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Mod 2 cohomology of 2-compact groups of low rank

By

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Abstract

We determine the mod 2 cohomology algebra over the Steenrod algebra $\mathcal{A}_2$ of the classifying space of loop groups LG where $G = Spin(7)$, $Spin(8)$, $Spin(9)$, F_4 and $DI(4)$. Then we show they are isomorphic as algebras over $\mathcal{A}_2$ to the mod 2 cohomology of the 2-compact groups of type G.

1. Introduction

Kuribayashi [Ku] considered the cohomology algebra of free loop spaces by developing "module derivation." Generalizing his method, Kishimoto and Kono [KK] have developed a method to calculate cohomology of certain free loop spaces and p-compact groups over the Steenrod algebra. Using their method we calculate the mod 2 cohomology over the Steenrod algebra $\mathcal{A}_2$ of BLG, the classifying space of loop groups and 2-compact groups of type G with $G = Spin(7), Spin(8), Spin(9), F_4$ and $DI(4)$, the finite loop space at prime 2 constructed by Dwyer and Wilkerson [DW].

Here we summarize the result [KK] necessary for our purpose. Let ϕ be a based self-map of a based space X. The twisted loop space of X, $\mathbb{L}_\phi X$ is defined in the following pull-back diagram:

$$\begin{array}{ccc} \mathbb{L}_\phi X & \longrightarrow & X^{[0,1]} \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e_0 \times e_1} \\ X & \xrightarrow{1\times\phi} & X \times X \end{array}$$

where e_i $(i = 0, 1)$ is the evaluation at i. The twisted tube of X, $\mathbb{T}_\phi X$ is defined by

$$\mathbb{T}_\phi X = \frac{[0,1] \times X}{(0,x) \simeq (1, \phi(x))}.$$

There is a canonical inclusion $\iota : X \hookrightarrow \mathbb{T}_\phi X$.

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Remark 1. When ϕ is the identity map, then $\mathbb{L}_\phi X$ is merely the free loop space of X and $\mathbb{T}_\phi X = S^1 \times X$.

The relation between the cohomology of $\mathbb{T}_\phi X$ and X can be obtained by the Wang exact sequence

(1.1)

$$\cdots H^{n-1}(X;R) \xrightarrow{1-\phi^*} H^{n-1}(X;R) \xrightarrow{\delta} H^n(\mathbb{T}_\phi X;R) \xrightarrow{\iota^*} H^n(X;R) \xrightarrow{1-\phi^*} H^n(X;R)\cdots,$$

where R is any commutative ring. Especially this exact sequence splits off to the short exact sequence when $H^*(\phi;R)$ is the identity map.

The twisted cohomology suspension is a map

$$\hat{\sigma}_\phi : H^*(\mathbb{T}_\phi X;R) \to H^{*-1}(\mathbb{L}_\phi X;R).$$

This together with the Wang sequence above relates the cohomology of X to that of $\mathbb{L}_\phi X$.

We consider the case when

(1.2)

$$\begin{cases} H^*(X;\mathbb{Z}/2) \text{ is a polynomial algebra } \mathbb{Z}/2[x_1,x_2,\dots,x_l], \\ H^*(\phi;\mathbb{Z}/2) \text{ is the identity map}, \\ \text{and } H^n(\phi;\mathbb{Z}/4) \text{ is the identity map for all odd } n \text{ and } n=4m\ (m\in\mathbb{Z}). \end{cases}$$

Under this condition, the result in [KK] specializes to the following Proposition:

Proposition 1.1. *Suppose that there is a section $r : H^*(X;\mathbb{Z}/2) \to H^*(\mathbb{T}_\phi X;\mathbb{Z}/2)$ of ι^*, which commutes with the Steenrod operations. Then we have*

1. $H^*(\mathbb{L}_\phi X;\mathbb{Z}/2) = \mathbb{Z}/2[e^*(x_1),e^*(x_2),\dots,e^*(x_n)] \otimes \Delta(\hat{\sigma}_\phi\circ r(x_1),\hat{\sigma}_\phi\circ r(x_2),\dots,\hat{\sigma}_\phi\circ r(x_n))$.
2. $\hat{\sigma}_\phi\circ r(xy) = \hat{\sigma}_\phi(r(x))(\iota\circ e)^*(y) + \hat{\sigma}_\phi(r(y))(\iota\circ e)^*(x)$ *for* $x,y \in H^*(\mathbb{T}_\phi X;\mathbb{Z}/2)$.
3. $\hat{\sigma}_\phi$ *commutes with the Steenrod operations.*

Let G be either $Spin(7), Spin(8), Spin(9), F_4$ or $DI(4)$, and X be BG.

