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The phase space of coupled Painlevé III system in dimension four

By

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Abstract

We find and study a 3-parameter family of coupled Painlevé III systems in dimension four, which can be obtained by a degeneration from the system of type $A_5^{(1)}$. We also give the phase space for this system.

1. Introduction

In 1998, Noumi and Yamada [2] proposed a system of autonomous ordinary differential equations for $l+1$ unknown functions $f_0, f_1, \dots, f_l$ involving complex parameters $\alpha_0, \alpha_1, \dots, \alpha_l$ satisfying $\alpha_0 + \alpha_1 + \dots + \alpha_l = 1$. This system's salient feature is that it has the symmetry under the affine Weyl group of type $A_l^{(1)}$, where $\alpha_0, \alpha_1, \dots, \alpha_l$ are considered as simple roots of the affine root system of type $A_l^{(1)}$. When $l = 3$, this system of type $A_3^{(1)}$ is equivalent to the fifth Painlevé equation P_V. When $l > 3$, the higher order Painlevé equations corresponding to these systems are not known to satisfy the Painlevé property, but it is widely believed that this is the case. They are considered to be higher order versions of P_V (resp. P_{IV}) when l is odd (resp. even).

It is well-known that P_V has a confluence to the third Painlevé equation P_{III}, where two accessible singularities come together into a single singularity. This suggests the possibility that there exist higher order versions of P_{III} as well, and furthermore, suggests a procedure for searching for such higher order versions. In this vein, the goal of this work is to find a fourth-order version of the Painlevé III equation. The purpose of this paper is to present a 3-parameter family of fourth-order algebraic ordinary differential equations that can be considered as a coupled Painlevé III system in dimension four, and which

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is given as follows:

$$
(1.1)\quad \begin{cases}
\dfrac{dx}{dt} = \dfrac{2x^2y + 2xzw - x^2 + (1-2\alpha_1)x}{t} + 1, \\
\dfrac{dy}{dt} = \dfrac{-2xy^2 - 2yzw + 2xy - (1-2\alpha_1)y + \alpha_0}{t}, \\
\dfrac{dz}{dt} = \dfrac{2z^2w + 2xyz - z^2 + (1-2\alpha_1)z}{t} + 1, \\
\dfrac{dw}{dt} = \dfrac{-2zw^2 - 2xyw + 2zw - (1-2\alpha_1)w + \beta_0}{t}.
\end{cases}
$$

Here x, y, z and w denote unknown complex variables, and α_0, α_1 and β_0 are complex parameters.

Our differential system is equivalent to a Hamiltonian system given by

$$
(1.2)\quad \begin{aligned}
H &= \frac{x^2y(y-1) + x((1-2\alpha_1)y - \alpha_0) + ty}{t} \\
&\quad + \frac{z^2w(w-1) + z((1-2\alpha_1)w - \beta_0) + tw}{t} + \frac{2xyzw}{t} \\
&= H_{III}(x, y, t; \alpha_0, \alpha_1) + H_{III}(z, w, t; \beta_0, \alpha_1) + \frac{2xyzw}{t}.
\end{aligned}
$$

Here $H_{III}(q, p, t; \alpha_0, \alpha_1)$ denotes the Hamiltonian of the second-order Painlevé III equations.

Our system (1.1) has the following symmetry.

Theorem 1.1. *The system* (1) *is invariant under the following transformations defined as follows*: *with the notation* $(*) = (x, y, z, w, t; \alpha_0, \alpha_1, \beta_0)$

$$
\begin{aligned}
s_1 &: (*) \to \left(x + \frac{\alpha_0}{y}, y, z, w, t; -\alpha_0, \alpha_1 + \alpha_0, \beta_0\right), \\
s_2 &: (*) \to \left(x, y, z + \frac{\beta_0}{w}, w, t; \alpha_0, \alpha_1 + \beta_0, -\beta_0\right), \\
s_3 &: (*) \to \left(x + \frac{1 - \alpha_0 - 2\alpha_1 - \beta_0}{(y + w - 1)}, y, z + \frac{1 - \alpha_0 - 2\alpha_1 - \beta_0}{(y + w - 1)}, w, t;\right. \\
&\qquad\quad \left.\alpha_0, 1 - \alpha_0 - \alpha_1 - \beta_0, \beta_0\right), \\
\pi_1 &: (*) \to \left(\frac{t}{x}, -\frac{x(xy + \alpha_0)}{t}, \frac{t}{z}, -\frac{z(zw + \beta_0)}{t}, t; \alpha_0, \frac{1}{2} - \alpha_1 - \alpha_0 - \beta_0, \beta_0\right), \\
\pi_2 &: (*) \to (z, w, x, y, t; \beta_0, \alpha_1, \alpha_0).
\end{aligned}
$$

Lemma 1.1. *The transformations described in Theorem* (1.1) *satisfy the following relations*:

$$
\begin{aligned}
&s_1^2 = s_2^2 = s_3^2 = \pi_1^2 = \pi_2^2 = (s_1s_2)^2 = (s_1s_3)^2 = (s_2s_3)^2 = (\pi_1\pi_2)^2 = 1, \\
&\pi_1(s_1, s_2) = (s_1, s_2)\pi_1, \quad \pi_2(s_1, s_2, s_3) = (s_2, s_1, s_3)\pi_2.
\end{aligned}
$$

Remark 1. The transformation $(s_3\pi_1)^2$ acts on the parameter as its translation:

$$(s_3\pi_1)^2 : (\alpha_0, \alpha_1, \beta_0) \rightarrow (\alpha_0, \alpha_1, \beta_0) + (0, -1, 0). \tag{1.3}$$

Moreover, we show that the system (1.1) has the following first integral.

Theorem 1.2. *The system* (1.1) *has the following first integral* I:

$$I = (x - z)(xyw - yzw + \alpha_0 w - \beta_0 y).$$

Theorems 1.1, 1.2 and Lemma 1.1 can be checked by a direct calculation.

As the fourth-order analogue of $P_V \rightarrow P_{III}$, we consider a degeneration from the system of type $A_5^{(1)}$. Here, we recall the system of type $A_5^{(1)}$ (see [2])

(1.4)

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{2x^2y + 2xzw}{t} + x^2 - \dfrac{2xy + 2zw}{t} - \left(1 + \dfrac{\alpha_1 + \beta_1 + \alpha_3}{t}\right) x + \dfrac{\alpha_1 + \beta_1}{t}, \\ \dfrac{dy}{dt} = -\dfrac{2xy^2 + 2yzw}{t} + \dfrac{y^2}{t} - 2xy + \left(1 + \dfrac{\alpha_1 + \beta_1 + \alpha_3}{t}\right) y - \alpha_2, \\ \dfrac{dz}{dt} = \dfrac{2z^2w + 2xyz}{t} + z^2 - \dfrac{2zw + 2yz}{t} - \left(1 + \dfrac{\alpha_1 + \beta_1 + \alpha_3}{t}\right) z + \dfrac{\beta_1}{t}, \\ \dfrac{dw}{dt} = -\dfrac{2zw^2 + 2xyw}{t} + \dfrac{w^2}{t} - 2zw + \dfrac{2yw}{t} + \left(1 + \dfrac{\alpha_1 + \beta_1 + \alpha_3}{t}\right) w - \beta_2, \end{cases}$$

where $\alpha_1, \alpha_2, \alpha_3$ and β_1, β_2 are complex parameters.

Theorem 1.3. *For the system* (1.4) *of type* $A_5^{(1)}$*, we make a change of parameters and variables*

$$\begin{aligned} &\alpha_1 = 0, \quad \alpha_2 = A_0, \quad \alpha_3 = 2A_1 - \varepsilon^{-1}, \quad \beta_1 = \varepsilon^{-1}, \quad \beta_2 = B_0, \\ &t = -\varepsilon T, \quad x = 1 + \frac{X}{\varepsilon T}, \quad y = \varepsilon TY, \quad z = 1 + \frac{Z}{\varepsilon T}, \quad w = \varepsilon TW, \end{aligned}$$

from $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, t, x, y, z, w$ *to* $A_0, A_1, B_0, \varepsilon, T, X, Y, Z, W$*. Then the system* (1.4) *can also be written in the new variables* T, X, Y, Z, W *and parameters* $A_0, A_1, B_0, \varepsilon$ *as a Hamiltonian system. This new system tends to the system* (1.1) *as* $\varepsilon \rightarrow 0$.

