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Functional limit theorems for occupation times of Lamperti's stochastic processes in discrete time

By

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Abstract

Two functional limit theorems for occupation times of Lamperti's stochastic processes are established. One is a generalization of Lamperti's result in 1957 in the null recurrent case, and the other is a limit theorem for the fluctuation in the positively recurrent case. The proofs are based on a limit theorem for i.i.d. random variables with common distribution function belonging to the domain of attraction of a stable law.

1. Introduction

The law of the occupation time on the positive side of a one-dimensional Brownian motion is well-known as P. Lévy's arc-sine law. He also showed that the law of the fraction of the occupation time of a one-dimensional simple random walk converges to the arc-sine law. This result was extended by J. Lamperti ([7], 1957). He studied the limit theorem of the law of the occupation time for a certain class of discrete time processes (we call them *Lamperti processes* here) which have the Markov property only at a special state. The possible limit laws, called Lamperti laws, and their variants appear in various literatures even recently, e.g., Barlow–Pitman–Yor ([1], 1989), S. Watanabe ([9], 1995) and Bertoin–Fujita–Roynette–Yor ([2], to appear).

S. Watanabe ([9], 1995) studied the similar problem for one-dimensional diffusion processes and proved that the class of possible limit laws coincides with that of Lamperti laws. Recently Kasahara–Watanabe ([6], 2006) studied the limit theorem for the fluctuation of the occupation time in the positively recurrent case, where the average of the occupation time has a degenerate limit in long time.

In the present paper we study limit theorems of the occupation time for a Lamperti process and generalize these results. In the null recurrent case we establish a functional limit theorem of Lamperti's result. In the positively recurrent case we establish a limit theorem for the fluctuation.

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One of the major contributions of the present paper is to extend Lamperti's limit theorem to a functional limit theorem (Theorem 2.2). While the proof of Lamperti's theorem was carried out analytically, we need a probabilistic method, a limit theorem of the partial sum process of i.i.d. random variables (see, e.g., [4] and [5]). In fact, the excursion intervals of a Lamperti process away from 0 are i.i.d. random variables. The key to the problem is the Williams formula (Proposition 2.4). The other is to prove a limit theorem for the fluctuation (Theorem 3.1) which is analogous to Kasahara–Watanabe's result (Theorem 4.1 in [6]). While Kasahara–Watanabe used several particular properties of one-dimensional diffusion processes, our proof is based on a limit theorem for i.i.d. random variables with common distribution function belonging to the domain of attraction of a stable law with index $1 < \alpha < 2$. The key tool is a function Φ which we shall introduce in (3.7).

The present paper is organized as follows. In Section 2, we shall review Lamperti's result and state our functional limit theorem in the null recurrent case with its proof. In Section 3, we will establish a limit theorem for the fluctuation in the positively recurrent case. We will introduce a function Φ and some lemmas and prove our theorem.

2. Lamperti's result and a functional limit theorem

We shall define a Lamperti process $X = \{X_n\}_{n\in\mathbb{Z}_+}$ on S as follows.

Definition 2.1. A *Lamperti process* is a discrete time process $X = \{X_n\}_{n\in\mathbb{Z}_+}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ whose state space S is divided into two sets, S_+ and S_-, and one special state 0, and satisfies the following properties.

(i) It starts at 0, i.e., $P(X_0 = 0) = 1$.

(ii) If $X_{n-1} \in S_+$ and $X_{n+1} \in S_-$ or if $X_{n-1} \in S_-$ and $X_{n+1} \in S_+$, then $X_n = 0$.

(iii) $P(\tau < \infty) = 1$ where τ is the first hitting time at 0 of X.

(iv) It starts afresh when it recurs the state 0, i.e.,

$$P((X_{\tau+k} : k > 0) \in \cdot|\mathcal{F}_\tau) = P((X_k : k > 0) \in \cdot) \quad \text{for} \quad k > 0$$

where $\mathcal{F}_n = \sigma(\{X_k : 0 \le k \le n\})$.

Let $A_+(n)$ denote the occupation time up to time n of the set S_+ of X. Occupation of the state 0 is counted or not according to whether the last other state occupied was in S_+.