When ϕ is the identity map, $\mathbb{L}_\phi BG$ is merely LBG, the free loop space LBG, which is homotopy equivalent to BLG. Now (1.2) is trivially satisfied. The projection $S^1\times X \to X$ is a section of ι. Hence we can calculate the cohomology of BLG by above Proposition.

For $G = Spin(7), Spin(8), Spin(9), F_4$ and a odd prime power q and $\phi = \psi^q$ the Adams operation of degree q [W], (1.2) is also satisfied. The Bousfield and Kan 2-completion [BK] of $\mathbb{L}_\phi X$ is known to be homotopy equivalent to that of the classifying space of Chevalley group of type $G(q)$ [F].

For $G = DI(4)$ and a odd prime power q, there is a self homotopy equivalence ψ^q of $BDI(4)$ also called the Adams operation of degree q [N]. When $\phi=\psi^q$, (1.2) is again satisfied. $\mathbb{L}_\phi BDI(4)$ is called $BSol(q)$ defined in [B].

In the following sections, our main observation is to construct the section r when $\phi = \psi^q$ and to show the following:

Theorem 1.1. *Let $G = Spin(7), Spin(8), Spin(9), F_4$ or $DI(4)$. Then $H^*(LBG;\mathbb{Z}/2) \simeq H^*(\mathbb{L}_{\psi^q}BG;\mathbb{Z}/2)$ as the algebras over the Steenrod algebra $\mathcal{A}_2$, where q is an odd prime power.*

2. The case $G = Spin(7), Spin(8), Spin(9)$ and F_4

The mod 2 cohomology over $\mathcal{A}_2$ of $BSpin(7), BSpin(8)$ and $BSpin(9)$ are well known [Q, K].

$H^*(BSpin(7);\mathbb{Z}/2) = \mathbb{Z}[w_4, w_6, w_7, w_8]$ and the action of $\mathcal{A}_2$ is determined by:

	w_4	w_6	w_7	w_8
Sq^1	0	w_7	0	0
Sq^2	w_6	0	0	0
Sq^4	w_4^2	w_4w_6	w_4w_7	w_4w_8

$H^*(BSpin(8);\mathbb{Z}/2) = \mathbb{Z}[w_4, w_6, w_7, w_8, e_8]$ and the action of $\mathcal{A}_2$ is determined by:

	w_4	w_6	w_7	w_8	e_8
Sq^1	0	w_7	0	0	0
Sq^2	w_6	0	0	0	0
Sq^4	w_4^2	w_4w_6	w_4w_7	w_4w_8	w_4e_8

$H^*(BSpin(9);\mathbb{Z}/2) = \mathbb{Z}[w_4, w_6, w_7, w_8, e_{16}]$ and the action of $\mathcal{A}_2$ is determined by:

	w_4	w_6	w_7	w_8	e_{16}
Sq^1	0	w_7	0	0	0
Sq^2	w_6	0	0	0	0
Sq^4	w_4^2	w_4w_6	w_4w_7	w_4w_8	0
Sq^8	0	0	0	w_8^2	$w_8e_{16} + w_4^2e_{16}$

Proposition 2.1. *$H^*(LBSpin(7);\mathbb{Z}/2) = \mathbb{Z}/2[v_4, v_6, v_7, v_8, y_3, y_5, y_7]/I(|v_i| = i, |y_i| = i)$, where I is the ideal generated by $\{y_3^4 + v_6y_3^2 + y_5v_7, y_5^2 + y_3v_7 + v_4y_3^2, y_7^2 + y_6w_8 + w_7y_7\}$. The action of $\mathcal{A}_2$ is determined by:*

	v_4	v_6	v_7	v_8	y_3	y_5	y_7
Sq^1	0	v_7	0	0	0	y_3^2	0
Sq^2	v_6	0	0	0	y_5	0	0
Sq^4	v_4^2	v_4v_6	v_4v_7	v_4v_8	0	$y_3v_6 + v_4y_5$	$y_3v_8 + v_4y_7$