We regard the system (1.1) as an algebraic vector field v defined on $\mathbb{C}^4 \times B$:

$$v = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z} + \frac{dw}{dt}\frac{\partial}{\partial w}, \quad (x, y, z, w, t) \in \mathbb{C}^4 \times B$$

with $B = \mathbb{C}$. If we take a relative compactification $\mathbb{P}^4 \times B$ of $\mathbb{C}^4 \times B$, the extended vector field $\tilde{v}$ satisfies the condition:

$$\tilde{v} \in H^0(\mathbb{P}^4, \Theta_{\mathbb{P}^4}(-\log \mathcal{H})(\mathcal{H})).$$

Here $\mathcal{H}$ is the boundary divisor in $\mathbb{P}^4$ and $\Theta_{\mathbb{P}^4}(-\log \mathcal{H})(\mathcal{H})$ is the subsheaf of $\Theta_{\mathbb{P}^4}$ whose local section v satisfies $v(f) \in (f)$ for any local equation f of $\mathcal{H}$. Let us extend the regular vector field v on $\mathbb{C}^4 \times B$ to a rational vector field $\tilde{v}$ on $\mathbb{P}^4 \times B$. Then $\tilde{v}$ has poles along the boundary divisor $\mathcal{H}$. Moreover, $\tilde{v}$ has accessible singularities along subvarieties in the boundary divisor $\mathcal{H}$. (For the definition of accessible singularities, see Definition 2.1.) In order to explain our main results, we recall the definition of a symplectic transformation and its properties (see [1], [10]). Let

$$\varphi : x = x(X,Y,Z,W,t),\ y = y(X,Y,Z,W,t),\ z = z(X,Y,Z,W,t),$$
$$w = w(X,Y,Z,W,t),\ t = t$$

be a biholomorphic mapping from a domain D in $\mathbb{C}^5 \ni (X,Y,Z,W,t)$ into $\mathbb{C}^5 \ni (x,y,z,w,t)$. We say that the mapping is symplectic if

$$dx \wedge dy + dz \wedge dw = dX \wedge dY + dZ \wedge dW,$$

where t is considered as a constant or a parameter, namely, if, for $t = t_0$, $\varphi_{t_0} = \varphi|_{t=t_0}$ is a symplectic mapping from the t_0-section D_{t_0} of D to $\varphi(D_{t_0})$. Suppose that the mapping is symplectic. Then any Hamiltonian system

$$dx/dt = \partial H/\partial y, \quad dy/dt = -\partial H/\partial x, \quad dz/dt = \partial H/\partial w, \quad dw/dt = -\partial H/\partial z$$

is transformed to

$$dX/dt = \partial K/\partial Y, \ \ dY/dt = -\partial K/\partial X, \ \ dZ/dt = \partial K/\partial W, \ \ dW/dt = -\partial K/\partial Z,$$

where

$$\text{(A)} \qquad dx \wedge dy + dz \wedge dw - dH \wedge dt = dX \wedge dY + dZ \wedge dW - dK \wedge dt.$$

Here t is considered as a variable. By this equation, the function K is determined by H uniquely modulo functions of t, namely, modulo functions independent of X, Y, Z and W. Regarding the vector field v in (1.1), we obtain the following theorem.

Theorem 1.4. *The phase space $\mathcal{X}$ over $B = \mathbb{C}^* = \mathbb{C} - \{0\}$ for the vector field v in* (1) *is obtained by gluing sixteen copies of* $\mathbb{C}^4 \times \mathbb{C}^*$:

$$U_0 \times \mathbb{C}^* = \mathbb{C}^4 \times \mathbb{C}^* \ni (x,y,z,w,t),$$
$$U_j \times \mathbb{C}^* = \mathbb{C}^4 \times \mathbb{C}^* \ni (x_j,y_j,z_j,w_j,t)\ (j = 1,2,\dots,15),$$

via the following birational and symplectic transformations:

$$1)\ x_1 = \frac{1}{x}, \quad y_1 = -x(xy+\alpha_0), \quad z_1 = z, \quad w_1 = w,$$

$$2)\ x_2 = x, \quad y_2 = y, \quad z_2 = \frac{1}{z}, \quad w_2 = -z(zw+\beta_0),$$

3) $x_3 = x, \quad y_3 = y + w + \dfrac{2(z-x)w - 2\alpha_1}{x} + \dfrac{t}{x^2}, \quad z_3 = \dfrac{z-x}{x^2}, \quad w_3 = x^2 w,$

4) $x_4 = \dfrac{x-z}{z^2}, \quad y_4 = z^2 y, \quad z_4 = z, \quad w_4 = w + y + \dfrac{2(x-z)y - 2\alpha_1}{z} + \dfrac{t}{z^2},$

5) $x_5 = \dfrac{1}{x}, \quad y_5 = -((y+w-1)x + 1 - \alpha_0 - 2\alpha_1 - \beta_0)x,$

$z_5 = z - x, \quad w_5 = w,$

6) $x_6 = -(x-z)y^2, \quad y_6 = \dfrac{1}{y}, \quad z_6 = z, \quad w_6 = y + w,$

7) $x_7 = \dfrac{1}{x}, \quad y_7 = -x(xy+\alpha_0), \quad z_7 = \dfrac{1}{z}, \quad w_7 = -z(zw+\beta_0),$

8) $x_8 = \dfrac{1}{x}, \quad y_8 = -((y+w-1)x + 1 - \alpha_0 - 2\alpha_1 - \beta_0)x,$

$z_8 = \dfrac{1}{(z-x)}, \quad w_8 = -(z-x)((z-x)w+\beta_0),$

9) $x_9 = \dfrac{1}{x}, \quad y_9 = -((y+w-1)x + 1 - \alpha_0 - 2\alpha_1 - \beta_0)x,$

$z_9 = -(z-x)w^2, \quad w_9 = \dfrac{1}{w},$

10) $x_{10} = \dfrac{x(x-z)(xy+\alpha_0)^2}{z}, \quad y_{10} = -\dfrac{1}{x(xy+\alpha_0)},$

$z_{10} = \dfrac{1}{z}, \quad w_{10} = -x^2 y - \alpha_0 x - z^2 w - \beta_0 z,$

11) $x_{11} = x, \quad y_{11} = y + w + \dfrac{2(z-x)w - 2\alpha_1}{x} + \dfrac{t}{x^2},$

$z_{11} = \dfrac{x^2}{z-x}, \quad w_{11} = -\dfrac{-(z-x)((z-x)w+\beta_0)}{x^2},$

12) $x_{12} = x, \quad y_{12} = y + w + \dfrac{2(z-x)w - 2\alpha_1}{x} + \dfrac{t}{x^2},$

$z_{12} = -(z-x)x^2 w^2, \quad w_{12} = \dfrac{1}{x^2 w},$

13) $x_{13} = \dfrac{1}{(x-z)}, \quad y_{13} = -(x-z)((x-z)y+\alpha_0),$

$z_{13} = \dfrac{1}{z}, \quad w_{13} = -z((w+y-1)z + 1 - \alpha_0 - 2\alpha_1 - \beta_0),$

14) $x_{14} = \dfrac{z^2}{(x-z)}, \quad y_{14} = -\dfrac{(x-z)((x-z)y+\alpha_0)}{z^2},$

$z_{14} = z, \quad w_{14} = w + y + \dfrac{2(x-z)y - 2\alpha_1}{z} + \dfrac{t}{z^2},$

15) $x_{15} = -(x-z)y^2 z^2, \quad y_{15} = \dfrac{1}{yz^2},$

$z_{15} = z, \quad w_{15} = w + y + \dfrac{2(x-z)y - 2\alpha_1}{z} + \dfrac{t}{z^2}.$

Because every coordinate transformation is symplectic, the Hamiltionian system H in $U_0 \times \mathbb{C}^*$ is also written as a Hamiltonian system in each $U_j \times \mathbb{C}^*$ $(j = 1, 2, \dots, 15)$. By direct calculation, we can verify the following theorem.

Theorem 1.5. *On each affine open set* $(x_i, y_i, z_i, w_i, t) \in U_i \times B$ *in Theorem* (1.4) *each corresponding Hamiltonian* H_i *on* $U_i \times B$ *is expressed as a polynomial in* x_i, y_i, z_i, w_i *and a rational function in* t *and satisfies the following condition*:

$$dx \wedge dy + dz \wedge dw - dH \wedge dt = dx_i \wedge dy_i + dz_i \wedge dw_i - dH_i \wedge dt.$$

This paper is organized as follows. In Section 1, we recall the system of type $A_5^{(1)}$ and prove Theorem 1.3. In Section 2, we review the notion of accessible singularity. In Section 3, we will prove Theorems 1.4 and 1.5 by giving an explicit birational transformation for each step.

2. The system of type $A_5^{(1)}$ and proof of Theorem 1.3

As is well-known, the second-order Painlevé III system can be obtained by the following degeneration from the Painlevé V system (see [2], [11]):

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{2x^2y}{t} + x^2 - \dfrac{2xy}{t} - \left(1 + \dfrac{\alpha_1 + \alpha_3}{t}\right) x + \dfrac{\alpha_1}{t}, \\ \dfrac{dy}{dt} = -\dfrac{2xy^2}{t} + \dfrac{y^2}{t} - 2xy + \left(1 + \dfrac{\alpha_1 + \alpha_3}{t}\right) y - \alpha_2 \end{cases}$$

with the Hamiltonian H_{P_V}

$$H_{P_V} = \frac{x^2y^2}{t} + x^2y - \frac{xy^2}{t} - \left(1 + \frac{\alpha_1 + \alpha_3}{t}\right) xy + \frac{\alpha_1 y}{t} + \alpha_2 x.$$

Here α_1, α_2 and α_3 are complex parameters (see [11], [13]). At first we set

$$\alpha_1 = \varepsilon^{-1}, \quad \alpha_2 = A_0, \quad \alpha_3 = 2A_1 - \varepsilon^{-1},$$
$$t = -\varepsilon T, \quad x = 1 + \frac{X}{\varepsilon T}, \quad y = \varepsilon T Y.$$

Since the change of variables is symplectic, we obtain the following system:

$$\begin{cases} \dfrac{dX}{dT} = \dfrac{2X^2Y - X^2}{T} + 2\varepsilon XY - \varepsilon X + \dfrac{(1 - 2A_1)X}{T} + 1 - 2\varepsilon A_1, \\ \dfrac{dY}{dT} = \dfrac{-2XY^2 + 2XY}{T} - \varepsilon Y^2 + \varepsilon Y - \dfrac{(1 - 2A_1)Y}{T} + \dfrac{A_0}{T} \end{cases}$$

with the Hamiltonian $\tilde{H}_{P_V}$

$$\tilde{H}_{P_V} = \frac{\{X(1 - 2A_1 + X(Y - 1)) + T(1 - 2A_1\varepsilon + \varepsilon X(Y - 1))\}Y - A_0 X}{T}.$$

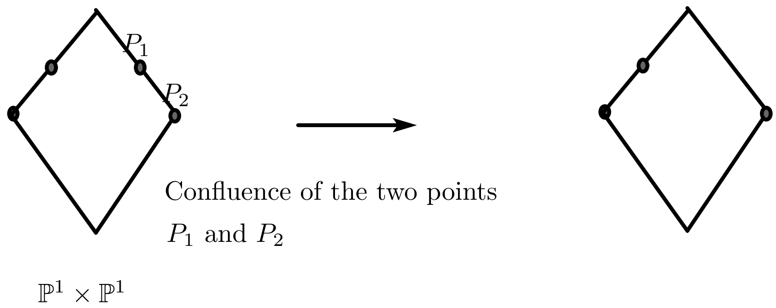

Figure 1.

