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The group homology and an algebraic version of the zero-in-the-spectrum conjecture

By

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Abstract

We introduce an algorithm which transforms a finitely presented group G into another one G_Ψ. By using this, we can get many finitely presented groups whose group homology with coefficients in the group von Neumann algebra vanish, that is, many counterexamples to an algebraic version of the zero-in-the-spectrum conjecture. Moreover we prove that the Baum-Connes conjecture does not imply the algebraic version of the zero-in-the-spectrum conjecture for finitely presented groups. Also we will show that for any $p \geq 3$ the p-th group homology of G_Ψ coming from free groups has infinite rank.

1. Introduction

In this paper we will give an algorithm Ψ which transforms a discrete group G into another one G_Ψ and its applications to the zero-in-the-spectrum conjecture and the group homology. G_Ψ is given by successive procedures: taking infinite sums of G, a semi-direct product with $\mathbb{Z}$ and an HNN-extension (Section 2). We see that when G is a finitely presented group, then G_Ψ is also finitely presented. In the case when G satisfies certain conditions, then G_Ψ shows particular phenomena in so-called the zero-in-the-spectrum conjecture by Gromov ([3]) and the group homology.

Firstly we will give an application to the zero-in-the-spectrum conjecture. The conjecture claims that for a closed, aspherical and connected Riemannian manifold M there always exists some $p \geq 0$, such that zero belongs to the spectrum of the Laplace-Beltrami operator Δ_p acting on square integrable p-forms on the universal covering $\widetilde{M}$ of M. In the non-aspherical case there exist counterexamples ([4], [5]). In this paper we discuss the case of non-manifolds. Let $H_p(G; \mathcal{N}(G))$ be the homology of G with coefficients in the group von Neumann algebra $\mathcal{N}(G)$. It is known that if BG is a closed manifold, then the

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original conjecture is equivalent to the algebraic condition that for some $p \geq 0$, $H_p\,(G;\mathcal{N}(G)) \neq 0$ holds ([6, p.438]).

Here we use the following notation.

Definition 1.1. Let n be a non-negative integer or $n = \infty$. Define $\mathcal{F}_n$ to be the class of groups for which BG are CW-complexes which have a finite number of p-dimensional cells for $p \leq n$. Moreover define $\mathcal{F}_{clm}$ to be the class of groups for which BG are closed manifolds.

For example, $G \in \mathcal{F}_0$ if and only if G is a discrete group, $G \in \mathcal{F}_1$ if and only if G is a finitely generated group, $G \in \mathcal{F}_2$ if and only if G is a finitely presented group, and $G \in \mathcal{F}_\infty$ if and only if G is a finite type group.

Formally we can generalize the zero-in-the-spectrum conjecture to an algebraic version of it. When n is a non-negative integer, ∞ or clm, we will call the following conjecture the zero-in-the-spectrum conjecture for $\mathcal{F}_n$.

Conjecture 1.1. *Let G be in $\mathcal{F}_n$. Then for some $p \geq 0$, $H_p\,(G;\mathcal{N}(G)) \neq 0$ holds.*

Several counterexamples to the zero-in-the-spectrum conjecture for $\mathcal{F}_1$ are easily constructed, but they are infinitely presented. In this paper we show that many G_Ψ do not satisfy the zero-in-the-spectrum conjecture for $\mathcal{F}_2$. Actually, the following theorem is proved in Section 3.

Theorem 1.1. *When G is a non-amenable group, then G_Ψ satisfies $H_*\,(G_\Psi;\mathcal{N}(G_\Psi)) = 0$.*

In particular when G is a finitely presented and non-amenable group, then G_Ψ is a counterexample to the zero-in-the-spectrum conjecture for $\mathcal{F}_2$.

