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A generalization of Matsushima's embedding theorem

By

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Abstract

Let M be a compact complex manifold and let $L \to M$ be a holomorphic line bundle whose curvature form is everywhere of signature (s_+, s_-). Under some conditions on the curvature form of L, it is shown that $K \otimes L^{\otimes m}$ admits, for some K and sufficiently large $m \in \mathbb{N}, C^\infty$ sections $t_0, \dots, t_N$ such that the ratio $(t_0 : \cdots : t_N)$ embeds M holomorphically in s_+ variables and antiholomorphically in s_- variables. The result extends the Kodaira's embedding theorem as well as a result of Matsushima for complex tori.

Introduction

In the theory of several complex variables, it often happens that certain cohomology classes on complex manifolds can be interpreted as holomorphic functions defined on other spaces. (For a general theory, see [B] for instance.) In 1979, Y. Matsushima [M] proved that, given a complex torus T equipped with a holomorphic line bundle $L \to T$ whose Chern class is represented by a parallel nondegenerate (1,1)-form of signature (s, r), one can find a family of polarized Abelian varieties over a Hermitian symmetric space $\mathcal{B}$, say $\{(T_b, H_b)\}_{b \in \mathcal{B}}$, diffeomorphisms φ_b between T and T_b, families of line bundles $L_{b,\psi} (\psi \in Pic_0 T_b)$ over T_b, and isomorphisms between $H^{0,r}(T, L)$ and $H^{0,0}(T_b, L_{b,\psi})$, in such a way that the composite of φ_b and the canonical embedding of T_b into $(H^{0,0}(T_b, L_{b,\psi}^{\otimes 3})^* \backslash \{0\})/\mathbb{C}^\times$ embeds T into $(H^{0,r}(T, L^{\otimes 3})^* \backslash \{0\})/\mathbb{C}^\times$ holomorphically in s variables and antiholomorphically in r variables.

Matsushima noted that the family $\{H^{0,0}(T_b, L_{b,\psi})\}$ contains Siegel's family of theta series which arose in the theory of automorphic functions (cf. [S]).

Recently, Matsushima's theorem was rediscovered by C. Birkenhake and H. Lange in a different context (cf. [B-L]).

The purpose of the present article is to complement the result of Matsushima by establishing the following. (For the terminology, see Section 1.)

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Theorem 0.1. *Let M be a compact and connected complex manifold equipped with two mutually transverse holomorphic foliations $\mathcal{F}_+$ and $\mathcal{F}_-$ with $\dim \mathcal{F}_+ + \dim \mathcal{F}_- = \dim M$. Assume that there exists a Hermitian holomorphic line bundle (L, h) over M whose curvature form splits into the sum of two $(1,1)$-forms Θ_+ and Θ_- such that $\sqrt{-1}\Theta_+ \geq 0$ (resp. $\sqrt{-1}\Theta_- \leq 0$), $\sqrt{-1}\Theta_+|T\mathcal{F}_+ > 0$ (resp. $\sqrt{-1}\Theta_-|T\mathcal{F}_- < 0$), and $\Theta_+|T\mathcal{F}_- = 0$ (resp. $\Theta_-|T\mathcal{F}_+ = 0$). Then there exists a positive integer m_0 such that, for any $m \in \mathbb{N}$ with $m \geq m_0$, there exist $N \in \mathbb{N}$ and C^∞ sections $t_0, \dots, t_N$ of $K_{\mathcal{F}_+} \otimes \bar{K}_{\mathcal{F}_-} \otimes L^{\otimes m}$ over M such that the ratio $(t_0 : \cdots : t_N)$ embeds M into $\mathbb{CP}^N$ holomorphically along $\mathcal{F}_+$ and antiholomorphically along $\mathcal{F}_-$. Here $K_{\mathcal{F}_\pm}$ denote canonical bundles along $\mathcal{F}_\pm$ and $\bar{K}_{\mathcal{F}_\pm}$ their conjugates.*

One may take $m_0 = 3$ for the above (T, L). The author would like to note that extension of Matsushima's theorem to general complex manifolds was a question raised by Mikio Sato after Matsushima's colloquium talk in Kyoto university (1977/autumn). Theorem 0.1 partially answers it.

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1. Preliminaries

Let M be a connected compact complex manifold of dimension n equipped with two holomorphic foliations $\mathcal{F}_+$ and $\mathcal{F}_-$ such that the tangent bundle of M is the direct sum of those of $\mathcal{F}_\pm$.