Proof. We take $v_i = e^*(w_i)$ and $y_i = \hat{\sigma}_\phi(w_i)$ $(i = 4, 6, 7, 8,\ \phi = Id)$. Then by Proposition 1.1 (1), we have $H^*(LBSpin(7);\mathbb{Z}/2) = \mathbb{Z}/2[v_4, v_6, v_7, v_8]$

$\otimes \Delta[y_3, y_5, y_6, y_7]$. By Proposition 1.1 (2) and (3), we have

$$
\begin{aligned}
&Sq^1 y_3 = Sq^1 \hat{\sigma}_\phi(w_4) = \hat{\sigma}_\phi(Sq^1 w_4) = 0\\
&Sq^2 y_3 = \hat{\sigma}_\phi(Sq^2 w_4) = \hat{\sigma}_\phi(w_6) = y_5\\
&Sq^1 y_5 = \hat{\sigma}_\phi(Sq^1 w_6) = y_6\\
&Sq^2 y_5 = \hat{\sigma}_\phi(Sq^2 w_6) = 0\\
&Sq^4 y_5 = \hat{\sigma}_\phi(Sq^4 w_6) = \hat{\sigma}_\phi(w_4 w_6) = e^*(w_4)\hat{\sigma}_\phi(w_6) + \hat{\sigma}_\phi(w_4)e^*(w_6) = v_4 y_5 + y_3 v_6\\
&Sq^1 y_7 = \hat{\sigma}_\phi(Sq^1 w_8) = 0\\
&Sq^2 y_7 = \hat{\sigma}_\phi(Sq^2 w_8) = 0\\
&Sq^4 y_7 = \hat{\sigma}_\phi(Sq^4 w_8) = \hat{\sigma}_\phi(w_4 w_8) = y_3 v_8 + v_4 y_7.
\end{aligned}
$$

By Adem relation, we have

$$
\begin{aligned}
&y_3^2 = Sq^3 y_3 = Sq^1 Sq^2 y_3 = Sq^1 y_5 = y_6\\
&y_5^2 = Sq^5 y_5 = Sq^1 Sq^4 y_5 = Sq^1(v_4 y_5 + y_3 v_6) = v_4 y_6 + y_3 v_7 = v_4 y_3^2 + y_3 v_7\\
&y_7^2 = Sq^7 y_7 = Sq^1(Sq^5 Sq^1 + Sq^2 Sq^4) y_7 = Sq^1(y_5 v_8 + v_6 y_7) = y_6 v_8 + v_7 y_7\\
&y_3^4 = y_6^2 = Sq^6 y_6 = (Sq^2 Sq^4 + Sq^5 Sq^1) y_6 = Sq^2(v_4 y_6 + y_3 v_7) = v_6 y_3^2 + y_5 v_7.
\end{aligned}
$$

□

Proposition 2.2. *For $\phi = \psi^q$, the Adams operation of degree an odd prime power q, $H^*(\mathbb{L}_\phi BSpin(7); \mathbb{Z}/2)$ is isomorphic to $H^*(LBSpin(7); \mathbb{Z}/2)$ as algebras over $\mathcal{A}_2$.*

Proof. By Proposition 1.1, we only have to construct a section r of the map $\iota^* : H^*(\mathbb{T}_\phi BSpin(7); \mathbb{Z}/2) \to H^*(BSpin(7); \mathbb{Z}/2)$ which commutes with the Steenrod operation.

As mentioned in the first section, the Wang sequence (1.1) splits to the short exact sequence

$$0 \to H^{*-1}(BSpin(7); R) \xrightarrow{\delta} H^*(\mathbb{T}_\phi BSpin(7); R) \xrightarrow{\iota^*} H^*(BSpin(7); R) \to 0,$$