Lamperti [7] determined the class of possible limit random variables in law of $A_+(n)/n$ as $n \to \infty$. Let $p(n)$ stand for the probability that the recurrence time of state 0 is just n ($\sum_{n=1}^{\infty} p(n) = 1$), i.e.,

$$p(n) = P(X_n = 0, X_i \neq 0, 0 < i < n).$$

Set $g(x) = \sum_{n=1}^{\infty} p(n)x^n$ be its generating function. Then Lamperti's result [7] is stated as follows (with a slight change of notation).

Theorem A ([7]).

$$\lim_{n\to\infty} P(A_+(n)/n \le x) = F(x) \tag{2.1}$$

exists if and only if

$$\lim_{n\to\infty} E[A_+(n)/n] = p, \quad 0 \le p \le 1 \tag{2.2}$$

exists and also

$$\lim_{x\to 1-} (1-x)g'(x)/(1-g(x)) = \alpha, \quad 0 \le \alpha \le 1 \tag{2.3}$$

exists.

Let us consider the case where the latter statements hold with $0 < p < 1$ and $0 < \alpha < 1$. It is known (see, for example, [1] and [9]) that the limit distribution $F(x) = F^{(\alpha,p)}(x)$ equals the distribution of the fraction of the occupation time $A_+^{(\alpha,p)}(t)$ on the positive side up to time t of a skew Bessel diffusion process of dimension $2-2\alpha$ with skew parameter p. It is also known by the Willams formula (see, for example, [9]) asserts that

$$(A_+^{(\alpha,p)})^{-1}(t) \overset{d}{=} t + \eta_-(\eta_+^{-1}(t)) \tag{2.4}$$

with $\eta_+(t) = S_+^{(\alpha)}(pt)$ and $\eta_-(t) = S_-^{(\alpha)}((1-p)t)$ where $S_+^{(\alpha)}$ is an α-stable process with Lévy measure $\alpha x^{-\alpha-1}dx$ and $S_-^{(\alpha)}$ is an independent copy of $S_+^{(\alpha)}$. Thus we rewrite (2.1) as

$$\frac{1}{n}A_+(n) \xrightarrow{d} A_+^{(\alpha,p)}(1) \overset{d}{=} \frac{1}{t}A_+^{(\alpha,p)}(t) \quad \text{as} \quad n\to\infty$$

for every $t > 0$ where $\xrightarrow{d}$ denotes the convergence in distribution.

We extend the occupation time process $A_+(n)$, $n = 0, 1, \ldots$ of a Lamperti process X to a process $[0,\infty) \ni t \mapsto A_+(t)$ by the linear interpolation, that is, for $n \le t < n+1$, $n \in \mathbb{Z}_+$,

$$A_+(t) = A_+(n) + (t-n)\{A_+(n+1) - A_+(n)\}.$$

Our purpose of this section is to prove the following functional limit theorem.

Theorem 2.1. *If Lamperti's limit distribution exists for $0 < \alpha < 1$, that is,*

$$\frac{1}{n}A_+(n) \xrightarrow{d} A_+^{(\alpha,p)}(1) \quad \textit{as} \quad n\to\infty, \tag{2.5}$$

then we have

$$\frac{1}{\lambda}A_+(\lambda t) \xrightarrow{\mathcal{L}} A_+^{(\alpha,p)}(t) \quad \textit{in} \quad C([0,\infty):[0,\infty)) \quad \textit{as} \quad \lambda\to\infty \tag{2.6}$$

where $\xrightarrow{\mathcal{L}}$ denotes the convergence in law over the function space.

Before proving Theorem 2.1, we first rewrite the conditions (2.2) and (2.3) following Lamperti [7]. Let $q_{\pm} = P(X_1 \in S_{\pm}) = P(X_{n+1} \in S_{\pm} | X_n = 0)$. Following Lamperti [7], we may suppose that the state 0 does not repeat itself, i.e., $q_+ + q_- = 1$. Put

$$p_{\pm}(n) = P(X_n = 0, X_i \neq 0, 0 < i < n | X_1 \in S_{\pm}),$$

$$g_{\pm}(x) = \sum_{n=1}^{\infty} p_{\pm}(n) x^n.$$

Throughout this paper, $f(x) \sim g(x)$ as $x \to a$ means that $\lim_{x \to a} f(x)/g(x) = 1$.

