mapping

By

Tomoko SHINOHARA*

Abstract

In this paper, we show that, for some birational mapping F of $\mathbf{P}^2$ with an indeterminate point I_1, there exists a *partial* horseshoe structure at I_1 and periodic points of F accumulate at I_1. This is a new dynamical model that gives a chaotic phenomenon in a neighbourhood of the indeterminate point I_1 at which F is not continuous.

1. Introduction

An indeterminate point of a rational mapping on the 2-dimensional complex projective space $\mathbf{P}^2$ naturally appears in the dynamics of the Newton method at a multiple root of a system of equations (cf. [3], [9]). Here let us recall briefly this. Let $R = (P, Q)$ be polynomials of variable $(x, y) \in \mathbf{C}^2$. Consider the solution of the system of equations $R(x, y) = (0, 0)$. The Newton method for $R(x, y) = (0, 0)$ is defined by rational mapping $F(x, y) = (x, y) - (JR_{(x,y)})^{-1} \circ R(x, y)$, where $(JR_{(x,y)})^{-1}$ is the inverse of Jacobian matrix of R at (x, y). If (x_0, y_0) is a multiple root of $R(x, y) = (0, 0)$, that is, $R(x_0, y_0) = (0, 0)$ and $\det(JR_{(x_0,y_0)}) = 0$, then it is an indeterminate point of F. Thus, to analyze local dynamical structure at an indeterminate point is closely related to the Newton method for a multiple root, and this is one of important problems on dynamical system of rational mappings of $\mathbf{P}^2$.

The investigation of the local dynamical structure at an indeterminate point originated with Y. Yamagishi [11], [12]; in which he constructed uncountably many stable manifolds of an indeterminate point. In view of his results, a chaotic phenomenon occurs in a neighbourhood of the indeterminate point at which the mapping is not continuous.

In this paper, we study exclusively the following birational mapping F of $\mathbf{P}^2$ having the form:

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$$(*) \qquad F : [x : y : t] \mapsto [x^3 + axt^2 - yt^2 : bx^2t : bxt^2] \ \text{ with } \ |a| > 1, \quad b \neq 0,$$

and give a *partial horseshoe structure* at its indeterminate point. Here, it should be remarked that the horseshoe structure is known as a typical model which induces a chaotic behavior (see [8]).

In order to state our Main Theorem, let us introduce some notations and terminology. Let $f_i(x, y, t)$ $(i = 0, 1, 2)$ be homogeneous polynomials of degree d. Then by setting

$$F([x : y : t]) = [f_0 : f_1 : f_2] \quad \text{and} \quad \hat{F}(x, y, t) = (f_0, f_1, f_2),$$

we have a rational mapping F on $\mathbf{P}^2$ and a polynomial mapping $\hat{F}$ on $\mathbf{C}^3$ with $\pi \circ \hat{F} = F \circ \pi$ on $\mathbf{C}^3$ outside some proper analytic sets, where $\pi : \mathbf{C}^3 \setminus \{(0,0,0)\} \to \mathbf{P}^2$ is the canonical projection. A point $p \in \mathbf{P}^2$ is said to be an *indeterminate point* of F if $\hat{F}(\hat{p}) = (0, 0, 0)$ for some point $\hat{p} \in \pi^{-1}(p)$. In general, if p is an indeterminate point, then F is not continuous at p and $\bigcap_{U_p} \overline{F(U_p \setminus \{p\})}$ is not a singleton, where the intersection is taken over all open neighborhoods U_p of p. A rational mapping F of $\mathbf{P}^2$ is said to be *birational* if there exists another rational mapping G of $\mathbf{P}^2$ such that $F \circ G = \mathrm{id}$ and $G \circ F = \mathrm{id}$ on $\mathbf{P}^2$ except some proper algebraic sets, where id is the identity mapping. In such a case, G is called the inverse mapping of F.

Let us now return to our mapping F appearing in $(*)$. Then it is easy to see that F has the inverse mapping G written in the form

$$G([x : y : t]) = [yt^2 : y^3 - bxyt + ayt^2 : t^3].$$

Moreover, a straightforward computation shows that $I_1 := [0 : 0 : 1]$ is an indeterminate point of F, $G(I_1) = I_1$ and the eigenvalues of the JG_{I_1} are 0 and a. Thus, in order to analyse the dynamical structure of F near the point I_1, it suffices to consider the behavior of G near the fixed point I_1.

Let U be an arbitrary small neighbourhood of I_1. Then, noting that I_1 is a saddle fixed point of G by our assumption $|a| > 1$, we define a *local stable set* $W^s_{loc}(I_1)$ and the *stable set* $W^s(I_1)$ of I_1 by

$$W^s_{loc}(I_1) = \left\{ q \in U \;\middle|\; G^n(q) \to I_1 \right\} \quad \text{and} \quad W^s(I_1) = \bigcup_{n \geq 0}^{\infty} G^{-n}(W^s_{loc}(I_1)),$$

respectively, and a *local unstable set* $W^u_{loc}(I_1)$ and the *unstable set* $W^u(I_1)$ of I_1 by

$$W^u_{loc}(I_1) = \left\{ q \in U \;\middle|\; F^n(q) \to I_1 \right\} \cup \{I_1\} \quad \text{and} \quad W^u(I_1) = \bigcup_{n \geq 0}^{\infty} G^n(W^u_{loc}(I_1)),$$

respectively, where $\{F^n\}$ and $\{G^n\}$ are, of course, the iteration of F and G, respectively. It is remarked here that the definition of (local) unstable set is slightly different from usual one (cf. [5, §6.4]), because F is not continuous at I_1. It then follows from the stable manifold theorem (see Theorem 3.2) that

$W^u(I_1) \neq \emptyset$ and $W^s(I_1) \neq \emptyset$. If $W^s(I_1)$ and $W^u(I_1)$ intersect at some point q with $q \neq I_1$, then q is said to be a *homoclinic point*. Moreover, q is said to be a *transversal homoclinic point* if $T_q\mathbf{C}^2$ is the direct sum of $T_qW^s(I_1)$ and $T_qW^u(I_1)$: $T_q\mathbf{C}^2 = T_qW^s(I_1) \oplus T_qW^u(I_1)$. Recall that, in general, if a C^r diffeomorphism H on a differentiable manifold has a saddle fixed point with a transversal homoclinic point, then some iteration H^k of H has a horseshoe structure and its dynamical structure is described by symbolic dynamics (see [8]). Notice that our G is not locally diffeomorphic near the point I_1, so that the general theory cannot be applied directly to our case. However, observing the orbits of critical sets of G carefully, we can obtain a similar conclusion in our situation. In fact, introducing the sets

$$\hat{\Sigma} := \left\{ s = (\ldots, s_{i-1}, s_i, s_{i+1}, \ldots) \;\middle|\; s_i = 0, 1 \text{ for all } i \in \mathbf{Z} \right\} \quad \text{and}$$
$$E := \left\{ s = (\ldots, s_{n-1}, s_n, 0, 0, \ldots) \in \hat{\Sigma} \;\middle|\; n \in \mathbf{Z} \right\},$$

we can prove the following:

Main Theorem. *Let $F : \mathbf{P}^2 \to \mathbf{P}^2$ be the birational mapping as in* $(*)$. *Then, we have the following*:
(1) *There exists a homoclinic point q_0 of I_1.*
(2) *Moreover, suppose that q_0 is a transversal homoclinic point with*

$$q_0 \in \left\{ [x : y : 1] \in \mathbf{P}^2 \;\middle|\; y = 0 \right\} \setminus \{[a : 0 : b]\}.$$

Then there exist a positive integer k, a set $X \subset \mathbf{P}^2$ and a homeomorphism $\hat{\Psi} : X \to \hat{\Sigma} \setminus E$ such that X is invariant under G and $\sigma \circ \hat{\Psi} = \hat{\Psi} \circ G^k$ on X, where σ is the shift mapping on $\hat{\Sigma} \setminus E$.
In particular, periodic points of F accumulate at its indeterminate point I_1.

As to the topological nature of X, see the remark in Section 4.

This paper is organized as follows. In Section 2, we collect some preliminary facts. Sections 3 and 4 are devoted to the proof of Main Theorem. In the final Section 5, we give an example of the parameter (a, b) such that G has a homoclinic point $q_0 \in \left\{ [x : y : 1] \in \mathbf{P}^2 \;\middle|\; y = 0 \right\} \setminus \{[a : 0 : b]\}$. It goes without saying that there are many rational mappings with the same point I_1 as one of their indeterminate points; and therefore, there exists the horseshoe structure at I_1. For the concrete description of such a mapping, see [10].

