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An infinitesimal deformation of the local system and the Beta function

By

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1. Introduction

Improper integrals are realized as some kinds of period integrals, that is, pairings between cohomology and homology (in fortunate cases). In fact, Euler's Beta function

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt \tag{1.1}$$

can be seen as a pairing between the twisted homology $H_1(X, \mathcal{L})$ and the twisted de Rham cohomology, where X is $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, $\mathcal{L}$ is the local system defined by the multi-valuedness of $t^{p-1}(1-t)^{q-1}$ and the twisted de Rham complex is defined by replacing a coboundary operator with $d + d\log(t^{p-1}(1-t)^{q-1})$ instead of the exterior derivative d. This is the Aomoto theory.

One of the most impressive aspects of the Aomoto theory allows one to regard the integration (1.1) as the pairing between the regularization of the open interval $(0, 1)$ and the cohomology class represented by dt. Here, the regularization of the open interval $(0, 1)$ is defined by

$$reg(0,1) := \frac{1}{c_0 - 1}\gamma_0 - \frac{1}{c_1 - 1}\gamma_1, \tag{1.2}$$

where $c_0 := e^{2\pi\sqrt{-1}(p-1)}$, $c_1 := e^{2\pi\sqrt{-1}(q-1)}$ and γ_i is a loop around $t = i$. However, as the formula (1.2) shows, the regularization is only possible unless $c_0, c_1 = 0$, that is the case where p and q are integers. In principle, this difficulty should be avoided by considering a family of twisted homologies and cohomologies parametrized by p, q because the Beta function $B(p,q)$ depends holomorphically on parameters p, q. To realize the value of $B(p,q)$ at integer points as a pairng, we consider 1-st order derivatives at integer points with respect to parameters p, q.

To carry out this more concretely, we introduce and calculate homology and cohomology with coefficients in $R_{\boldsymbol{\varepsilon}} := \mathbb{C}[\varepsilon_0]/(\varepsilon_0^2) \otimes \mathbb{C}[\varepsilon_1]/(\varepsilon_1^2)$-valued local

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system. Instead of the cycle $reg(0,1)$, we define an *infinitesimal deformation* of $(c_0-1)(c_1-1)reg(0,1)$, that is, we introduce

$$Reg(0,1) := (\overline{c_1}-1)\gamma_0 - (\overline{c_0}-1)\gamma_1,$$

where $\overline{c_0} := e^{2\pi\sqrt{-1}(p-1+\varepsilon_0)}$, $\overline{c_1} := e^{2\pi\sqrt{-1}(q-1+\varepsilon_1)}$. Also in case $p,\ q \in \mathbb{Z}$, $Reg(0,1)$ survives!

We shall describe the contents of this paper. In Section 2, we give the definition of $R_{\boldsymbol{\varepsilon}}$-valued local system $\mathcal{L}_{R_{\boldsymbol{\varepsilon}}}$. In Section 3, we calculate the homology with coefficients in $\mathcal{L}_{R_{\boldsymbol{\varepsilon}}}$. In Section 4, we give the definition of the twisted de Rham cohomology and calculate it. In Section 5, we calculate the pairing between the twisted homology and de Rham cohomology. Consequently, we get Euler's Beta function. Note that our method in this paper is valid for hypergeometric functions associated to the configuration space of points on $\mathbb{P}^1$.

2. The local system

Let $R_{\boldsymbol{\varepsilon}}$ be $\mathbb{C}[\varepsilon_0]/(\varepsilon_0^2) \otimes \mathbb{C}[\varepsilon_1]/(\varepsilon_1^2)$. We fix a 3-tuple of complex numbers $(\alpha_0,\alpha_1,\alpha_\infty) \in \mathbb{C}^3$ with $\alpha_0+\alpha_1+\alpha_\infty = 0$. Put $\overline{\alpha_i} = \alpha_i + \varepsilon_i$, where $\varepsilon_\infty := -\varepsilon_0 - \varepsilon_1$. (Note that $\varepsilon_\infty^2 \notin (\varepsilon_0^2, \varepsilon_1^2) \subset \mathbb{C}[\varepsilon_0,\varepsilon_1]$.) We introduce the following sheaf:

$$\mathcal{L}_{R_{\boldsymbol{\varepsilon}}} := \ker\big(d-\omega_{\boldsymbol{\varepsilon}} : \mathcal{O}_X \otimes_{\mathbb{C}_X} R_{\boldsymbol{\varepsilon}} \longrightarrow \Omega_X^1 \otimes_{\mathbb{C}_X} R_{\boldsymbol{\varepsilon}}\big),$$

where

$$\omega_{\boldsymbol{\varepsilon}} = \overline{\alpha_0}\frac{dt}{t} + \overline{\alpha_1}\frac{d(1-t)}{1-t}.$$

Lemma 2.1. *The sheaf $\mathcal{L}_{R_{\boldsymbol{\varepsilon}}}$ is an $R_{\boldsymbol{\varepsilon}}$-valued local system of rank 1, whose local sections are generated by $t^{\overline{\alpha_0}}(1-t)^{\overline{\alpha_1}}$, where $z^{\overline{\alpha_i}} := \exp(\overline{\alpha_i}\log z) := z^{\alpha_i}(1+\varepsilon_i \log z)$.*

Proof. We can pick an open covering $\{U_i\}_i$ of X such that $t^{\overline{\alpha_0}}(1-t)^{\overline{\alpha_1}}$ is single-valued on U_i. Thus we choose a branch ς_i of $t^{\overline{\alpha_0}}(1-t)^{\overline{\alpha_1}}$ on U_i. Note that $\frac{d\varsigma_i}{\varsigma_i} = \omega_{\boldsymbol{\varepsilon}}$. We shall prove that $\Gamma(U_i,\mathcal{L}_{R_{\boldsymbol{\varepsilon}}}) = R_{\boldsymbol{\varepsilon}}\varsigma_i$. If $\varsigma \in \Gamma(U_i,\mathcal{L}_{R_{\boldsymbol{\varepsilon}}})$, then $0 = (d-\omega_{\boldsymbol{\varepsilon}})\varsigma = \varsigma_i d(\varsigma_i^{-1}\varsigma)$. So $d(\varsigma_i^{-1}\varsigma) = 0$ and hence $\varsigma_i^{-1}\varsigma \in R_{\boldsymbol{\varepsilon}}$. This implies that $\Gamma(U_i,\mathcal{L}_{R_{\boldsymbol{\varepsilon}}}) = R_{\boldsymbol{\varepsilon}}\varsigma_i$. □