Moreover we should study the relation of G_Ψ to the Baum-Connes conjecture because it is known that the Baum-Connes conjecture implies the zero-in-the-spectrum conjecture for $\mathcal{F}_{clm}$ ([8, p.61]). The Baum-Connes conjecture identifies G-equivariant K-homology with G-compact supports of the classifying space $\underline{E}G$ for proper actions of G and the K-theory of the reduced C^*-algebra $C_r^*(G)$ ([8]). We show that the situation is completely different when BG is far from being a manifold. Assume that G is finitely presented, non-amenable and has the Haagerup property. For example, G could be a free group of rank $m \geq 2$. Then G_Ψ satisfies the Baum-Connes conjecture, but does not satisfy the Conjecture 1.1 (Section 3). Therefore,

Theorem 1.2. *The Baum-Connes conjecture does not imply the the zero-in-the-spectrum conjecture for $\mathcal{F}_2$.*

Finally we will calculate the group homology of G_Ψ coming from free groups in Section 4. Let $H_p\,(G;\mathbb{Z})$ be the group homology of G.

Theorem 1.3. *Suppose that G is a free group of rank $m \geq 1$, then G_Ψ*

satisfies the following.

$$
\begin{aligned}
&H_p(G_\Psi;\mathbb{Z}) hasinfinite\ rank\ (p \geq 3),\\
&H_2(G_\Psi;\mathbb{Z}) \cong \mathbb{Z}^{2m+m^2},\\
&H_1(G_\Psi;\mathbb{Z}) \cong \mathbb{Z}^{m+1},\\
&H_0(G_\Psi;\mathbb{Z}) \cong \mathbb{Z}.
\end{aligned}
$$

In particular G_Ψ is in $\mathcal{F}_2 \setminus \mathcal{F}_3$
Moreover Ψ is injective on the class of free groups.

This theorem shows that there exist many finitely presented groups most far from finite type groups by using only the rational homology.

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2. Construction of the algorithm Ψ

We will use $[g,h] := g^{-1}h^{-1}gh$ and $g^h := h^{-1}gh$ for $g, h \in G$.
We construct the algorithm

$$\Psi : \mathcal{F}_0 \to \mathcal{F}_0; G \mapsto G_\Psi$$

passing through three steps.

Let $G^{(k)}$ $(k \in \mathbb{Z})$ be infinite copies of G and $g^{(k)}$ be an element in $G^{(k)}$. We identify $G^{(0)}$ with G. Let us put

$$G_0 := \bigoplus_{k\in\mathbb{Z}} G^{(k)}, H_0 := \bigoplus_{l\in\mathbb{Z}} G^{(2l)} \oplus G^{(2l+1)}, K_0 := \bigoplus_{l\in\mathbb{Z}} G^{(3l)} \oplus G^{(3l+1)}.$$

$G_1 := G_0 \rtimes \mathbb{Z}$ is the HNN-extension of $G_0 = \bigoplus_{k\in\mathbb{Z}} G^{(k)}$ by the isomorphism

$$G_0 \xrightarrow{\sim} G_0; g^{(k)} \mapsto g^{(k+1)}.$$

$H_1 := H_0 \rtimes \mathbb{Z}$ is the HNN-extension of $H_0 = \bigoplus_{l\in\mathbb{Z}} G^{(2l)} \oplus G^{(2l+1)}$ by the isomorphism

$$H_0 \xrightarrow{\sim} H_0; g^{(k)} \mapsto g^{(k+2)}.$$

$K_1 := K_0 \rtimes \mathbb{Z}$ is the HNN-extension of $K_0 = \bigoplus_{l\in\mathbb{Z}} G^{(3l)} \oplus G^{(3l+1)}$ by the isomorphism

$$K_0 \xrightarrow{\sim} K_0; g^{(k)} \mapsto g^{(k+3)}.$$

Then we have presentations as:

$$G_1 = \langle G, a \mid [G, G^{a^k}](0 \neq k \in \mathbb{Z})\rangle,$$
$$H_1 = \left\langle G^{(0)}, G^{(1)}, b \;\middle|\; \begin{array}{l} [G^{(0)}, (G^{(1)})^{b^k}](k \in \mathbb{Z}), \\ [G^{(0)}, (G^{(0)})^{b^k}], [G^{(1)}, (G^{(1)})^{b^k}](0 \neq k \in \mathbb{Z}) \end{array} \right\rangle,$$
$$K_1 = \left\langle G^{(0)}, G^{(1)}, c \;\middle|\; \begin{array}{l} [G^{(0)}, (G^{(1)})^{c^k}](k \in \mathbb{Z}), \\ [G^{(0)}, (G^{(0)})^{c^k}], [G^{(1)}, (G^{(1)})^{c^k}](0 \neq k \in \mathbb{Z}) \end{array} \right\rangle.$$