2. Fundamental properties of mappings F and G

In this section, we fix the notation which will be used throughout this paper, and collect some preliminary facts on our F and G. First of all, we fix an homogeneous coordinate system $[x : y : t]$ in $\mathbf{P}^2$ once and for all; and we shall often use the natural identification given by

$$\mathbf{C}^2 = \left\{ [x : y : t] \in \mathbf{P}^2 \;\middle|\; t \neq 0 \right\} \quad \text{and} \quad (x, y) = [x : y : 1].$$

If $z_0 \in \mathbf{C}$ and $r > 0$, we set, as usual,

$$\Delta_r(z_0) = \left\{z \in \mathbf{C} \;\middle|\; |z - z_0| < r\right\},\ \Delta_r(z_0)^* = \Delta_r(z_0) \setminus \{z_0\},\ \Delta_r = \Delta_r(0),$$
$$\Delta_r^2(z_0) = \Delta_r(z_0) \times \Delta_r(z_0) \quad \text{and} \quad \Delta_r^2 = \Delta_r^2(0).$$

We define the canonical projections

$$\pi_i : \mathbf{C}^2 \to \mathbf{C}\ (i = 1, 2) \ \text{ by } \ \pi_1(x, y) = x \ \text{ and } \ \pi_2(x, y) = y,$$

and also define three points I_j and three curves C_j in $\mathbf{P}^2$ by

$$I_1 = [0 : 0 : 1],\ I_2 = [0 : 1 : 0],\ I_3 = [1 : 0 : 0];$$
$$\text{and} \quad C_1 = \left\{[x : y : t] \in \mathbf{P}^2 \;\middle|\; x = 0\right\}, \quad C_2 = \left\{[x : y : t] \in \mathbf{P}^2 \;\middle|\; t = 0\right\},$$
$$C_3 = \left\{[x : y : t] \in \mathbf{P}^2 \;\middle|\; y = 0\right\}.$$

Finally, we denote by I_F, I_G the sets of indeterminate points of F and G, respectively. The verification of the following proposition is straightforward; therefore, the proof is left to the reader.

Proposition 2.1. *With the notation above, we have:*
(1) $I_F = \{I_1, I_2\}$ *and* $I_G = \{I_3\}$;
(2) $I_F = \overline{\bigcup_{j=0}^{\infty} F^{-j}(I_F)}$ *and* $I_G = \overline{\bigcup_{j=0}^{\infty} G^{-j}(I_G)}$;
(3) $F\big(C_1 \cup C_2 \setminus \{I_F\}\big) = \{I_3\}$, $G\big(C_2 \setminus \{I_G\}\big) = \{I_2\}$ *and* $G\big(C_3 \setminus \{I_G\}\big) = \{I_1\}$;
(4) $F : \mathbf{P}^2 \setminus \{C_1 \cup C_2\} \to \mathbf{P}^2 \setminus \{C_2 \cup C_3\}$ *and* $G : \mathbf{P}^2 \setminus \{C_2 \cup C_3\} \to \mathbf{P}^2 \setminus \{C_1 \cup C_2\}$ *are biholomorphic mappings.*

3. Proof of (1) of Main Theorem

Throughout this section, we concentrate our attention on dynamics of G in the chart $\mathbf{C}^2$. Observe that $I_1 = (0, 0)$ and the restriction of G to $\mathbf{C}^2$, which we denote also by G, is written as $G(x, y) = (y, y^3 - bxy + ay)$. As an immediate consequence of Proposition 2.1, (4), we have the following:

Proposition 3.1. *Let* $n \geq 1$. *Then*
(1) $C_3 \setminus I_G \subset W^s(I_1)$;
(2) $G^n\left(\bigcup_{k=0}^{n-1} G^{-k}(C_3)\right) = \{I_1\}$;
(3) $G^n : \mathbf{C}^2 \setminus \bigcup_{k=0}^{n-1} G^{-k}(C_3) \to \mathbf{C}^2 \setminus C_1$ *is a biholomorphic mapping.*

In order to prove the assertion (1) of Main Theorem, we assume the contrary that

(3.1) there is no homoclinic point of I_1.

For the proof, we need the following well-known result:

Theorem 3.2 ([5, Theorem 6.4.3])**.** *Let G be a holomorphic mapping from an open subset U of $\mathbf{C}^2$ to $\mathbf{C}^2$ with a fixed point $p \in U$. Let α, β be the eigenvalues of JG_p and suppose that $|\beta| < 1 < |\alpha|$. Then there exists a holomorphic mapping $\tilde{H} : \Delta_\rho \to U$ such that*

$$\tilde{H}(0) = p \quad \text{and} \quad G \circ \tilde{H}(z) = \tilde{H}(\alpha z) \quad \text{for} \quad z,\ \alpha z \in \Delta_\rho. \tag{3.2}$$

In particular, we have $\tilde{H}(\Delta_\rho) \subset W^u(p)$.

Applying Theorem 3.2 to our G, we obtain an entire holomorphic mapping $H : \mathbf{C} \to \mathbf{C}^2$ satisfying the following:

$$H \text{ satisfies (3.2) and is injective on } \Delta_\rho \text{ and } JH_0 = {}^t(1, a). \tag{3.3}$$

Here, ${}^t(1, a)$ is the transpose of $(1, a)$. Indeed, let us define the holomorphic mappings $P,\ \tilde{G} : \mathbf{C}^2 \to \mathbf{C}^2$ by

$$P(x, y) = (x + y, ax), \qquad \tilde{G} = P^{-1} \circ G \circ P.$$

Then it is easy to see that

$$\tilde{G}(I_1) = I_1 \ \text{ and } \ J\tilde{G}_{I_1} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, by applying Theorem 3.2 to $\tilde{G}$ and I_1, we obtain a holomorphic mapping $\tilde{H} : \Delta_\rho \to \mathbf{C}^2$ satisfying (3.2) for $\tilde{G}$ and $\tilde{H}$ such that $\tilde{H}(0) = I_1$ and $J\tilde{H}_0 = {}^t(1, 0)$ (for detail, see the proof of [5, Theorem 6.4.3]). Consider the composition $H_0 := P \circ \tilde{H} : \Delta_\rho \to \mathbf{C}^2$. Owing to the fact (3.2), one can now define an entire holomorphic mapping H by setting

$$H(z) = G^n \circ H_0(z/a^n) \ \text{ for } z \in \Delta_{|a|^n \rho},\ n = 1, 2, 3, \ldots.$$

Then, it is easy to see that H satisfies the conditions required in (3.3).

Proposition 3.3. (1) *$H(z) \in W^u(I_1)$ for $z \in \mathbf{C}$.*
(2) *If $H(\Delta_\rho) \cap \bigcup_{n \geq 0}^{\infty} G^{-n}(C_3 \setminus I_G) = \{I_1\}$, then H is injective on $\mathbf{C}$.*

Proof. (1) is a direct consequence of the definition of H and (3.2). Since H is injective on Δ_ρ, the assertion (2) follows from Proposition 3.1. □

Here, if H is not injective on $\mathbf{C}$, then there exists a homoclinic point by Propositions 3.1 and 3.3. Therefore, recalling our assumption (3.1), we assume in the reminder of this section that

$$H \text{ is injective on } \mathbf{C}. \tag{3.4}$$

Set

$$P_1(x, y) = y^2 - bx + a,\ P_2(x, y) = y^2 (P_1)^2 - by + a, \ \text{ and}$$
$$\Sigma_i = \left\{ (x, y) \in \mathbf{C}^2 \ \middle|\ P_i(x, y) = 0 \right\} \ \ (i = 1, 2).$$

It should be remarked here that Σ_i are irreducible components of the algebraic sets $G^{-i}(C_3)$ and $\Sigma_i \subset W^s(I_1)$. Next, we write $H = (h_1, h_2)$ by coordinates. Then, both functions h_1 and h_2 are non-constant by (3.3). In the following part of the proof of Main Theorem, (1), we will go along the same line as in Jin [4, §2].

Lemma 3.4. *At least one of h_1 or h_2 is a non-constant transcendental entire function on* $\mathbf{C}$.