3. The twisted homology

We calculate the *twisted homology* $H_\bullet(X,\mathcal{L}_{R_{\boldsymbol{\varepsilon}}})$. Take a point $o \in X$. Let γ_i be a loop around $t = i$ $(i = 0,1)$ both of whose ends are o. The inclusion map $o \cup \gamma_0 \cup \gamma_1 \hookrightarrow X$ is a deformation retract. Then X is homotopic to $o \cup \gamma_0 \cup \gamma_1$. Let

$$C_0(X,\mathcal{L}_{R_{\boldsymbol{\varepsilon}}}) := R_{\boldsymbol{\varepsilon}} o,$$
$$C_1(X,\mathcal{L}_{R_{\boldsymbol{\varepsilon}}}) := R_{\boldsymbol{\varepsilon}}\gamma_0 \oplus R_{\boldsymbol{\varepsilon}}\gamma_1.$$

The boundary operator $\partial : C_1(X, \mathcal{L}_{R_\varepsilon}) \longrightarrow C_0(X, \mathcal{L}_{R_\varepsilon})$ is defined by the following:

$$\partial(\gamma_i) = (\overline{c_i} - 1)o, \qquad (i = 0, 1)$$

where $\overline{c_i} := \exp(2\pi\sqrt{-1}\overline{\alpha_i}) = c_i(1 + 2\pi\sqrt{-1}\varepsilon_i)$, $c_i = \exp(2\pi\sqrt{-1}\alpha_i)$. Homologies of this chain complex $\{C_\bullet(X, \mathcal{L}_{R_\varepsilon}), \partial\}$ coincide with $H_\bullet(X, \mathcal{L}_{R_\varepsilon})$.

Theorem 3.1. *Put* $Reg(0,1) := (\overline{c_1} - 1)\gamma_0 - (\overline{c_0} - 1)\gamma_1$.

1. *If both* α_0 *and* α_1 *are integers,* $H_0(X, \mathcal{L}_{R_\varepsilon}) \cong R_\varepsilon/(\varepsilon_0, \varepsilon_1) \cong \mathbb{C}$. *Otherwise* $H_0(X, \mathcal{L}_{R_\varepsilon}) = 0$.
2. *If both* α_0 *and* α_1 *are integers,* $H_1(X, \mathcal{L}_{R_\varepsilon})$ *is generated by* $Reg(0,1)$, $\varepsilon_0\gamma_0$ *and* $\varepsilon_1\gamma_1$. *Otherwise* $H_1(X, \mathcal{L}_{R_\varepsilon})$ *is generated by* $Reg(0,1)$ *only.*

Proof.

1. $H_0(X, \mathcal{L}_{R_\varepsilon}) \cong R_\varepsilon/(\overline{c_0} - 1, \overline{c_1} - 1)$. If both α_0 and α_1 are integers, then $(\overline{c_0} - 1, \overline{c_1} - 1) = (\varepsilon_0, \varepsilon_1)$. Otherwise $(\overline{c_0} - 1, \overline{c_1} - 1) = R_\varepsilon$. We have thus proved the statement.
2. If $k_0\gamma_0 + k_1\gamma_1$ is a cycle, then

$$(\overline{c_0} - 1)k_0 + (\overline{c_1} - 1)k_1 = 0.$$

If both α_0 and α_1 are integers, then $\overline{c_0} - 1 = \varepsilon_0$, $\overline{c_1} - 1 = \varepsilon_1$. By using lemma A.3, we get the following:

$$\begin{bmatrix} k_0 \\ k_1 \end{bmatrix} \in R_\varepsilon \begin{bmatrix} \varepsilon_1 \\ -\varepsilon_0 \end{bmatrix} + R_\varepsilon \begin{bmatrix} \varepsilon_0 \\ 0 \end{bmatrix} + R_\varepsilon \begin{bmatrix} 0 \\ \varepsilon_1 \end{bmatrix}.$$

If α_0 is not an integer, then $\overline{c_0} - 1$ is invertible. Hence

$$k_0 = -(\overline{c_0} - 1)^{-1}(\overline{c_1} - 1)k_1.$$

If α_1 is not an integer, then $\overline{c_1} - 1$ is invertible. Hence

$$k_1 = -(\overline{c_1} - 1)^{-1}(\overline{c_0} - 1)k_0.$$

We have thus proved the theorem.

□

4. The twisted de Rham cohomology

We calculate the *twisted algebraic de Rham cohomology* $H^\bullet_{d+\omega_\varepsilon}(X^{alg}, R_\varepsilon)$. Let

$$C^0(X^{alg}, R_\varepsilon) := R_\varepsilon\left[t, \frac{1}{t}, \frac{1}{1-t}\right],$$
$$C^1(X^{alg}, R_\varepsilon) := R_\varepsilon\left[t, \frac{1}{t}, \frac{1}{1-t}\right] dt.$$

The coboundary operator is defined by $d+\omega_{\varepsilon}$. The k-th cohomology of this cochain complex $\left\{C^{\bullet}(X^{alg},R_{\varepsilon}), d+\omega_{\varepsilon}\right\}$ is denoted by $H^k_{d+\omega_{\varepsilon}}(X^{alg},R_{\varepsilon})$ and is said to be the *k-th twisted algebraic de Rham cohomology*. Note that $C^1(X^{alg},R_{\varepsilon})$ are generated by $t^k\frac{dt}{t}$, $(1-t)^k\frac{d(1-t)}{1-t}$ $(k=0,\pm1,\pm2,\ldots)$. We agree that η_0^k and η_1^k denote the cohomology classes represented by $t^k\frac{dt}{t}$ and $(1-t)^k\frac{d(1-t)}{1-t}$, respectively.