Let us regard H_1 and K_1 as subgroups of G_1 by

$$H_1 \hookrightarrow G_1; g^{(0)}, g^{(1)}, b \mapsto g, g^a, a^2,$$
$$K_1 \hookrightarrow G_1; g^{(0)}, g^{(1)}, c \mapsto g, g^a, a^3.$$

Definition 2.1. G_Ψ is the HNN-extension of G_1 by the isomorphism

$$H_1 \xrightarrow{\sim} K_1; g^{(0)}, g^{(1)}, b \mapsto g^{(0)}, g^{(1)}, c.$$

Then we have a presentation as:

$$G_\Psi = \left\langle G, a, t \;\middle|\; \begin{array}{l} [G, G^{a^k}](0 \neq k \in \mathbb{Z}), \\ g^t = g, (g^a)^t = g^a(g \in G), (a^2)^t = a^3 \end{array} \right\rangle.$$

Here we claim the following.

Lemma 2.1. *The relations*

$$g^t = g, (g^a)^t = g^a(g \in G), (a^2)^t = a^3, 1 = [G, G^a]$$

imply the following relations

$$1 = [G, G^{a^k}](0 \neq k \in \mathbb{Z}).$$

Proof. We have

$$1 = [G, G^a]^{ata^{-1}} = [G^a, G^{a^2}]^{ta^{-1}} = [G^a, G^{a^3}]^{a^{-1}} = [G, G^{a^2}]$$

and

$$1 = [G, G^{a^2}]^t = [G, G^{a^3}].$$

Suppose $1 = [G, G^{a^k}]$ for $1 \leq k \leq 3N$ $(N \geq 1)$. Then since $2N+1 \leq 3N$ we have

$$1 = [G, G^{a^{2N+1}}]^t = [G, G^{a^{3N+1}}],$$
$$1 = [G, G^{a^{2N+1}}]^{ata^{-1}} = [G^a, G^{a^{2(N+1)}}]^{ta^{-1}} = [G^a, G^{a^{3(N+1)}}]^{a^{-1}} = [G, G^{a^{3N+2}}].$$

Then since $2(N+1) \leq 3N+1$ we have

$$1 = [G, G^{a^{2(N+1)}}]^t = [G, G^{a^{3(N+1)}}].$$

Hence we have $1 = [G, G^{a^k}]$ for $1 \le k \le 3(N+1)$ $(N \ge 1)$. Consequently we have $1 = [G, G^{a^k}]$ for $k \ge 1$. Moreover we have

$$1 = ([G, G^{a^k}]^{a^{-k}})^{-1} = [G^{a^{-k}}, G]^{-1} = [G, G^{a^{-k}}]$$

for $k \ge 1$. Thus we get $1 = [G, G^{a^k}]$ for $0 \ne k \in \mathbb{Z}$. $\square$

Hence we get the most important property of the algorithm Ψ.

Corollary 2.1. *Let*

$$G = \langle s_i (1 \le i \le m) \mid r_i (1 \le i \le n) \rangle$$

be a presentation. Then we have

$$G_\Psi = \left\langle s_i (1 \le i \le m), a, t \;\middle|\; \begin{array}{l} r_i (1 \le i \le n), [s_i, s_j^a] (1 \le i, j \le m), \\ s_i^t = s_i, (s_i^a)^t = s_i^a (1 \le i \le m), (a^2)^t = a^3 \end{array} \right\rangle .$$

In particular when G is finitely presented or generated, G_Ψ has the same property respectively.

If G is a finitely generated free group of rank $m \ge 1$, then we have

$$H_1(G_\Psi; \mathbb{Z}) \cong G_\Psi / [G_\Psi, G_\Psi] = \langle s_i (1 \le i \le m), t \rangle \cong \mathbb{Z}^{m+1}.$$

Accordingly we have the following.