Proof. By the assumption (3.1), one can see that $H(\mathbf{C}) \cap \Sigma_i = \emptyset$ for $i = 1, 2$. So, $P_i \circ H$ are non-zero constants or transcendental entire functions with the exceptional value 0. Suppose that $P_i \circ H$ are constants, say, $P_i \circ H \equiv \alpha_i$ for some $\alpha_i \in \mathbf{C}^*$. Then

$$h_2(z)^2\alpha_1^2 - bh_2(z) + a = \alpha_2 \quad \text{for all} \quad z \in \mathbf{C},$$

which contradicts the fact that h_2 is a non-constant holomorphic function. Thus, at least one of $P_i \circ H$ is a transcendental entire function, as desired. □

Without loss of generality, we may assume that

(3.5) h_2 is a non-constant transcendental entire function.

Lemma 3.5. *$H(\mathbf{C})$ is not contained in any algebraic hypersurface in* $\mathbf{C}^2$.

Proof. Assume the contrary. Then, there exists a non-trivial polynomial $Q(x, y)$ such that

$$H(\mathbf{C}) \subset \left\{(x, y) \in \mathbf{C}^2 \,\middle|\, Q(x, y) = 0\right\}.$$

It should be remarked here that Q is a genuine two variables polynomial because h_1 and h_2 are non-constant holomorphic functions. From (3.5) there are some constant γ and infinitely many distinct points z_ν in $\mathbf{C}$ such that $h_2(z_\nu) = \gamma$ for all ν. Set $\delta_\nu = h_1(z_\nu)$. Then, $H(z_\nu) = (\delta_\nu, \gamma)$ are infinitely many distinct points by (3.4). On the other hand, $Q(\delta_\nu, \gamma) = Q \circ H(z_\nu) = 0$ for all ν. This contradicts the fact that Q is a genuine two variable polynomial. □

Let us return to the proof of (1) of Main Theorem. Set

$$P_3(x, y) = \{P_2(x, y)\}^2 - bP_1(x, y) + a, \ \Sigma_3 = \left\{(x, y) \in \mathbf{C}^2 \,\middle|\, P_3(x, y) = 0\right\}.$$

Then, a simple computation shows that $\Sigma_3 \subset G^{-3}(C_3)$. Here, we assert that polynomials P_i $(i = 1, 2, 3)$ are non-constant, irreducible and relatively prime, after rechoosing some irreducible components in place of P_i $(i = 2, 3)$, if necessary. Indeed, this follows from the assumptions that $|a| > 1$ and $b \neq 0$ and from the facts that $(a/b, 0) \in \Sigma_1$, $(a/b - a/b^2, 0) \in \Sigma_3$ and $\Sigma_2 \cap \{(x, y) \in \mathbf{C}^2 | y = 0\} = \emptyset$.

Finally, recall the following:

Theorem 3.6 ([6, Theorem 5.6], [7])**.** *Let $H : \mathbf{C} \to \mathbf{C}^2$ be an entire holomorphic mapping. Assume that the set of exceptional values of H contains algebraic surfaces $\Sigma_i = \left\{(x,y) \in \mathbf{C}^2 \;\middle|\; P_i(x,y) = 0\right\}$ $(i = 1,2,3)$, where P_i are non-constant, irreducible and relatively prime polynomials. Then there exists some polynomial $Q(x,y)$ such that $H(\mathbf{C}) \subset \left\{(x,y) \in \mathbf{C}^2 \;\middle|\; Q(x,y) = 0\right\}$.*

Applying this theorem to our H and Σ_i, we conclude that $H(\mathbf{C})$ is contained in some algebraic surface. But this contradicts the fact in Lemma 3.5; completing the proof of (1) in Main Theorem.

4. Proof of (2) of Main Theorem

We retain the terminology and notation in the previous section. Our construction of a horseshoe mapping on a neighborhood of I_1 is based on the arguments in [5, §7.4] developed in the case of Hénon mapping. The proof will be divided into four steps. Here, we need a few preparations. Since $h_2(0) = 0$ and $h_2'(0) = a$ by (3.3), the inverse of h_2 can be defined on some small disk Δ_R, and $H(\Delta_\rho)$ can be locally described as

$$H(\Delta_\rho) = \left\{(x,y) \in \mathbf{C}^2 \;\middle|\; x = h_1 \circ h_2^{-1}(y), \quad y \in \Delta_R\right\}.$$

Define the mappings $\phi^u : \Delta_R \to \mathbf{C}$ and $\Phi : \Delta_R^2 \to \mathbf{C}^2$ by

$$\phi^u(y) = h_1 \circ h_2^{-1}(y) \quad \text{and} \quad \Phi(x,y) = (x - \phi^u(y), y),$$

respectively. Clearly, Φ is a biholomorphic mapping and $\Phi(I_1) = I_1$. Thus, the inverse mapping $\Psi = \Phi^{-1}$ can be defined on $\Delta_{R'}^2$ for some R' with $0 < R' < R$. *In what follows, the index j will run over $1, 2, \dots$ and the indices i and i_j will run over $0, 1$, unless specified otherwise.*

Step 1. The purpose of this step is to construct some fundamental domains V_i and W_i on which a horseshoe mapping is defined. We now proceed to define the vertical set l_ξ and the mapping $\Psi_\xi : \Delta_{R'} \to \mathbf{C}^2$ for $\xi \in \Delta_{R'}$ by setting

$$l_\xi = \left\{(x,y) \in \Delta_{R'}^2 \;\middle|\; x = \xi\right\} \quad \text{and} \quad \Psi_\xi(y) = \Psi(\xi, y),\ y \in \Delta_{R'}.$$

Clearly, Ψ_ξ is injective on $\Delta_{R'}$ and $\Psi_0(\Delta_{R'}) = H(\Delta_\rho)$. Set $\gamma_\xi = \Psi(l_\xi)$ for $\xi \in \Delta_{R'}$. By (1) of Main Theorem, there exists a homoclinic point $p_0 \in W^u(I_1) \cap W^s(I_1) \setminus \{I_1\}$. Without loss of generality, we may assume that $p_0 \in H(\Delta_\rho)$; accordingly, one can choose a point $(0, y_0) \in l_0$ such that $\Psi_0(y_0) = p_0$. By the argument in Section 3, we know that

$$\left\{H(\Delta_\rho) \cap \bigcup_{n\geq 0}^{\infty} G^{-n}(C_3 \setminus I_G)\right\} \setminus \{I_1\} \neq \emptyset,\ G(C_3 \setminus I_G) = \{I_1\},\ C_3 \setminus I_G \subset W^s(I_1).$$

Hence, there is a unique positive integer n_0 such that

$$G^{n_0}(p_0) \in W^u(I_1) \cap \Big[C_3 \setminus \{I_1, I_3\}\Big].$$

Then, putting $q_0 = G^{n_0}(p_0)$, we have the following:

Lemma 4.1. *There is a positive constant r_2 such that*

$$G^{n_0} \circ \Psi_0\big(\Delta_{r_2}(y_0)\big) \cap C_3 = \big\{q_0\big\}.$$

Proof. If such an r_2 does not exist, then $G^{n_0} \circ \Psi_0(\Delta_{R'}) \subset C_3$ by the identity theorem. Then, $H(\Delta_\rho) \subset G^{n_0} \circ H(\Delta_\rho) = G^{n_0} \circ \Psi_0(\Delta_{R'}) \subset C_3$. This contradicts the fact that h_2 is non-constant by (3.3); completing the proof. □

From now on, we assume that

(4.1) $\quad q_0$ is a transverse homoclinic point of I_1 and $q_0 \neq (a/b, 0)$.

Put $q_0 = (x_0, 0)$, so that $x_0 \neq a/b$. From Proposition 3.1, (2), (3) and Lemma 4.1, $G^{n_0} \circ \Psi_0 : \Delta_{r_2}(y_0) \to \mathbf{C}^2$ is an injective holomorphic mapping and $\gamma^1_{0,n_0} = G^{n_0} \circ \Psi_0(\Delta_{r_2}(y_0)) \ni q_0$. Moreover, $T_{q_0}\gamma^1_{0,n_0} \oplus T_{q_0}C_3 = T_{q_0}\mathbf{C}^2$ from the transversality condition. Hence, there exists a tangent vector $v = (\alpha, \beta) \in T_{q_0}\gamma^1_{0,n_0}$ with $\beta \neq 0$; and accordingly,

$$JG_{q_0}(v) = \begin{pmatrix} 0 & 1 \\ 0 & -bx_0 + a \end{pmatrix} v = \begin{pmatrix} \beta \\ (-bx_0 + a)\beta \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This implies that, for some $r > 0$, γ^1_{0,n_0+1} is a one-dimensional submanifold of $\mathbf{C}^2$ given by the graph of the holomorphic function $x = (\phi^1_0)_{n_0+1}(y)$ on Δ_r. On the other hand, it is easy to see that the family of functions $\{\pi_2 \circ G^{n_0} \circ \Psi_\xi\}_{\xi \in \Delta_{R'}}$ converges to the function $\pi_2 \circ G^{n_0} \circ \Psi_0$ as $|\xi| \to 0$, uniformly on every compact subset of $\Delta_{r_2}(y_0)$. Together with Lemma 4.1, Hurwitz's theorem guarantees that:

$$\text{(4.2)} \quad \begin{cases} \text{There exists a constant } r_1 \text{ with } 0 < r_1 < R' \text{ such that, for each} \\ \xi \in \Delta_{r_1},\ \pi_2 \circ G^{n_0} \circ \Psi_\xi(y) \text{ has a unique zero point } y_\xi \in \Delta_{r_2}(y_0). \end{cases}$$