Let $T\mathcal{F}_\pm$ denote the tangent bundles of $\mathcal{F}_\pm$ and $T^{1,0}\mathcal{F}_\pm$ (resp. $T^{0,1}\mathcal{F}_\pm$) the holomorphic (resp. antiholomorphic) tangent bundles of $\mathcal{F}_\pm$. For any C^∞ complex vector bundle $E \to M$, E^* will denote the dual bundle of E and $C^\infty(E)$ the space of C^∞ sections of E. Then we put

$$
\begin{aligned}
&C^{a,b}_{p,q}(E) \\
&\quad = C(E \otimes \wedge^a (T^{1,0}\mathcal{F}_+)^* \otimes \wedge^b (T^{0,1}\mathcal{F}_+)^* \otimes \wedge^p (T^{1,0}\mathcal{F}_-)^* \otimes \wedge^q (T^{0,1}\mathcal{F}_-)^*),
\end{aligned}
$$

and $K_{\mathcal{F}_\pm} = (\det T^{1,0}\mathcal{F}_\pm)^*$ and $\bar{K}_{\mathcal{F}_\pm} = (\det T^{0,1}\mathcal{F}_\pm)^*$ for simplicity.

Since the holomorphic tangent bundle of M is the direct sum of $T^{1,0}\mathcal{F}_+$ and $T^{1,0}\mathcal{F}_-$, $C^{a,b}_{p,q}(E)$ is naturally identified with a subspace of the space $C^{a+p,b+q}(E)$ of E-valued $(a+p, b+q)$-forms on M. E will not be referred to if it is trivial.

Let $L \to M$ be a holomorphic line bundle equipped with a C^∞ fiber metric h.

Let $\bar\partial : C^{k,l}(L) \to C^{k,l+1}(L)$ be the $\bar\partial$-operator, i.e. the complex exterior derivative of type $(0,1)$, let $\partial : C^{k,l}(\bar{L}^*) \to C^{k+1,l}(\bar{L}^*)$ be its complex conjugate, and let $\partial_h : C^{k,l}(L) \to C^{k+1,l}(L)$ be defined by $\partial_h = h^{-1} \cdot \partial \cdot h$, where h is regarded as an element of $C^\infty(\mathrm{hom}(L, \bar{L}^*))$.

Let $\bar\partial_+ : C^{a,b}_{p,q}(L) \to C^{a,b+1}_{p,q}(L)$ and $\bar\partial_- : C^{a,b}_{p,q}(L) \to C^{a,b}_{p,q+1}(L)$ be the leafwise $\bar\partial$-operator along $\mathcal{F}_+$ (resp. $\mathcal{F}_-$), defined in such a way that $\partial = \bar\partial_+ + \bar\partial_-$ holds on $C^{a,b}_{p,q}(L)$. Similarly $\partial_\pm$ and $\partial_{\pm,h}$ are defined on $C^{a,b}_{p,q}(\bar{L}^*)$ and $C^{a,b}_{p,q}(L)$, respectively.

Let us assume that there exist $\omega_+ \in C^{1,1}_{0,0}$ and $\omega_- \in C^{0,0}_{1,1}$ such that ω_+

(resp. ω_-) is closed and positive on the leaves of $\mathcal{F}_+$ (resp. $\mathcal{F}_-$). We put $s_\pm = \dim \mathcal{F}_\pm$.

With respect to h and the metrics of $(T^{1,0}\mathcal{F}_+)^*$ and $(T^{0,1}\mathcal{F}_-)^*$ induced from $\omega_\pm$, the pointwise inner product is defined between any two elements $u, v \in C^{a,b}_{p,q}(L)$, which we denote by $\langle u, v\rangle_h$.

Then we put

$$(u, v)_h = \int_M \langle u, v\rangle_h \omega_+^{s_+} \wedge \omega_-^{s_-}$$

and

$$\|u\|_h = (u, u)_h^{1/2}.$$

With respect to the inner product $(\,,)_h$, we define the formal adjoint T^*_h of the operators $T = \bar\partial_+, \partial_{+,h}$, etc. by requiring $(Tu, v)_h = (u, T^*_h v)_h$ to hold for any $u, v \in C^{a,b}_{p,q}(L)$.

For simplicity we put $\vartheta_{\pm,h} = (\bar\partial_\pm)^*_h$ and $\bar\vartheta_\pm = (\partial_{\pm,h})^*_h$, since $(\partial_{\pm,h})^*_h$ does not depend on the choice of h. For any $\eta \in C^{a',b'}_{p',q'}$, we denote by $e(\eta)$ the exterior multiplication by η from the left hand side. $e(\omega_\pm)^*_h$ will be denoted by $\Lambda_\pm$. Let Θ_h denote the curvature form of h and $\Theta_{h,+}$ (resp. $\Theta_{h,-}$) its $C^{1,1}_{0,0}$ (resp. $C^{0,0}_{1,1}$) component.