Corollary 2.2. *Ψ is injective on the class of free groups.*

Also we can confirm easily that the algorithm Ψ has the following properties.

Proposition 2.1. *G_Ψ is torsion-free if and only if G is torsion-free.*

Proof. G_Ψ is an HNN-extension of G_1 and G_1 is an HNN-extension of G_0. Thus this proposition is clear by the torsion theorem for HNN-extensions ([7, p.185]). $\square$

Proposition 2.2. *The cohomological dimension of G_Ψ is infinite if and only if G is not trivial.*

Proof. G has a torsion element if and only if G_Ψ has a torsion element by Proposition 2.1. Then the cohomological dimension of each is infinite. If G is torsion-free and not trivial, then we have $G \supset \mathbb{Z}$. Thus we have $G_\Psi \supset \bigoplus_{k \in \mathbb{Z}} \mathbb{Z}$. Consequently the cohomological dimension of G_Ψ is infinite. If G is trivial, then we have $G_\Psi = \langle a, t \mid (a^2)^t = a^3 \rangle$. Hence G_Ψ is a torsion-free one-relator group. Therefore the cohomological dimension of G_Ψ is two. $\square$

3. Counterexamples to the zero-in-the-spectrum conjecture for $\mathcal{F}_2$

In this section we will get counterexamples to the zero-in-the-spectrum conjecture for $\mathcal{F}_2$.

Definition 3.1. Let d be a non-negative integer or ∞. Define $\mathcal{Z}_d$ to be the class of groups for which $H_p(G; \mathcal{N}(G)) = 0$ hold for $p \leq d$.

Lemma 3.1. *Let d, e be a non-negative integer or ∞. Then*

(1) *Let G be the directed union $\bigcup_{i \in I} G_i$ of subgroups $G_i \subset G$. Suppose that $G_i \in \mathcal{Z}_d$ for each $i \in I$. Then we have $G \in \mathcal{Z}_d$.*

(2) *If G contains a normal subgroup $H \subset G$ with $H \in \mathcal{Z}_d$, then we have $G \in \mathcal{Z}_d$.*

(3) *If $G \in \mathcal{Z}_d$ and $H \in \mathcal{Z}_e$, then we have $G \times H \in \mathcal{Z}_{d+e+1}$.*

(4) *$\mathcal{Z}_0$ is the class of non-amenable groups.*

(5) *Let $G = G_1 *_A G_2$ where $A \hookrightarrow G_1$ and $A \hookrightarrow G_2$. Suppose that $G_1, G_2 \in \mathcal{Z}_d$ and $A \in \mathcal{Z}_{d-1}$. Then we have $G \in \mathcal{Z}_d$.*

(6) *Let $G = H*_A = \langle H, t \mid \theta(a) = a^t \rangle$ where $A \subset H$ and $\theta : A \hookrightarrow H$. Suppose that $H \in \mathcal{Z}_d$ and $A \in \mathcal{Z}_{d-1}$. Then we have $G \in \mathcal{Z}_d$.*

Proof. $(1) \sim (4)$ are proved in [6, p.448] and $(5), (6)$ are clear by Mayer-Vietoris sequences ([1, p.178]), where we use that induction with an injective homomorphism between two groups is a faithfully flat functor between two categories of group von Neumann algebra modules ([6, p.253]). □

Now we can prove Theorem 1.1 by using Corollary 2.1 and the above Lemma.

Proof. When G is non-amenable, then we have $G_0, H_0 \in \mathcal{Z}_\infty$ by Lemma 3.1 (1), (3), (4). Moreover we have $G_1, H_1 \in \mathcal{Z}_\infty$ by Lemma 3.1 (2) or (6). Accordingly we have $G_\Psi \in \mathcal{Z}_\infty$ by Lemma 3.1 (6).

In particular when G is finitely presented and non-amenable, G_Ψ is a counterexample to the zero-in-the-spectrum conjecture for $\mathcal{F}_2$ by Corollary 2.1. Also we note that G_0 is a counterexample to the zero-in-the-spectrum conjecture for $\mathcal{F}_0$ and G_1 is a counterexample to the zero-in-the-spectrum conjecture for $\mathcal{F}_1$. □

We note that there exist many counterexamples to the zero-in-the-spectrum conjecture for $\mathcal{F}_2$ by Corollary 2.2.