Let us put

$$V_0 = \Delta_{r_1} \times \Delta_{r_2},\ V_1 = \Delta_{r_1} \times \Delta_{r_2}(y_0) \ \text{ and } \ \gamma^i_{\xi,n} = G^n \circ \Psi(l_\xi \cap V_i).$$

As in the case of γ^i_{0,n_0+1}, after rechoosing r_1 and r_2 if necessary, one can see that γ^i_{ξ,n_0+1} is given by the graph of the holomorphic function $x = (\phi^i_\xi)_{n_0+1}(y)$ on Δ_r for every $\xi \in \Delta_{r_1}$. Since $I_1 \in \gamma^i_{\xi,n_0+1}$, one can set $l^i_{\xi,n_0+1} = \Phi\big(\gamma^i_{\xi,n_0+1}\big)$.

Before proceeding, we need to state a λ−lemma. To this end, define a holomorphic mapping

$$\tilde{G} : \Delta^2_{R'} \to \mathbf{C}^2 \ \text{ by } \ \tilde{G} = \Phi \circ G \circ \Psi.$$

Then, $\tilde{G}(I_1) = I_1$ and I_1 is a saddle fixed point of $\tilde{G}$. In particular, the local stable and unstable sets of I_1 for $\tilde{G}$ are contained in the x-axis and in the y-axis, respectively. Moreover, $\tilde{G}^n$ is well-defined on some neighbourhood U_n of I_1 such that

$$\text{(4.3)} \qquad \tilde{G}^n \text{ is injective on } U_n \setminus \bigcup_{k=0}^{n-1} \tilde{G}^{-k}(C_3) \text{ and } \tilde{G}^n\left(\bigcup_{k=0}^{n-1} \tilde{G}^{-k}(C_3)\right) = I_1.$$

Let $\phi : \Delta_r \to \mathbf{C}^2$ be an injective holomorphic mapping with $\phi(0) = I_1$. Write $\phi(z) = (\phi_1(z), \phi_2(z))$ and put $D = \phi(\Delta_r)$. Then, we can now state our λ-lemma as follows:

Lemma 4.2 (λ-lemma for $\tilde{G}$ at I_1)**.** *Assume that $D \cap C_3 = \{I_1\}$ and $\phi_2'(0) \neq 0$. Then, there exists a family of holomorphic functions $\phi_n : \Delta_{R'} \to \mathbf{C}$ such that*

$$\phi_n(0) = 0 \quad \textit{and} \quad \tilde{G}^n(D) \cap \Delta_{R'}^2 \supset \left\{(x, y) \in \Delta_{R'}^2 \;\middle|\; x = \phi_n(y), \quad y \in \Delta_{R'}\right\}$$

for all sufficiently large n. In particular, $\{\phi_n\}$ converges locally uniformly to the constant function $x \equiv 0$ on $\Delta_{R'}$.

Since this lemma can be proved with exactly the same argument as in [8, Lemma 7.1], we omit it. From Lemma 4.2, we obtain the following:

Lemma 4.3. *There is an integer m_0 such that*

$$\pi_1 \circ \tilde{G}^{m_0}\bigl(l^i_{\xi, n_0+1}\bigr) \subset \Delta_{r_1/2}, \quad \pi_2 \circ \tilde{G}^{m_0}\bigl(l^i_{\xi, n_0+1}\bigr) \supset \Delta_{R'} \quad \textit{for all} \quad \xi \in \Delta_{r_1}.$$

To simplify discussion, we change notation and write G, F in place of $\Phi \circ G^{n_0+m_0+1} \circ \Psi$, $\Phi \circ F^{n_0+m_0+1} \circ \Psi$, respectively.

Noting that $G(I_1) = I_1$ and the iteration G^n is well-defined at I_1 for every n, *we here concentrate our attention on dynamics of G in a local neighbourhood $\Delta_{R'}^2$ of I_1*. Rechoosing V_i if necessary, we may assume by Lemma 4.3 that

$$\text{(4.4)} \qquad \pi_2 \circ G(V_i) = \Delta_{R'}.$$

Let us here define the functions ψ_0^i on Δ_{r_1} by

$$\psi_0^0(x) = 0, \quad \psi_0^1(x) = y_x \quad \text{for} \quad x \in \Delta_{r_1},$$

where y_x is the zero point appearing in (4.2), and set

$$\text{(4.5)} \qquad \hat{l}_0^i = \left\{(x, y) \in V_i \;\middle|\; y = \psi_0^i(x), \quad x \in \Delta_{r_1}\right\}.$$

Then, one can see that $y = \psi_0^i(x)$ is holomorphic on Δ_{r_1}. Indeed, from the construction of V_i, we know that $G^{-1}(I_1) = \bigcup_{i=0}^1 \hat{l}_0^i$ and $\hat{l}_0^i$ is an analytic subset of pure dimension one; and hence, $y = \psi_0^i(x)$ is holomorphic on Δ_{r_1} (cf. [7, Theorem 4.4.1]). Therefore, putting $W_i = G(V_i)$, we obtain the following:

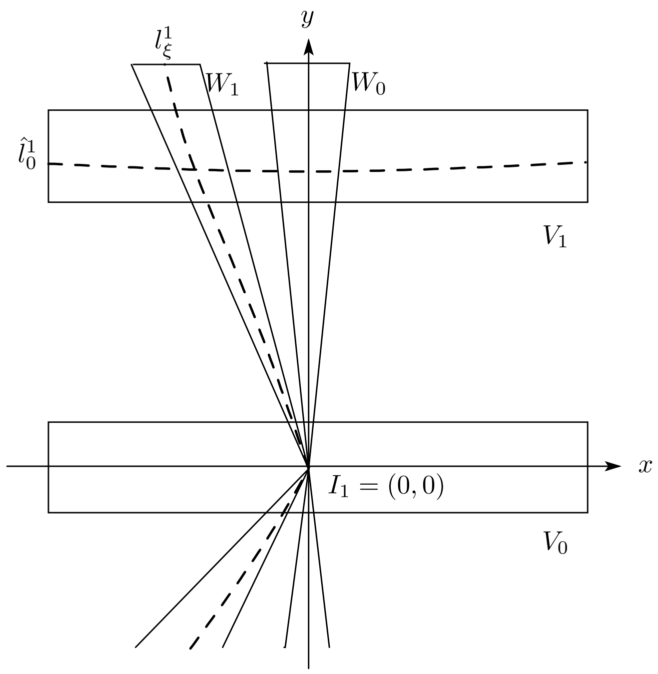

Figure 1

Lemma 4.4. *$G : V_0 \cup V_1 \setminus \bigcup_{i=0}^{1} \hat{l}^i_0 \to W_0 \cup W_1 \setminus \{I_1\}$ is a biholomorphic mapping and $G\left(\bigcup_{i=0}^{1} \hat{l}^i_0\right) = \{I_1\}$.*

As seen in Figure 1, $W_0 \cup W_1$ is pinched at I_1. In this way, we have constructed two domains V_i and W_i, which will play a crucial role in our proof.

Before proceeding, we need to introduce some terminology. Let U be an open subset of $\mathbf{C}^2$. Then we say that U *is foliated by the leaves* $\{\ell_\xi\}_{\xi \in \Delta_r}$ if

(i) ℓ_ξ is a one-dimensional complex submanifold of U for every $\xi \in \Delta_r$;

(ii) $U = \bigcup_{\xi \in \Delta_r} \ell_\xi$; and

(iii) $\ell_\xi \cap \ell_{\xi'} = \emptyset$ for $\xi,\ \xi' \in \Delta_r$ with $\xi \neq \xi'$.