Then we have the following straightforward variant of Nakano's formula.

Formula 1.1.

$$\bar\partial_\pm\vartheta_{\pm,h} + \vartheta_{\pm,h}\bar\partial_\pm = \partial_{\pm,h}\bar\vartheta_\pm + \bar\vartheta_\pm\partial_{\pm,h} + \sqrt{-1}(e(\Theta_{h,\pm})\Lambda_\pm - \Lambda_\pm e(\Theta_{h,\pm}))$$

2. L^2 estimates for the mixed complex

Let $(M, \mathcal{F}_\pm, L, h, \omega_\pm)$ be as in Section 1. To prove Theorem 0.1, we must analyze the solutions of

$$(\bar\partial_+ + \partial_{-,h})f = 0, \qquad f \in C^{s_+,0}_{0,s_-}(L). \tag{2.1}$$

For that we shall assume below that $\pm\sqrt{-1}\Theta_{h,\pm} = \omega_\pm$ and

$$\Theta_h = \Theta_{h,+} + \Theta_{h,-} \tag{2.2}$$

hold. Clearly (2.2) is equivalent to

$$(\bar\partial_+ + \partial_{-,h})^2 = 0. \tag{2.3}$$

Thus we are naturally led to study the complex

$$C^{s_+,0}_{0,s_-}(L) \xrightarrow{\bar\partial_+ + \partial_{-,h}} C^{s_+,1}_{0,s_-}(L) + C^{s_+,0}_{1,s_-}(L) \xrightarrow{\bar\partial_+ + \partial_{-,h}} C^{s_+,2}_{0,s_-}(L) + C^{s_+,1}_{1,s_-}(L) + C^{s_+,0}_{2,s_-}(L)$$

We note that

$$(\bar\partial_+ + \partial_{-,h})(\bar\partial_+ + \partial_{-,h})^*_h = \bar\partial_+\vartheta_{+,h} + \bar\partial_+\bar\vartheta_- + \partial_{-,h}\vartheta_{+,h} + \partial_{-,h}\bar\vartheta_- \tag{2.4}$$

and

$$(\partial_+ + \partial_{-,h})^*_h(\bar\partial_+ + \partial_{-,h}) = \vartheta_{+,h}\bar\partial_+ + \bar\vartheta_-\bar\partial_+ + \vartheta_{+,h}\partial_{-,h} + \bar\vartheta_-\partial_{-,h}. \tag{2.5}$$

Since $d\omega_\pm = 0$ and $\omega_\pm|\mathcal{F}_\mp = 0$, we have

$$\begin{aligned} &\bar\partial_+\bar\vartheta_- + \bar\vartheta_-\bar\partial_+ \\ &\quad = -\sqrt{-1}(\bar\partial_+[\bar\partial_-, \Lambda_-] + [\bar\partial_-, \Lambda_-]\bar\partial_+) \\ &\quad = -\sqrt{-1}(\bar\partial_+\bar\partial_-\Lambda_- + \bar\partial_-\Lambda_-\bar\partial_+ - \bar\partial_+\Lambda_-\bar\partial_- - \Lambda_-\bar\partial_-\bar\partial_+) \\ &\quad = -\sqrt{-1}((\bar\partial_+\bar\partial_- + \bar\partial_-\bar\partial_+)\Lambda_- - \Lambda_-(\bar\partial_+\bar\partial_- + \bar\partial_-\bar\partial_+)) \\ &\quad = 0. \end{aligned} \tag{2.6}$$

Similarly

$$\partial_{-,h}\vartheta_{+,h} + \vartheta_{+,h}\partial_{-,h} = 0. \tag{2.7}$$

Therefore

$$\begin{aligned} &(\bar\partial_+ + \partial_{-,h})(\bar\partial_+ + \partial_{-,h})^*_h + (\bar\partial_+ + \partial_{-,h})^*_h(\bar\partial_+ + \partial_{-,h}) \\ &\quad = \bar\partial_+\vartheta_{+,h} + \vartheta_{+,h}\bar\partial_+ + \partial_{-,h}\bar\vartheta_- + \bar\vartheta_-\partial_{-,h}. \end{aligned} \tag{2.8}$$