Theorem B ([7]).
(*i*) *The condition* (2.3) *for* $0 < \alpha < 1$ *is equivalent to*

$$1 - g(x) \sim (1-x)^{\alpha} L\left(\frac{1}{1-x}\right) \quad \textit{as} \quad x \to 1-. \tag{2.7}$$

(*ii*) *The conditions* (2.2) *and* (2.3) *for* $0 < p < 1$ *and* $0 < \alpha < 1$ *are equivalent to the conditions*

$$1 - g_{\pm}(x) \sim (1-x)^{\alpha} L_{\pm}\left(\frac{1}{1-x}\right) \quad \textit{as} \quad x \to 1- \tag{2.8}$$

where $L_{\pm}(x)$ *are slowly varying functions at* ∞ *such that*

$$\lim_{x \to \infty} \frac{L_+(x)}{L_-(x)} = \frac{q_-}{q_+} \cdot \frac{p}{1-p}. \tag{2.9}$$

(*iii*) *The conditions* (2.8) *are equivalent to the conditions*

$$\sum_{k=n+1}^{\infty} p_{\pm}(k) \sim \frac{1}{\Gamma(1-\alpha)} n^{-\alpha} L_{\pm}(n) \quad \textit{as} \quad n \to \infty. \tag{2.10}$$

Second, we rewrite the conditions (2.10) in terms of excursion intervals. Let τ_m be the m-th hitting time at 0 of X, i.e., $\tau_m = \inf\{n > \tau_{m-1} : X_n = 0\}$ ($\tau_0 = 0$), and let $\xi(m) = \tau_m - \tau_{m-1} (> 0)$. Then $\{\xi(m)\}_{m \in \mathbb{N}}$ is a sequence of nonnegative i.i.d. random variables by the property (ii) of a Lamperti process. For $m = 1, 2, \ldots$, put

$$e_m(n) = \begin{cases} X_{n+\tau_{m-1}} & \text{for} \quad 0 \le n \le \xi(m), \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\xi_+(m) = \begin{cases} \xi(m) & \text{if} \quad e_m(1) \in S_+, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\xi_-(m) = \begin{cases} \xi(m) & \text{if} \quad e_m(1) \in S_-, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{(\xi_+(m), \xi_-(m))\}_{m \in \mathbb{N}}$ is a sequence of i.i.d. random variables.

Lemma 2.1. *The conditions* (2.10) *are equivalent to the conditions*

$$P(\xi_+(1) > n) \sim \frac{p}{\varphi(n)} \quad \text{and} \quad P(\xi_-(1) > n) \sim \frac{1-p}{\varphi(n)} \tag{2.11}$$

as $n \to \infty$ where $\varphi(x)$ is a regularly varying function with index α, $0 < \alpha < 1$.

Proof. Set $\varphi(n) = p\Gamma(1-\alpha)n^\alpha/\{q_+ L_+(n)\}$. Then $\varphi(n) \sim (1-p)\Gamma(1-\alpha)n^\alpha/\{q_- L_-(n)\}$ as $n \to \infty$. Hence, by the equalities

$$P(\xi_\pm(1) > n) = q_\pm \cdot \sum_{k=n+1}^{\infty} p_\pm(k),$$

we obtain the desired equivalence. □

Third, we prepare a functional limit theorem for the partial sum process of excursion intervals. For $t \geq 0$, put

$$T_\pm(t) = \sum_{k=1}^{[t]} \xi_\pm(k) \quad \text{and} \quad T(t) = T_+(t) + T_-(t).$$

Lemma 2.2. *Under the condition* (2.11), *it holds that*

$$\left(\frac{1}{\lambda}T_+(\varphi(\lambda)t), \frac{1}{\lambda}T_-(\varphi(\lambda)t)\right) \xrightarrow{\mathcal{L}} (S_+^{(\alpha)}(pt), S_-^{(\alpha)}((1-p)t)) \quad \text{in} \quad D([0,\infty) : [0,\infty)^2) \quad \text{as} \quad \lambda \to \infty \tag{2.12}$$

where $D([0,\infty) : [0,\infty)^2)$ is the space of càdlàg functions with Skorokhod's J_1-topology.

For Skorokhod's J_1-topology, see, e.g., [3] and [8].