If we take the limit $\varepsilon \to 0$, we obtain the third Painlevé system.

The above process can be considered as the confluence of two accessible singular points P_1, P_2 (see Figure 1):

$$P_1 = \{(u, v) = (0, 0)\}, \quad P_2 = \{(u, v) = (-\varepsilon T, 0)\}.$$

Here the coordinate system (u, v) is the boundary coordinate system of $\mathbb{P}^1 \times \mathbb{P}^1$ with the rational transformations $(u, v) = (X, 1/Y)$.

Let us recall the system (1.4) of type $A_5^{(1)}$. This system is equivalent to a Hamiltonian system given by

$$\begin{aligned} H_{A_5^{(1)}} &= \frac{x(x-1)y(y+t) - (\alpha_1 + \beta_1 + \alpha_3)xy + (\alpha_1 + \beta_1)y + \alpha_2 tx}{t} \\ &\quad + \frac{z(z-1)w(w+t) - (\alpha_1 + \beta_1 + \alpha_3)zw + \beta_1 w + \beta_2 tz}{t} + \frac{2(x-1)yzw}{t} \\ &= H_V(x, y, t; \alpha_1 + \beta_1, \alpha_2, \alpha_3) + H_V(z, w, t; \beta_1, \beta_2, \alpha_3 + \alpha_1) \\ &\quad + \frac{2(x-1)yzw}{t}. \end{aligned}$$

Here $H_V(q, p, t; \alpha_1, \alpha_2, \alpha_3)$ denotes the Hamiltonian of the second-order Painlevé V equations. This system is invariant under the affine Weyl group of type $A_5^{(1)} =< s_0, s_1, s_2, s_3, s_4, s_5 >$, explicitly written as follows (see [2, 3]).

The affine Weyl group of type $A_5^{(1)} =< s_0, s_1, s_2, s_3, s_4, s_5 >$
By using the notation

$$(*) = (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2) \quad (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 = 1),$$

we define

$$\begin{aligned} &s_0 : (*) \to (x, y - \alpha_3/(x-1), z, w, t; \alpha_0 + \alpha_3, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \beta_1, \beta_2), \\ &s_1 : (*) \to \left(x + \frac{\alpha_2}{y}, y, z, w, t; \alpha_0, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \beta_1, \beta_2\right), \\ &s_2 : (*) \to \left(x, y - \frac{\alpha_1}{x-z}, z, w + \frac{\alpha_1}{x-z}, t; \alpha_0, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \beta_1, \beta_2 + \alpha_1\right), \end{aligned}$$

$$s_3 : (*) \to \left(x, y, z + \frac{\beta_2}{w}, w, t; \alpha_0, \alpha_1 + \beta_2, \alpha_2, \alpha_3, \beta_1 + \beta_2, -\beta_2\right),$$
$$s_4 : (*) \to \left(x, y, z, w - \frac{\beta_1}{z}, t; \alpha_0 + \beta_1, \alpha_1, \alpha_2, \alpha_3, -\beta_1, \beta_2 + \beta_1\right),$$
$$s_5 : (*) \to \left(x + \frac{\alpha_0}{y+w+t}, y, z + \frac{\alpha_0}{y+w+t}, w, t; \right.$$
$$\left. -\alpha_0, \alpha_1, \alpha_2, \alpha_3 + \alpha_0, \beta_1 + \alpha_0, \beta_2\right).$$

We note that the generators s_i satisfy the following relations:

$$s_0^2 = s_1^2 = s_2^2 = s_3^2 = s_4^2 = s_5^2 = 1,$$
$$(s_0 s_2)^2 = (s_0 s_3)^2 = (s_0 s_4)^2 = (s_1 s_3)^2$$
$$= (s_1 s_4)^2 = (s_1 s_5)^2 = (s_2 s_4)^2 = (s_2 s_5)^2 = (s_3 s_5)^2 = 1,$$
$$(s_0 s_1)^3 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_3 s_4)^3 = (s_4 s_5)^3 = (s_5 s_0)^3 = 1.$$

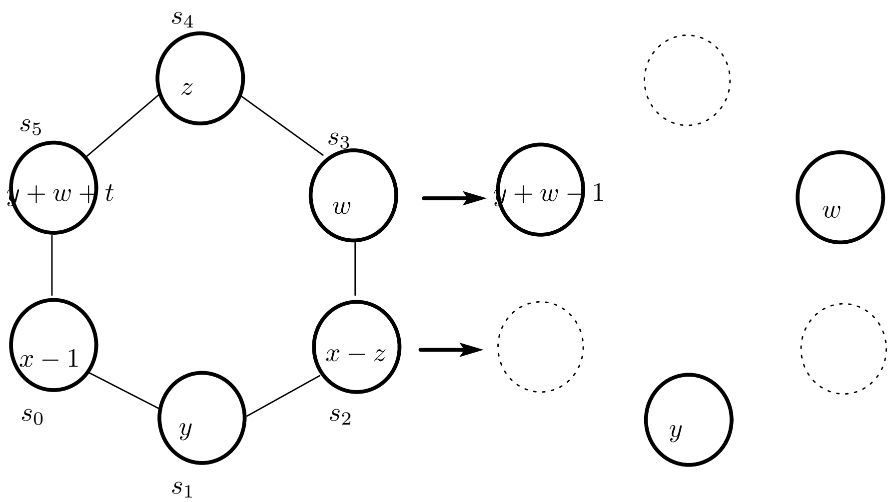

Figure 2. Dynkin diagram

Here let us explain Figure 2:

1. The left hand side of Figure 2 denotes the Dynkin diagram of type $A_5^{(1)}$. The symbol in each circle denotes the invariant divisor of the system of type $A_5^{(1)}$.

2. The process from the left side to the right side denotes the confluence process of the system of type $A_5^{(1)}$ to the system (1.1) in Theorem 1.3.

As the fourth-order analogue of $P_V \to P_{III}$, we consider the following degeneration from the system of type $A_5^{(1)}$. At first we set

$$\alpha_1 = 0, \quad \alpha_2 = A_0, \quad \alpha_3 = 2A_1 - \varepsilon^{-1}, \quad \beta_1 = \varepsilon^{-1}, \quad \beta_2 = B_0,$$
$$t = -\varepsilon T, \quad x = 1 + \frac{X}{\varepsilon T}, \quad y = \varepsilon T Y, \; z = 1 + \frac{Z}{\varepsilon T}, \quad w = \varepsilon T W.$$

Since the change of variables is symplectic, we obtain the following system:

(2.1)
$$\begin{cases} \dfrac{dX}{dT} = \dfrac{2X^2Y + 2XZW - X^2}{T} + 2\varepsilon X(Y+W) - \varepsilon X \\ \qquad + \dfrac{X - 2A_1X}{T} + 1 - 2\varepsilon A_1, \\ \dfrac{dY}{dT} = \dfrac{-2XY^2 - 2YZW}{T} - \varepsilon Y^2 + \dfrac{2XY}{T} - 2\varepsilon YW + \varepsilon Y \\ \qquad + \dfrac{-Y + 2A_1Y}{T} + \dfrac{A_0}{T}, \\ \dfrac{dZ}{dT} = \dfrac{2Z^2W + 2XYZ - Z^2}{T} + 2\varepsilon ZW + 2\varepsilon XY - \varepsilon Z \\ \qquad + \dfrac{Z - 2A_1Z}{T} + 1 - 2\varepsilon A_1, \\ \dfrac{dW}{dT} = \dfrac{-2ZW^2 - 2XYW}{T} - \varepsilon W^2 + \dfrac{2ZW}{T} + \varepsilon W + \dfrac{-W + 2A_1W}{T} + \dfrac{B_0}{T} \end{cases}$$

with the Hamiltonian $\tilde{H}_{A_5^{(1)}}$

$$\begin{aligned} \tilde{H}_{A_5^{(1)}} = & -\frac{A_0X - TY + 2A_1\varepsilon TY - XY + 2A_1XY + \varepsilon TXY - 2\varepsilon TXYW}{T} \\ & - \frac{-X^2Y - \varepsilon TXY^2 - X^2Y^2 - 2XYZW - TW + 2A_1\varepsilon TW}{T} \\ & - \frac{B_0Z - ZW + 2A_1ZW + \varepsilon TZW - \varepsilon TZW^2 + Z^2W - Z^2W^2}{T}. \end{aligned}$$

If we take the limit $\varepsilon \to 0$, we obtain the system (1.1) with the Hamiltonian (1.2). The proof of Theorem 1.3 is now complete.