Interchanging the roles of $\mathcal{F}_+$ and $\mathcal{F}_-$ we have

$$\begin{aligned} &(\partial_{+,h} + \bar\partial_-)(\partial_{+,h} + \bar\partial_-)^*_h + (\partial_{+,h} + \bar\partial_-)^*_h(\partial_{+,h} + \bar\partial_-) \\ &\quad = \partial_{+,h}\bar\vartheta_+ + \bar\vartheta_+\partial_{+,h} + \bar\partial_-\vartheta_{-,h} + \vartheta_{-,h}\bar\partial_-. \end{aligned} \tag{2.9}$$

Hence, by Formula 1.1 and by integration by parts, (2.8) and (2.9) yield

$$\begin{aligned} &\|(\bar\partial_+ + \partial_{-,h})^*_h(u+v)\|^2_h + \|(\bar\partial_+ + \partial_{-,h})(u+v)\|^2_h - \|(\partial_{+,h} + \bar\partial_-)^*_h(u+v)\|^2_h \\ &\qquad - \|(\partial_{+,h} + \bar\partial_-)(u+v)\|^2_h \\ &\quad = (\sqrt{-1}[e(\Theta_{h,+}), \Lambda_+]u, u)_h - (\sqrt{-1}[e(\Theta_{h,-}), \Lambda_-]v, v)_h \\ &\quad = \|u\|^2_h + \|v\|^2_h = \|u+v\|^2_h \end{aligned} \tag{2.10}$$

for any $u \in C^{s_+,1}_{0,s_-}(L)$ and $v \in C^{s_+,0}_{1,s_-}(L)$.

From (2.10) it is easy to deduce the following existence theorem. For the standard argument of the proof, see [H].

Theorem 2.1. *In the above situation, let w be any element of $(C^{s_+,1}_{0,s_-}(L) + C^{s_+,0}_{1,s_-}(L)) \in \mathrm{Ker}(\bar\partial_+ + \partial_{-,h})$. Then there exists an element f of $C^{s_+,0}_{0,s_-}(L)$ satisfying*

$$(\bar\partial_+ + \partial_{-,h})f = w \tag{2.11}$$

and

$$\|f\|_h \le \|w\|_h. \tag{2.12}$$

3. Proof of Theorem 0.1

Let $(M, \mathcal{F}_\pm, L, h)$ be as in the assumption, and let $x \in M$ be any point. We shall show that there exist $m \in \mathbb{N}$ and $t \in C^{s_+,0}_{0,s_-}(L^{\otimes m}) \cap \mathrm{Ker}(\bar\partial_+ + \partial_{-,h^m})$ such that $t(x) \neq 0$.

For that, let $z = (z_+, z_-) = (z_{+,1}, \ldots, z_{+,s_+}, z_{-,1}, \ldots, z_{-,s_-})$ be a holomorphic local coordinate around x such that $dz_\pm|\mathcal{F}_\mp = 0$, $\Theta_{h,\pm} = \pm \sum_{i=1}^{s_\pm} dz_{\pm,i} \wedge d\bar z_{\pm,i}$ and that a neighbourhood U of x is mapped biholomorphically onto $\mathbb{D}^n$ by (z_+, z_-). Here $\mathbb{D} = \{w \in \mathbb{C} | |w| < 1\}$.

We may assume that a fiber coordinate ζ_m of $L^{\otimes m}$ over U is specified in such a way that the length of the point of $L^{\otimes m}$ represented by (z, ζ_m) is given by

$$|\zeta_m|^2 \exp m(-\|z_+\|^2 + \|z_-\|^2 + \varepsilon(z)), \tag{3.1}$$

where $\varepsilon(z)$ is independent of m and at least of the third order in z.

Then we choose $0 < \delta < 1$ so that

$$|\varepsilon(z)|^{1/3} + |\nabla\varepsilon(z)|^{1/2} + |\nabla^2\varepsilon(z)| \leq \|z\|/\delta \tag{3.2}$$

holds on $\{z | \|z\|^2 < 3\delta\}$.

Let $\varrho : \mathbb{R} \to [0,1]$ be a C^∞ function satisfying $\varrho = 1$ on $(-\infty, \delta)$ and $\varrho = 0$ on $(2\delta, \infty)$, and let $\chi(z) = \varrho(\|z\|^2)$. Then we define $t_m \in C^{s_+,0}_{0,s_-}(L^{\otimes m})$ by letting

$$t_m(z) = \chi(z)\exp(-m\|z_-\|^2) dz_+ \wedge d\bar z_- \qquad \text{on} \quad U$$

and

$$t_m = 0 \qquad \text{on} \quad M \backslash U.$$

Here we put $dz_{+\wedge} d\bar z_- = dz_{+,1} \wedge \cdots \wedge dz_{+,s_+} \wedge d\bar z_{-,1} \wedge \cdots \wedge d\bar z_{-,s_-}$.