Proposition 4.1. *Let i be 0 or 1.*
1. $(k-1+\overline{\alpha_i})\eta_i^{k-1}=(k-\overline{\alpha_\infty})\eta_i^k$.
2. *Let k be a positive integer. Then the equalities*

$$\eta_1^k=-\sum_j\binom{k-1}{j}(-1)^j\eta_0^{j+1}$$

and

$$\eta_0^k=-\sum_j\binom{k-1}{j}(-1)^j\eta_1^{j+1}$$

hold.

Proof.
1. We have $(d+\omega_{\varepsilon})\left(t^k\frac{1}{t}\right)=(k-1+\overline{\alpha_0})t^{k-1}\frac{dt}{t}+\overline{\alpha_1}t^{k-1}\frac{d(1-t)}{1-t}$. This implies that $\overline{\alpha_1}\left[t^{k-1}\frac{d(1-t)}{1-t}\right]=-(k-1+\overline{\alpha_0})\eta_0^{k-1}$, where $\left[t^{k-1}\frac{d(1-t)}{1-t}\right]$ denotes the cohomology class whose representative is $t^{k-1}\frac{d(1-t)}{1-t}$. We shall indicate by [ ] a cohomology class in latter discussion. Since $-\overline{\alpha_1}\eta_0^k=\overline{\alpha_1}t^{k-1}(1-t)\frac{d(1-t)}{1-t}=\overline{\alpha_1}\left[t^{k-1}\frac{d(1-t)}{1-t}\right]-\overline{\alpha_1}\left[t^k\frac{d(1-t)}{1-t}\right]=-(k-1+\overline{\alpha_0})\eta_0^{k-1}+(k+\overline{\alpha_0})\eta_0^k$, we have $(k-1+\overline{\alpha_0})\eta_0^{k-1}=(k-\overline{\alpha_\infty})\eta_0^k$. In a similar fashion, the equality $(k-1+\overline{\alpha_1})\eta_1^{k-1}=(k-\overline{\alpha_\infty})\eta_1^k$ is proved.
2. We use the binomial expansion: $(1-t)^{k-1}=\sum_j\binom{k-1}{j}(-t)^j$.

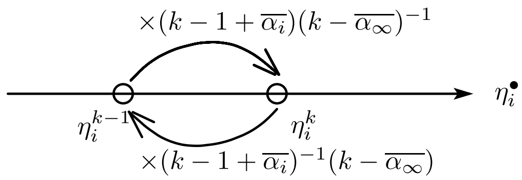
□

Theorem 4.1. *Let $J:=\left\{j \mid \alpha_j\in\mathbb{Z}\right\}\subset\{0,1,\infty\}$.*
1. *If $J=\phi$, then $H^1_{d+\omega_{\varepsilon}}(X^{alg},R_{\varepsilon})$ is generated by η_0^1, that is, generated by $[dt]$.*
2. *If $J=\{j_0\}$, then $H^1_{d+\omega_{\varepsilon}}(X^{alg},R_{\varepsilon})$ is generated by $\eta_{j_0}^{-\alpha_{j_0}}$, where $\eta_\infty^{-\alpha_\infty}:=\eta_0^{\alpha_\infty}$.*
3. *If $J=\{0,1,\infty\}$, then one of the following statements holds:*
   (a) *If $\alpha_0\geq 0$ and $\alpha_1\geq 0$, then $H^1_{d+\omega_{\varepsilon}}(X^{alg},R_{\varepsilon})$ is generated by $\eta_0^{-\alpha_0}$ and $\eta_1^{-\alpha_1}$, between which the relation $\varepsilon_0\eta_0^{-\alpha_0}+\varepsilon_1\eta_1^{-\alpha_1}=0$ holds.*
   (b) *If $\alpha_0\geq 0$ and $\alpha_1<0$, then $H^1_{d+\omega_{\varepsilon}}(X^{alg},R_{\varepsilon})$ is generated by $\eta_0^{-\alpha_0}$ and $\eta_0^{\alpha_\infty}$, between which the relation $\varepsilon_0\eta_0^{-\alpha_0}-(-1)^{-\alpha_1}\varepsilon_\infty\eta_0^{\alpha_\infty}=0$ holds.*

(c) *If* $\alpha_0 < 0$ *and* $\alpha_1 \geq 0$*, then* $H^1_{d+\omega_{\boldsymbol{\varepsilon}}}(X^{alg}, R_{\boldsymbol{\varepsilon}})$ *is generated by* $\eta_1^{-\alpha_1}$ *and* $\eta_1^{\alpha_\infty}$*, between which the relation* $\varepsilon_1\eta_1^{-\alpha_1} - (-1)^{-\alpha_0}\varepsilon_\infty\eta_1^{\alpha_\infty} = 0$ *holds.*

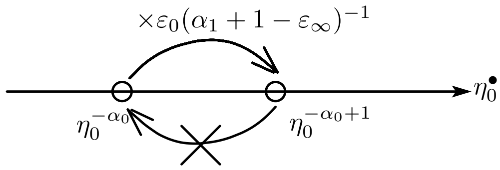
(d) *If* $\alpha_0 < 0$ *and* $\alpha_1 < 0$*, then* $H^1_{d+\omega_{\boldsymbol{\varepsilon}}}(X^{alg}, R_{\boldsymbol{\varepsilon}})$ *is generated by* $\eta_0^{-\alpha_0}$ *and* $\eta_0^{\alpha_\infty}$*, between which the relation* $\varepsilon_0\eta_0^{-\alpha_0} - (-1)^{-\alpha_1}\varepsilon_\infty\eta_0^{\alpha_\infty} = 0$ *holds.*

Proof. Proposition 4.1 implies that η_i^{k-1} is generated by η_i^k unless $k - 1 + \overline{\alpha_i}$ is not invertible. Similarly, η_i^k is generated by η_i^{k-1} unless $k - \overline{\alpha_\infty}$ is not invertible. If η_0^k is generated for every positive integer k, then η_1^k is also generated for every positive integer k. The converse also holds.