We see the relation of G_Ψ to the Baum-Connes conjecture. We refer to [8] about the Baum-Connes conjecture and Haagerup property.

Proposition 3.1. *Suppose that G has Haagerup property, then G_Ψ satisfies the Baum-Connes conjecture.*

Proof. If G has Haagerup property, then $\bigoplus_{-K \leq k \leq K} G^{(k)}$ has Haagerup property, too. So $\bigoplus_{-K \leq k \leq K} G^{(k)}$ satisfies the Baum-Connes conjecture ([8, p.43]). G_0 and H_0 satisfy the Baum-Connes conjecture because G_0 and H_0 are directed unions of $\bigoplus_{-K \leq k \leq K} G^{(k)}$ for all $K \in \mathbb{Z}$ ([8, p.38]). G_1 and H_1 satisfy the Baum-Connes conjecture because G_1 and H_1 are HNN-extensions of

G_0 and H_0 respectively ([8, p.40]). Therefore G_Ψ satisfies the Baum-Connes conjecture because G_Ψ is an HNN-extension of G_1 on H_1 ([8, p.40]). $\square$

When G is a free group of rank $m \geq 2$, this is finitely presented, non-amenable and has the Haagerup property. Hence Theorem 1.2 follows by using Theorem 1.1 and the above proposition.

Unfortunately none of G_Ψ is a counterexample to the zero-in-the-spectrum conjecture for $\mathcal{F}_{clm}$ because if G is not trivial, then the cohomological dimension of G_Ψ is infinite and if G is trivial, G_Ψ satisfies the Baum-Connes conjecture.

The author believes that neither the zero-in-the-spectrum conjecture for $\mathcal{F}_n$ for any $n = 3, 4, \ldots, \infty$ is true nor the Baum-Connes conjecture implies the zero-in-the-spectrum conjecture for $\mathcal{F}_n$ for any $n = 3, 4, \ldots, \infty$. We may expect that G_Ψ can be a counterexample for $\mathcal{F}_n$ for $n = 3, 4, \ldots, \infty$, but for example G_Ψ coming from a free group is not contained in $\mathcal{F}_3$. In fact we calculate the group homology of G_Ψ coming from free groups by using Künneth formula and a Mayer-Vietoris sequence in the next section.

4. The group homology of G_Ψ coming from a free group

In this section, we calculate the group homology of G_Ψ coming from a free group G, that is, we give the proof of Theorem 1.3. Let the generators of G be $s_i (1 \leq i \leq m)$.

We will follow five steps.

Firstly we can decide the group homology of G_0, H_0 and K_0 by

$$\begin{aligned} &H_n(G; \mathbb{Z}) \cong 0 \ (n \geq 2), \\ &H_1(G; \mathbb{Z}) = \langle s_i (1 \leq i \leq m) \rangle, \\ &H_0(G; \mathbb{Z}) \cong \mathbb{Z}. \end{aligned}$$

and Künneth formula. In fact we have

$$H_n(G_0 = H_0; \mathbb{Z}) = \left\langle \begin{array}{l} s_{i_1}^{(k_1)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)} \\ (1 \leq i_1, i_2, \ldots, i_n \leq m, \ k_1 < k_2 < \cdots < k_n) \end{array} \right\rangle \ (n \geq 1),$$

$$H_0(G_0 = H_0; \mathbb{Z}) \cong \mathbb{Z}.$$

$$H_n(K_0; \mathbb{Z}) = \left\langle \begin{array}{l} s_{i_1}^{(k_1)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)} (1 \leq i_1, i_2, \ldots, i_n \leq m, \\ k_1 < k_2 < \cdots < k_n, \ k_j \equiv 0, \ 1 \mod 3) \end{array} \right\rangle \ (n \geq 1),$$

$$H_0(K_0; \mathbb{Z}) \cong \mathbb{Z}.$$

Secondly we will decide the group homology of G_1. $G_1 = G_0 \rtimes \mathbb{Z}$ is the HNN-extension of $G_0 = \bigoplus_{k \in Z} G^{(k)}$ by the isomorphism