Then we have

$$\bar\partial_+ t_m = \bar\partial_+ \chi(z) \exp(-m\|z_-\|^2) \wedge dz_+ \wedge d\bar z_- \tag{3.3}$$

and

(3.4)

$$\begin{aligned}
&\partial_{-,h^m} t_m \\
&= \exp m(\|z_+\|^2 - \|z_-\|^2 - \varepsilon(z)) \partial_-((\exp m(-\|z_+\|^2 + \|z_-\|^2 + \varepsilon(z)) \\
&\quad \times \chi(z)\exp(-m\|z_-\|^2)) dz_{+\wedge} d\bar z_- \\
&= m\partial_-\varepsilon(z) \wedge \exp(-m\|z_-\|^2)\chi(z) dz_{+\wedge} d\bar z_- + \exp(-m\|z_-\|^2)\partial_-\chi(z) \wedge dz_{+\wedge} d\bar z_-.
\end{aligned}$$

From (3.3) and (3.4) it is clear that $\|(\bar\partial_+ + \partial_{-,h^m}) t_m\|_{h^m}$ decays as $m \to \infty$ at least as fast as vol $(\{\|z\| < 1/\sqrt{m}\}) \sim m^{-n}$. Here the L^2 norm is measured with respect to h^m and $\sqrt{-1}(\Theta_+ - \Theta_-)$.

Hence, by solving the equation

$$(\bar\partial_+ + \partial_{-,h^m}) f_m = (\bar\partial_+ + \partial_{-,h^m}) t_m$$

on M with the L^2 norm estimate as in Theorem 2.1, we have $\lim_{m\to\infty} |f_m(x)|_{h^m}$

$= 0$ by the Cauchy's estimate. In fact, since $\|(\bar{\partial}_+ + \partial_{-,h^m})t_m\|_{h^m} = O(m^{-n})$ and $\Theta_{h^m} = m\Theta_h$, Therem 2.1 implies that one can choose f_m in such a way that $\sqrt{m}\|f_m\|_{h^m} = O(m^{-n})$. Hence the comparison between $|f_m(x)|_{h^m}$ and the integral of $|f_m|^2_{h^m}$ on $\{\|z\| < 1/\sqrt{m}\}$ yields the result. Therefore we my take $t_m - f_m$ as t for sufficiently large m.

Since the Cauchy's estimate is available to control the derivatives of any order for holomorphic or antiholomorphic functions, one can see similarly as above that there exists $t_1, \ldots, t_n \in C^{s_+,0}_{0,s_-}(L^{\otimes m}) \cap \mathrm{Ker}(\bar{\partial}_+ + \partial_{-,h^m})$ such that $t_* := (t_1/t, \ldots, t_n/t)$ is a C^∞ local coordinate around x. Clearly t_* is holomorphic in z_+ and antiholomorphic in z_-.

Similarly, for any two distinct points $x, y \in M$, there exist $t' \in C^{s_+,0}_{0,s_-}(L^{\otimes m}) \cap \mathrm{Ker}(\bar{\partial}_+ + \partial_{-,h^m})$ for sufficiently large m such that $t'(x) = 0$ and $t'(y) \neq 0$.

Hence M is embeddable into $\mathbb{CP}^N$ in the required way. □

Note. Let M' be the complex manifold obtained by changing the complex structure of M along $\mathcal{F}_-$ by replacing z_- by $\bar{z}_-$ around each $x \in M$. Then $\sqrt{-1}\Theta_h > 0$ on M' and $[\frac{\sqrt{-1}}{2\pi}\Theta_h] \in H^2(M', Z)$, so that there exists a positive line bundle $L' \to M'$ and a fiber metric h' of L' such that $\Theta_{h'} = \Theta_h$. At this point, however, it is totally unclear to the author whether or not one can find L' in such a way that the space

$$C^{s_+,0}_{0,s_-}(L) \cap \mathrm{Ker}\,\bar{\partial}/C^{s_+,0}_{0,s_-}(L) \cap \mathrm{Image}\,\bar{\partial}$$

is canonically isomorphic to $H^{n,0}(M', L')$, which is true if $(M, \mathcal{F}_\pm)$ is a torus with flat foliations or a product manifold such that the leaves of $\mathcal{F}_\pm$ are compact.

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