1. If $J = \phi$, then both $k - 1 + \overline{\alpha_i}$ and $k - \overline{\alpha_\infty}$ are invertible for every integer k. Consequently η_0^1 generates $H^1_{d+\omega_{\boldsymbol{\varepsilon}}}(X^{alg}, R_{\boldsymbol{\varepsilon}})$.

2. If $j_0 = 0$, then $-\alpha_0 + \overline{\alpha_0} = \varepsilon_0$ is not invertible. So $\eta_0^{-\alpha_0}$ cannot be generated by $\eta_0^{-\alpha_0+1}$. However $k + \overline{\alpha_0}$ is invertible except for $k = -\alpha_0$. For every integer k, both $k - \overline{\alpha_\infty}$ and $k + \overline{\alpha_1}$ are also invertible. Consequently $\eta_0^{-\alpha_0}$ generates $H^1_{d+\omega_{\boldsymbol{\varepsilon}}}(X^{alg}, R_{\boldsymbol{\varepsilon}})$.



In the same way, we see that $\eta_1^{-\alpha_1}$ generates $H^1_{d+\omega_{\boldsymbol{\varepsilon}}}(X^{alg}, R_{\boldsymbol{\varepsilon}})$ in case of $j_0 = 1$. If $j_0 = \infty$, then $\alpha_\infty - \overline{\alpha_\infty} = -\varepsilon_\infty$ is not invertible. So $\eta_0^{\alpha_\infty}$ cannot be generated by $\eta_0^{\alpha_\infty - 1}$. However $k - \overline{\alpha_\infty}$ is invertible except for $k = \alpha_\infty$. For every integer k, both $k + \overline{\alpha_0}$ and $k + \overline{\alpha_1}$ are also invertible. Consequently η_0^k is generated by $\eta_0^{\alpha_\infty}$ for each integer k. We shall prove that η_1^k is generated by $\eta_0^{\alpha_\infty}$ for each integer k. It is sufficient to prove that $\eta_1^{\alpha_\infty}$ is generated by $\{\eta_0^k\}_{k=0,\pm1,\pm2,\ldots}$. In case $\alpha_\infty < 1$,

$$\begin{aligned}
\eta_1^{\alpha_\infty} &= (1-\varepsilon_\infty)(2-\varepsilon_\infty)\cdots(-\alpha_\infty+1-\varepsilon_\infty) \\
&\quad \times \overline{\alpha_1}^{-1}(1+\overline{\alpha_1})^{-1}\cdots(-\alpha_\infty+\overline{\alpha_1})^{-1}\eta_1^1 \\
&= (1-\varepsilon_\infty)(2-\varepsilon_\infty)\cdots(-\alpha_\infty+1-\varepsilon_\infty) \\
&\qquad \overline{\alpha_1}^{-1}(1+\overline{\alpha_1})^{-1}\cdots(-\alpha_\infty+\overline{\alpha_1})^{-1}(-\eta_0^1).
\end{aligned}$$

In case $\alpha_\infty \geq 1$,

$$\eta_1^{\alpha_\infty} = -\sum_{j=0}^{\alpha_\infty - 1} \begin{pmatrix} \alpha_\infty - 1 \\ j \end{pmatrix} (-1)^j \eta_0^{j+1}.$$

Consequently $\eta_0^{\alpha_\infty}$ generates $H^1_{d+\omega_{\boldsymbol{\varepsilon}}}(X^{alg}, R_{\boldsymbol{\varepsilon}})$.

3. For $j = 0,\ 1,\ -\alpha_j + \overline{\alpha_j} = \varepsilon_j$ is not invertible. So $\eta_j^{-\alpha_j}$ cannot be generated by $\eta_j^{-\alpha_j+1}$. Similarly, $\eta_0^{\alpha_\infty}$ (resp. $\eta_1^{\alpha_\infty}$) cannot be generated by $\eta_0^{\alpha_\infty - 1}$ (resp. $\eta_1^{\alpha_\infty - 1}$).

(a) In case $\alpha_0 \geq 0$, $\alpha_1 \geq 0$:
In this case, $\alpha_\infty \leq -\alpha_0$, $\alpha_\infty \leq -\alpha_1$. So $\eta_j^{\alpha_\infty}$ is generated by $\eta_j^{-\alpha_j}$. In fact, if $\alpha_\infty < -\alpha_j$,

$$\begin{aligned}\eta_j^{\alpha_\infty} &= (\alpha_\infty + 1 + \overline{\alpha_\infty})(\alpha_\infty + 2 + \overline{\alpha_\infty}) \cdots (-\alpha_j + \overline{\alpha_\infty}) \\ &\quad \times (\alpha_\infty + \overline{\alpha_j})^{-1}(\alpha_\infty + 1 + \overline{\alpha_j})^{-1} \cdots (-\alpha_j - 1 + \overline{\alpha_j})^{-1}\eta_j^{-\alpha_j}.\end{aligned}$$

If $\alpha_\infty = -\alpha_j$, then $\eta_j^{\alpha_\infty}$ coincides with $\eta_j^{-\alpha_j}$. We shall prove the relation: $\varepsilon_0\eta_0^{-\alpha_0} + \varepsilon_1\eta_1^{-\alpha_1} = 0$.