Step 2. In this step, we want to show that both the domains V_i and W_i have the structure of foliation. For this purpose, we put $l^i_\xi = G(V_i \cap l_\xi)$ for $\xi \in \Delta_{r_1}$. Then, by Lemma 4.3, there is a holomorphic function ϕ^i_ξ on $\Delta_{R'}$ such that

(4.6) $$|\phi^i_\xi(y)| < r_1/2 \text{ on } \Delta_{R'} \text{ and } l^i_\xi = \left\{(x, y) \in W_i \;\middle|\; x = \phi^i_\xi(y),\ y \in \Delta_{R'}\right\}.$$

Thus, $W_i \setminus \{I_1\}$ is foliated by the *vertical leaves* $\left\{l^i_\xi \setminus \{I_1\}\right\}_{\xi \in \Delta_{r_1}}$.

Also, we wish to show that V_i has the structure of foliation. To this end, define the horizontal sets

$$\hat{l}_\eta = \left\{(x, y) \in \Delta^2_{R'} \;\middle|\; y = \eta\right\} \quad \text{for} \quad \eta \in \Delta_{R'},$$
$$\hat{l}^{i_1}_\eta = F(\hat{l}_\eta \cap W_{i_1}) \quad \text{for} \quad \eta \in \Delta^*_R \quad \text{and} \quad \hat{l}^{i_1}_0 = G^{-1}(\hat{l}_0 \cap W_{i_1}) \cap V_{i_1}.$$

Then, we have the following:

Lemma 4.5. *For each $\eta \in \Delta_{R'}$ and i_1, there exists a holomorphic function $\psi^{i_1}_\eta$ on Δ_{r_1} such that*

$$\hat{l}^{i_1}_\eta = \left\{(x, y) \in V_{i_1} \;\middle|\; y = \psi^{i_1}_\eta(x), \quad x \in \Delta_{r_1}\right\}$$

and V_{i_1} is foliated by the horizontal leaves $\{\hat{l}^{i_1}_\eta\}_{\eta \in \Delta_{R'}}$.

Proof. If $\eta = 0$, we can construct $\hat{l}^{i_1}_0$ by (4.5). So, consider the case where $\eta \in \Delta^*_{R'}$. Then, by Lemma 4.4, G^{-1} is an injective mapping on $W_{i_1} \setminus \{I_1\}$ and

$$\hat{l}^{i_1}_\eta = F(\hat{l}_\eta \cap W_{i_1}) = G^{-1}\left(\hat{l}_\eta \cap \bigcup_{\xi \in \Delta_{r_1}} l^{i_1}_\xi\right) = \bigcup_{\xi \in \Delta_{r_1}} G^{-1}(\phi^{i_1}_\xi(\eta), \eta).$$

Since $G^{-1}\big(\phi^{i_1}_\xi(\eta), \eta\big)$ is a single point contained in $l_\xi \cap V_{i_1}$ for every $\xi \in \Delta_{r_1}$, if one defines a function $\psi^{i_1}_\eta$ on Δ_{r_1} by the relation

$$G^{-1}(\phi^{i_1}_\xi(\eta), \eta) = (\xi, \psi^{i_1}_\eta(\xi)) \quad \text{for} \quad \xi \in \Delta_{r_1},$$

then

$$\hat{l}^{i_1}_\eta = \left\{(x, y) \in V_{i_1} \;\middle|\; y = \psi^{i_1}_\eta(x), \quad x \in \Delta_{r_1}\right\}.$$

On the other hand, $G^{-1}(\hat{l}_\eta \cap W_{i_1})$ is an analytic subset of pure dimension one; accordingly, $\psi^{i_1}_\eta$ is a holomorphic function on Δ_{r_1} by [7, Theorem 4.4.1]. □

Step 3. In this step, we shall show that the mappings G^n and F^n satisfy the *horseshoe condition* for every n. For this purpose, writing $G^n = (g^n_1, g^n_2)$ and $F^n = (f^n_1, f^n_2)$ by coordinates, we define inductively the sets $V_{i_n \cdots i_1}$ and the holomorphic mappings $\mathcal{G}^{i_n \cdots i_1} : V_{i_n \cdots i_1} \to \Delta_{r_1} \times \Delta_{R'}$ by

$$V_{i_{n+1} i_n \cdots i_1} = V_{i_{n+1}} \cap G^{-1}(V_{i_n \cdots i_1}) \quad \text{and} \quad \mathcal{G}^{i_n \cdots i_1}(x, y) = (x, g^n_2(x, y))$$

for all i_j with $1 \le j \le n$. Here, we say that G^n satisfies the *horseshoe condition* if $\mathcal{G}^{i_n \cdots i_1}$ are biholomorphic mappings for all i_j with $1 \le j \le n$. With this terminology, we have the following:

Lemma 4.6. *G^n satisfies the horseshoe condition for every n.*

Proof. To prove this lemma, we proceed by induction on n. Let $n = 1$ and assume that $\mathcal{G}^{i_1}$ is not injective. So, there are points (ξ, η), $(\xi, \eta') \in V_{i_1}$ with $\eta \neq \eta'$ such that $g_2(\xi, \eta) = g_2(\xi, \eta')$. On the other hand, since $l_\xi^{i_1} = G(l_\xi \cap V_{i_1})$ is given by the graph of the function $x = \phi_\xi^{i_1}(y)$, it then follows that $g_1(\xi, \eta) = g_1(\xi, \eta')$; which contradicts the fact that G is injective on $l_\xi \cap V_{i_1}$. Thus $\mathcal{G}^{i_1}$ is injective. On the other hand, it is clear by (4.4) that $\mathcal{G}^{i_1}$ is surjective, and the proof is completed in the case when $n = 1$.

Assume the lemma is proved for $n \geq 1$. Then, since $\mathcal{G}^{i_n \cdots i_1}$ is biholomorphic, there exists a holomorphic function $\psi_\eta^{i_n \cdots i_1}$ on Δ_{r_1} such that

$$\Big\{(x, y) \in V_{i_n \ldots i_1} \;\Big|\; g_2^n(x, y) = \eta\Big\} = \Big\{(x, y) \in V_{i_n \ldots i_1} \;\Big|\; y = \psi_\eta^{i_n \cdots i_1}(x),\ x \in \Delta_{r_1}\Big\}.$$

Denoting this set by $\hat{l}_\eta^{i_n \cdots i_1}$, we see that $V_{i_n \ldots i_1}$ is foliated by the *horizontal leaves* $\{\hat{l}_\eta^{i_n \cdots i_1}\}_{\eta \in \Delta_{R'}}$. Define the holomorphic mapping

$$\tilde{\mathcal{G}}^{i_n \cdots i_1} : V_{i_n \ldots i_1} \cap G(V_{i_{n+1}}) \setminus \hat{l}_0^{i_n \cdots i_1} \to \Delta_{r_1} \times \Delta_{R'}^* \;\text{ by }\; (x, y) \mapsto \big(f_1(x, y), g_2^n(x, y)\big)$$

and claim that this is biholomorphic. To do this, it is enough to show that the set

$$\begin{aligned} l_\xi^{i_{n+1}} \cap \hat{l}_\eta^{i_n \cdots i_1} &= \Big\{(x, y) \in G(V_{i_{n+1}}) \cap V_{i_n \ldots i_1} \;\Big|\; f_1(x, y) = \xi, \quad g_2^n(x, y) = \eta\Big\} \\ &= \Big\{(x, y) \in G(V_{i_{n+1}}) \cap V_{i_n \ldots i_1} \;\Big|\; x = \phi_\xi^{i_{n+1}}(y), \quad y = \psi_\eta^{i_n \cdots i_1}(x)\Big\} \end{aligned}$$

consists of a single point. Indeed, since $\phi_\xi^{i_{n+1}} \circ \psi_\eta^{i_n \cdots i_1}(\Delta_{r_1}) \subset \Delta_{r_1/2}$ by (4.6), one can see that there exists a unique fixed point $\tilde{x} \in \Delta_{r_1}$ of $\phi_\xi^{i_{n+1}} \circ \psi_\eta^{i_n \cdots i_1}$ (cf. [5, Theorem 6.3.5]). Thus, for any $(\xi, \eta) \in \Delta_{r_1} \times \Delta_{R'}^*$, there exists a unique point $(\tilde{x}, \psi_\eta^{i_n \cdots i_1}(\tilde{x})) \in V_{i_n \ldots i_1} \cap G(V_{i_{n+1}}) \setminus \hat{l}_0^{i_n \cdots i_1}$ with $\tilde{\mathcal{G}}^{i_n \cdots i_1}\big(\tilde{x}, \psi_\eta^{i_n \cdots i_1}(\tilde{x})\big) = (\xi, \eta)$. Consequently, $\tilde{\mathcal{G}}^{i_n \cdots i_1}$ is a biholomorphic mapping.