3. Accessible singularities

Let us review the notion of accessible singularity in accordance with [6]. Let B be a connected open domain in $\mathbb{C}$ and $\pi : \mathcal{W} \longrightarrow B$ a smooth proper holomorphic map. We assume that $\mathcal{H} \subset \mathcal{W}$ is a normal crossing divisor which is flat over B. Let us consider a rational vector field $\tilde{v}$ on $\mathcal{W}$ satisfying the condition

$$\tilde{v} \in H^0(\mathcal{W}, \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H})).$$

Fixing $t_0 \in B$ and $P \in \mathcal{W}_{t_0}$, we can take a local coordinate system $(x_1, x_2, \ldots, x_n)$ of $\mathcal{W}_{t_0}$ centered at P such that $\mathcal{H}_{\text{smooth}}$ can be defined by the local equation

$x_1 = 0$. Since $\tilde{v} \in H^0(\mathcal{W}, \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H}))$, we can write down the vector field $\tilde{v}$ near $P = (0, 0, \ldots, 0, t_0)$ as follows:

$$\tilde{v} = \frac{\partial}{\partial t} + a_1 \frac{\partial}{\partial x_1} + \frac{a_2}{x_1} \frac{\partial}{\partial x_2} + \cdots + \frac{a_n}{x_1} \frac{\partial}{\partial x_n}.$$

This vector field defines the following system of differential equations

$$\tag{3.1} \begin{cases} \dfrac{dx_1}{dt} = a_1(x_1, x_2, \ldots, x_n, t), \\ \dfrac{dx_2}{dt} = \dfrac{a_2(x_1, x_2, \ldots, x_n, t)}{x_1}, \\ \vdots \\ \dfrac{dx_n}{dt} = \dfrac{a_n(x_1, x_2, \ldots, x_n, t)}{x_1}. \end{cases}$$

Here $a_i(x_1, x_2, \ldots, x_n, t)$, $i = 1, 2, \ldots, n$, are holomorphic functions defined near $P = (0, \ldots, 0, t_0)$.

Definition 3.1. With the above notation, assume that the rational vector field $\tilde{v}$ on $\mathcal{W}$ satisfies the condition

$$\tilde{v} \in H^0(\mathcal{W}, \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H})).$$

We say that $\tilde{v}$ has an *accessible singularity* at $P = (0, 0, \ldots, 0, t_0)$ if

$$x_1 = 0 \text{ and } a_i(0, 0, \ldots, 0, t_0) = 0 \text{ for every } i,\ 2 \le i \le n.$$

If $P \in \mathcal{H}_{\text{smooth}}$ is not an accessible singularity, all solutions of the ordinary differential equation passing through P are vertical solutions, that is, the solutions are contained in the fiber $\mathcal{W}_{t_0}$ over $t = t_0$. If $P \in \mathcal{H}_{\text{smooth}}$ is an accessible singularity, there may be a solution of (3.1) which passes through P and goes into the interior $\mathcal{W} - \mathcal{H}$ of $\mathcal{W}$.

Let us recall the notion of local index. When we construct the phase spaces of the higher order Painlevé equations, an object, called the local index, is the key to determining when we need to make a blowing-up of an accessible singularity or a blowing-down to a minimal phase space. In the case of equations of higher order with favorable properties, for example the systems of type $A_4^{(1)}$ [2], the local index at the accessible singular point corresponds to the set of orders that appears in the free parameters of formal solutions passing through that point [12].

Definition 3.2. Let v be an algebraic vector field which is given by (3.1) and (X, Y, Z, W) be a boundary coordinate system in a neighborhood of an accessible singularity $P = (0, 0, 0, 0, t)$. Assume that the system is written as

$$\begin{cases} \dfrac{dX}{dt} = a + f_1(X, Y, Z, W, t), \\ \dfrac{dY}{dt} = \dfrac{bY + f_2(X, Y, Z, W, t)}{X}, \\ \dfrac{dZ}{dt} = \dfrac{cZ + f_3(X, Y, Z, W, t)}{X}, \\ \dfrac{dW}{dt} = \dfrac{dW + f_4(X, Y, Z, W, t)}{X} \end{cases}$$

near the accessible singularity P, where a, b, c and d are nonzero constants. We say that the vector field v has the *local index* (a, b, c, d) at P if $f_1(X, Y, Z, W, t)$ is a polynomial which vanishes at $P = (0, 0, 0, 0, t)$ and $f_i(X, Y, Z, W, t),\ i = 2, 3, 4$, are polynomials of order 2 in X, Y, Z, W. Here $f_i \in \mathbb{C}[X, Y, Z, W, t]$ for $i = 1, 2, 3, 4$.

Remark 2. We are interested in the case with local index $(1, b/a, c/a, d/a) \in \mathbb{Z}^4$. If each component of $(1, b/a, c/a, d/a)$ has the same sign, we may resolve the accessible singularity by blowing-up finitely many times. However, when different signs appear, we may need to both blow up and blow down.

4. Proof of Theorems 1.4 and 1.5

Comparing the resolution of singularities for the differential equations of Painlevé type, there are important differences between the second-order Painlevé equations and those of higher order. Unlike the second-order case, in higher order cases there may exist some meromorphic solution spaces with codimension 2. In 1979, K. Okamoto constructed the spaces of initial conditions of Painlevé equations, which can be considered as the parametrized spaces of all solutions, including the meromorphic solutions (see [1], [4], [5], [7], [8], [9], [10]). They are constructed by means of successive blowing-up procedures at singular points. For second-order Painlevé equations, we can obtain the entire space of initial conditions by adding subvarieties of codimension 1 (equivalently, of dimension 1) to the space of initial conditions of holomorphic solutions. However, in the case of fourth-order differential equations, we need to add codimension 2 subvarieties to the space in addition to codimension 1 subvarieties (see [12]). In order to resolve singularities, we need to both blow up and blow down. Moreover, to obtain a smooth variety by blowing-down, we need to resolve for a pair of singularities (see [8], [9], [12]). In this section, we will give the canonical coordinate systems of the system (1.1). Each of them corresponds to a 3-parameter or 2-parameter family of meromorphic solutions.

4.1. Accessible singularities of the system (1.1)

In order to consider a family of phase spaces for the system (1.1), let us take the compactification $([z_0 : z_1 : z_2 : z_3 : z_4], t) \in \mathbb{P}^4 \times B$ of $(x, y, z, w, t) \in \mathbb{C}^4 \times B$ with the natural embedding

$$(x, y, z, w) = (z_1/z_0, z_2/z_0, z_3/z_0, z_4/z_0).$$

Moreover, we denote the boundary divisor in $\mathbb{P}^4$ by $\mathcal{H}$. Fixing the parameters $\alpha_0, \alpha_1, \beta_0$, consider the product $\mathbb{P}^4 \times B$ and extend the regular vector field on $\mathbb{C}^4 \times B$ to a rational vector field $\tilde{v}$ on $\mathbb{P}^4 \times B$. It is easy to see that $\mathbb{P}^4 \times B$ is covered by five copies of $\mathbb{C}^4 \times B$:

$$U_0 \times B = \mathbb{C}^4 \times B \ni (x, y, z, w, t),$$
$$U_j \times B = \mathbb{C}^4 \times B \ni (X_j, Y_j, Z_j, W_j, t) \ (j = 1, 2, 3, 4),$$

via the following rational transformations

$$\begin{aligned}
&1)\ X_1 = 1/x, \quad Y_1 = y/x, \quad Z_1 = z/x, \quad W_1 = w/x,\\
&2)\ X_2 = x/z, \quad Y_2 = y/z, \quad Z_2 = 1/z, \quad W_2 = w/z,\\
&3)\ X_3 = x/y, \quad Y_3 = 1/y, \quad Z_3 = z/y, \quad W_3 = w/y,\\
&4)\ X_4 = x/w, \quad Y_4 = y/w, \quad Z_4 = z/w, \quad W_4 = 1/w.
\end{aligned}$$

The following Lemma 3.1 shows that this rational vector field $\tilde{v}$ has six accessible singular loci on the boundary divisor $\mathcal{H} \times t \subset \mathbb{P}^4 \times t$ for each $t \in B$.

Lemma 4.1. *The rational vector field $\tilde{v}$ has the following accessible singular loci*:

$$\begin{cases}
P_i = \{(X_i, Y_i, Z_i, W_i) | X_i = Y_i = Z_i = W_i = 0\} \ (1 = 1, 2, 3, 4),\\
P_5 = \{(X_1, Y_1, Z_1, W_1) | X_1 = Y_1 = W_1 = 0, Z_1 = 1\},\\
P_6 = \{(X_3, Y_3, Z_3, W_3) | X_3 = Y_3 = Z_3 = 0, W_3 = -1\}.
\end{cases}$$

This lemma can be proven by a direct calculation. □

Remark 3. By the symmetry

$$\pi_2 : (x, y, z, w; \alpha_0, \alpha_1, \beta_0) \longrightarrow (z, w, x, y; \beta_0, \alpha_1, \alpha_0),$$

it is easy to see that

$$\pi_2(P_1) = P_2, \quad \pi_2(P_2) = P_1, \quad \pi_2(P_3) = P_4, \quad \pi_2(P_4) = P_3.$$

Let us explain Figure 3. This figure denotes the four-dimensional projective space $\mathbb{P}^4 = \mathbb{C}^4 \sqcup \mathbb{P}^3$. $\mathbb{P}^4$ is covered by five open sets $\mathbb{C}^4$ around the points P_i $(i = 0, 1, 2, 3, 4)$. We also remark that the figure spanned by the points P_i $(i = 1, 2, 3, 4)$ denotes the three-dimensional projective space $\mathbb{P}^3 = \mathbb{C}^3 \sqcup \mathbb{P}^2$.

Now we are ready to prove Theorems 1.4 and 1.5.

4.2. Resolution of the accessible singular point P_1

In this subsection, we give an explicit resolution process for the accessible singular point P_1 by giving a convenient coordinate system at each step.