(4.1)
$$\begin{aligned}\eta_j^1 &= \varepsilon_j(\varepsilon_j + 1) \cdots (\varepsilon_j + \alpha_j) \\ &\quad \times (-\alpha_j + 1 - \overline{\alpha_\infty})^{-1}(-\alpha_j + 2 - \overline{\alpha_\infty})^{-1} \cdots (1 - \overline{\alpha_\infty})^{-1}\eta_j^{-\alpha_j}.\end{aligned}$$

Then

$$\begin{aligned}&\varepsilon_0(\varepsilon_0 + 1) \cdots (\varepsilon_0 + \alpha_0) \\ &\qquad \times (-\varepsilon_\infty + \alpha_0 + 1)(-\varepsilon_\infty + \alpha_0 + 2) \cdots (-\varepsilon_\infty + \alpha_0 + \alpha_1 + 1)\eta_0^{-\alpha_0} \\ &\qquad + \varepsilon_1(\varepsilon_1 + 1) \cdots (\varepsilon_1 + \alpha_1) \\ &\qquad \times (-\varepsilon_\infty + \alpha_1 + 1)(-\varepsilon_\infty + \alpha_1 + 2) \cdots (-\varepsilon_\infty + \alpha_0 + \alpha_1 + 1)\eta_1^{-\alpha_1} \\ &\quad = 0,\end{aligned}$$

that is,

$$\varepsilon_0\left((\alpha_0 + \alpha_1 + 1)! + b_0\right)\eta_0^{-\alpha_0} + \varepsilon_1\left((\alpha_0 + \alpha_1 + 1)! + b_1\right)\eta_1^{-\alpha_1} = 0,$$

where $b_0, b_1 \in (\varepsilon_0, \varepsilon_1)$. By applying lemma A.2 we can complete the proof.

(b) In case $\alpha_0 \geq 0$, $\alpha_1 < 0$:
For each integer k, η_0^k is generated by $\eta_0^{-\alpha_0}$ and $\eta_0^{\alpha_\infty}$. Then η_1^k is generated for each positive integer k. So $\eta_1^{-\alpha_1}$ is generated. If $\alpha_\infty > 0$, then $\eta_1^{\alpha_\infty}$ is also generated. If $\alpha_\infty < 0$, then

$$\begin{aligned}\eta_1^{\alpha_\infty} &= -\eta_0^{\alpha_\infty} \times (-\alpha_1 + \varepsilon_0)(-\alpha_1 + 1 + \varepsilon_0) \cdots (\alpha_0 + \varepsilon_0) \\ &\quad \times (-\alpha_0 + \varepsilon_1)^{-1}(-\alpha_0 + 1 + \varepsilon_1)^{-1} \cdots (\alpha_1 + \varepsilon_1)^{-1}.\end{aligned}$$

Hence $\eta_1^{\alpha_\infty}$ is generated. We shall prove the relation: $\varepsilon_0\eta_0^{-\alpha_0} + \varepsilon_\infty\eta_0^{\alpha_\infty} = 0$.

$$\begin{aligned}(-\alpha_0+\overline{\alpha_0})(-\alpha_0+1+\overline{\alpha_0})\cdots(\alpha_\infty-2+\overline{\alpha_0})\\ \times(-\alpha_0+1-\overline{\alpha_\infty})^{-1}(-\alpha_0+2-\overline{\alpha_\infty})^{-1}\cdots(\alpha_\infty-1-\overline{\alpha_\infty})^{-1}\eta_0^{-\alpha_0}\\ =(\alpha_\infty-\overline{\alpha_\infty})(\alpha_\infty-1+\overline{\alpha_0})^{-1}\eta_0^{\alpha_\infty}.\end{aligned}$$

Then

$$\begin{aligned}\varepsilon_0(\varepsilon_0+1)\cdots(\varepsilon_0-\alpha_1-1)\eta_0^{-\alpha_0}\\ =(-\varepsilon_\infty)(-\varepsilon_\infty-1)\cdots(-\varepsilon_\infty+\alpha_1+1)\eta_0^{\alpha_\infty},\end{aligned}$$

that is,

$$\varepsilon_0\left((-\alpha_1-1)!+b_0\right)\eta_0^{-\alpha_0}-(-1)^{-\alpha_1}\varepsilon_\infty\left((-\alpha_1-1)!+b_\infty\right)\eta_0^{\alpha_\infty}=0,$$

where $b_0, b_\infty \in (\varepsilon_0, \varepsilon_1)$. By applying lemma A.2 we can complete the proof.

(c) In case $\alpha_0 < 0$, $\alpha \geq 0$:
In exactly the same way as case (b), we can obtain the desired result.

(d) In case $\alpha_0 < 0$, $\alpha_1 < 0$:
For each integer k, η_0^k is generated by $\eta_0^{-\alpha_0}$ and $\eta_0^{\alpha_\infty}$. Then η_1^k is generated for each positive integer k. So both $\eta_1^{-\alpha_1}$ and $\eta_1^{\alpha_\infty}$ are generated. In exactly the same way as case (b), we obtain the relation between $\eta_0^{-\alpha_0}$ and $\eta_0^{\alpha_\infty}$.

□

5. The euler type integral

We shall consider the *euler type integral*.

Definition 5.1. Let γ be a singular 1-simplex, $\varsigma \in \Gamma(\gamma, \mathcal{L}_{R_\varepsilon})$ and $\eta \in \Gamma(X, \Omega_X^1)$. We pick $\mathbb{C}$-valued functions a, a_0, a_1, a_{01} over γ such that $\varsigma\eta = adt + \varepsilon_0 a_0 dt + \varepsilon_1 a_1 dt + \varepsilon_0\varepsilon_1 a_{01} dt$. We define

$$\int_{\gamma\otimes\varsigma} t^{\overline{\alpha_0}}(1-t)^{\overline{\alpha_1}}\eta := \int_\gamma adt + \varepsilon_0\int_\gamma a_0dt + \varepsilon_1\int_\gamma a_1dt + \varepsilon_0\varepsilon_1\int_\gamma a_{01}dt.$$