Next, put $\hat{l}_\eta^{i_{n+1} \cdots i_1} = G^{-1}\big(\hat{l}_\eta^{i_n \cdots i_1} \cap G(V_{i_{n+1}})\big)$ for every $\eta \in \Delta_{R'}$ and i_{n+1}. As in the proof of Lemma 4.5, we here assert that there exists a holomorphic function $\psi_\eta^{i_{n+1} \cdots i_1}$ on Δ_{r_1} such that

$$\hat{l}_\eta^{i_{n+1} \cdots i_1} = \Big\{(x, y) \in V_{i_{n+1}} \cap G^{-1}(V_{i_n \ldots i_1}) \;\Big|\; y = \psi_\eta^{i_{n+1} \cdots i_1}(x), \quad x \in \Delta_{r_1}\Big\}.$$

Indeed, by construction we have

$$\hat{l}_\eta^{i_{n+1} \cdots i_1} = G^{-1}(\hat{l}_\eta^{i_n \cdots i_1} \cap G(V_{i_{n+1}})) = G^{-1}\left(\bigcup_{\xi \in \Delta_{r_1}} \hat{l}_\eta^{i_n \cdots i_1} \cap l_\xi^{i_{n+1}}\right).$$

So, repeating the same argument as above, one can see that, for any given $\xi \in \Delta_{r_1}$, $l_\xi^{i_{n+1}}$ intersects $\hat{l}_\eta^{i_n \cdots i_1}$ at only one point, which we denote by $\big(\phi_\xi^{i_{n+1}}(y_\eta), y_\eta\big) \in \hat{l}_\eta^{i_n \cdots i_1} \cap l_\xi^{i_{n+1}}$. If $y_\eta \neq 0$, then $G^{-1}\big(\phi_\xi^{i_{n+1}}(y_\eta), y_\eta\big)$ is given by a single point contained in $G^{-1}(V_{i_n \ldots i_1}) \cap l_\xi$, so it can be written in the form $\big(\xi, \psi_\eta^{i_{n+1} \cdots i_1}(\xi)\big)$, and

$\psi_\eta^{i_{n+1}\cdots i_1}$ is a holomorphic function on Δ_{r_1} by [7, Theorem 4.4.1]. If $y_\eta = 0$, then $l_\xi^{i_{n+1}} \cap \hat{l}_\eta^{i_n\cdots i_1} = \{I_1\}$. By (4.5), we have that $(\xi, \psi_\eta^{i_{n+1}\cdots i_1}(\xi)) = (\xi, \psi_0^{i_{n+1}}(\xi))$ and $\hat{l}_\eta^{i_{n+1}\cdots i_1} = \hat{l}_0^{i_{n+1}}$.

Finally, consider the mapping

$$\tilde{\mathcal{G}}^{i_n\cdots i_1} \circ G : G^{-1}\big(V_{i_n\dots i_1} \cap G(V_{i_{n+1}}) \setminus \hat{l}_0^{i_n\cdots i_1}\big) \cap V_{i_{n+1}} \to \Delta_{r_1} \times \Delta^*_{R'}.$$

Then, by Lemma 4.4, it follows that

$$G^{-1}\big(V_{i_n\dots i_1} \cap G(V_{i_{n+1}}) \setminus \hat{l}_0^{i_n\cdots i_1}\big) = G^{-1}(V_{i_n\dots i_1}) \cap V_{i_{n+1}} \setminus \hat{l}_0^{i_{n+1}\cdots i_1} \quad \text{and}$$
$$\mathcal{G}^{i_{n+1}\cdots i_1} = \tilde{\mathcal{G}}^{i_{n+1}\cdots i_1} \circ G : G^{-1}(V_{i_n\dots i_1}) \cap V_{i_{n+1}} \setminus \hat{l}_0^{i_{n+1}\cdots i_1} \to \Delta_{r_1} \times \Delta^*_{R'}$$

is a biholomorphic mapping. It is now an easy matter to see that $\mathcal{G}^{i_{n+1}\cdots i_1}$ naturally extends to a biholomorphic mapping

$$\mathcal{G}^{i_{n+1}\cdots i_1} : G^{-1}(V_{i_n\dots i_1}) \cap V_{i_{n+1}} \to \Delta_{r_1} \times \Delta_{R'},$$

by setting $\mathcal{G}^{i_{n+1}\cdots i_1}(x, y) = (x, 0)$ for $(x, y) \in \hat{l}_0^{i_{n+1}\cdots i_1}$, and the proof of Lemma 4.6 is completed. □

Now, replacing G by F in the argument above, we define inductively the sets $W_{i_n\dots i_1}$ and the holomorphic mappings $\mathcal{F}^{i_n\cdots i_1} : W_{i_n\dots i_1} \to \Delta_{r_1} \times \Delta^*_{R'}$ by

$$W_{i_2 i_1} = (W_{i_2} \setminus \{I_1\}) \cap F^{-1}(W_{i_1} \setminus \{I_1\}),$$
$$W_{i_{n+1}\dots i_1} = \big(W_{i_{n+1}} \setminus \{I_1\}\big) \cap F^{-1}(W_{i_n\dots i_1}) \quad \text{for} \quad n \geq 2 \quad \text{and}$$
$$\mathcal{F}^{i_n\cdots i_1}(x, y) = (f_1^n(x, y), y) \quad \text{for every} \quad i_n, \dots, i_1 \quad \text{with} \quad n \geq 1.$$

We also say that F^n satisfies the *horseshoe condition* if $\mathcal{F}^{i_n\cdots i_1}$ are biholomorphic mappings for every $i_n, \dots, i_1$ with $n \geq 1$.

Lemma 4.7. *F^n satisfies the horseshoe condition for every n.*

Proof. The proof is almost identical to that of Lemma 4.6. Consider first the case $n = 1$. To prove that F satisfies the horseshoe condition, we need to show that $\mathcal{F}^{i_1}$ is injective. If not, there are distinct points (ξ, η), $(\xi', \eta) \in \hat{l}_\eta \cap W_{i_1} \setminus \{I_1\}$ with $f_1(\xi, \eta) = f_1(\xi', \eta)$. Since $\hat{l}_\eta^{i_1} = F(\hat{l}_\eta \cap W_{i_1})$ is given by the graph of the function $y = \psi_\eta^{i_1}(x)$, it follows that $f_2(\xi, \eta) = f_2(\xi', \eta)$; contradicting the fact that F is injective on $\hat{l}_\eta \cap W_{i_1}$ for $\eta \neq 0$. Therefore, $\mathcal{F}^{i_1}$ is injective.

Assume that F^n satisfies the horseshoe condition for some $n \geq 1$. Then, since $\mathcal{F}^{i_n\cdots i_1}$ is a biholomorphic mapping, there exists a holomorphic function $\phi_\xi^{i_n\cdots i_1}$ on $\Delta^*_{R'}$ for every $\xi \in \Delta_{r_1}$ such that

$$\Big\{(x, y) \in W_{i_n\dots i_1} \;\Big|\; f_1^n(x, y) = \xi\Big\}$$
$$= \Big\{(x, y) \in W_{i_n\dots i_1} \;\Big|\; x = \phi_\xi^{i_n\cdots i_1}(y), \quad y \in \Delta^*_{R'}\Big\}.$$

Hence, denoting this set by $l_\xi^{i_n \cdots i_1}$, we can see that $W_{i_n \cdots i_1}$ is foliated by the *vertical leaves* $\{l_\xi^{i_n \cdots i_1}\}_{\xi \in \Delta_{r_1}}$. Next, define the holomorphic mapping

$$\tilde{\mathcal{F}}^{i_n \cdots i_1} : W_{i_n \cdots i_1} \cap F(W_{i_{n+1}} \setminus \{I_1\}) \to \Delta_{r_1} \times \Delta^*_{R'} \text{ by } (x,y) \mapsto (f_1^n(x,y), g_2^1(x,y)).$$

Then, noting that $\phi_\xi^{i_n \cdots i_1}$ can be extended to a holomorphic function on $\Delta_{R'}$ as $\phi_\xi^{i_n \cdots i_1}(0) = 0$ and repeating the same argument as in the proof of Lemma 4.6, we can check that $\tilde{\mathcal{F}}^{i_n \cdots i_1}$ is biholomorphic. Moreover, it follows from Lemma 4.4 that

(i) F is biholomorphic on $W_{i_{n+1}} \setminus \{I_1\}$;

(ii) $F^{-1}\left(W_{i_n \cdots i_1} \cap F(W_{i_{n+1}} \setminus \{I_1\})\right)$
$$= F^{-1}(W_{i_n \cdots i_1}) \cap \left\{W_{i_{n+1}} \setminus \{I_1\}\right\} = W_{i_{n+1} \cdots i_1}; \quad \text{and}$$

(iii) $\mathcal{F}^{i_{n+1} \cdots i_1} = \tilde{\mathcal{F}}^{i_n \cdots i_1} \circ F : W_{i_{n+1} \cdots i_1} \to \Delta_{r_1} \times \Delta^*_{R'}$ is biholomorphic.