$$\theta : G_0 \xrightarrow{\sim} G_0; s_i^{(k)} \mapsto s_i^{(k+1)}.$$

Thus we can use a Mayer-Vietoris sequence

$$\cdots \to H_n(G_0;\mathbb{Z}) \overset{\alpha_n}{\to} H_n(G_0;\mathbb{Z}) \to H_n(G_1;\mathbb{Z}) \to H_{n-1}(G_0;\mathbb{Z}) \to \cdots$$

where $\alpha_* := \theta_* - id_*$.

Lemma 4.1. *α_n is injective for $n \geq 1$.*

Proof. Let us put $\mathbf{k} := (k_1, k_2, \ldots, k_n), \mathbf{1} := (1, 1, \ldots, 1), \mathbf{i} := (i_1, i_2, \ldots, i_n)$ and $s_{\mathbf{i}}^{\mathbf{k}} := s_{i_1}^{(k_1)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}$.

Now we have $\alpha_n(s_{\mathbf{i}}^{\mathbf{k}}) = s_{\mathbf{i}}^{\mathbf{k+1}} - s_{\mathbf{i}}^{\mathbf{k}}$. If we have $\alpha_n(\sum \lambda_{\mathbf{k}}^{\mathbf{i}} s_{\mathbf{i}}^{\mathbf{k}}) = 0$, then we have $\sum(\lambda_{\mathbf{k-1}}^{\mathbf{i}} - \lambda_{\mathbf{k}}^{\mathbf{i}}) s_{\mathbf{i}}^{\mathbf{k}}) = 0$. Hence we have $\lambda_{\mathbf{k}}^{\mathbf{i}} = \lambda_{\mathbf{k-1}}^{\mathbf{i}}$. Because $H_n(G_0;\mathbb{Z})$ is finitely generated, we have $\lambda_{\mathbf{k}}^{\mathbf{i}} = 0$ $(\forall \mathbf{i}, \forall\ \mathbf{k})$. □

Because we have $\alpha_n(s_{\mathbf{i}}^{\mathbf{k}}) = s_{\mathbf{i}}^{\mathbf{k+1}} - s_{\mathbf{i}}^{\mathbf{k}}$ and

$$H_n(G_1;\mathbb{Z}) \cong H_n(G_0;\mathbb{Z})/\alpha_n(H_n(G_0;\mathbb{Z}))$$

for $n \geq 2$, we have

$$\begin{aligned} &H_n(G_1;\mathbb{Z}) \cong \left\langle \begin{array}{l} [s_{i_1}^{(0)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}] \\ (1 \leq i_1, i_2, \ldots, i_n \leq m,\ 0 < k_2 < \cdots < k_n) \end{array} \right\rangle \ (n \geq 2), \\ &H_1(G_1;\mathbb{Z}) \cong G_1/[G_1, G_1] = \langle s_i (1 \leq i \leq m), a \rangle, \\ &H_0(G_1;\mathbb{Z}) \cong \mathbb{Z}, \end{aligned}$$

where $[s_{i_1}^{(0)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}]$ denotes the equivalence class of $s_{i_1}^{(0)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}$ in $H_n(G_0;\mathbb{Z})/\alpha_n(H_n(G_0;\mathbb{Z}))$.

Thirdly we will decide the group homology of H_1. $H_1 := H_0 \rtimes \mathbb{Z}$ is the HNN-extension of $H_0 = \bigoplus_{l \in \mathbb{Z}} G^{(2l)} \oplus G^{(2l+1)}$ by the isomorphism

$$\theta' : H_0 \overset{\sim}{\to} H_0; s_i^{(k)} \mapsto s_i^{(k+2)}.$$

Thus we can use a Mayer-Vietoris sequence

$$\cdots \to H_n(H_0;\mathbb{Z}) \overset{\alpha_n'}{\to} H_n(H_0;\mathbb{Z}) \to H_n(H_1;\mathbb{Z}) \to H_{n-1}(H_0;\mathbb{Z}) \to \cdots$$

where $\alpha_*' := \theta_*' - id_*$. We have the following by the same argument as that in the proof of Lemma 4.1.