For $\Xi = \sum \gamma_i \otimes \varsigma_i$, we define

$$\int_\Xi t^{\overline{\alpha_0}}(1-t)^{\overline{\alpha_1}}\eta := \sum\int_{\gamma_i\otimes\varsigma_i} t^{\overline{\alpha_0}}(1-t)^{\overline{\alpha_1}}\eta.$$

Example 5.1.

$$
\begin{aligned}
&\int_{Reg(0,1)} t^{\overline{\alpha_0}}(1-t)^{\overline{\alpha_1}}\eta_0^k \\
&\quad = \left[(c_1-1)\int_{\gamma_0} t^{\alpha_0+k-1}(1-t)^{\alpha_1}dt - (c_0-1)\int_{\gamma_1} t^{\alpha_0+k-1}(1-t)^{\alpha_1}dt\right] \\
&\qquad + \varepsilon_0\left[(c_1-1)\int_{\gamma_0} t^{\alpha_0+k-1}(1-t)^{\alpha_1}\log t dt\right. \\
&\qquad\qquad - (c_0-1)\int_{\gamma_1} t^{\alpha_0+k-1}(1-t)^{\alpha_1}\log t dt \\
&\qquad\qquad \left. - 2\pi\sqrt{-1}c_0\int_{\gamma_1} t^{\alpha_0+k-1}(1-t)^{\alpha_1}dt\right] \\
&\qquad + \varepsilon_1\left[(c_1-1)\int_{\gamma_0} t^{\alpha_0+k-1}(1-t)^{\alpha_1}\log(1-t)dt\right. \\
&\qquad\qquad + 2\pi\sqrt{-1}c_1\int_{\gamma_0} t^{\alpha_0+k-1}(1-t)^{\alpha_1}dt \\
&\qquad\qquad \left. - (c_0-1)\int_{\gamma_1} t^{\alpha_0+k-1}(1-t)^{\alpha_1}\log(1-t)dt\right] \\
&\qquad + \varepsilon_0\varepsilon_1\left[(c_1-1)\int_{\gamma_0} t^{\alpha_0+k-1}(1-t)^{\alpha_1}\log t\log(1-t)dt\right. \\
&\qquad\qquad + 2\pi\sqrt{-1}c_1\int_{\gamma_0} t^{\alpha_0+k-1}(1-t)^{\alpha_1}\log t dt \\
&\qquad\qquad - (c_0-1)\int_{\gamma_1} t^{\alpha_0+k-1}(1-t)^{\alpha_1}\log t\log(1-t)dt \\
&\qquad\qquad \left. - 2\pi\sqrt{-1}c_0\int_{\gamma_1} t^{\alpha_0+k-1}(1-t)^{\alpha_1}\log(1-t)dt\right].
\end{aligned}
\tag{5.1}
$$

We shall consider the case $\alpha_0, \alpha_1 \in \mathbb{Z}$ in latter discussions.

Lemma 5.1. *Let a, b be integers.*

1. *If* $\mathrm{res}_{t=0} t^a(1-t)^b dt = 0$*, then there exists a meromorphic function* $F(a,b;t)$ *over* $\mathbb{P}^1 \setminus [1,\infty]$ *such that* $dF = t^a(1-t)^b dt$*,* $F(a,b;o) = 0$*, and the following formula holds*:

$$\int_{\gamma_0} t^a(1-t)^b \log t dt = -2\pi\sqrt{-1}\mathrm{res}_{t=0}\frac{F(a,b;t)}{t}dt. \tag{5.2}$$

2. *If* $\mathrm{res}_{t=1} t^a(1-t)^b dt = 0$*, then there exists a meromorphic function* $F(a,b;t)$ *over* $\mathbb{P}^1 \setminus [-\infty,0]$ *such that* $dF = t^a(1-t)^b dt$*,* $F(a,b;o) = 0$*, and the following formula holds*:

$$\int_{\gamma_1} t^a(1-t)^b \log(1-t) dt = -2\pi\sqrt{-1}\mathrm{res}_{t=1}\frac{F(a,b;t)}{t-1}dt. \tag{5.3}$$

3. *If* $\mathrm{res}_{t=0} t^a(1-t)^b dt = 0$ *and* $\mathrm{res}_{t=1} t^a(1-t)^b dt = 0$, *then there exists a meromorphic function* $F(a,b;t)$ *over* $\mathbb{P}^1$ *(and holomorphic on* X*) such that* $dF = t^a(1-t)^b dt$, $F(a,b;o) = 0$, *and two formulae* (5.2), (5.3) *hold.*

Proof.
1. Let l_t be a path in $\mathbb{P}\setminus[0,\infty]$ whose initial point is o and whose terminal point is t. Put

$$F(a,b;t) := \int_{l_t} t^a(1-t)^b dt.$$

$F(a,b;t)$ is well-defined, that is, defined independently of the choice of l_t because $\mathrm{res}_{t=0} t^a(1-t)^b dt = 0$.

$$\begin{aligned}
&\int_{\gamma_0} t^a(1-t)^b \log t dt = \int_{\gamma_0} \log t dF = \int_{\gamma_0} \left(d(F(a,b;t)\log t) - F(a,b;t)\frac{dt}{t} \right) \\
&= [F(a,b;t)\log t]_{\partial\gamma_0} - 2\pi\sqrt{-1}\mathrm{res}_{t=0} \frac{F(a,b;t)}{t} dt \\
&= 2\pi\sqrt{-1}F(a,b;o) - 2\pi\sqrt{-1}\mathrm{res}_{t=0} \frac{F(a,b;t)}{t} dt \\
&= -2\pi\sqrt{-1}\mathrm{res}_{t=0} \frac{F(a,b;t)}{t} dt.
\end{aligned}$$

2. In exactly the same way as 1., we can obtain the proof.
3. In exactly the same way as 1., we can obtain the proof.

□

By using this lemma, we get the *Beta function*:

Theorem 5.1. *Let* α_0, α_1 *be integers. We assume that* $\mathrm{res}_{t=0} t^{\alpha_0}(1-t)^{\alpha_1} dt = 0$ *and* $\mathrm{res}_{t=1} t^{\alpha_0}(1-t)^{\alpha_1} dt = 0$. *Then there exists a meromorphic function* $F(\alpha_0,\alpha_1;t)$ *over* $\mathbb{P}^1$ *(and holomorphic over* $X = \mathbb{P}^1 \setminus \{0,1,\infty\}$*) such that* $dF = t^{\alpha_0}(1-t)^{\alpha_1} dt$, $F(\alpha_0,\alpha_1;o) = 0$, *and the following formula holds*:

$$\begin{aligned}
&\int_{Reg(0,1)} t^{\overline{\alpha_0}}(1-t)^{\overline{\alpha_1}}\eta_0^1 \\
&\quad = -\varepsilon_0\varepsilon_1 \cdot (2\pi\sqrt{-1})^2 \left(\mathrm{res}_{t=0} \frac{F(\alpha_0,\alpha_1;t)}{t} dt - \mathrm{res}_{t=1} \frac{F(\alpha_0,\alpha_1;t)}{t-1} dt \right).
\end{aligned} \tag{5.4}$$

Especially in case $\alpha_0 \geq 0$, $\alpha_1 \geq 0$,

$$\int_{Reg(0,1)} t^{\overline{\alpha_0}}(1-t)^{\overline{\alpha_1}}\eta_0^1 = \varepsilon_0\varepsilon_1 \cdot (2\pi\sqrt{-1})^2 B(\alpha_0+1,\alpha_1+1), \tag{5.5}$$

where $B(\alpha_0+1,\alpha_1+1)$ *is a classical Beta function of Euler.*

Proof. Because α_0, $\alpha_1 \in \mathbb{Z}$, we have $c_0 = c_1 = 1$. By the assumption,

$$\int_{\gamma_0} t^{\alpha_0}(1-t)^{\alpha_1} dt = \int_{\gamma_1} t^{\alpha_0}(1-t)^{\alpha_1} dt = 0.$$

Then we get the following by formula (5.1):

$$\begin{aligned}&\int_{Reg(0,1)} t^{\overline{\alpha_0}}(1-t)^{\overline{\alpha_1}}\eta_0^1\\ &= \varepsilon_0\varepsilon_1\cdot 2\pi\sqrt{-1}\left[\int_{\gamma_0} t^{\alpha_0+k-1}(1-t)^{\alpha_1}\log t dt\right.\\ &\qquad\left. -\int_{\gamma_1} t^{\alpha_0+k-1}(1-t)^{\alpha_1}\log(1-t)dt\right].\end{aligned}$$

By using lemma5.1, we can get (5.4). In case $\alpha_0\ge 0$, $\alpha_1\ge 0$,

$$\begin{aligned}\mathrm{res}_{t=0}\frac{F(\alpha_0,\alpha_1;t)}{t}dt &= F(\alpha_0,\alpha_1;0)=\int_{l_0} t^{\alpha_0}(1-t)^{\alpha_1}dt,\\ \mathrm{res}_{t=1}\frac{F(\alpha_0,\alpha_1;t)}{t-1}dt &= F(\alpha_0,\alpha_1;1)=\int_{l_1} t^{\alpha_0}(1-t)^{\alpha_1}dt.\end{aligned}$$

Then

$$\begin{aligned}&\mathrm{res}_{t=0}\frac{F(\alpha_0,\alpha_1;t)}{t}dt-\mathrm{res}_{t=1}\frac{F(\alpha_0,\alpha_1;t)}{t-1}dt\\ &=\int_{l_0} t^{\alpha_0}(1-t)^{\alpha_1}dt-\int_{l_1} t^{\alpha_0}(1-t)^{\alpha_1}dt\\ &=-\int_0^1 t^{\alpha_0}(1-t)^{\alpha_1}dt\\ &=-B(\alpha_0+1,\alpha_1+1).\end{aligned}$$

□

Example 5.2. We assume that both α_0 and α_1 are non-negative integers. We can calculate $B(\alpha_0+1,\alpha_1+1)$ by using generators of $H^1_{d+\omega_{\boldsymbol{\varepsilon}}}(X^{alg},R_{\boldsymbol{\varepsilon}})$ which are given by theorem 4.1. By the formula (4.1), we get

$$\begin{aligned}(-\varepsilon_\infty+\alpha_1+1)(-\varepsilon_\infty+\alpha_1+2)\cdots(-\varepsilon_\infty+\alpha_1+\alpha_0+1)\eta_0^1\\ =\varepsilon_0(\varepsilon_0+1)\cdots(\varepsilon_0+\alpha_0)\eta_0^{-\alpha_0}.\end{aligned}\tag{5.6}$$

We shall consider the pairing between $Reg(0,1)$ and each side of (5.6). By the formula (5.5), we get

$$\begin{aligned}(-\varepsilon_\infty+\alpha_1+1)(-\varepsilon_\infty+\alpha_1+2)\cdots(-\varepsilon_\infty+\alpha_1+\alpha_0+1)\int_{Reg(0,1)} t^{\overline{\alpha_0}}(1-t)^{\overline{\alpha_1}}\eta_0^1\\ =\varepsilon_0\varepsilon_1\cdot(2\pi\sqrt{-1})^2(\alpha_1+1)(\alpha_1+2)\cdots(\alpha_1+\alpha_0+1)B(\alpha_0+1,\alpha_1+1).\end{aligned}$$

On the other hand, we can get the following by using formula (5.1):

$$\begin{aligned}&\varepsilon_0(\varepsilon_0+1)\cdots(\varepsilon_0+\alpha_0)\int_{Reg(0,1)} t^{\overline{\alpha_0}}(1-t)^{\overline{\alpha_1}}\eta_0^{-\alpha_0}\\ &=\varepsilon_0\varepsilon_1\cdot 2\pi\sqrt{-1}\alpha_0!\int_{\gamma_0}(1-t)^{\alpha_1}\frac{dt}{t}\\ &=\varepsilon_0\varepsilon_1\cdot(2\pi\sqrt{-1})^2\alpha_0!.\end{aligned}$$

Hence

$$(\alpha_1+1)(\alpha_1+2)\cdots(\alpha_1+\alpha_0+1)B(\alpha_0+1,\alpha_1+1)=\alpha_0!,$$

that is,

$$B(\alpha_0+1,\alpha_1+1)=\frac{\alpha_0!\alpha_1!}{(\alpha_0+\alpha_1+1)!},$$

which is a familiar formula.