Therefore, the proof is completed. □

Step 4. In this final step we define an invariant set X on which F and G are conjugate to the shift mapping on $\hat{\Sigma}$. First, we classify the points $p \in \bigcap_{n=0}^{\infty} G^{-n}(V_0 \cup V_1)$ by using the fact that the j-th orbit of p is contained in V_0 or V_1. To this end, let us introduce some notation from symbol dynamics. A sequence $(s_0, \ldots, s_{n-1})$ with terms $s_j = 0, 1$ is said to be a *symbol sequence of length* n and the set of all symbol sequences of length n is denoted by $\{0,1\}^n$. For each $(s_0, \ldots, s_{n-1}) \in \{0,1\}^n$, define the set $V_{s_0 \ldots s_{n-1}}$ by

$$V_{s_0 \ldots s_{n-1}} = \left\{(x,y) \in \Delta_{r_1} \times \Delta_{R'} \;\middle|\; G^j(x,y) \in V_{s_j},\ 0 \le j \le n-1\right\}.$$

Then, from Lemma 4.6, we have the following:

Lemma 4.8. *$V_{s_0 \ldots s_{n-1}} = \bigcup_{\eta \in \Delta_{R'}} \hat{l}_\eta^{s_0 \ldots s_{n-1}}$ and $G(V_{s_0 \ldots s_{n-1}}) \subset V_{s_1 \ldots s_{n-1}}$ for every $(s_0, \ldots, s_{n-1}) \in \{0,1\}^n$.*

As in [5, §7.4], define the space Σ of all infinite symbol sequences by

$$\Sigma = \left\{s_+ = (s_0, s_1, \ldots) \;\middle|\; s_i = 0, 1\right\}$$

and set

$$\Gamma(s_+) = \bigcap_{n=0}^{\infty} V_{s_0 \ldots s_n} \quad \text{for every } s_+ \in \Sigma.$$

Then, we have the following:

Lemma 4.9. *For every $s_+ \in \Sigma$, there exists a holomorphic function $\psi_{s_+} : \Delta_{r_1} \to \Delta_{R'}$ such that*

$$\Gamma(s_+) = \left\{(x,y) \in \Delta_{r_1} \times \Delta_{R'} \;\middle|\; y = \psi_{s_+}(x), \quad x \in \Delta_{r_1}\right\}.$$

To prove Lemma 4.9, we start with the following general fact:

Lemma 4.10 ([5, Lemma 6.3.7]). *Let $K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$ be a decreasing sequence of compact sets in $\mathbf{C}^N$. Suppose that there exist a domain $V \subset \mathbf{C}^M$, a compact set $L \subset V$ and a sequence of holomorphic mappings $\Phi_n : V \to \mathbf{C}^N$ such that*

$$K_n \supset \Phi_n(V) \supset \Phi_n(L) \supset K_{n+1} \quad \textit{for every} \quad n \in \mathbf{N}.$$

Then the intersection $\bigcap_{n=1}^{\infty} K_n$ consists of a single point.

Proof of Lemma 4.9. Now putting $s_+ = (s_0, s_1, \ldots)$, we assert that $l_\xi \cap \Gamma(s_+)$ consists of a single point for every $\xi \in \Delta_{r_1}$. Indeed, taking into account the fact $l_\xi \cap \Gamma(s_+) = \bigcap_{n=0}^{\infty} (V_{s_0 \ldots s_n} \cap l_\xi)$, we define the holomorphic function

$$(g_2^{n+1})_\xi : \pi_2(l_\xi \cap V_{s_0 \ldots s_n}) \to \Delta_{R'} \quad \text{by} \quad y \mapsto (g_2^{n+1})_\xi(y) = g_2^{n+1}(\xi, y).$$

Then by the horseshoe condition, $(g_2^{n+1})_\xi$ is a univalent function with inverse $(g_2^{n+1})_\xi^{-1} : \Delta_{R'} \to \pi_2(l_\xi \cap V_{s_0 \ldots s_{n+1}})$. Moreover, since $G(V_{s_0 \ldots s_{n+1}}) \subset V_{s_1 \ldots s_{n+1}}$, it follows that $G^{n+1}(V_{s_0 \ldots s_{n+1}}) \subset V_{s_{n+1}}$ and

$$(g_2^{n+1})_\xi \circ \pi_2(l_\xi \cap V_{s_0 \ldots s_{n+1}}) \subset \pi_2(V_{s_{n+1}}) \subset \Delta_{r_2} \cup \Delta_{r_2}(y_0) \subset \Delta_{R''}$$

for some constant R'' with $0 < R'' < R'$. Thus, applying Lemma 4.10 to the case where $V = \Delta_{R'}$, $L = \overline{\Delta_{R''}}$, $K_n = \pi_2(l_\xi \cap V_{s_0 \ldots s_n})$ and $\Phi_n = (g_2^{n+1})_\xi^{-1}$, one can see that $\bigcap_{n=0}^{\infty} \pi_2(l_\xi \cap V_{s_0 \ldots s_n})$ consists of a unique point. So, denoting it by $\psi_{s_+}(\xi)$, we have that

$$\Gamma(s_+) = \left\{ (x, y) \in \Delta_{r_1} \times \Delta_R \;\middle|\; y = \psi_{s_+}(x), \quad x \in \Delta_{r_1} \right\}.$$

Remark here that, for each fixed point $\xi \in \Delta_{r_1}$, the sequence $\{\psi_0^{s_0 \ldots s_n}(\xi)\}$ converges to $\psi_{s_+}(\xi)$ as $n \to \infty$. Moreover, $\{\psi_0^{s_0 \cdots s_n}\}_{n \geq 0}$ is a normal family, since it is uniformly bounded on Δ_{r_1}. Therefore, $\{\psi_0^{s_0 \cdots s_n}\}_{n \geq 0}$ converges to the holomorphic function ψ_{s_+} on Δ_{r_1}; completing the proof of Lemma 4.9. □

Put $V = \bigcup_{s_+ \in \Sigma} \Gamma(s_+)$ and define the mappings

$$\begin{aligned} \psi_+ : V \to \Sigma \quad &\text{by} \quad (x, y) \mapsto s_+ \quad \text{if} \quad (x, y) \in \Gamma(s_+), \\ \Psi_+ : V \to \Delta_{r_1} \times \Sigma \quad &\text{by} \quad (x, y) \mapsto (x, \psi_+(x, y)), \quad \text{and} \\ \sigma : \Sigma \to \Sigma \quad &\text{by} \quad s_+ = (s_0, s_1, \ldots) \mapsto (s_1, s_2, \ldots). \end{aligned}$$

Then, Lemmas 4.8 and 4.9 yield the following lemma (cf. [5, Theorem 7.4.12]):

Lemma 4.11. *Ψ_+ is a homeomorphism and $\sigma \circ \psi_+ = \psi_+ \circ G$ on V.*

Next, replacing G and $V_{s_0, \ldots s_{n-1}}$ by F and $W_{s_{-1} \ldots s_{-n}}$, respectively, in the argument above, we can repeat the same process. Notice that

$$W_{s_{-1} \ldots s_{-n}} = \left\{ (x, y) \in W_0 \cup W_1 \setminus \{I_1\} \;\middle|\; F^j(x, y) \in W_{s_{-j}}, \quad 1 \leq j \leq n \right\}$$

for every symbol sequence in the form $(s_{-1}, \ldots, s_{-n})$. Then, from the definitions of $W_{s_{-1} \ldots s_{-n}}$ and F, the following lemma is obvious:

Lemma 4.12. *$W_{s_{-1}\dots s_{-(n+1)}} \subset W_{s_{-1}\dots s_{-n}}$, $F(W_{s_{-1}\dots s_{-n}}) \subset W_{s_{-2}\dots s_{-n}}$ for every symbol sequences $(s_{-1}, \dots, s_{-(n+1)}) \in \Sigma$.*

Let us now put

$$\Lambda(s_-) = \bigcap_{n=1}^{\infty} W_{s_{-1}\dots s_{-n}} \quad \text{for every} \quad s_- = (s_{-1}, \dots, s_{-n}, \dots) \in \Sigma.$$