Lemma 4.2. *α_n' is injective for $n \geq 1$.*

Because we have $\alpha_n'(s_{\mathbf{i}}^{\mathbf{k}}) = s_{\mathbf{i}}^{\mathbf{k+2}} - s^{\mathbf{k}}_{textbfi}$ and for $n \geq 2$

$$H_n(H_1;\mathbb{Z}) \cong H_n(H_0;\mathbb{Z})/\alpha_n'(H_n(H_0;\mathbb{Z})),$$

we have

$$H_n(H_1;\mathbb{Z}) \cong \left\langle \begin{array}{l} [s_{i_1}^{(0)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}]' \\ (1 \le i_1, i_2, \ldots, i_n \le m,\ 0 < k_2 < \cdots < k_n) \\ [s_{i_1}^{(1)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}]' \\ (1 \le i_1, i_2, \ldots, i_n \le m,\ 1 < k_2 < \cdots < k_n) \end{array} \right\rangle \ (n \ge 2),$$

$$H_1(H_1;\mathbb{Z}) \cong H_1/[H_1,H_1] = \langle s_i^{(0)}, s_i^{(1)} (1 \le i \le m), b\rangle,$$
$$H_0(H_1;\mathbb{Z}) \cong \mathbb{Z},$$

where $[s_{i_1}^{(0)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}]'$ and $[s_{i_1}^{(1)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}]'$ denote the equivalence classes of $s_{i_1}^{(0)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}$ and $s_{i_1}^{(1)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}$ in $H_n(H_0;\mathbb{Z})/\alpha_n^{'}(H_n(H_0;\mathbb{Z}))$ respectively.

Fourthly we will decide the group homology of K_1. $K_1 := K_0 \rtimes \mathbb{Z}$ is the HNN-extension of $K_0 = \bigoplus_{l\in\mathbb{Z}} G^{(3l)} \oplus G^{(3l+1)}$ by the isomorphism

$$\theta^{''} : K_0 \xrightarrow{\sim} K_0; s_i^{(k)} \mapsto s_i^{(k+3)}.$$

Thus we can use a Mayer-Vietoris sequence

$$\cdots \to H_n(K_0;\mathbb{Z}) \xrightarrow{\alpha_n^{''}} H_n(K_0;\mathbb{Z}) \to H_n(K_1;\mathbb{Z}) \to H_{n-1}(K_0;\mathbb{Z}) \to \cdots$$

where $\alpha_*^{''} := \theta_n^{''} - id_*$. We have the following by the same argument as that in the proof of Lemma 4.1.

Lemma 4.3. *$\alpha_n^{''}$ is injective for $n \ge 1$.*

Because we have $\alpha_n^{''}(s_{\mathbf{i}}^{\mathbf{k}}) = s_{\mathbf{i}}^{\mathbf{k+3}} - s_{\mathbf{i}}^{\mathbf{k}} (k_1 < k_2 < \cdots < k_n,\ k_j \equiv 0,\ 1 \mod 3)$ and

$$H_n(K_1;\mathbb{Z}) \cong H_n(K_0;\mathbb{Z})/\alpha_n^{''}(H_n(K_0;\mathbb{Z}))$$

for $n \ge 2$, we have

$$H_n(K_1;\mathbb{Z}) \cong \left\langle \begin{array}{l} [s_{i_1}^{(0)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}]'' (1 \le i_1, i_2, \ldots, i_n \le m, \\ 0 < k_2 < \cdots < k_n,\ k_j \equiv 0,\ 1 \mod 3) \\ [s_{i_1}^{(1)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}]'' (1 \le i_1, i_2, \ldots, i_n \le m, \\ 1 < k_2 < \cdots < k_n,\ k_j \equiv 0,\ 1 \mod 3) \end{array} \right\rangle \ (n \ge 2),$$

$$H_1(K_1;\mathbb{Z}) \cong K_1/[K_1,K_1] = \langle s_i^{(0)}, s_i^{(1)} (1 \le i \le m), c\rangle,$$
$$H_0(K_1;\mathbb{Z}) \cong \mathbb{Z},$$

where $[s_{i_1}^{(0)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}]''$ and $[s_{i_1}^{(1)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}]''$ denote the equivalence classes of $s_{i_1}^{(0)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}$ and $s_{i_1}^{(1)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}$ in $H_n(K_0;\mathbb{Z})/\alpha_n^{''}(H_n(K_0;\mathbb{Z}))$ respectively.