By the following steps, we can resolve the accessible singular point P_1.

At first, we take the coordinate system centered at $\{(X_1, Y_1, Z_1, W_1)|$ $X_1 = Y_1 = Z_1 = W_1 = 0\}$.

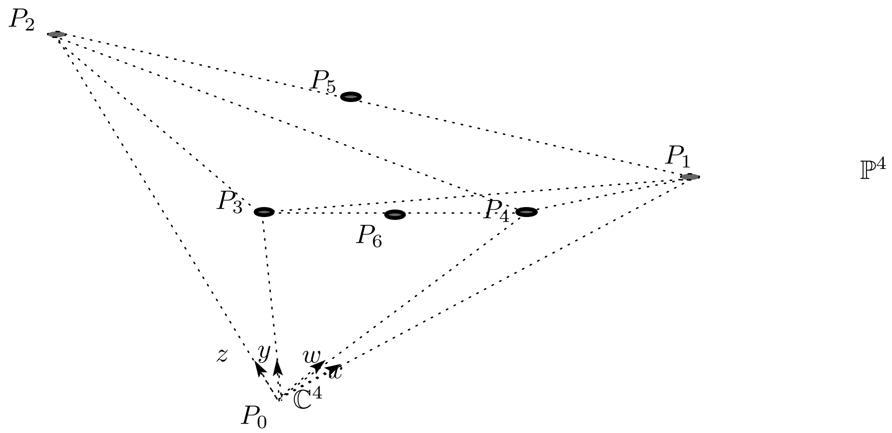

Figure 3. Four-dimensional projective space

Step 1: We blow up at the point

$$P_1 = \{(X_1, Y_1, Z_1, W_1)|X_1 = Y_1 = Z_1 = W_1 = 0\} :$$
$$x_1{}^{(1)} = X_1, \quad y_1{}^{(1)} = \frac{Y_1}{X_1}, \quad z_1{}^{(1)} = \frac{Z_1}{X_1}, \quad w_1{}^{(1)} = \frac{W_1}{X_1}.$$

Step 2: We blow up along the surface

$$\{(x_1{}^{(1)}, y_1{}^{(1)}, z_1{}^{(1)}, w_1{}^{(1)})|y_1{}^{(1)} = x_1{}^{(1)} = 0\} :$$
$$x_1{}^{(2)} = x_1{}^{(1)}, \quad y_1{}^{(2)} = \frac{y_1{}^{(1)}}{x_1{}^{(1)}}, \quad z_1{}^{(2)} = z_1{}^{(1)}, \quad w_1{}^{(2)} = w_1{}^{(1)}.$$

Step 3: We blow up along the surface

$$\{(x_1{}^{(2)}, y_1{}^{(2)}, z_1{}^{(2)}, w_1{}^{(2)})|y_1{}^{(2)} + \alpha_0 = x_1{}^{(2)} = 0\} :$$
$$x_1{}^{(3)} = x_1{}^{(2)}, \quad y_1{}^{(3)} = \frac{y_1{}^{(2)} + \alpha_0}{x_1{}^{(2)}}, \quad z_1{}^{(3)} = z_1{}^{(2)}, \quad w_1{}^{(3)} = w_1{}^{(2)}.$$

We have resolved the accessible singular point P_1. The coordinate system $(x_1{}^{(3)}, -y_1{}^{(3)}, z_1{}^{(3)}, w_1{}^{(3)})$ corresponds to the coordinate system (x_1, y_1, z_1, w_1) in Theorem 1.4. By a direct calculation, it is easy to see that the differential system with respect to the coordinate system (x_1, y_1, z_1, w_1) is a Hamiltonian system, whose Hamiltonian H_1 satisfies the following condition:

$$dx_1 \wedge dy_1 + dz_1 \wedge dw_1 - dH_1 \wedge dt = dx \wedge dy + dz \wedge dw - dH \wedge dt.$$

Let us explain Figure 4. The first picture denotes the boundary divisor $\mathcal{H} \cong \mathbb{P}^3$ of $\mathbb{P}^4$. By Step 1, the point P_1 is transformed to $\mathbb{P}^3$. By Step 2, each point on the surface $\{(x_1{}^{(1)}, y_1{}^{(1)}, z_1{}^{(1)}, w_1{}^{(1)})|y_1{}^{(1)} = x_1{}^{(1)} = 0\}$ is transformed to $\mathbb{P}^1$.

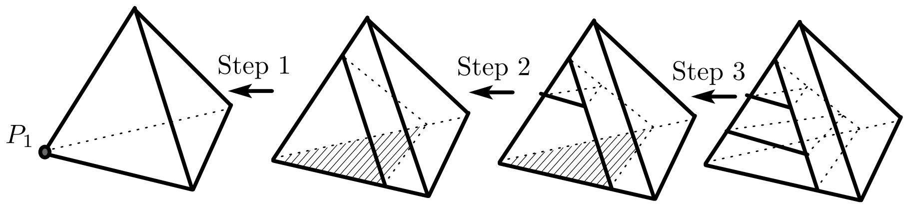

Figure 4. Resolution process from Step 1 to Step 3

4.3. Resolution of the accessible singular locus S_7

By using the coordinate system (x_1, y_1, z_1, w_1), we now make a coordinate system associated with small meromorphic solution spaces. Let us explain our approach [12]. At first, we can take the coordinate system $(x_1, y_1, z_1, w_1) = (1/x, -x(xy+\alpha_0), z, w)$. As a boundary coordinate system of this system (x_1, y_1, z_1, w_1), we can take the coordinate system

$$(X_7, Y_7, Z_7, W_7) = (x_1, y_1, 1/z_1, w_1).$$

It is easy to see that there is an accessible singular locus along the surface

$$S_7 = \{(X_7, Y_7, Z_7, W_7) | Z_7 = W_7 = 0\}.$$

Now we blow up along the accessible singularity S_7.

Step 1: We blow up along the surface S_7:

$$x_7^{(1)} = X_7, \quad y_7^{(1)} = Y_7, \quad z_7^{(1)} = Z_7, \quad w_7^{(1)} = \frac{W_7}{Z_7}.$$

Step 2: We blow up along the surface

$$\{(x_7^{(1)}, y_7^{(1)}, z_7^{(1)}, w_7^{(1)}) | w_7^{(1)} + \beta_0 = z_7^{(1)} = 0\} :$$

$$x_7^{(2)} = x_7^{(1)}, \quad y_7^{(2)} = y_7^{(1)}, \quad z_7^{(2)} = z_7^{(1)}, \quad w_7^{(2)} = \frac{w_7^{(1)} + \beta_0}{z_7^{(1)}}.$$

We have resolved the accessible singular locus S_7. The coordinate system $(x_7^{(2)}, y_7^{(2)}, z_7^{(2)}, -w_7^{(2)})$ corresponds to the coordinate system (x_7, y_7, z_7, w_7) in Theorem 1.4. By a direct calculation, it is easy to see that the differential system with respect to the coordinate system (x_7, y_7, z_7, w_7) is a Hamiltonian system, whose Hamiltonian H_7 satisfies the following condition:

$$dx_7 \wedge dy_7 + dz_7 \wedge dw_7 - dH_7 \wedge dt = dx \wedge dy + dz \wedge dw - dH \wedge dt.$$

Remark 4. By the following blowing-ups and blowing-down, we can take the canonical coordinate system (X_7, Y_7, Z_7, W_7) of the coordinate system (x_1, y_1, z_1, w_1).

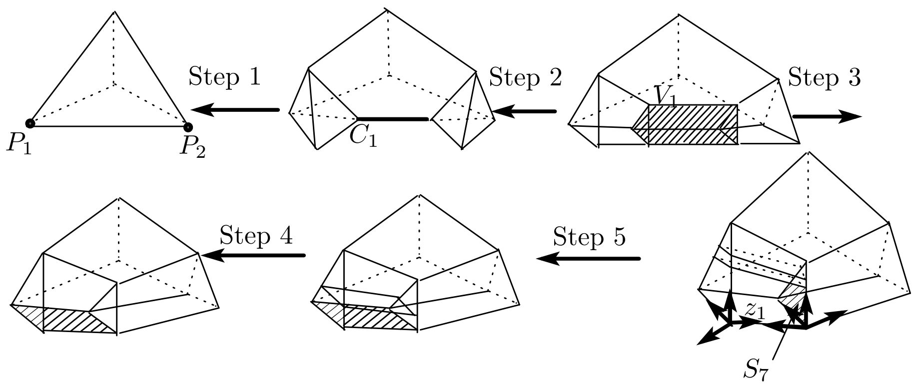

Figure 5.

Step 1: We blow up at the points P_1, P_2 (see Figure 5 for each step).
Step 2: We blow up along the curve $C_1 \cong \mathbb{P}^1$.
Step 3: We blow down the 3−fold $V_1 \cong \mathbb{P}^2 \times \mathbb{P}^1$ along the $\mathbb{P}^1$−fiber.
Step 4: We blow up along the surface.
Step 5: We blow up along the surface.

Let us explain Figure 5. The first picture denotes the boundary divisor $\mathcal{H} \cong \mathbb{P}^3$ of $\mathbb{P}^4$. The resolution process from Step 2 to Step 3 is well-known as $\mathbb{P}^2$−flop. By Step 1, each of points P_1, P_2 is transformed to $\mathbb{P}^3$. By Step 2, each point on C_1 is transformed to $\mathbb{P}^2$. The 3-fold V_1 is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^1$. By Step 3, we blow down the 3-fold V_1 along the $\mathbb{P}^1$−fiber. After Step 5, we take the boundary coordinate system $(X_7, Y_7, Z_7, W_7) = (x_1, y_1, 1/z_1, w_1)$.