when coefficient R have the property $H^*(\phi; R) = 1$. Let $u_4 \in H^4(\mathbb{T}_\phi BSpin(7); \mathbb{Z}/2) \simeq \mathbb{Z}/2$ be the generator. Then we define $u_6 = Sq^2 u_4$, $u_7 = Sq^1 u_6$. By the Wang sequence for $R = \mathbb{Z}/4$ and the Bockstein spectral sequence, we have that $\ker(Sq^1) \subset (\iota^*)^{-1}(w_8) \subset H^8(\mathbb{T}_\phi BSpin(7); \mathbb{Z}/2)$ is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. We take u_8 to be a generator of $\ker(Sq^1) \subset (\iota^*)^{-1}(w_8)$ such that $\ker(Sq^1)$ is generated by u_8 and $\delta(w_7)$. Then $Sq^2 u_8 = 0$ since $H^9(BSpin(7)) = 0$ and $Sq^2 w_8 = 0$. Moreover we have $Sq^4 u_8 = u_4 u_8 + \epsilon\delta(w_4 w_7)$, where $\epsilon = 0$ or 1. Since $\delta(w_4 w_7) = Sq^4 \delta(w_7)$, we can assume $\epsilon = 0$.

Take r to be the ring homomorphism $r(w_i) = u_i$ $(i = 4, 6, 7, 8)$, then r is a section of ι^* which commutes with the Steenrod operations. □

Proposition 2.3. *$H^*(LBSpin(8); \mathbb{Z}/2) = \mathbb{Z}/2[v_4, v_6, v_7, v_8, f_8, y_3, y_5, y_7, z_7]/I(|v_i| = i, |y_i| = i, |v_8| = 8, |z_7| = 7)$, where I is the ideal generated* *by* $\{y_3^4+v_6y_3^2+y_5v_7, y_5^2+y_3v_7+v_4y_3^2, y_7^2+y_3^2v_8+w_7y_7, z_7^2+y_3^2f_8+w_7z_7\}$. *The action of* $\mathcal{A}_2$ *is determined by*:

	v_4	v_6	v_7	v_8	f_8	y_3	y_5	y_7	z_7
Sq^1	0	v_7	0	0	0	0	y_3^2	0	0
Sq^2	v_6	0	0	0	0	y_5	0	0	0
Sq^4	v_4^2	v_4v_6	v_4v_7	v_4v_8	v_4f_8	0	$y_3v_6+v_4y_5$	$y_3v_8+v_4y_7$	$y_3f_8+v_4z_7$

Proof. Completely parallel to the case of $Spin(7)$ since the generator $e_8 \in H^8(BSpin(8);\mathbb{Z}/2)$ is looks same as w_8. □

Proposition 2.4. *For* $\phi=\psi^q$*, the Adams operation of degree an odd prime power* q*,* $H^*(\mathbb{L}_\phi BSpin(8);\mathbb{Z}/2)$ *is isomorphic to* $H^*(LBSpin(8);\mathbb{Z}/2)$ *as algebras over* $\mathcal{A}_2$.

Proof. We can construct a section r completely same as in the case of $BSpin(7)$. □

Proposition 2.5. $H^*(LBSpin(9);\mathbb{Z}/2) = \mathbb{Z}/2[v_4,v_6,v_7,v_8,f_{16},y_3,y_5,y_7,z_{15}]/I(|v_i|=i, |y_i|=i, |f_{16}|=16, |z_{16}|=16)$*, where* I *is the ideal generated by*

$$\{y_3^4+v_6y_3^2+y_5v_7, y_5^2+y_3v_7+v_4y_3^2, y_7^2+y_3^2w_8+w_7y_7,$$
$$z_{15}^2+v_7v_8z_{15}+w_7y_7f_{16}+y_3^2v_8f_{16}\}.$$

The action of $\mathcal{A}_2$ *is determined by*:

	v_4	v_6	v_7	v_8	f_{16}	y_3	y_5	y_7	z_{15}
Sq^1	0	v_7	0	0	0	0	y_3^2	0	0
Sq^2	v_6	0	0	0	0	y_5	0	0	0
Sq^4	v_4^2	v_4v_6	v_4v_7	v_4v_8	0	0	$y_3v_6+v_4y_5$	$y_3v_8+v_4y_7$	0
Sq^8	0	0	0	v_8^2	$v_8f_{16}+v_4^2f_{16}$	0	0	0	J_1

where $J_1 = y_7f_{16}+v_8z_{15}+v_4^2z_{15}$.