Finally we will calculate the group homology of G_Ψ. G_Ψ is the HNN-extension of G_1 by the isomorphism

$$\phi : H_1 \xrightarrow{\sim} K_1; s_i^{(0)}, s_i^{(1)}, b \mapsto s_i^{(0)}, s_i^{(1)}, c.$$

Thus we can use a Mayer-Vietoris sequence

$$\cdots \to H_n(H_1;\mathbb{Z}) \xrightarrow{\beta_n} H_n(G_\Psi;\mathbb{Z}) \to H_n(G_\Psi;\mathbb{Z}) \to H_{n-1}(H_1;\mathbb{Z}) \to \cdots$$

where $\beta_* := \phi_* - i_*$. We use $\mathbf{l} := (0, l_2, \ldots, l_n), \mathbf{q} := (q_1, q_2, \ldots, q_n), (q_1, q_2, \ldots, q_n = 0,\ 1,\ q_1 < 2l_2 + q_2 < \ldots < 2l_n + q_n)$. Since we have $\beta_n([s_{\mathbf{i}}^{\mathbf{2l+q}}]') = [s_{\mathbf{i}}^{\mathbf{3l+q}}] - [s_{\mathbf{i}}^{\mathbf{2l+q}}]$, we have $\beta_n([s_{\mathbf{i}}^{\mathbf{2l}}]') = \beta_n([s_{\mathbf{i}}^{\mathbf{2l+1}}]')$. Thus we have $\operatorname{Ker}\beta_n \supset \langle [s_{\mathbf{i}}^{\mathbf{2l+1}}]' - [s_{\mathbf{i}}^{\mathbf{2l}}]'(0 < 2l_2 < \ldots < 2l_n)\rangle$. Hence $\operatorname{Ker}\beta_n$ has infinite rank for $n \geq 2$. Thus $H_{n+1}(G_\Psi;\mathbb{Z})$ has infinite rank, too. Also since we have $\operatorname{Ker}\beta_1 = \langle s_i^{(0)}, s_i^{(1)}\rangle \cong \mathbb{Z}^{2m}$ and $H_2(G_1;\mathbb{Z})/\beta_2(H_2(H_1;\mathbb{Z})) \cong \langle [s_{i_1}^{(0)} \times s_{i_2}^{(1)}]\rangle \cong \mathbb{Z}^{m^2}$, we have $H_2(G_\Psi;\mathbb{Z}) \cong \mathbb{Z}^{2m+m^2}$. Hence we observe

$$\begin{aligned}
&H_n(G_\Psi;\mathbb{Z}) has infinite\ rank\ (n \geq 3),\\
&H_2(G_\Psi;\mathbb{Z}) \cong \mathbb{Z}^{2m+m^2},\\
&H_1(G_\Psi;\mathbb{Z}) \cong G_\Psi/[G_\Psi, G_\Psi] = \langle s_i(1 \leq i \leq m), t\rangle,\\
&H_0(G_\Psi;\mathbb{Z}) \cong \mathbb{Z}.
\end{aligned}$$

Let G_2 be G_Ψ. In this section, we proved that for $n = 0, 1, 2$ G_n coming from a free group of rank $m \geq 1$ is in $\mathcal{F}_n$ and the p-th group homology of G_n has infinite rank for any $p \geq n+1$. It is known when n is a non-negative integer, then $\mathcal{F}_n \supsetneq \mathcal{F}_{n+1}$ ([2]). Here we will formulate the following conjecture.

Conjecture 4.1. *When n is a non-negative integer, then there is $G \in \mathcal{F}_n$ whose p-th group homology has infinite rank for any $p \geq n+1$.*

The author does not know whether this holds or not except for the case $n = 0, 1, 2$.

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