4.4. Resolution of the accessible singular point P_5

In this subsection, we give an explicit resolution process for the accessible singular point P_5 by giving a convenient coordinate system at each step.

By the following steps, we can resolve the accessible singular point P_5.

Step 0: We take the coordinate system centered at P_5:

$$x_5{}^{(0)} = X_1, \quad y_5{}^{(0)} = Y_1, \quad z_5{}^{(0)} = Z_1 - 1, \quad w_5{}^{(0)} = W_1.$$

Step 1: We blow up at the point

$$P_5 = \{(x_5{}^{(0)}, y_5{}^{(0)}, z_5{}^{(0)}, w_5{}^{(0)}) | x_5{}^{(0)} = y_5{}^{(0)} = z_5{}^{(0)} = w_5{}^{(0)} = 0\} :$$

$$x_5{}^{(1)} = x_5{}^{(0)}, \quad y_5{}^{(1)} = \frac{y_5{}^{(0)}}{x_5{}^{(0)}}, \quad z_5{}^{(1)} = \frac{z_5{}^{(0)}}{x_5{}^{(0)}}, \quad w_5{}^{(1)} = \frac{w_5{}^{(0)}}{x_5{}^{(0)}}.$$

Step 2: We blow up along the surface

$$\{(x_5{}^{(1)}, y_5{}^{(1)}, z_5{}^{(1)}, w_5{}^{(1)}) | x_5{}^{(1)} = y_5{}^{(1)} + w_5{}^{(1)} - 1 = 0\} :$$

$$x_5{}^{(2)} = x_5{}^{(1)}, \quad y_5{}^{(2)} = \frac{y_5{}^{(1)} + w_5{}^{(1)} - 1}{x_5{}^{(1)}}, \quad z_5{}^{(2)} = z_5{}^{(1)}, \quad w_5{}^{(2)} = w_5{}^{(1)}.$$

Step 3: We blow up along the surface

$$\{(x_5{}^{(2)}, y_5{}^{(2)}, z_5{}^{(2)}, w_5{}^{(2)}) | x_5{}^{(2)} = 0, y_5{}^{(2)} = -1 + \alpha_0 + 2\alpha_1 + \beta_0\} :$$
$$x_5{}^{(3)} = x_5{}^{(2)}, \quad y_5{}^{(3)} = \frac{y_5{}^{(2)} + 1 - \alpha_0 - 2\alpha_1 - \beta_0}{x_5{}^{(2)}}$$
$$z_5{}^{(3)} = z_5{}^{(2)}, \quad w_5{}^{(3)} = w_5{}^{(2)}.$$

We have resolved the accessible singular point P_5. The coordinate system $(x_5{}^{(3)}, -y_5{}^{(3)}, z_5{}^{(3)}, w_5{}^{(3)})$ corresponds to the coordinate system (x_5, y_5, z_5, w_5) in Theorem 1.4. By a direct calculation, it is easy to see that the differential system with respect to the coordinate system (x_5, y_5, z_5, w_5) is a Hamiltonian system, whose Hamiltonian H_5 satisfies the following condition:

$$dx_5 \wedge dy_5 + dz_5 \wedge dw_5 - dH_5 \wedge dt = dx \wedge dy + dz \wedge dw - dH \wedge dt.$$

Remark 5. Taking the boundary coordinate system centered at the accessible singular point P_5 as $(\tilde{x_5}^{(0)}, \tilde{y_5}^{(0)}, \tilde{z_5}^{(0)}, \tilde{w_5}^{(0)}) = (X_2 - 1, Y_2, Z_2, W_2)$, we can resolve the accessible singular point P_5 by the following steps.

Step 0: We take the coordinate system centered at P_5:

$$\tilde{x_5}^{(0)} = X_2 - 1, \quad \tilde{y_5}^{(0)} = Y_2, \quad \tilde{z_5}^{(0)} = Z_2, \quad \tilde{w_5}^{(0)} = W_2.$$

Step 1: We blow up at the point

$$P_5 = \{(\tilde{x_5}^{(0)}, \tilde{y_5}^{(0)}, \tilde{z_5}^{(0)}, \tilde{w_5}^{(0)}) | \tilde{x_5}^{(0)} = \tilde{y_5}^{(0)} = \tilde{z_5}^{(0)} = \tilde{w_5}^{(0)} = 0\} :$$
$$\tilde{x_5}^{(1)} = \frac{\tilde{x_5}^{(0)}}{\tilde{z_5}^{(0)}}, \quad \tilde{y_5}^{(1)} = \frac{\tilde{y_5}^{(0)}}{\tilde{z_5}^{(0)}}, \quad \tilde{z_5}^{(1)} = \tilde{z_5}^{(0)}, \quad \tilde{w_5}^{(1)} = \frac{\tilde{w_5}^{(0)}}{\tilde{z_5}^{(0)}}.$$

Step 2: We blow up along the surface

$$\{(\tilde{x_5}^{(1)}, \tilde{y_5}^{(1)}, \tilde{z_5}^{(1)}, \tilde{w_5}^{(1)}) | \tilde{z_5}^{(1)} = \tilde{w_5}^{(1)} + \tilde{y_5}^{(1)} - 1 = 0\} :$$
$$\tilde{x_5}^{(2)} = \tilde{x_5}^{(1)}, \quad \tilde{y_5}^{(2)} = \tilde{y_5}^{(1)}, \quad \tilde{z_5}^{(2)} = \tilde{z_5}^{(1)}, \quad \tilde{w_5}^{(2)} = \frac{\tilde{w_5}^{(1)} + \tilde{y_5}^{(1)} - 1}{\tilde{z_5}^{(1)}}.$$

Step 3: We blow up along the surface

$$\{(\tilde{x_5}^{(2)}, \tilde{y_5}^{(2)}, \tilde{z_5}^{(2)}, \tilde{w_5}^{(2)}) | \tilde{z_5}^{(2)} = 0, \tilde{w_5}^{(2)} = -1 + \alpha_0 + 2\alpha_1 + \beta_0\} :$$
$$\tilde{x_5}^{(3)} = \tilde{x_5}^{(2)}, \quad \tilde{y_5}^{(3)} = \tilde{y_5}^{(2)}$$
$$\tilde{z_5}^{(3)} = \tilde{z_5}^{(2)}, \quad \tilde{w_5}^{(3)} = \frac{\tilde{w_5}^{(2)} + 1 - \alpha_0 - 2\alpha_1 - \beta_0}{\tilde{z_5}^{(2)}}.$$

We have resolved the accessible singular point P_5. The transition function between $(\tilde{x}_5, \tilde{y}_5, \tilde{z}_5, \tilde{w}_5) := (\tilde{x_5}^{(3)}, \tilde{y_5}^{(3)}, \tilde{z_5}^{(3)}, -\tilde{w_5}^{(3)})$ and (x, y, z, w) is symplectic, which is explicitly written as follows:

$$(\tilde{x}_5, \tilde{y}_5, \tilde{z}_5, \tilde{w}_5) = (x - z, y, 1/z, -((w + y - 1)z + 1 - \alpha_0 - 2\alpha_1 - \beta_0)z).$$

4.5. Resolution of the accessible singular point P_6

In this subsection, we give an explicit resolution process for the accessible singular point P_6 by giving a convenient coordinate system at each step.

By the following steps, we can resolve the accessible singular point P_6.

Step 0: We take the coordinate system centered at P_6:

$$x_6{}^{(0)} = X_3, \quad y_6{}^{(0)} = Y_3, \quad z_6{}^{(0)} = Z_3, \quad w_6{}^{(0)} = W_3 + 1.$$

Step 1: We blow up along the surface

$$\{(x_6{}^{(0)}, y_6{}^{(0)}, z_6{}^{(0)}, w_6{}^{(0)}) | x_6{}^{(0)} = y_6{}^{(0)} = z_6{}^{(0)} = w_6{}^{(0)} = 0\} :$$

$$x_6{}^{(1)} = \frac{x_6{}^{(0)}}{y_6{}^{(0)}}, \quad y_6{}^{(1)} = y_6{}^{(0)}, \quad z_6{}^{(1)} = \frac{z_6{}^{(0)}}{y_6{}^{(0)}}, \quad w_6{}^{(1)} = \frac{w_6{}^{(0)}}{y_6{}^{(0)}}.$$

Step 2: We blow up along the surface

$$\{(x_6{}^{(1)}, y_6{}^{(1)}, z_6{}^{(1)}, w_6{}^{(1)}) | x_6{}^{(1)} - z_6{}^{(1)} = y_6{}^{(1)} = 0\} :$$

$$x_6{}^{(2)} = \frac{x_6{}^{(1)} - z_6{}^{(1)}}{y_6{}^{(1)}}, \quad y_6{}^{(2)} = y_6{}^{(1)}, \quad z_6{}^{(2)} = z_6{}^{(1)}, \quad w_6{}^{(2)} = w_6{}^{(1)}.$$

Step 3: We blow up along the surface

$$\{(x_6{}^{(2)}, y_6{}^{(2)}, z_6{}^{(2)}, w_6{}^{(2)}) | x_6{}^{(2)} = y_6{}^{(2)} = 0\} :$$

$$x_6{}^{(3)} = \frac{x_6{}^{(2)}}{y_6{}^{(2)}}, \quad y_6{}^{(3)} = y_6{}^{(2)}, \quad z_6{}^{(3)} = z_6{}^{(2)}, \quad w_6{}^{(3)} = w_6{}^{(2)}.$$

We have resolved the accessible singular point P_6. The coordinate system $(-x_6{}^{(3)}, y_6{}^{(3)}, z_6{}^{(3)}, w_6{}^{(3)})$ corresponds to the coordinate system (x_6, y_6, z_6, w_6) in Theorem 1.4. By a direct calculation, it is easy to see that the differential system with respect to the coordinate system (x_6, y_6, z_6, w_6) is a Hamiltonian system, whose Hamiltonian H_6 satisfies the following condition:

$$dx_6 \wedge dy_6 + dz_6 \wedge dw_6 - dH_6 \wedge dt = dx \wedge dy + dz \wedge dw - dH \wedge dt.$$

4.6. Resolution of the accessible singular point P_3

In this subsection, we give an explicit resolution process for the accessible singular point P_3 by giving a convenient coordinate system at each step.