Proof. In dimension lower than 9, calculation is completely same as in the case of $BSpin(7)$. We have only to calculate the following:

$$Sq^8z_{15} = \hat{\sigma}_\phi(Sq^8f_{16}) = \hat{\sigma}_\phi(v_8f_{16}+v_4^2f_{16}) = y_7f_{16}+v_8z_{15}+v_4^2z_{15}.$$
$$z_{15}^2 = Sq^{15}z_{15} = \hat{\sigma}_\phi Sq^{15}f_{16} = \hat{\sigma}_\phi(v_7v_8f_{16}) = v_7v_8z_{15}+w_7y_7f_{16}+y_3^2v_8f_{16}.$$

□

Proposition 2.6. *For* $\phi=\psi^q$*, the Adams operation of degree an odd prime power* q*,* $H^*(\mathbb{L}_\phi BSpin(9);\mathbb{Z}/2)$ *is isomorphic to* $H^*(LBSpin(9);\mathbb{Z}/2)$ *as algebras over* $\mathcal{A}_2$.

Proof. In dimension lower than 9, we can construct a section r completely same as in the case of $BSpin(7)$.

Choose an element $f'_{16} \in \ker(Sq^1) \subset (\iota^*)^{-1}(e_{16})$. Then by the Wang sequence, we have $Sq^2 f'_{16} = \epsilon_1\delta(w_4w_6w_7)$ since $H^{17}(BSpin(9);\mathbb{Z}/2) \simeq \mathbb{Z}/2$ is generated by $w_4w_6w_7$. Then $Sq^2Sq^2 f'_{16} = \epsilon_1\delta(Sq^2(w_4w_6w_7)) = \epsilon_1\delta(w_6^2w_7)$. By Adem relation this must be 0 since $Sq^2Sq^2 = Sq^3Sq^1$ and $Sq^1 f'_{16} = 0$. Therefore we have $\epsilon_1 = 0$.

Similarly we have $Sq^4 f'_{16} = \epsilon_2\delta(w_4^3w_7) + \epsilon_3\delta(w_6^2w_7) + \epsilon_4\delta(w_4w_7w_8)$. Then we have $Sq^4Sq^4 f'_{16} = \epsilon_2\delta(w_4w_6^2w_7) + \epsilon_3\delta(w_4w_6^2w_7) + \epsilon_4(w_4^2w_7w_8)$. By Adem relation we have $Sq^4Sq^4 f_{16} = (Sq^7Sq^1 + Sq^6Sq^2)f_{16} = 0$. Therefore we have $\epsilon_2 = \epsilon_3, \epsilon_4 = 0$. Put $f_{16} = f'_{16} - \epsilon_2 w_4^2 w_7$, then we have $Sq^i f_{16} = 0$ $(i = 1, 2, 4)$ since $Sq^4 w_4^2 w_7 = w_4^3 w_7 + w_6^2 w_7$.

Similarly we have $Sq^8 f_{16} = w_8 f_{16} + w_4^2 f_{16} + \epsilon_5\delta(w_4^4w_7) + \epsilon_6\delta(w_4^2w_7w_8) + \epsilon_7\delta(w_4w_6^2w_7) + \epsilon_8\delta(w_7w_8^2) + \epsilon_9\delta(w_7f_{16})$. By Adem relation $Sq^8Sq^8 f_{16} = 0$ and we have $\epsilon_5 = \epsilon_7 = \epsilon_9 = 0, \epsilon_6 = \epsilon_8$. Replacing f_{16} by $f_{16} - \epsilon_7\delta(w_7w_8)$ we have $Sq^8 f_{16} = w_8 f_{16} + w_4^2 f_{16}$ and $Sq^i f_{16} = 0$ $(i < 8)$. □

3. The case $G = F_4$

Denote the classifying map of the canonical inclusion $Spin(9) \hookrightarrow F_4$ by i. Then by [K], $H^*(F_4;\mathbb{Z}/2) = \mathbb{Z}[x_4, x_6, x_7, x_{16}, x_{24}]$ where $i^*(x_4) = w_4, i^*(x_6) = w_6, i^*(x_7) = w_7, i^*(x_{16}) = e_{16} + w_8^2, i^*(x_{24}) = w_8e_{16}$. Then the action of $\mathcal{A}_2$ is determined by:

	x_4	x_6	x_7	x_{16}	x_{24}
Sq^1	0	x_7	0	0	0
Sq^2	x_6	0	0	0	0
Sq^4	x_4^2	x_4x_6	x_4x_7	0	x_4x_{24}
Sq^8	0	0	0	$x_{24} + x_4^2x_{16}$	$x_4^2x_{24}$
Sq^{16}	0	0	0	x_{16}^2	$x_{16}x_{24} + x_4x_6^2x_{24}$.