Proof. We first note that $(T_+(t), T_-(t))_{t\geq 0}$ is the partial sum process of an i.i.d. random vectors $\{(\xi_+(k), \xi_-(k))\}_{k\in\mathbb{N}}$ and that $(S_+^{(\alpha)}(pt), S_-^{(\alpha)}((1-p)t))_{t\geq 0}$ is an $\mathbb{R}^2$-valued Lévy process with Lévy measure

$$\nu(dx, dy) = \alpha\{px^{-\alpha-1}dx\delta_0(dy) + (1-p)y^{-\alpha-1}dy\delta_0(dx)\}.$$

Therefore it suffices to show that

$$\varphi(\lambda)P((\xi_+(1), \xi_-(1)) \in (\lambda dx, \lambda dy)) \longrightarrow \nu(dx, dy) \tag{2.13}$$

vaguely on $[-\infty,\infty] \times [-\infty,\infty] \setminus \{(0,0)\}$. Since all measures in (2.13) are concerned on the set $\{(x,y)|xy = 0\}$, (2.13) can be rewritten as

$$\varphi(\lambda)P(\xi_+(1) > \lambda x) \longrightarrow px^{-\alpha}$$

and

$$\varphi(\lambda)P(\xi_-(1) > \lambda y) \longrightarrow (1-p)y^{-\alpha}$$

as $\lambda \to \infty$ for every $x, y > 0$. However, this is an immediate consequence of Lemma 2.1. $\square$

Fourth, we need a discrete version of the Williams formula (2.4). Let $A_+^{-1}(t)$ be the right-continuous inverse of $A_+(t)$, that is,

$$A_+^{-1}(t) = \inf\{s : A_+(s) > t\}, \quad 0 \le t < \infty$$

with the obvious convention that $\inf \phi = \infty$.

Proposition 2.1. *For a general Lamperti process X, it holds that*

$$A_+^{-1}(t) = t + T_-(T_+^{-1}(t)), \quad 0 \le t < \infty \tag{2.14}$$

where T_+^{-1} is the right-continuous inverse of $t \mapsto T_+(t)$.

Proof. Put $k = T_+^{-1}(t)$. Then $T_+(k-1) \le t < T_+(k)$. Now we suppose that $\xi(k) = \xi_+(k)$ and put $k_0 = \sup\{m < k : \xi(m) = \xi_+(m)\}$. Then $\xi(k_0+1) = \xi_-(k_0+1), \xi(k_0+2) = \xi_-(k_0+2), \dots, \xi(k-1) = \xi_-(k-1)$ and we have $T_+(k_0) \le t < T_+(k)$. Hence we obtain $A_+^{-1}(t) = T(k-1) + (t - T_+(k_0)) = \{T_+(k-1) + T_-(k-1)\} + (t - T_+(k_0)) = T_-(k-1) + t = T_-(k) + t$. In the case that $\xi(k) = \xi_-(k)$, the proof is similar, so we omit the detail. $\square$

Let D_0 be the space of càdlàg functions f that are non-decreasing and satisfy $f(0) = 0$ and $\lim_{t\to\infty} f(t) = \infty$. Let Λ denote the class of strictly increasing, continuous functions λ on $[0, \infty)$ with $\lambda(0) = 0$ and $\lim_{x\to\infty} \lambda(x) = \infty$.

Lemma 2.3. *Let $x_n, x \in D_0([0,\infty) : [0,\infty))$ and let $y_n, y \in D([0,\infty) : [0,\infty))$ be non-decreasing functions. If $x(t)$ is strictly increasing in t and if*

$$(x_n, y_n) \longrightarrow (x, y) \quad \text{in} \quad D([0,\infty) : [0,\infty)^2), \tag{2.15}$$

then it holds that

$$y_n(x_n^{-1}(t)) \longrightarrow y(x^{-1}(t)) \tag{2.16}$$

at every t for which $x^{-1}(t)$ is a continuity point of $y(\cdot)$.