By the following steps, we can resolve the accessible singular point P_3. At first, we take the coordinate system centered at $\{(X_3, Y_3, Z_3, W_3) | X_3 = Y_3 = Z_3 = W_3 = 0\}$.

Step 1: We blow up along the curve

$$\{(X_3, Y_3, Z_3, W_3) | X_3 = Y_3 = Z_3 = 0\} :$$

$$x_3{}^{(1)} = \frac{X_3}{Y_3}, \quad y_3{}^{(1)} = Y_3, \quad z_3{}^{(1)} = \frac{Z_3}{Y_3}, \quad w_3{}^{(1)} = W_3.$$

Step 2: We blow up along the curve

$$\{(x_3{}^{(1)},y_3{}^{(1)},z_3{}^{(1)},w_3{}^{(1)})|x_3{}^{(1)}=y_3{}^{(1)}=z_3{}^{(1)}=0\}:$$
$$x_3{}^{(2)}=x_3{}^{(1)},\quad y_3{}^{(2)}=\frac{y_3{}^{(1)}}{x_3{}^{(1)}},\quad z_3{}^{(2)}=\frac{z_3{}^{(1)}}{x_3{}^{(1)}},\quad w_3{}^{(2)}=w_3{}^{(1)}.$$

Step 3: We blow up along the curve

$$\{(x_3{}^{(2)},y_3{}^{(2)},z_3{}^{(2)},w_3{}^{(2)})|x_3{}^{(1)}=y_3{}^{(1)}=0,z_3{}^{(1)}=1\}:$$
$$x_3{}^{(3)}=x_3{}^{(2)},\quad y_3{}^{(3)}=\frac{y_3{}^{(2)}}{x_3{}^{(2)}},\quad z_3{}^{(3)}=\frac{z_3{}^{(2)}-1}{x_3{}^{(2)}},\quad w_3{}^{(3)}=w_3{}^{(2)}.$$

Step 4: We blow up along the surface

$$\{(x_3{}^{(3)},y_3{}^{(3)},z_3{}^{(3)},w_3{}^{(3)})|y_3{}^{(3)}=w_3{}^{(3)}=0\}:$$
$$x_3{}^{(4)}=x_3{}^{(3)},\quad y_3{}^{(4)}=y_3{}^{(3)},\quad z_3{}^{(4)}=z_3{}^{(3)},\quad w_3{}^{(4)}=\frac{w_3{}^{(3)}}{y_3{}^{(3)}}.$$

Step 5: We make a change of variables

$$x_3{}^{(5)}=x_3{}^{(4)},\quad y_3{}^{(5)}=\frac{1}{y_3{}^{(4)}},\quad z_3{}^{(5)}=z_3{}^{(4)},\quad w_3{}^{(5)}=w_3{}^{(4)}.$$

This change of variables is necessary for making the transition functions in the description of $\mathcal{X}$ symplectic [1].

Step 6: We blow up along the surface

$$\{(x_3{}^{(5)},y_3{}^{(5)},z_3{}^{(5)},w_3{}^{(5)})|x_3{}^{(5)}=y_3{}^{(5)}+w_3{}^{(5)}+t=0\}:$$
$$x_3{}^{(6)}=x_3{}^{(5)},\quad y_3{}^{(6)}=\frac{y_3{}^{(5)}+w_3{}^{(5)}+t}{x_3{}^{(5)}},\quad z_3{}^{(6)}=z_3{}^{(5)},\quad w_3{}^{(6)}=w_3{}^{(5)}.$$

Step 7: We blow up along the surface

$$\{(x_3{}^{(6)},y_3{}^{(6)},z_3{}^{(6)},w_3{}^{(6)})|y_3{}^{(6)}+2z_3{}^{(6)}w_3{}^{(6)}-2\alpha_1=x_3{}^{(6)}=0\}:$$
$$x_3{}^{(7)}=x_3{}^{(6)},\quad y_3{}^{(7)}=\frac{y_3{}^{(6)}+2z_3{}^{(6)}w_3{}^{(6)}-2\alpha_1}{x_3{}^{(6)}}$$
$$z_3{}^{(7)}=z_3{}^{(6)},\quad w_3{}^{(7)}=w_3{}^{(6)}.$$

We have resolved the accessible singular point P_3. The coordinate system $(x_3{}^{(7)},y_3{}^{(7)},z_3{}^{(7)},w_3{}^{(7)})$ corresponds to the coordinate system (x_3,y_3,z_3,w_3) in Theorem 1.4. By a direct calculation, it is easy to see that the differential system with respect to the coordinate system (x_3,y_3,z_3,w_3) is a Hamiltonian system, whose Hamiltonian H_3 satisfies the following condition:

$$dx_3\wedge dy_3+dz_3\wedge dw_3-dH_3\wedge dt=dx\wedge dy+dz\wedge dw-d\left(H+\frac{1}{x}\right)\wedge dt.$$

4.7. Resolution of the accessible singular locus S_8

By using the coordinate system (x_5, y_5, z_5, w_5), we now make a coordinate system associated with small meromorphic solution spaces. At first, we can take the coordinate system $(x_5, y_5, z_5, w_5) = (1/x, -x((y+w-1)x+1-\alpha_0-2\alpha_1-\beta_0), z-x, w)$. As a boundary coordinate system of this system (x_5, y_5, z_5, w_5), we can take the coordinate system

$$(X_8, Y_8, Z_8, W_8) = (x_5, y_5, 1/z_5, w_5).$$

It is easy to see that there is an accessible singular locus along the surface

$$S_8 = \{(X_8, Y_8, Z_8, W_8) | Z_8 = W_8 = 0\}.$$

Now we blow up along the accessible singularity S_8.

Step 1: We blow up along the surface S_8:

$$x_8{}^{(1)} = X_8, \quad y_8{}^{(1)} = Y_8, \quad z_8{}^{(1)} = Z_8, \quad w_8{}^{(1)} = \frac{W_8}{Z_8}.$$

Step 2: We blow up along the surface

$$\{(x_8{}^{(1)}, y_8{}^{(1)}, z_8{}^{(1)}, w_8{}^{(1)}) | z_8{}^{(1)} = w_8{}^{(1)} + \beta_0 = 0\} :$$

$$x_8{}^{(2)} = x_8{}^{(1)}, \quad y_8{}^{(2)} = y_8{}^{(1)}, \quad z_8{}^{(2)} = z_8{}^{(1)}, \quad w_8{}^{(2)} = \frac{w_8{}^{(1)} + \beta_0}{z_8{}^{(1)}}.$$

We have resolved the accessible singular locus S_8. The coordinate system $(x_8{}^{(2)}, y_8{}^{(2)}, z_8{}^{(2)}, -w_8{}^{(2)})$ corresponds to the coordinate system (x_8, y_8, z_8, w_8) in Theorem 1.4. By a direct calculation, it is easy to see that the differential system with respect to the coordinate system (x_8, y_8, z_8, w_8) is a Hamiltonian system, whose Hamiltonian H_8 satisfies the following condition:

$$dx_8 \wedge dy_8 + dz_8 \wedge dw_8 - dH_8 \wedge dt = dx \wedge dy + dz \wedge dw - dH \wedge dt.$$

4.8. Resolution of the accessible singular locus S_9

By using the coordinate system (x_5, y_5, z_5, w_5), we now make a coordinate system associated with the small meromorphic solution spaces other than (x_8, y_8, z_8, w_8). At first, we can take the coordinate system $(x_5, y_5, z_5, w_5) = (1/x, -x((y+w-1)x+1-\alpha_0-2\alpha_1-\beta_0), z-x, w)$. As a boundary coordinate system of this system $(x_5, y_5, z_5, 1/w_5)$, we can take the coordinate system

$$(X_9, Y_9, Z_9, W_9) = (x_5, y_5, z_5, 1/w_5).$$

It is easy to see that there is an accessible singular locus along the surface

$$S_9 = \{(X_9, Y_9, Z_9, W_9) | Z_9 = W_9 = 0\}.$$

Now we blow up along the accessible singularity S_9.