Proposition 3.1. *$H^*(LBF_4;\mathbb{Z}/2) = \mathbb{Z}/2[v_4, v_6, v_7, v_{16}, v_{24}, y_3, y_5, y_{15}, y_{23}]/I(|v_i| = i, |y_i| = i)$, where I is the ideal generated by*

$$\{y_3^4 + v_6y_3^2 + y_5v_7, y_5^2 + y_3v_7 + v_4y_3^2, y_{15}^2 + v_7y_{23} + v_{24}y_3^2, y_{23}^2 + y_3^2v_{16}v_{24} + v_7v_{24}y_{15} + v_7v_{16}y_{23}\}.$$

The action of $\mathcal{A}_2$ is determined by:

	v_4	v_6	v_7	v_{16}	v_{24}
Sq^1	0	v_7	0	0	0
Sq^2	v_6	0	0	0	0
Sq^4	v_4^2	v_4v_6	v_4v_7	0	v_4v_{24}
Sq^8	0	0	0	$v_{24} + v_4^2v_{16}$	$v_4^2v_{24}$
Sq^{16}	0	0	0	v_{16}^2	$v_{16}v_{24} + v_4v_6^2v_{24}$

	y_3	y_5	y_{15}	y_{23}
Sq^1	0	y_3^2	0	0
Sq^2	y_5	0	0	0
Sq^4	0	$y_3v_6 + v_4y_5$	0	$y_3v_{24} + v_4y_{23}$
Sq^8	0	0	$y_{23} + v_4^2y_{15}$	$v_4^2y_{23}$
Sq^{16}	0	0	0	J_2

where $J_2 = v_{24}y_{15} + v_{16}y_{23} + y_3v_6^2v_{24} + v_4v_6^2y_{23}$.

Proof. In dimension lower than 9, calculation is completely same as in the case of $BSpin(9)$.

$$
\begin{aligned}
Sq^1y_{15} &= \hat{\sigma}_\phi(Sq^1v_{16}) = 0\\
Sq^2y_{15} &= \hat{\sigma}_\phi(Sq^2v_{16}) = 0\\
Sq^4y_{15} &= \hat{\sigma}_\phi(Sq^4v_{16}) = 0\\
Sq^8y_{15} &= \hat{\sigma}_\phi(Sq^8v_{16}) = \hat{\sigma}_\phi(v_24 + v_4^2v_{16}) = y_{23} + v_4^2y_{15}\\
Sq^1y_{23} &= \hat{\sigma}_\phi(Sq^1v_{24}) = 0\\
Sq^2y_{23} &= \hat{\sigma}_\phi(Sq^2v_{24}) = 0\\
Sq^4y_{23} &= \hat{\sigma}_\phi(Sq^4v_{24}) = \hat{\sigma}_\phi(v_4v_{24}) = y_3v_{24} + v_4y_{23}\\
Sq^8y_{23} &= \hat{\sigma}_\phi(Sq^8v_{24}) = \hat{\sigma}_\phi(v_4^2v_{24}) = v_4^2y_{23}\\
Sq^{16}y_{23} &= \hat{\sigma}_\phi(Sq^{16}v_{24}) = \hat{\sigma}_\phi(v_{16}v_{24} + v_4v_6^2v_{24})\\
&= y_{15}v_{24} + v_{16}y_{23} + y_3v_6^2v_{24} + v_4v_6^2y_{23}\\
y_{15}^2 &= Sq^{15}y_{15} = \hat{\sigma}_\phi(Sq^{15}v_{16}) = \hat{\sigma}_\phi(v_7v_{24}) = v_7y_{23} + y_3^2v_{24}\\
y_{23}^2 &= Sq^{23}y_{23} = \hat{\sigma}_\phi(Sq^{23}v_{24}) = \hat{\sigma}_\phi(v_7v_{16}v_{24})\\
&= y_3^2v_{16}v_{24} + v_7v_{24}y_{15} + v_7v_{16}y_{23}.
\end{aligned}
$$