Proof. By the assumption (2.15), there exist functions $\lambda_n \in \Lambda$ such that

$$x_n \circ \lambda_n \longrightarrow x \quad \text{and} \quad y_n \circ \lambda_n \longrightarrow y$$

uniformly on every compact set. Since $(x_n \circ \lambda_n)^{-1}$ are non-decreasing and x^{-1} is continuous, we have

$$(x_n \circ \lambda_n)^{-1} \longrightarrow x^{-1}$$

uniformly on every compact set. Let t be a point for which $x^{-1}(t)$ is a continuity point of $y(\cdot)$ and fixed. Then we have

$$\begin{aligned} y_n(x_n^{-1}(t)) &= y_n \circ \lambda_n((x_n \circ \lambda_n)^{-1}(t)) \\ &= y_n \circ \lambda_n((x_n \circ \lambda_n)^{-1}(t)) - y((x_n \circ \lambda_n)^{-1}(t)) + y((x_n \circ \lambda_n)^{-1}(t)) \\ &\longrightarrow y(x^{-1}(t)), \end{aligned}$$

which completes the proof. □

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Under our assumption, we can apply Theorem B, Lemmas 2.1 and 2.2 to obtain the functional limit (2.12). By Skorokhod's theorem, we can realize the convergence (2.12) by an almost-sure convergence; There exist processes $\hat{T}_\pm$ and $\hat{S}^{(\alpha)}_\pm$ on a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ such that $T_\pm \stackrel{d}{=} \hat{T}_\pm$, $S^{(\alpha)}_\pm \stackrel{d}{=} \hat{S}^{(\alpha)}_\pm$ and

$$\frac{1}{\lambda}\left(\hat{T}_+(\varphi(\lambda)t), \hat{T}_-(\varphi(\lambda)t)\right) \longrightarrow (\hat{S}^{(\alpha)}_+(pt), \hat{S}^{(\alpha)}_-((1-p)t)) =: (\hat{\eta}_+(t), \hat{\eta}_-(t))$$

in $D([0,\infty) : [0,\infty)^2)$ almost surely.

Define two continuous nondecreasing functions $\hat{A}_+(t)$ and $\tilde{A}_+(t)$ by

$$(\hat{A}_+)^{-1}(t) = t + \hat{T}_-(\hat{T}_+^{-1}(t)) \quad \text{and} \quad (\tilde{A}_+)^{-1}(t) = t + \hat{\eta}_-(\hat{\eta}_+^{-1}(t)).$$

Then, by Lemma 2.3, we see that, for almost all paths, $(\hat{A}_+)^{-1}(\lambda t)/\lambda \to (\tilde{A}_+)^{-1}(t)$ at every fixed t for which the value $\hat{\eta}_+^{-1}(t)$ is a continuity point of $\hat{\eta}_-$. Hence we obtain $\hat{A}_+(\lambda t)/\lambda \to \tilde{A}_+(t)$ in the function space $C([0,\infty) : [0,\infty))$ for almost all paths; In fact, the left-hand side is non-decreasing in t and the right-hand side is continuous in t, so that the convergence is uniform on every compact set.

By (2.4) and Proposition 2.1, we see that $\hat{A}_+ \stackrel{d}{=} A_+$ and $\tilde{A}_+ \stackrel{d}{=} A_+^{(\alpha,p)}$. Therefore we obtain (2.6), and hence we conclude the assertion. □

3. Limit theorem for the fluctuation

In this section, we study the extreme case of Lamperti's result, $\alpha = 1$, that is, the limit distribution F degenerates. In this case, it holds that

$$A_+(n)/n \xrightarrow{p} p \quad \text{as} \quad n \to \infty \tag{3.1}$$

for some constant $p \in (0,1)$ where $\xrightarrow{p}$ denotes the convergence in probability. Our aim of this section is to evaluate the fluctuation

$$\frac{1}{\lambda}A_+(\lambda t) - pt \quad \text{as} \quad \lambda \to \infty.$$

Theorem 3.1. *Suppose that*

$$P(\xi_+(1) > n) \sim \frac{c_+}{\varphi(n)} \quad \textit{and} \quad P(\xi_-(1) > n) \sim \frac{c_-}{\varphi(n)} \quad \textit{as} \quad n \to \infty \tag{3.2}$$

where $\varphi(n)$ is a regularly varying function at ∞ with index $1 < \alpha < 2$. Then it holds that

$$\frac{1}{\varphi^{-1}(\lambda)}(A_+(\lambda t) - p\lambda t) \xrightarrow{f.d.} (1-p)C_+S^{(\alpha)}_+(t) - pC_-S^{(\alpha)}_-(t) \tag{3.3}$$

where $S_+^{(\alpha)}$ and $S_-^{(\alpha)}$ are independent α-stable processes whose Lévy measures are both given by $\alpha x^{-\alpha-1}dx$ and where the constants p and $C_\pm$ are such that

$$\gamma_\pm = E[\xi_\pm(1)] < \infty, \tag{3.4}$$

$$p = \frac{\gamma_+}{\gamma_+ + \gamma_-} \tag{3.5}$$

and

$$C_\pm = \left(\frac{c_\pm}{\gamma_+ + \gamma_-}\right)^{1/\alpha}. \tag{3.6}$$

Here and throughout $\xrightarrow{f.d.}$ denotes the convergence of all finite-dimensional marginal distributions.