Step 1: We blow up along the surface S_9:

$$x_9{}^{(1)} = X_9, \quad y_9{}^{(1)} = Y_9, \quad z_9{}^{(1)} = \frac{Z_9}{W_9}, \quad w_9{}^{(1)} = W_9.$$

Step 2: We blow up along the surface

$$\{(x_9{}^{(1)}, y_9{}^{(1)}, z_9{}^{(1)}, w_9{}^{(1)})|z_9{}^{(1)} = w_9{}^{(1)} = 0\} :$$
$$x_9{}^{(2)} = x_9{}^{(1)}, \quad y_9{}^{(2)} = y_9{}^{(1)}, \quad z_9{}^{(2)} = \frac{z_9{}^{(1)}}{w_9{}^{(1)}}, \quad w_9{}^{(2)} = w_9{}^{(1)}.$$

We have resolved the accessible singular locus S_9. The coordinate system $(x_9{}^{(2)}, y_9{}^{(2)}, -z_9{}^{(2)}, w_9{}^{(2)})$ corresponds to the coordinate system (x_9, y_9, z_9, w_9) in Theorem 1.4. By a direct calculation, it is easy to see that the differential system with respect to the coordinate system (x_9, y_9, z_9, w_9) is a Hamiltonian system, whose Hamiltonian H_9 satisfies the following condition:

$$dx_9 \wedge dy_9 + dz_9 \wedge dw_9 - dH_9 \wedge dt = dx \wedge dy + dz \wedge dw - dH \wedge dt.$$

4.9. Resolution of the accessible singular locus S_{10}

By using the coordinate system (x_7, y_7, z_7, w_7), we now make a coordinate system associated with small meromorphic solution spaces. At first, we can take the coordinate system $(x_7, y_7, z_7, w_7) = (1/x, -x(xy + \alpha_0), 1/z, -z(zw + \beta_0))$. As a boundary coordinate system of this system (x_7, y_7, z_7, w_7), we can take the coordinate system

$$(X_{10}, Y_{10}, Z_{10}, W_{10}) = (x_7, 1/y_7, z_7, w_7).$$

It is easy to see that there is an accessible singular locus along the surface

$$S_{10} = \{(X_{10}, Y_{10}, Z_{10}, W_{10})|X_{10} = Y_{10} = 0\}.$$

Now we blow up along the accessible singularity S_{10}.

Step 1: We blow up along the surface S_{10}:

$$x_{10}{}^{(1)} = \frac{X_{10}}{Y_{10}}, \quad y_{10}{}^{(1)} = Y_{10}, \quad z_{10}{}^{(1)} = Z_{10}, \quad w_{10}{}^{(1)} = W_{10}.$$

Step 2: We blow up along the surface

$$\{(x_{10}{}^{(1)}, y_{10}{}^{(1)}, z_{10}{}^{(1)}, w_{10}{}^{(1)})|x_{10}{}^{(1)} = y_{10}{}^{(1)} = 0\} :$$
$$x_{10}{}^{(2)} = \frac{x_{10}{}^{(1)}}{y_{10}{}^{(1)}}, \quad y_{10}{}^{(2)} = y_{10}{}^{(1)}, \quad z_{10}{}^{(2)} = z_{10}{}^{(1)}, \quad w_{10}{}^{(2)} = w_{10}{}^{(1)}.$$

We have resolved the accessible singular locus S_{10}. The coordinate system $(-x_{10}{}^{(2)}, y_{10}{}^{(2)}, z_{10}{}^{(2)}, w_{10}{}^{(2)})$ corresponds to the coordinate system $(x_{10}, y_{10}, z_{10}, w_{10})$ in Theorem 1.4. By a direct calculation, it is easy to see that the differential system with respect to the coordinate system $(x_{10}, y_{10}, z_{10}, w_{10})$ is a Hamiltonian system, whose Hamiltonian H_{10} satisfies the following condition:

$$dx_{10} \wedge dy_{10} + dz_{10} \wedge dw_{10} - dH_{10} \wedge dt = dx \wedge dy + dz \wedge dw - dH \wedge dt.$$

4.10. Resolution of the accessible singular locus S_{11}

By using the coordinate system (x_3, y_3, z_3, w_3), we now make a coordinate system associated with small meromorphic solution spaces. At first, we can take the coordinate system $(x_3, y_3, z_3, w_3) = (x, y+w+\frac{2(z-x)w-2\alpha_1}{x}+\frac{t}{x^2}, \frac{z-x}{x^2}, x^2w)$. As a boundary coordinate system of this system (x_3, y_3, z_3, w_3), we can take the coordinate system

$$(X_{11}, Y_{11}, Z_{11}, W_{11}) = (x_3, y_3, 1/z_3, w_3).$$

It is easy to see that there is an accessible singular locus along the surface

$$S_{11} = \{(X_{11}, Y_{11}, Z_{11}, W_{11}) | Z_{11} = W_{11} = 0\}.$$

Now we blow up along the accessible singularity S_{11}.

Step 1: We blow up along the surface S_{11}:

$$x_{11}{}^{(1)} = X_{11}, \quad y_{11}{}^{(1)} = Y_{11}, \quad z_{11}{}^{(1)} = Z_{11}, \quad w_{11}{}^{(1)} = \frac{W_{11}}{Z_{11}}.$$

Step 2: We blow up along the surface

$$\{(x_{11}{}^{(1)}, y_{11}{}^{(1)}, z_{11}{}^{(1)}, w_{11}{}^{(1)}) | z_{11}{}^{(1)} = w_{11}{}^{(1)} + \beta_0 = 0\} :$$
$$x_{11}{}^{(2)} = x_{11}{}^{(1)}, \quad y_{11}{}^{(2)} = y_{11}{}^{(1)}, \quad z_{11}{}^{(2)} = z_{11}{}^{(1)}, \quad w_{11}{}^{(2)} = \frac{w_{11}{}^{(1)} + \beta_0}{z_{11}{}^{(1)}}.$$

We have resolved the accessible singular locus S_{11}. The coordinate system $(x_{11}{}^{(2)}, y_{11}{}^{(2)}, z_{11}{}^{(2)}, -w_{11}{}^{(2)})$ corresponds to the coordinate system $(x_{11}, y_{11}, z_{11}, w_{11})$ in Theorem 1.4. By a direct calculation, it is easy to see that the differential system with respect to the coordinate system $(x_{11}, y_{11}, z_{11}, w_{11})$ is a Hamiltonian system, whose Hamiltonian H_{11} satisfies the following condition:

$$dx_{11} \wedge dy_{11} + dz_{11} \wedge dw_{11} - dH_{11} \wedge dt = dx \wedge dy + dz \wedge dw - d\left(H + \frac{1}{x}\right) \wedge dt.$$

4.11. Resolution of the accessible singular locus S_{12}

By using the coordinate system (x_3, y_3, z_3, w_3), we now make a coordinate system associated with the small meromorphic solution spaces other than $(x_{11}, y_{11}, z_{11}, w_{11})$. At first, we can take the coordinate system $(x_3, y_3, z_3, w_3) = (x, y + w + \frac{2(z-x)w-2\alpha_1}{x} + \frac{t}{x^2}, \frac{z-x}{x^2}, x^2w)$. As a boundary coordinate system of this system (x_3, y_3, z_3, w_3), we can take the coordinate system

$$(X_{12}, Y_{12}, Z_{12}, W_{12}) = (x_3, y_3, z_3, 1/w_3).$$

It is easy to see that there is an accessible singular locus along the surface

$$S_{12} = \{(X_{12}, Y_{12}, Z_{12}, W_{12}) | Z_{12} = W_{12} = 0\}.$$

Now we blow up along the accessible singularity S_{12}.

Step 1: We blow up along the surface S_{12}:

$$x_{12}{}^{(1)} = X_{12}, \quad y_{12}{}^{(1)} = Y_{12}, \quad z_{12}{}^{(1)} = \frac{Z_{12}}{W_{12}}, \quad w_{12}{}^{(1)} = W_{12}.$$

Step 2: We blow up along the surface

$$\{(x_{12}{}^{(1)}, y_{12}{}^{(1)}, z_{12}{}^{(1)}, w_{12}{}^{(1)}) | z_{12}{}^{(1)} = w_{12}{}^{(1)} = 0\} :$$

$$x_{12}{}^{(2)} = x_{12}{}^{(1)}, \quad y_{12}{}^{(2)} = y_{12}{}^{(1)}, \quad z_{12}{}^{(2)} = \frac{z_{12}{}^{(1)}}{w_{12}{}^{(1)}}, \quad w_{12}{}^{(2)} = w_{12}{}^{(1)}.$$

We have resolved the accessible singular locus S_{12}. The coordinate system $(x_{12}{}^{(2)}, y_{12}{}^{(2)}, -z_{12}{}^{(2)}, w_{12}{}^{(2)})$ corresponds to the coordinate system $(x_{12}, y_{12}, z_{12}, w_{12})$ in Theorem 1.4. By a direct calculation, it is easy to see that the differential system with respect to the coordinate system $(x_{12}, y_{12}, z_{12}, w_{12})$ is a Hamiltonian system, whose Hamiltonian H_{12} satisfies the following condition:

$$dx_{12} \wedge dy_{12} + dz_{12} \wedge dw_{12} - dH_{12} \wedge dt = dx \wedge dy + dz \wedge dw - d\left(H + \frac{1}{x}\right) \wedge dt.$$

4.12. Resolution of the remaining accessible singular points

Each procedure is the same as that given in the preceding sections 3.2 through 3.11, provided the variables and parameters $x, y, z, w, \alpha_0, \alpha_1, \beta_0$ are replaced by the transformation

$$\pi_2 : (x, y, z, w, t; \alpha_0, \alpha_1, \beta_0) \longmapsto (z, w, x, y, t; \beta_0, \alpha_1, \alpha_0).$$

Each coordinate system (x_j, y_j, z_j, w_j) for $j = 13, 14, 15$ is explicitly given as follows:

$$(x_j, y_j, z_j, w_j) = \pi_2(x_k, y_k, z_k, w_k), \quad k = 8, 11, 12, \text{ respectively.}$$

Each Hamiltonian H_j for $j = 13, 14, 15$ is explicitly given as follows:

$$H_{13} = \pi_2(H_8),\ H_{14} = \pi_2(H_{11}),\ H_{15} = \pi_2(H_{12}).$$

In Sections 3.2 through 3.12, we have resolved all the accessible singularities for the system (1.1), thus completing the proof of Theorems 1.4 and 1.5.

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