□

Proposition 3.2. *For $\phi = \psi^q$, the Adams operation of degree an odd prime power q, $H^*(\mathbb{L}_\phi BF_4; \mathbb{Z}/2)$ is isomorphic to $H^*(LBF_4; \mathbb{Z}/2)$ as algebras over $\mathcal{A}_2$.*

Proof. By [JMO] the following diagram is homotopy commutative.

$$
\begin{array}{ccc}
BSpin(9)^\wedge_2 & \xrightarrow{\psi^q} & BSpin(9)^\wedge_2 \\
\downarrow{\scriptstyle Bi} & & \downarrow{\scriptstyle Bi} \\
(BF_4)^\wedge_2 & \xrightarrow{\psi^q} & (BF_4)^\wedge_2
\end{array}
$$

By the naturality of the construction of the twisted tube, there is a map $\mathbb{T}_\phi BSpin(9) \to \mathbb{T}_\phi BF_4$ and we have the Proposition. □

4. The case $G = DI(4)$

In [DW] they constructed a space called $BDI(4)$ with the cohomology isomorphic to the mod 2 Dickson invariant of rank 4, that is, $H^*(BDI(4); \mathbb{Z}/2) = \mathbb{Z}/2[x_8, x_{12}, x_{14}, x_{15}]$, where $|x_j| = j$. The action of $\mathcal{A}_2$ is determined by:

	x_8	x_{12}	x_{14}	x_{15}
Sq^1	0	0	x_{15}	0
Sq^2	0	x_{14}	0	0
Sq^4	x_{12}	0	0	0
Sq^8	x_8^2	x_8x_{12}	x_8x_{14}	x_8x_{15}

Notbohm [N] shows there is a self homotopy equivalence ψ^q of $BDI(4)$ for odd prime power q called the Adams operation of degree q with the property $H^{2r}(\psi^q;\mathbb{Q}_p)$ is multiplication by q^r. Using this, Benson [B] defined an exotic 2-compact group $BSol(q)$ as $L_{\psi^q}BDI(4)$ which can be called "the classifying space" of Solomon's non-existent finite group [S].

Recently Grbic [G] calculated the mod 2 cohomology of $BSol(q)$ over $\mathcal{A}_2$ by using Eilenberg-Moore spectral sequence. Here we calculate it by the method of Kishimoto and Kono.

Proposition 4.1. *$H^*(LBDI(4);\mathbb{Z}/2) = \mathbb{Z}/2[v_8, v_{12}, v_{14}, v_{15}, y_7, y_{11}, y_{13}]/I(|v_i| = i, |y_i| = i)$, where I is the ideal generated by*

$$\{y_7^4 + y_{13}v_{15} + v_{14}y_7^2, y_{11}^2 + v_7y_{15} + v_8y_7^2, y_{13}^2 + y_{11}v_{15} + v_{12}y_7^2\}.$$

The action of $\mathcal{A}_2$ is determined by:

	v_8	v_{12}	v_{14}	v_{15}	y_7	y_{11}	y_{13}
Sq^1	0	0	v_{15}	0	0	0	y_7^2
Sq^2	0	v_{14}	0	0	0	y_{13}	0
Sq^4	v_{12}	0	0	0	y_{11}	0	0
Sq^8	v_8^2	v_8v_{12}	v_8v_{14}	v_8v_{15}	0	$y_{11}v_8 + v_{12}y_7$	$y_{13}v_8 + v_{14}y_7$

Remark 2. Kuribayashi has also this result in [Ku].