In order to prove Theorem 3.1, we introduce a function Φ as follows. Let $x \in D_0([0,\infty) : [0,\infty))$ and let $y \in D([0,\infty) : \mathbb{R})$. Let x^{-1} be the right-continuous inverse of x, that is,

$$x^{-1}(t) = \inf\{s : x(s) > t\}.$$

Put

$$t^* = x(x^{-1}(t)),$$
$$t_* = x(x^{-1}(t) - 0).$$

Define

$$\Phi[x,y](t) = \begin{cases} \dfrac{t-t_*}{t^*-t_*} y(x^{-1}(t)) + \dfrac{t^*-t}{t^*-t_*} y(x^{-1}(t)-0) & \text{if} \quad t^* \neq t_*, \\ y(x^{-1}(t)) & \text{if} \quad t^* = t_*. \end{cases} \tag{3.7}$$

Then the function $\Phi[x,y](\cdot) \in D([0,\infty) : \mathbb{R})$. We note that the function $\Phi[x,y](\cdot)$ is obtained from the graph $\{(x(t), y(t)) | t \geq 0\}$ with points where $x(t)$ and $y(t)$ jump simultaneously complemented by a line segment.

Proposition 3.1. *Let $x \in D_0([0,\infty) : [0,\infty))$ and $y, z \in D([0,\infty) : \mathbb{R})$. The function Φ has the following properties.*

(*i*) *$\Phi[x,x](t) = t$ for any $t \geq 0$.*

(*ii*) *For $a > 0$, $\Phi[x,y](at) = \Phi\left[\dfrac{x}{a}, y\right](t)$ for any $t \geq 0$.*

(*iii*) *For a and $b \in \mathbb{R}$,*

$$\Phi[x, ay+bz] = a\Phi[x,y] + b\Phi[x,z].$$

(*iv*) *Let $\lambda \in \Lambda$. Then*

$$\Phi[x \circ \lambda, y \circ \lambda] = \Phi[x,y].$$

(v) Let $x_+, x_- \in D_0([0,\infty):[0,\infty))$ *and suppose that* x_+ *and* x_- *do not jump simultaneously. Then*

$$\Phi[x_+ + x_-, x_+]^{-1}(t) = t + x_-(x_+^{-1}(t)) \quad \textit{for} \quad t \geq 0.$$

Proof. We only prove (v); The rest can be checked easily. Set $x = x_+ + x_-$. Let $t \geq 0$ and set $s = t + x_-(x_+^{-1}(t))$. It suffices to show that $\Phi[x, x_+](s) = t$.

First, we consider the case where the value $x_+^{-1}(t)$ is a continuity point of x_+. Then $t = x_+(x_+^{-1}(t))$ and hence $s = x_+(x_+^{-1}(t)) + x_-(x_+^{-1}(t)) = x(x_+^{-1}(t))$. Now we have $x^{-1}(s) = x_+^{-1}(t)$ and therefore we obtain $\Phi[x, x_+](s) = x_+(x^{-1}(s)) = x_+(x_+^{-1}(t)) = t$.

Second, we consider the case where the value $x_+^{-1}(t)$ is a jump point of x_+. Then $x_+(x_+^{-1}(t) - 0) \leq t < x_+(x_+^{-1}(t))$. Since $x_-(x_+^{-1}(t) - 0) \leq x_-(x_+^{-1}(t))$, we have $x(x_+^{-1}(t) - 0) \leq s < x(x_+^{-1}(t))$. Hence we obtain $x^{-1}(s) = x_+^{-1}(t)$. Set $t_* = x_+(x_+^{-1}(t) - 0)$ and $t^* = x_+(x_+^{-1}(t))$. Noting that the value $x^{-1}(s) = x_+^{-1}(t)$ is a continuity point of x_-, we have