Appendix A. Appendix

Let $R_{\boldsymbol{\varepsilon}}$ be $\mathbb{C}[\varepsilon_0]/(\varepsilon_0^2)\otimes\mathbb{C}[\varepsilon_1]/(\varepsilon_1^2)$, and M be a $R_{\boldsymbol{\varepsilon}}$-module.

Lemma A.1. *Let $s,t\in M$ and $a,b\in R_{\boldsymbol{\varepsilon}}$.*

1. *There exists a unit $u\in R_{\boldsymbol{\varepsilon}}$ such that $\varepsilon_0 s+\varepsilon_1 t+\varepsilon_0\varepsilon_1(as+bt)=u(\varepsilon_0 s+\varepsilon_1 t)$.*
2. *There exists a unit $u\in R_{\boldsymbol{\varepsilon}}$ such that $\varepsilon_0 s+\varepsilon_\infty t+\varepsilon_0\varepsilon_\infty(as+bt)=u(\varepsilon_0 s+\varepsilon_\infty t)$.*
3. *If $\varepsilon_0 s+\varepsilon_1 t+\varepsilon_0\varepsilon_1(as+bt)=0$, then $\varepsilon_0 s+\varepsilon_1 t=0$ and $\varepsilon_0\varepsilon_1(as+bt)=0$.*
4. *If $\varepsilon_0 s+\varepsilon_\infty t+\varepsilon_0\varepsilon_\infty(as+bt)=0$, then $\varepsilon_0 s+\varepsilon_\infty t=0$ and $\varepsilon_0\varepsilon_\infty(as+bt)=0$.*

Proof.
1. Put $u=1+b\varepsilon_0+a\varepsilon_1$. This is a unit.
2. Put $u=1+(a+b)\varepsilon_0-a\varepsilon_1$. This is a unit.
3. This fact immediately follows from 1.
4. This fact immediately follows from 2.

□

Lemma A.2. *Let $s,t\in M$, $a_j,a_k\in R_{\boldsymbol{\varepsilon}}^\times$, $b_j,b_k\in R_{\boldsymbol{\varepsilon}}\setminus R_{\boldsymbol{\varepsilon}}^\times=(\varepsilon_0,\varepsilon_1)$ and $j,k\in\{0,1,\infty\}$.*

1. *There exists a unit $u\in R_{\boldsymbol{\varepsilon}}$ such that $\varepsilon_j(a_j+b_j)s+\varepsilon_k(a_k+b_k)t=u(\varepsilon_j a_j s+\varepsilon_k a_k t)$.*
2. *If $\varepsilon_j(a_j+b_j)s+\varepsilon_k(a_k+b_k)t=0$, then $\varepsilon_j a_j s+\varepsilon_k a_k t=0$ and $\varepsilon_j b_j s+\varepsilon_k b_k t=0$.*

Proof.
1. There exists an element $b_i'\in R_{\boldsymbol{\varepsilon}}$ such that $b_i'\varepsilon_j\varepsilon_k=\varepsilon_i b_i a_i^{-1}$ for $i=j$ or k because $b_i a_i^{-1}\in(\varepsilon_0,\varepsilon_1)$. Then

$$\varepsilon_j(a_j+b_j)s+\varepsilon_k(a_k+b_k)t=\varepsilon_j\cdot a_j s+\varepsilon_k\cdot a_k t+\varepsilon_j\varepsilon_k(b_j'\cdot a_j s+b_k'\cdot a_k t).$$

By applying lemma A.1 we can complete the proof.

2. By applying statement 1., $\varepsilon_j a_j s+\varepsilon_k a_k t=0$. Hence $\varepsilon_j b_j s+\varepsilon_k b_k t=\varepsilon_j(a_j+b_j)s+\varepsilon_k(a_k+b_k)t-(\varepsilon_j a_j s+\varepsilon_k a_k t)=0$.

□

Lemma A.3. *Let $a_0, a_1 \in R_{\boldsymbol{\varepsilon}}$. If $\varepsilon_0 a_0 + \varepsilon_1 a_1 = 0$, then*

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \in R_{\boldsymbol{\varepsilon}} \begin{bmatrix} \varepsilon_1 \\ -\varepsilon_0 \end{bmatrix} + R_{\boldsymbol{\varepsilon}} \begin{bmatrix} \varepsilon_0 \\ 0 \end{bmatrix} + R_{\boldsymbol{\varepsilon}} \begin{bmatrix} 0 \\ \varepsilon_1 \end{bmatrix}.$$

Proof. $\varepsilon_0\varepsilon_1 a_0 = \varepsilon_1(\varepsilon_0 a_0 + \varepsilon_1 a_1) = 0$. This implies that $a_0 \in (\varepsilon_0, \varepsilon_1)$. Then we can pick $s, t_0 \in R_{\boldsymbol{\varepsilon}}$ such that $a_0 = s\varepsilon_1 + t_0\varepsilon_0$ and hence $\varepsilon_1(s\varepsilon_0 + a_1) = 0$. This implies that $s\varepsilon_0 + a_1 \in (\varepsilon_1)$. Consequently we can pick $t_1 \in R_{\boldsymbol{\varepsilon}}$ such that $a_1 = -s\varepsilon_0 + t_1\varepsilon_1$. We have completed the proof of the lemma. □

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