Proof.

$$\begin{aligned}
y_7^2 &= Sq^7y_7 = \hat{\sigma}_\phi(Sq^7v_8) = \hat{\sigma}_\phi(v_{15}) = y_{14}\\
Sq^1y_i &= \hat{\sigma}_\phi(Sq^1v_{i+1}) = 0\ (i = 7, 11)\\
Sq^1y_{13} &= \hat{\sigma}_\phi(Sq^1v_{14}) = \hat{\sigma}_\phi(v_{15}) = y_{14} = y_7^2\\
Sq^2y_i &= \hat{\sigma}_\phi(Sq^2v_{i+1}) = 0\ (i = 7, 13)\\
Sq^2y_{11} &= \hat{\sigma}_\phi(Sq^2v_{12}) = \hat{\sigma}_\phi(v_{14}) = y_{13}\\
Sq^4y_i &= \hat{\sigma}_\phi(Sq^4v_{i+1}) = 0\ (i = 11, 13)\\
Sq^4y_7 &= \hat{\sigma}_\phi(Sq^4v_8) = \hat{\sigma}_\phi(v_{12}) = y_{11}\\
Sq^8y_7 &= \hat{\sigma}_\phi(Sq^8v_8) = 0\\
Sq^8y_{11} &= \hat{\sigma}_\phi(Sq^8v_{12}) = \hat{\sigma}_\phi(v_8v_{12}) = y_7v_{12} + v_8y_{11}\\
Sq^8y_{13} &= \hat{\sigma}_\phi(Sq^8v_{14}) = \hat{\sigma}_\phi(v_8v_{14}) = y_7v_{14} + v_8y_{13}\\
y_{11}^2 &= Sq^{11}y_{11} = \hat{\sigma}_\phi(Sq^{11}v_{12}) = \hat{\sigma}_\phi(Sq^1Sq^2Sq^8v_{12}) = \hat{\sigma}_\phi(v_8v_{15})\\
&= v_8y_7^2 + y^7v_{15}\\
y_{13}^2 &= Sq^{13}y_{13} = \hat{\sigma}_\phi(Sq^{13}v_{14}) = \hat{\sigma}_\phi((Sq^5Sq^8 + Sq^{11}Sq^2)v_{14})\\
&= \hat{\sigma}_\phi(Sq^5v_8v_{14}) = \hat{\sigma}_\phi(v_{12}v_{15}) = y_{11}v_{15} + v_{12}y_7^2\\
y_7^4 &= y_{14}^2 = Sq^{14}y_{14} = \hat{\sigma}_\phi(Sq^{14}v_{15}) = \hat{\sigma}_\phi(v_{14}v_{15}) = y_{13}v_{15} + v_{14}y_7^2.
\end{aligned}$$

□

Now we proceed to show that mod 2 cohomology of $BSol(q)$ over $\mathcal{A}_2$ is isomorphic to that of $LBDI(4)$.

Proposition 4.2. *For $\phi = \psi^q$, the Adams operation of degree an odd prime power q, $H^*(\mathbb{L}_\phi BDI(4);\mathbb{Z}/2)$ is isomorphic to $H^*(LBDI(4);\mathbb{Z}/2)$ as algebras over $\mathcal{A}_2$.*

Proof. Choose an element $v_8 \in \ker(Sq^1)\cap\ker(Sq^1Sq^4) \subset (\iota^*)^{-1}(x_8)$. Put $v_{12} = Sq^4v_8, v_{14} = Sq^2v_{12}$ and $v_{15} = Sq^1v_{14}$. Then we have $Sq^1v_i = 0$ $(i = 8, 12, 15)$. By dimensional reason

$$Sq^2v_8 = 0, Sq^4v_8 = v_{12}.$$

$Sq^4v_{12} = Sq^4Sq^4v_8 = 0$. $Sq^8v_{12} = v_8v_{12}$ since $H^{19}(BDI(4);\mathbb{Z}/2) = 0$ in the Wang sequence. Other operations are calculated as follows.

$$\begin{aligned}
&Sq^2v_{14} = Sq^2Sq^2v_{12} = 0\\
&Sq^4v_{14} = Sq^4Sq^6v_8 = Sq^2Sq^8v_8 = 0\\
&Sq^8v_{14} = Sq^8Sq^2v_{12} = (Sq^4Sq^6 + Sq^2Sq^8)v_{12} = v_8v_{14}\\
&Sq^2v_{15} = Sq^2Sq^7v_8 = Sq^9v_8 = 0\\
&Sq^4v_{15} = Sq^4Sq^7v_8 = Sq^{11}v_8 = 0\\
&Sq^8v_{15} = Sq^8Sq^1v_{14} = (Sq^9 + Sq^2Sq^7)v_{14} = Sq^1Sq^8v_{14} = v_8v_{15}
\end{aligned}$$

Hence we can construct the section r by $x_i \to v_i$. □

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