$$\begin{aligned}
s^* :=& x(x^{-1}(s)) = x(x_+^{-1}(t)) = t^* + x_-(x_+^{-1}(t)) = t^* + s - t, \\
s_* :=& x(x^{-1}(s) - 0) = x(x_+^{-1}(t) - 0) = t_* + x_-(x_+^{-1}(t)) = t_* + s - t.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
\Phi[x, x_+](s) &= \frac{s - s_*}{s^* - s_*} x_+(x^{-1}(s)) + \frac{s^* - s}{s^* - s_*} x_+(x^{-1}(s) - 0) \\
&= \frac{t - t_*}{t^* - t_*} x_+(x_+^{-1}(t)) + \frac{t^* - t}{t^* - t_*} x_+(x_+^{-1}(t) - 0) \\
&= t.
\end{aligned}$$

The proof is completed. □

The following continuity lemma will play an important role in the proof of Theorem 3.1.

Lemma 3.1. *Let* $x_n, x \in D_0([0,\infty):[0,\infty))$ *and let* $y_n, y \in D([0,\infty):\mathbb{R})$*. Assume that*

$$(x_n, y_n) \longrightarrow (x, y) \quad \textit{in} \quad D([0,\infty):[0,\infty) \times \mathbb{R}). \tag{3.8}$$

If $x(t)$ *is strictly increasing in* t*, then it holds that*

$$\Phi[x_n, y_n](t) \longrightarrow y(x^{-1}(t)) \tag{3.9}$$

at every $t \in [0,\infty)$ *for which* $x^{-1}(t)$ *is a continuity point of* $y(\cdot)$*.*

The proof of Lemma 3.1 is similar to that of Lemma 2.3, so we omit it.

Let X be a Lamperti process. We keep the notations such as $\xi_\pm(m)$, $T_\pm(t)$, $T(t)$, $A_+(t)$ etc. in the previous section. The function Φ provides us with a direct representation of the occupation time in terms of the partial sum processes T and T_+ of excursion intervals as follows.

Proposition 3.2. *For a general Lamperti process X, it holds that*

$$A_+(t) = \Phi[T, T_+](t). \tag{3.10}$$

Proof. This is obvious from Propositions 2.1 and (v) of Proposition 3.1. □

We prepare the following lemma of limit theorems of the partial sum processes $T_\pm$.

Lemma 3.2. *Under the conditions* (3.2)*, the following statements hold.*
(*i*) *For every t,*

$$\frac{1}{\lambda}T_+(\lambda t) \longrightarrow \gamma_+ t, \quad \frac{1}{\lambda}T_-(\lambda t) \longrightarrow \gamma_- t \quad \text{and} \quad \frac{1}{\lambda}T(\lambda t) \longrightarrow \gamma t \tag{3.11}$$

as $\lambda \to \infty$ almost surely where $\gamma = \gamma_+ + \gamma_-$.
(*ii*) *Let $\tilde{T}_\pm(t) = T_\pm(t) - \gamma_\pm \cdot [t]$. Then*

$$\left(\frac{1}{\varphi^{-1}(\lambda)} \tilde{T}_+(\lambda t), \frac{1}{\varphi^{-1}(\lambda)} \tilde{T}_-(\lambda t) \right) \xrightarrow{\mathcal{L}} (S_+^{(\alpha)}(c_+ t), S_-^{(\alpha)}(c_- t)) \quad \text{in} \quad D([0,\infty) : \mathbb{R}^2) \quad \text{as} \quad \lambda \to \infty. \tag{3.12}$$

The proof of Lemma 3.2 is similar to that of Lemma 2.2, so that we omit it.

For every $t \in [0, \infty)$, we have

$$\begin{aligned} \frac{1}{\lambda}A_+(\lambda t) &= \frac{1}{\lambda}\Phi[T, T_+](\lambda t) = \Phi\left[\frac{1}{\lambda}T, \frac{1}{\lambda}T_+\right](t) \\ &= \Phi\left[\frac{1}{\lambda}T(\lambda \cdot), \frac{1}{\lambda}T_+(\lambda \cdot)\right](t) \\ &\xrightarrow{p} \frac{\gamma_+}{\gamma} t = pt \end{aligned}$$

by (i) of Lemma 3.2, by Lemma 3.1 and by (3.5).