Then, in exactly the same way as in the proof of Lemma 4.9, one can show the following:

Lemma 4.13. *For every $s_- \in \Sigma$, there exists a holomorphic function $\phi_{s_-} : \Delta^*_{R'} \to \mathbf{C}$ such that*

$$\Lambda(s_-) = \left\{ (x, y) \in W_0 \cup W_1 \setminus \{I_1\} \;\middle|\; x = \phi_{s_-}(y), \quad y \in \Delta^*_{R'} \right\}.$$

Set $W = \bigcup_{s_- \in \Sigma} \Lambda(s_-)$ and define the mappings $\psi_- : W \to \Sigma$ and $\Psi_- : W \to \Sigma \times \Delta^*_{R'}$ by

$$\psi_-(x, y) = s_- \text{ if } (x, y) \in \Lambda(s_-), \quad \text{and} \quad \Psi_-(x, y) = (\psi_-(x, y), y),$$

respectively. Then, by Lemmas 4.12 and 4.13, we have the following:

Lemma 4.14. *Ψ_- is a homeomorphism and $\sigma \circ \psi_- = \psi_- \circ F$ on W.*

Finally, we set

$$X = V \cap W \setminus \bigcup_{n=0}^{\infty} G^{-n}(I_1).$$

Then, Proposition 2.1 together with the definitions of V and W gives the following:

Lemma 4.15. *X is an invariant set of F and G. Moreover, F and G are bijective self-mappings of X.*

Let us now consider the space of *bi-infinite symbol sequences*

$$\hat{\Sigma} = \left\{ s = (s_-, s_+) \in \Sigma \times \Sigma \;\middle|\; s = (\dots, s_{-1}, s_0, s_1, \dots) \right\}$$

and its subset

$$E = \left\{ s \in \hat{\Sigma} \;\middle|\; \text{there is an integer } n_0 \text{ such that } s_n = 0 \text{ for } n \geq n_0 \right\}.$$

And, define a function $\rho : \hat{\Sigma} \times \hat{\Sigma} \to \mathbf{R}$ by

$$\rho(s, t) = \sum_{n=-\infty}^{\infty} \frac{|s_n - t_n|}{2^{|n|}} \quad \text{for} \quad s, t \in \hat{\Sigma}.$$

Then, it is easy to verify that ρ is a metric on $\hat{\Sigma}$ and

$$(4.7) \qquad \rho(s,t) < 2^{-k+1} \quad \text{if and only if} \quad s_i = t_i \quad \text{for all} \quad |i| \leq k.$$

In the following, we will always consider $\hat{\Sigma}$ equipped with the topology induced by this metric ρ.

From the construction of X, one can define the mappings $\hat{\Psi} : X \to \hat{\Sigma} \setminus E$ and $\sigma : \hat{\Sigma} \to \hat{\Sigma}$ by

$$\hat{\Psi}(x,y) = (\psi_-(x,y), \psi_+(x,y)) \quad \text{and} \quad \sigma(\ldots, s_1, \hat{s_0}, s_1, \ldots) = (\ldots, s_0, \hat{s_1}, s_2, \ldots).$$

Lemma 4.16. *$\hat{\Psi} : X \to \hat{\Sigma} \setminus E$ is a homeomorphism such that $\sigma \circ \hat{\Psi} = \hat{\Psi} \circ G$ and $\sigma^{-1} \circ \hat{\Psi} = \hat{\Psi} \circ F$ on X.*

Proof. To show that $\hat{\Psi}$ is bijective, we claim that:

$$(4.8) \qquad \text{For every } (s_-, s_+) \in \hat{\Sigma} \setminus E, \quad \Gamma(s_+) \cap \Gamma(s_-) \quad \text{consists of one point.}$$

Indeed, if $s_+ \neq (0, 0, \cdots)$, then $\psi_{s_+}(x) \neq 0$ for all $x \in \Delta_{r_1}$. So, the mapping $\phi_{s_-} \circ \psi_{s_+} : \Delta_{r_1} \to \Delta_{r_1}$ is well-defined and $\phi_{s_-} \circ \psi_{s_+}(\Delta_{r_1}) \subset \Delta_{r_1/2}$. Hence, there exists a unique fixed point $x_0 \in \Delta_{r_1}$ of $\phi_{s_-} \circ \psi_{s_+}(x)$ and $\Gamma(s_+) \cap \Gamma(s_-) = (x_0, \psi_{s_+}(x_0))$, required in (4.8). □

By Lemma 4.16, there exists a one-to-one correspondence between the sets of periodic points of F and σ. On the other hand, it is well-known that the set of periodic points of σ is dense in $\hat{\Sigma}$. Hence, the periodic points of F accumulate at I_1. Therefore, the proof of Main Theorem is completed.

Remark. The set $\hat{\Sigma} \setminus E$ is neither closed nor open in $\hat{\Sigma}$, and both $\hat{\Sigma} \setminus E$ and E are dense in $\hat{\Sigma}$. More precisely, $\hat{\Sigma} \setminus E$ is a *residual set*, that is, it can be represented as the intersection of at most countably many open dense subsets in $\hat{\Sigma}$. Indeed, putting

$$U_n = \left\{ s \in \hat{\Sigma} \;\middle|\; \textit{there exists an integer } m_0 > n \textit{ such that } s_{m_0} \neq 0 \right\}$$

for every n, we see that U_n are open dense subsets of $\hat{\Sigma}$ by (4.7) and $\hat{\Sigma} \setminus E = \bigcap_{n=-\infty}^{\infty} U_n$.

5. Examples

We have already known from Section 3 that there exists a homoclinic point q_0 of I_1 such that $q_0 \in C_3 \setminus \{I_G\}$. In this section, we give an example such that G has a homoclinic point $q_0 \in C_3 \setminus \{[a/b : 0 : 1], I_G\}$. To construct such a mapping, it is enough to find a condition on the parameter (a, b) which implies that $[a/b : 0 : 1]$ is not a homoclinic point.

On the chart $\left\{ [x : y : t] \in \mathbf{P}^2 \;\middle|\; x \neq 0 \right\}$, F can be written in the form

$$F(y,t) = \left(\frac{bt}{1 + at^2 - yt^2}, \frac{bt^2}{1 + at^2 - yt^2} \right) \quad \text{and} \quad [a/b : 0 : 1] = (0, b/a).$$

Put $(y_1, t_1) = F(y, t)$. A direct calculation shows that $[1 : 0 : 0] = (0, 0)$ is an attracting fixed point of F. Moreover, we can prove the following:

Lemma 5.1. *Assume that the parameter (a, b) satisfies the inequality $64|b| < 63 - 4|a|$. Then, we have*:
(1) *For each fixed (a, b), there exists a constant $0 < \epsilon_0 < 1$ such that*

$$|y_1| < (1 - \epsilon_0)|t|, \quad |t_1| < (1 - \epsilon_0)|t|/4 \quad \textit{for every} \quad (y, t) \in \Delta^2_{1/4};$$

(2) *For every $(y, t) \in \Delta^2_{1/4}$, $F^n(y, t) \to (0, 0)$ as $n \to \infty$.*

Proof. Let $(y, t) \in \Delta^2_{1/4}$. Then, it is easily seen that

$$|y_1| < \left| \frac{bt}{1 + at^2 - yt^2} \right| < \frac{|bt|}{1 - |at^2| - |yt^2|} < \frac{64|b|}{63 - 4|a|}|t| \ \text{ and } \ |t_1| < |y_1||t|,$$

from which we have (1). Applying (1) to $(y, t) \in \Delta^2_{1/4}$ inductively, we have the assertion (2). ☐

Put

$$A = \left\{ (a, b) \in \mathbf{C}^2 \ \middle| \ 0 < 64|b| < 63 - 4|a|, \quad |b/a| < 1/4, \quad |a| > 1 \right\}.$$

Then, it follows from Lemma 5.1 that $F^n(0, b/a) \to (0, 0)$ as $n \to \infty$ for every $(a, b) \in A$. Hence, if we choose a parameter $(a, b) \in A$, then $[a/b : 0 : 1]$ cannot be a homoclinic point of G; and there must be a homoclinic point q_0 in $C_3 \setminus \{[a/b : 0 : 1], I_G\}$, as required.

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Tokyo Metropolitan College of
Industrial Technology
1-10-40 Higashi-ooi, Shinagawa-ku
Tokyo 140-0011, Japan
e-mail: sinohara@tokyo-tmct.ac.jp

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