Now we proceed to prove Theorem 3.1.

Proof of Theorem 3.1. Using Propositions 3.2 and 3.1, we have

$$\begin{aligned} A_+(t) - pt &= \Phi[T, T_+](\lambda t) - pt \\ &= \Phi[T, T_+](t) - p\Phi[T, T](t) \\ &= \Phi[T, T_+ - pT](t). \end{aligned}$$

By the relationship between $T_\pm$ and $\tilde{T}_\pm$, we have

$$\begin{aligned} T_+(t) - pT(t) &= (1-p)T_+(t) - pT_-(t) \\ &= (1-p)\tilde{T}_+(t) - p\tilde{T}_-(t) + \{(1-p)\gamma_+ - p\gamma_-\} \cdot [t]. \end{aligned}$$

By (3.5), it holds that $(1-p)\gamma_+ - p\gamma_- = 0$ and we have

$$T_+(t) - pT(t) = (1-p)\tilde{T}_+(t) - p\tilde{T}_-(t).$$

Hence we have

$$\begin{aligned} A_+(t) - pt &= \Phi[T, (1-p)\tilde{T}_+ - p\tilde{T}_-](t) \\ &= (1-p)\Phi[T, \tilde{T}_+](t) - p\Phi[T, \tilde{T}_-](t), \end{aligned}$$

and thus we obtain

$$\begin{aligned} \frac{1}{\varphi^{-1}(\lambda)}(A_+(\lambda t) - p\lambda t) = (1-p)\Phi\left[\frac{1}{\lambda}T(\lambda\cdot), \frac{1}{\varphi^{-1}(\lambda)}\tilde{T}_+(\lambda\cdot)\right](t) \\ - p\Phi\left[\frac{1}{\lambda}T(\lambda\cdot), \frac{1}{\varphi^{-1}(\lambda)}\tilde{T}_-(\lambda\cdot)\right](t). \end{aligned}$$

By (3.11) and (3.12), we have

$$\left(\frac{1}{\lambda}T(\lambda t), \frac{1}{\varphi^{-1}(\lambda)}\tilde{T}_+(\lambda t), \frac{1}{\varphi^{-1}(\lambda)}\tilde{T}_-(\lambda t)\right) \xrightarrow{\mathcal{L}} (\gamma t, S_+^{(\alpha)}(c_+ t), S_-^{(\alpha)}(c_- t))$$

in $D([0,\infty) : [0,\infty) \times \mathbb{R} \times \mathbb{R})$. Now Lemma 3.1 implies that

$$\begin{aligned} &\left(\Phi\left[\frac{1}{\lambda}T(\lambda\cdot), \frac{1}{\varphi^{-1}(\lambda)}\tilde{T}_+(\lambda\cdot)\right](t), \Phi\left[\frac{1}{\lambda}T(\lambda\cdot), \frac{1}{\varphi^{-1}(\lambda)}\tilde{T}_-(\lambda\cdot)\right](t)\right) \\ &\qquad \xrightarrow{f.d.} \left(S_+^{(\alpha)}\left(\frac{c_+ t}{\gamma}\right), S_-^{(\alpha)}\left(\frac{c_- t}{\gamma}\right)\right) \overset{d}{=} \left(C_+ S_+^{(\alpha)}(t), C_- S_-^{(\alpha)}(t)\right). \end{aligned}$$

Therefore we obtain

$$\begin{aligned} \frac{1}{\varphi^{-1}(\lambda)}(A_+(\lambda t) - p\lambda t) &= (1-p)\Phi\left[\frac{1}{\lambda}T(\lambda\cdot), \frac{1}{\varphi^{-1}(\lambda)}\tilde{T}_+(\lambda\cdot)\right](t) \\ &\qquad - p\Phi\left[\frac{1}{\lambda}T(\lambda\cdot), \frac{1}{\varphi^{-1}(\lambda)}\tilde{T}_-(\lambda\cdot)\right](t) \\ &\xrightarrow{f.d.} (1-p)C_+ S_+^{(\alpha)}(t) - pC_- S_-^{(\alpha)}(t), \end{aligned}$$

which competes the proof. □

Remark 1. Thanks to the function $\Phi[x,y](t)$, we can give a direct proof of the functional limit theorem in the null recurrent case stated as Theorem 2.1, but we shall not go into details here.

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